JOURNAL OF COMMUTATIVE ALGEBRA
Volume 9, Number 1, Spring 2017

ON CANONICAL MODULES OF IDEALIZATIONS

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ABSTRACT. Let $(R, \mathfrak{m})$ be a Noetherian local ring which is a quotient of a Gorenstein local ring. Let M be a finitely generated R-module. In this paper, we study the structure of the canonical module $K(R \ltimes M)$ of the idealization $R \ltimes M$ via the polynomial type introduced by Cuong **[5]**. In particular, we give a characterization for $K(R \ltimes M)$ being Cohen-Macaulay and generalized Cohen-Macaulay.

1. Introduction. Throughout this paper, $(R, \mathfrak{m})$ denotes an r-dimensional Noetherian local ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module with dimension d. The concept of principle of idealization was introduced by Nagata **[12]**. In the Cartesian product $R \times M$, we introduce the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. These operations give a structure of a commutative ring to $R \times M$. This ring is called the *idealization of* M and is denoted by $R \ltimes M$. The purpose of idealization is to put M inside the commutative ring $R \ltimes M$ so that the structure of M as an R-module is essentially the same as that of M as an ideal of $R \ltimes M$. The notion of principle of idealization plays an important role in the study of Noetherian rings and modules. Idealization is useful for reducing results concerning submodules to the ideal case, generalizing results from rings to modules and constructing examples of commutative rings with zero divisors, cf., **[1, 12, 17]**.

The notion of a canonical module of a Noetherian local ring is due to Grothendieck, who called it a module of dualizing differentials, cf., **[6]**. The term "a canonical module" was first adopted by Herzog, et al. **[7]**, in which they defined the notion of a canonical module for general local

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2010 AMS *Mathematics subject classification.* Primary 13C14, 13E05.

Keywords and phrases. Idealization, Cohen-Macaulay canonical module, generalized Cohen-Macaulay canonical module.

This paper was written while the author visited Vietnam Institute for Advanced Study in Mathematics (VIASM).

Received by the editors on May 26, 2014, and in revised form on November 4, 2014.

DOI:10.1216/JCA-2017-9-1-107

rings. We note that a local ring R has a canonical module if and only if R is a homomorphic image of a Gorenstein local ring. Schenzel [**15**] introduced the canonical module $K(M)$ of an R-module, M.

The polynomial type introduced by Cuong [**5**] plays an important role in the study of finitely generated modules, cf., [**5**]. Let $\underline{a} = (a_1, \ldots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \ldots, a_d^{n_d})$. Then the difference between length and multiplicity

$$I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \ldots, a_d^{n_d})M) - n_1 \cdots n_d e(\underline{a}; M)$$

can be considered as a function in $\underline{n}$. It is well known that M is Cohen-Macaulay, respectively, generalized Cohen-Macaulay, if and only if $I(\underline{a}(\underline{n}); M) = 0$, respectively, there exists a constant C such that $I(\underline{a}(\underline{n}); M) \leqslant C$ for all $\underline{a}$ and $\underline{n}$. In general, $I(\underline{a}(\underline{n}); M)$ is not a polynomial for $n_1, \ldots, n_d \gg 0$, but it takes nonnegative values and is bounded above by polynomials. The least degree of all polynomials bounding above this function does not depend on the choice of $\underline{a}$, cf., [**5,** Theorem 2.3]. This least degree is called the *polynomial type of* M and is denoted by $p(M)$. It should be mentioned that $p(M)$ gives much information on the structure of M. For example, if we stipulate the degree of the zero polynomial to be $-\infty$, then M is Cohen-Macaulay if and only if $p(M) = -\infty$, and M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$. We denote by $\widehat{R}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completion of R and M, respectively. In general,

$$p(M) = p(\widehat{M}) = \max_{i<d} \dim \widehat{R}/\mathrm{Ann}_{\widehat{R}} H^i_{\widehat{\mathfrak{m}}}(\widehat{M}).$$

And, if R is a quotient of a Gorenstein local ring and M is equidimensional, then $p(M) = \dim \mathrm{nCM}(M)$, cf., [**5,** Theorems 3.1, 3.3], where $\mathrm{nCM}(M)$ is the non Cohen-Macaulay locus of M.

The purpose of this paper is to study the polynomial type of the canonical module of the idealization $R \ltimes M$. In particular, we give a criterion for the canonical module $K(R \ltimes M)$ being Cohen-Macaulay, respectively, generalized Cohen-Macaulay. Techniques used in this paper are the associativity formula of multiplicity of local cohomology modules given by Brodmann and Sharp [**3**] (see also [**14**]) and the extension of idealization introduced by Yamagishi [**17**].

The main result of this paper is the next theorem.

Theorem 1.1. *The following statements are true*:

(i) *If* $\dim M = \dim R$, *then* $p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}$;
(ii) *If* $\dim M < \dim R$, *then* $p(K(R \ltimes M)) = p(K(R))$.

In Section 2, we shall outline some properties of polynomial type and idealization which will be needed later. The proof of Theorem 1.1 will be shown in Section 3 (see Theorem 3.3).

2. Preliminaries. Firstly, we recall the notion of polynomial type which was introduced by Cuong [**5**]. Let $\underline{a} = (a_1, \ldots, a_d)$ be a system of parameters of M and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers. Set $\underline{a}(\underline{n}) = (a_1^{n_1}, \ldots, a_d^{n_d})$ and

$$I(\underline{a}(\underline{n}); M) = \ell(M/(a_1^{n_1}, \ldots, a_d^{n_d})M) - n_1 \ldots n_d e(\underline{a}; M).$$

Then $I(\underline{a}(\underline{n}); M)$ can be considered as a function in $\underline{n}$. Note that this function is nonnegative and ascending, i.e., $I(\underline{a}(\underline{n}); M) \geq I(\underline{a}(\underline{m}); M)$ for $\underline{n} = (n_1, \ldots, n_d)$, $\underline{m} = (m_1, \ldots, m_d)$ with $n_i \geq m_i$, $i = 1, \ldots, d$. This function is bounded above by a polynomial in $\underline{n}$. Moreover, we have the next important property.

Lemma 2.1 ([**5,** Theorem 2.3])**.** *The least degree of all polynomials in* $\underline{n}$ *bounding above the function* $I(\underline{a}(\underline{n}); M)$ *does not depend on the choice of* $\underline{a}$.

Definition 2.2 ([**5,** Definition 2.4])**.** The numerical invariant of M given in Theorem 2.1 is called the *polynomial type* of M and is denoted by $p(M)$.

Lemma 2.3 ([**5,** Lemma 2.6])**.** *The polynomial type is preserved by* $\mathfrak{m}$*-adic completion, i.e.,* $p(M) = p(\widehat{M})$.

Next, we recall the concept of the principle of idealization introduced by Nagata [**12**]. We make the Cartesian product $R \times M$ become a commutative ring under the componentwise addition and the multiplication defined by $(a, x)(b, y) = (ab, ay + bx)$. This ring is called the *idealization* of M over R and denoted by $R \ltimes M$.

Note that the idealization $R \ltimes M$ is again a Noetherian local ring with the unique maximal ideal $\mathfrak{m} \times M$ and $\dim R \ltimes M = \dim R$. Moreover the $\mathfrak{m} \times M$-adic completion $\widehat{R \ltimes M}$ of $R \ltimes M$ is naturally isomorphic to $\widehat{R} \ltimes \widehat{M}$, cf., [**1**]. In particular, $(0, x_1)(0, x_2) = (0, 0)$ for all $x_1, x_2 \in M$, and hence, $0 \times M$ is an ideal whose square is zero. Furthermore, $R \ltimes M/0 \times M \cong R$.

There are a canonical projection $\rho : R \ltimes M \to R$ defined by $\rho((a, x)) = a$ and a canonical inclusion $\sigma : R \to R \ltimes M$ defined by $\sigma(a) = (a, 0)$. Note that ρ and σ are local homomorphisms, and we can regard any R-module (respectively, $R \ltimes M$-module) as an $R \ltimes M$-module (respectively, R-module) by ρ (respectively, σ). Moreover, the structure of R-modules induced by the composition $\rho\sigma$ coincides with the original one. Let $\epsilon : M \to R \ltimes M$ be the canonical inclusion defined by $\epsilon(x) = (0, x)$. Then we have an exact sequence of $R \ltimes M$-modules

$$0 \longrightarrow M \xrightarrow{\epsilon} R \ltimes M \xrightarrow{\rho} R \longrightarrow 0.$$

3. The proof of Theorem 1.1. Before proving the main result of this paper, we need to recall the notions of canonical module and idealization. Let R be a quotient of an n-dimensional Gorenstein local ring $(R', \mathfrak{m}')$. We denote by $K^i(M) = \operatorname{Ext}_{R'}^{n-i}(M, R')$. Then $K^i(M)$ is a finitely generated R-module. Following Schenzel [**16**], $K^i(M)$ is called the ith *deficiency module* of M for $i = 0, \ldots, d-1$, and $K(M) = K^d(M)$ is called the *canonical module* of M. By local duality, cf., [**2,** 11.2.6], we have an isomorphism

$$H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(K^i(M), E(R/\mathfrak{m})),$$

for all i, where $E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$.

Definition 3.1 ([**10, 16**]). An R-module M is called a *Cohen-Macaulay canonical*, respectively, *generalized Cohen-Macaulay canonical*, *module* if the canonical R-module $K(M)$ of M is Cohen-Macaulay, respectively, generalized Cohen-Macaulay. If R itself is a Cohen-Macaulay canonical, respectively, generalized Cohen-Macaulay canonical, module, then it is called a *Cohen-Macaulay canonical*, respectively, *generalized Cohen-Macaulay canonical ring.*

The Cohen-Macaulay canonical property is related to some important questions. For example, if M is Cohen-Macaulay canonical, then the monomial conjecture raised by Hochster [**8**] is valid for the ring $R/\operatorname{Ann}_R M$. Furthermore, if R is a domain, then R is Cohen-Macaulay canonical if and only if R possesses a birational "Macaulayfication" R_1, i.e., an extension ring $R \subseteq R_1 \subseteq Q$ (where Q is the field of fractions of R) such that R_1 is finitely generated as an R-module and R_1 is a Cohen-Macaulay ring, cf., [**16,** Theorem 1.1].

Remark 3.2.

(i) It is easy to see that $\widehat{K(M)} \cong K(\widehat{M})$ as an $\widehat{R}$-module. Therefore M is a Cohen-Macaulay canonical, respectively, generalized Cohen-Macaulay canonical, R-module if and only if $\widehat{M}$ is a Cohen-Macaulay canonical, respectively, generalized Cohen-Macaulay canonical, $\widehat{R}$-module.

(ii) Let E and F be R-modules. Yamagishi [**17**] extended the concept of the idealization as follows: given an R-linear map $\phi : M \otimes_R E \to F$, it can make the Cartesian product $E \times F$ into an $R \ltimes M$-module with respect to componentwise addition and multiplication defined by

$$(a, x)(e, f) = (ae, af + \phi(x \otimes e)).$$

We denote this $R \ltimes M$-module by $E \overset{\phi}{\ltimes} F$.

Theorem 3.3. *The following statements are true*:

(i) *If* $\dim M = \dim R$, *then* $p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}$;

(ii) *If* $\dim M < \dim R$, *then* $p(K(R \ltimes M)) = p(K(R))$.

Proof. Note that $\widehat{R} \ltimes \widehat{M}$ is isomorphic to the $\mathfrak{m} \times M$-adic completion of $R \ltimes M$. Moreover, the polynomial type is preserved by the completion, i.e., $p(K(M)) = p(K(\widehat{M}))$, $p(K(R)) = p(K(\widehat{R}))$ and $p(R \ltimes M) = p(\widehat{R} \ltimes \widehat{M}))$ (see Lemma 2.3). Therefore, without any loss of generality, we may assume that R is complete with respect to $\mathfrak{m}$-adic completion.

Let $\mathfrak{Q}$ be an ideal of $R \ltimes M$, and put $\mathfrak{q} = \rho(\mathfrak{Q})$, where $\rho : R \ltimes M \to R$ is the map defined by $\rho(a, x) = a$ for all $(a, x) \in R \ltimes M$. Note that

$\mathfrak{Q}$ is $\mathfrak{m} \times M$-primary if and only if $\mathfrak{q}$ is $\mathfrak{m}$-primary, cf., [**17,** Remark 2.1].

Firstly, we claim the following fact.

Claim 3.4. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. Then we have*

$$e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M).$$

Proof of Claim 3.4. To prove this claim, we need to recall some notions and facts on multiplicities for the Artinian module. Suppose that A is an Artinian R-module. Let $\mathfrak{a}$ be an ideal of R such that $\ell(0 :_A \mathfrak{a}) < \infty$. Then $\ell(0 :_A \mathfrak{a}^n)$ is a polynomial with rational coefficients for $n \gg 0$. Since R is complete, the degree of this polynomial is equal to $t := \dim R/\mathrm{Ann}_R A$, cf., Kirby [**9**]. Following Brodmann and Sharp [**3**], the multiplicity of A with respect to $\mathfrak{a}$, denoted by $e'(\mathfrak{a}; A)$, is defined by the formula $e'(\mathfrak{a}; A) = a_t t!$ where a_t is the leading coefficient of the polynomial $\ell(0 :_A \mathfrak{a}^n)$.

Let $D(-)$ be the Matlis dual functor. Since A is Artinian and R is complete, $D(A)$ is a finitely generated R-module. Since $\ell_R(0 :_A \mathfrak{a}) < \infty$ and $D(0 :_A \mathfrak{a}^n) \cong D(A)/\mathfrak{a}^n D(A)$, we have $\ell_R(0 :_A \mathfrak{a}^n) = \ell_R(D(A)/\mathfrak{a}^n D(A))$ for all $n \in \mathbb{N}$. It follows that $e'(\mathfrak{a}; A) = e(\mathfrak{a}; D(A))$. Now, we apply this fact for the Artinian module $H^d_{\mathfrak{m}}(M)$ and the $\mathfrak{m}$-primary ideal $\mathfrak{q}$. As R is complete, we have $K(M) \cong D(H^d_{\mathfrak{m}}(M))$. Now, we get

$$e'(\mathfrak{q}; H^d_{\mathfrak{m}}(M)) = e(\mathfrak{q}; K(M)).$$

For each integer $i \geq 0$, let

$$\mathrm{Psupp}^i_R(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}$$

be the ith pseudo-support of M defined by Brodmann and Sharp [**3**]. Then, we get by [**14,** Corollary 3.4] that

$$e'(\mathfrak{q}; H^d_{\mathfrak{m}}(M)) = \sum_{\substack{\mathfrak{p} \in \mathrm{Supp}_R(M) \\ \dim R/\mathfrak{p} = d}} \ell_{R_{\mathfrak{p}}}(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})) e(\mathfrak{q}; R/\mathfrak{p}).$$

Since R is complete, R is catenary. Therefore, we get by [**14,** Corollary

3.4] that

$$\begin{aligned}\operatorname{Psupp}_R^d(M) = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \\ \text{there exists } \mathfrak{p}' \in \operatorname{Ass}_R(M), \dim R/\mathfrak{p}' = d, \mathfrak{p}' \subseteq \mathfrak{p}\}.\end{aligned}$$

Hence,

$$\{\mathfrak{p} \in \operatorname{Psupp}_R^d(M) \mid \dim R/\mathfrak{p} = d\} = \{\mathfrak{p} \in \operatorname{Supp}_R M \mid \dim R/\mathfrak{p} = d\}.$$

So, by the associativity formula for multiplicity of M with respect to $\mathfrak{q}$, cf., [**11,** 14.7], we have

$$\begin{aligned}e'(\mathfrak{q}; H_{\mathfrak{m}}^d(M)) &= \sum_{\substack{\mathfrak{p}\in\operatorname{Supp}_R(M)\\ \dim R/\mathfrak{p}=d}} \ell_{R_\mathfrak{p}}(H^0_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}))e(\mathfrak{q}; R/\mathfrak{p}) \\ &= \sum_{\substack{\mathfrak{p}\in\operatorname{Supp}_R(M)\\ \dim R/\mathfrak{p}=d}} \ell_{R_\mathfrak{p}}(M_\mathfrak{p})e(\mathfrak{q}; R/\mathfrak{p}) \\ &= e(\mathfrak{q}; M).\end{aligned}$$

Therefore, $e(\mathfrak{q}; K(M)) = e(\mathfrak{q}; M)$, and the claim is proved. □

From now on, let $\underline{a} = (a_1, \dots, a_r)$ be a system of parameters of R. Set $\underline{u} = (u_1, \dots, u_r)$ with $u_i = (a_i, 0)$ for $i = 1, \dots, r$. It is easy to see that $\underline{u}$ is a system of parameters of $R \ltimes M$. Set $\mathfrak{q} = (a_1, \dots, a_r)R$ and $\mathfrak{Q} = \sum_{i=1}^{r} u_i(R \ltimes M) \subseteq R \ltimes M$. Then $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R and $\mathfrak{Q}$ is an $\mathfrak{m} \times M$-primary ideal of $R \ltimes M$. Moreover, $\mathfrak{Q} = \mathfrak{q} \times \mathfrak{q}M$ and $\mathfrak{q} = \rho(\mathfrak{Q})$.

Claim 3.5. *With the above notation, if $d = r$, i.e., $\dim M = \dim R$, then*

$$\begin{aligned}\ell_{R \ltimes M}(K(R \ltimes M)/\mathfrak{Q}K(R \ltimes M)) \\ = \ell_R(K(R)/\mathfrak{q}K(R)) + \ell_R(K(M)/\mathfrak{q}K(M)).\end{aligned}$$

Otherwise, we have

$$\ell_{R \ltimes M}(K(R \ltimes M)/\mathfrak{Q}K(R \ltimes M)) = \ell_R(K(R)/\mathfrak{q}K(R)).$$

Proof of Claim 3.5. By [**7,** 5.14], we have an isomorphism

$$K(R \ltimes M) \cong \mathrm{Hom}_R(R \ltimes M, K(R))$$

of $R \ltimes M$-modules. Moreover, there is an isomorphism of R-modules

$$\mathrm{Hom}_R(R \ltimes M, K(R)) \longrightarrow \mathrm{Hom}_R(M, K(R)) \oplus K(R)$$

defined by $\alpha \mapsto (\alpha\epsilon, \alpha((1,0)))$ for each $\alpha \in \mathrm{Hom}_R(R \ltimes M, K(R))$, where $\epsilon : M \to R \ltimes M$ is defined by $\epsilon(x) = (0, x)$ for all $x \in M$. Then, by Remark 3.2 (ii), we can make the R-module $\mathrm{Hom}_R(M, K(R)) \oplus K(R)$ into an $R \ltimes M$-module, which is denoted by

$$\mathrm{Hom}_R(M, K(R)) \overset{\phi}{\ltimes} K(R)$$

with respect to the R-linear map $\phi : M \otimes_R \mathrm{Hom}_R(M, K(R)) \to K(R)$ such that $\phi(x \otimes f) = f(x)$ for every $x \in M$ and $f \in \mathrm{Hom}_R(M, K(R))$. Therefore,

$$K(R \ltimes M) \cong \mathrm{Hom}_R(M, K(R)) \overset{\phi}{\ltimes} K(R)$$

as $R \ltimes M$-modules. By [**4,** 3.5.10], there is an isomorphism

$$\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) \cong \mathrm{Hom}_R(M, K(R))$$

of R-modules. Now, suppose $d = r$. Then $\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) \cong K(M)$ as R is complete, and hence, $\mathrm{Hom}_R(M, K(R)) \cong K(M)$. Therefore, we get an isomorphism $K(R \ltimes M) \cong K(M) \overset{\phi}{\ltimes} K(R)$ as $R \ltimes M$-modules. It follows that

$$\begin{aligned}
\mathfrak{Q}K(R \ltimes M) &\cong (\mathfrak{q} \times \mathfrak{q}M)\big(K(M) \overset{\phi}{\ltimes} K(R)\big) \\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \phi(\mathfrak{q}M \otimes K(M))\big) \\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \mathfrak{q}\phi(M \otimes K(M))\big) \\
&\cong \mathfrak{q}K(M) \times \big(\mathfrak{q}K(R) + \mathfrak{q}\mathrm{Im}\phi\big) \\
&\cong \mathfrak{q}K(M) \times \mathfrak{q}K(R).
\end{aligned}$$

Then we obtain that

$$\begin{aligned}
&\ell_{R \ltimes M}(K(R \ltimes M)/\mathfrak{Q}K(R \ltimes M)) \\
&\quad = \ell_{R \ltimes M}((K(M) \overset{\phi}{\ltimes} K(R))/(\mathfrak{q}K(M) \times \mathfrak{q}K(R))) \\
&\quad = \ell_R(K(M)/\mathfrak{q}K(M)) + \ell_R(K(R)/\mathfrak{q}K(R)).
\end{aligned}$$

Suppose that $d < r$. Then $\mathrm{Hom}_R(H^r_{\mathfrak{m}}(M), E_R(R/\mathfrak{m})) = 0$. Therefore, we have $\mathrm{Hom}_R(M, K(R)) = 0$. It follows that $K(R \ltimes M) \cong 0 \overset{\phi}{\ltimes} K(R)$ as $R \ltimes M$-modules, and therefore, $\mathfrak{Q}K(R \ltimes M) \cong 0 \times \mathfrak{q}K(R)$. Then, we obtain that

$$\begin{aligned}
&\ell_{R \ltimes M}\big(K(R \ltimes M)/\mathfrak{Q}K(R \ltimes M)\big) \\
&\quad = \ell_{R \ltimes M}\big(0 \overset{\phi}{\ltimes} K(R)/0 \times \mathfrak{q}K(R)\big) \\
&\quad = \ell_R\big(K(R)/\mathfrak{q}K(R)\big),
\end{aligned}$$

and Claim 3.5 is proved. □

Now, we consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\epsilon} R \ltimes M \xrightarrow{\rho} R \longrightarrow 0.$$

If $d = r$, then $e(\mathfrak{Q}; R \ltimes M) = e(\mathfrak{q}; R) + e(\mathfrak{q}; M)$, and therefore,

$$e(\mathfrak{Q}; K(R \ltimes M)) = e(\mathfrak{q}; K(R)) + e(\mathfrak{q}; K(M)),$$

by Claim 3.4. On the other hand, if $d < r$, then $e(\mathfrak{Q}; K(R \ltimes M)) = e(\mathfrak{q}; R) = e(\mathfrak{q}; K(R))$ by Claim 3.4.

Let $\underline{n} = (n_1, \ldots, n_r)$ be a set of positive integers, let $\underline{a}(\underline{n}) := (a_1^{n_1}, \ldots, a_r^{n_r})$ and $\underline{u}(\underline{n}) := (u_1^{n_1}, \ldots, u_r^{n_r}) = ((a_1^{n_1}, 0), \ldots, (a_r^{n_r}, 0))$. Set

$$\mathfrak{Q}(\underline{n}) = \sum_{i=1}^{r} u_i^{n_i}(R \ltimes M) \subseteq R \ltimes M$$

and $\mathfrak{q}(\underline{n}) = \underline{a}(\underline{n})R$.

(i) If $d = r$, then $\underline{a}(\underline{n})$ is a system of parameters of R, $K(R), M$ and $K(M)$. Moreover, $\underline{u}(\underline{n})$ is a system of parameters of $R \ltimes M$ and $K(R \ltimes M)$. Therefore, we get by Claim 3.5 and the above facts that

$$I(\mathfrak{Q}(\underline{n}); K(R \ltimes M)) = I(\mathfrak{q}(\underline{n}); K(R)) + I(\mathfrak{q}(\underline{n}); K(M)).$$

So, we obtain by Lemma 2.1

$$p(K(R \ltimes M)) = \max\{p(K(R)), p(K(M))\}.$$

(ii) Suppose $d < r$. Then, by Claim 3.5 we notice that $e(\mathfrak{Q}; K(R \ltimes M)) = e(\mathfrak{q}; K(R))$ and obtain

$$I(\mathfrak{Q}; K(R \ltimes M)) = I(\mathfrak{q}; K(R)).$$

Thus, $p(K(R \ltimes M)) = p(K(R))$ by Lemma 2.1. $\square$

Note that M is Cohen-Macaulay if and only if $p(M) = -\infty$ and M is generalized Cohen-Macaulay if and only if $p(M) \leqslant 0$. Therefore, we have the following characterization for $R \ltimes M$ being Cohen-Macaulay canonical (respectively, generalized Cohen-Macaulay).

Corollary 3.6. *The following statements are true*:

(i) *If* $\dim M = \dim R$, *then* $R \ltimes M$ *is Cohen-Macaulay canonical, respectively, generalized Cohen-Macaulay canonical, if and only if so are* R *and* M.

(ii) *If* $\dim M < \dim R$, *then* $R \ltimes M$ *is Cohen-Macaulay canonical, respectively, generalized Cohen-Macaulay canonical, if and only if so is* R.

Acknowledgments. The author would like to thank VIASM for the hospitality. The author would like to thank Professor Nguyen Tu Cuong, Professor Le Thanh Nhan and especially the referee for his/her useful suggestions.

REFERENCES

1. D.D. Anderson and W. Winders, *Idealization of a module*, J. Commutative Algebra **1** (2009), 3–56.

2. M. Brodmann and R.Y. Sharp, *Local cohomology, An algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998.

3. ———, *On the dimension and multiplicity of local cohomology modules*, Nagoya Math. J. **167** (2002), 217–233.

4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, Cambridge, 1993.

5. N.T. Cuong, *On the least degree of polynomials bounding above the differences between lengths and multiplicities of certain systems of parameters in local rings*, Nagoya Math. J. **125** (1992), 105–114.

6. A. Grothendieck, *Local cohomology*, Lect. Notes Math. **41**, Springer Verlag, Berlin, 1967.

7. J. Herzog and E. Kunz, et al., *Der kanonische modul eines Cohen-Macaulay-Rings*, Lect. Notes Math. **238**, Springer Verlag, Berlin, 1971.

8. M. Hochster, *Contraced ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973), 25–43.

9. D. Kirby, *Dimension and length of Artinian modules*, Quart. J. Math. Oxford **41** (1990), 419–429.

10. N.T.H. Loan and L.T. Nhan, *On generalized Cohen-Macaulay canonical modules*, Comm. Algebra **41** (2013), 4453–4462.

11. H. Matsumura, *Commutative rings theory*, Cambridge University Press, Cambridge, 1986.

12. M. Nagata, *Local rings*, Tracts Pure Appl. Math. **13**, 1962.

13. L.T. Nhan, *A remark on the monomial conjecture and Cohen-Macaulay canonical modules*, Proc. Amer. Math. Soc. **134** (2006), 2785–2794.

14. L.T. Nhan and T.N. An, *On the unmixedness and universal catenaricity of local rings and local cohomology modules*, J. Algebra **321** (2009), 303–311.

15. P. Schenzel, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, Lect. Notes Math. **907**, Springer-Verlag, Berlin, 1982.

16. ———, *On birational macaulayfications and Cohen-Macaulay canonical modules*, J. Algebra **275** (2004), 751–770.

17. K. Yamagishi, *Idealizations of maximal Buchsbaum modules over a Buchsbaum ring*, Math. Proc. Camb. Phil. Soc. **104** (1988), 451–478.

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