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ANTI-HOMOMORPHISMS BETWEEN MODULE LATTICES

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ABSTRACT. We examine the properties of certain mappings between the lattice $\mathcal{L}(R)$ of ideals of a commutative ring R and the lattice $\mathcal{L}({}_RM)$ of submodules of an R-module M, in particular considering when these mappings are lattice anti-homomorphisms. The mappings in question are the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ defined by setting for each ideal B of R, $\alpha(B)$ to be the submodule of M consisting of all elements m in M with $Bm = 0$ and the mapping $\beta : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ defined by $\beta(N)$ is the annihilator in R of N, for each submodule N of M.

1. Introduction. This paper is concerned with mappings, in particular anti-homomorphisms, between certain lattices. Let L and L' be lattices. As usual, given a and b in L, the least upper bound and the greatest lower bound of a and b are denoted by $a \vee b$ and $a \wedge b$, respectively. Given mappings φ, θ from a lattice L to a lattice L' we write $\varphi \leq \theta$ to mean that $\varphi(a) \leq \theta(a)$ for all $a \in L$. Clearly, $\varphi = \theta$ if and only if $\varphi \leq \theta$ and $\theta \leq \varphi$.

A mapping φ from a lattice L to a lattice L' is an *anti-homomorphism*, provided

$$\varphi(a \vee b) = \varphi(a) \ \wedge \ \varphi(b) \quad \text{and} \quad \varphi(a \wedge b) = \varphi(a) \ \vee \ \varphi(b),$$

for all $a, b \in L$. In other words, a mapping φ from a lattice L to a lattice L' is an anti-homomorphism if and only if the mapping φ is a homomorphism from L to the opposite lattice of L'. A bijective (respectively, injective, surjective) anti-homomorphism is called an *anti-isomorphism* (respectively, *anti-monomorphism, anti-epimorphism*). The first result is absolutely standard and easy to prove.

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Lemma 1.1. *The following statements are equivalent for a bijection φ from a lattice L to a lattice L'.*

(i) *φ is an anti-isomorphism.*
(ii) *$\varphi(a \vee b) = \varphi(a) \wedge \varphi(b)$ for all $a, b \in L$.*
(iii) *$\varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$.*

Moreover, in this case the inverse mapping $\varphi^{-1} : L' \to L$ is also an anti-isomorphism.

Throughout this note, all rings will be commutative with identity, and all modules will be unital. Let R be a ring, and let M be any R-module. The collection of submodules of M form a lattice which we shall denote by $\mathcal{L}({}_RM)$ with respect to the following definitions:

$$L \vee N = L + N \quad \text{and} \quad L \wedge N = L \cap N,$$

for all submodules L and N of M. Note that $\mathcal{L}({}_RM)$ is a lattice with least element the zero submodule, greatest element M and, for any given submodules L and N of M,

$$L \leq N \text{ in } \mathcal{L}({}_RM) \Longleftrightarrow L \subseteq N \text{ in } M.$$

In particular, we shall denote the lattice $\mathcal{L}({}_RR)$ of ideals of R by $\mathcal{L}(R)$. We shall be interested in mappings between $\mathcal{L}(R)$ and $\mathcal{L}({}_RM)$.

For any ideal B of the ring R we shall denote by $\mathrm{ann}_M(B)$ the set of all elements $m \in M$ such that $Bm = 0$. Note that $\mathrm{ann}_M(B)$ is a submodule of M. In addition, for any submodule N of M we denote by $\mathrm{ann}_R(N)$ the set of all elements $r \in R$ such that $rN = 0$ and note that $\mathrm{ann}_R(N)$ is an ideal of R. Let $A = \mathrm{ann}_R(M)$, the *annihilator* of M in R. By defining

$$(r + A)m = rm \quad (r \in R,\, m \in M),$$

M becomes a faithful (R/A)-module with the property that a subset X of M is an R-submodule of M if and only if X is an (R/A)-submodule of M. Thus, the lattice $\mathcal{L}({}_RM)$ is identical to the lattice $\mathcal{L}({}_{R/A}M)$. Note that

$$\mathrm{ann}_{R/A}(N) = \mathrm{ann}_R(N)/A$$

for every submodule N of M. In addition,

$$\mathrm{ann}_M((B + A)/A) = \mathrm{ann}_M(B),$$

for every ideal B of R. This implies that

$$\operatorname{ann}_M(\operatorname{ann}_R(N)) = \operatorname{ann}_M(\operatorname{ann}_{R/A}(N)),$$

for every submodule N of M.

Let L and N be any submodules of a module M over a general ring R. Then $(L :_R N)$ will denote the set of elements $r \in R$ such that $rN \subseteq L$. Thus, $(L :_R N) = \operatorname{ann}_R((N + L)/L)$ which is an ideal of the ring R. In **[17, 18]**, we investigate the mapping $\lambda : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ defined by $\lambda(B) = BM$ for every ideal B of R and the mapping $\mu : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ defined by $\mu(N) = (N :_R M)$ for every submodule N of M. It is proved that $\lambda = \lambda\mu\lambda$ and $\mu = \mu\lambda\mu$. In **[17, 18]**, we examine when the mappings λ and μ are (lattice) homomorphisms with different properties. The module M is called a λ-*module* in case the mapping λ is a homomorphism and is called a μ-*module* if μ is a homomorphism.

Now we define a mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ by

$$\alpha(B) = \operatorname{ann}_M(B),$$

for every ideal B of R and we define a mapping $\beta : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ by

$$\beta(N) = \operatorname{ann}_R(N),$$

for every submodule N of M. Let $A = \operatorname{ann}_R(M)$. We shall denote by $\overline{\alpha}$ the mapping from $\mathcal{L}(R/A)$ to $\mathcal{L}({}_RM)$ defined by $\overline{\alpha}(B/A) = \operatorname{ann}_M(B/A)$ for every ideal B of R containing A. Note that, by our above remarks, $\overline{\alpha}(B/A) = \operatorname{ann}_M(B)$ for every ideal B containing A. In addition, we denote by $\overline{\beta}$ the mapping from $\mathcal{L}({}_{R/A}M)$ to $\mathcal{L}(R/A)$ defined by $\overline{\beta}(N) = \operatorname{ann}_{R/A}(N)$ for every submodule N of M. Our above remarks show that $\overline{\beta}(N) = \operatorname{ann}_R(N)/A$ for every submodule N of M. It follows that, if $\pi : R \to R/A$ is the canonical projection, then $\alpha = \overline{\alpha}\pi$ and $\overline{\beta} = \pi\beta$.

Note the following result.

Lemma 1.2. *Let R be a ring, and let M be any R-module. Then, with the above notation,*

(a) *$\alpha(C) \leq \alpha(B)$ for all ideals B and C of R with $B \leq C$ in $\mathcal{L}(R)$.*
(b) *$\beta(N) \leq \beta(L)$ for all submodules L and N of M with $L \leq N$ in $\mathcal{L}({}_RM)$.*

(c) $\alpha\beta\alpha = \alpha$.
(d) $\beta\alpha\beta = \beta$.

Proof. (a), (b). Clear.

(c). Let B be any ideal of R. Then $B\mathrm{ann}_M(B) = 0$ so that

$$B \subseteq \mathrm{ann}_R(\mathrm{ann}_M(B)) = \beta\alpha(B).$$

It follows that $1 \leq \beta\alpha$. Similarly, $1 \leq \alpha\beta$. By (a), $\alpha\beta\alpha(B) = \alpha(\beta\alpha(B)) \leq \alpha(B)$. Thus, $\alpha\beta\alpha \leq \alpha$. On the other hand, $1 \leq \alpha\beta$ implies that $\alpha(B) \leq \alpha\beta(\alpha(B)) = \alpha\beta\alpha(B)$. This gives that $\alpha \leq \alpha\beta\alpha$, and hence, $\alpha = \alpha\beta\alpha$.

(d). Similar to (c). □

Lemma 1.2 has as an easy consequence the following result.

Lemma 1.3. *With the above notation, the following statements are equivalent*:

(i) α *is a surjection.*
(ii) $\alpha\beta = 1$.
(iii) $N = \mathrm{ann}_M(\mathrm{ann}_R(N))$ *for every submodule* N *of* M.
(iv) β *is an injection.*
(v) $\overline{\alpha}$ *is a surjection.*
(vi) $\overline{\alpha}\overline{\beta} = 1$.
(vii) $\overline{\beta}$ *is an injection.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 1.2(c).

(ii) $\Rightarrow$ (i), (iv). Clear.

(ii) $\Leftrightarrow$ (iii). Clear.

(iv) $\Rightarrow$ (ii). By Lemma 1.2(d).

(iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). We have noted above that

$$\mathrm{ann}_M(\mathrm{ann}_R(N)) = \mathrm{ann}_M(\mathrm{ann}_{R/A}(N)),$$

for every submodule N of M and thus (iii) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) all follow from the equivalence of (i)–(iv). □

The proof of the next result is similar to the proof of Lemma 1.3. For the last part, note that if M is an R-module with $A = \mathrm{ann}_R(M)$, then

$$\alpha(A) = \mathrm{ann}_M(A) = M = \alpha(0).$$

Lemma 1.4. *With the above notation, the following statements are equivalent*:

(i) *α is an injection.*
(ii) *$\beta\alpha = 1$.*
(iii) *$B = \mathrm{ann}_R(\mathrm{ann}_M(B))$ for every ideal B of R.*
(iv) *β is a surjection.*

Moreover, in this case M is faithful.

Proposition 1.5. *With the above notation, the mapping α is a bijection if and only if β is a bijection. In this case α and β are inverses of each other and are both anti-isomorphisms.*

Proof. By Lemmas 1.3 and 1.4 α is a bijection if and only if β is a bijection and, in this case, α and β are inverses of each other. Moreover, for all ideals B and C of R,

$$\begin{aligned}\alpha(B \vee C) &= \mathrm{ann}_M(B + C) = \mathrm{ann}_M(B) \cap \mathrm{ann}_M(C) \\ &= \alpha(B) \wedge \alpha(C).\end{aligned}$$

By Lemma 1.1, α is an anti-isomorphism. The same proof proves that β is also an anti-isomorphism. □

Next we recall the definition of trivial extensions. These will provide a fruitful source of examples later. Let R be a ring, and let M be an R-module. Then the *trivial extension of M by R* is the ring $M \ltimes R$ defined as follows. The set $M \ltimes R$ consists of all ordered pairs (r, m), with $r \in R$ and $m \in M$, and addition and multiplication are defined by

$$(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2),$$

and

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1),$$

for all $r_1, r_2 \in R$ and $m_1, m_2 \in M$. It is well known that $M \ltimes R$ is a commutative ring with zero $(0, 0)$ and identity $(1, 0)$. For any ideal B

of R and submodule N of the R-module M, $N \ltimes B$ will denote the set of elements (b, x) with $b \in B$ and $x \in N$. Note that $N \ltimes B$ is an ideal of $M \ltimes R$ if and only if $BM \subseteq N$.

Let R be any ring, and let M be an R-module. We shall call M an *α-module* in case the above mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_R M)$ is an anti-homomorphism. The R-module M is called a *comultiplication module* in the case where α is a surjection. Thus, M is a comultiplication α-module precisely when α is an anti-epimorphism. Note that M is a comultiplication module provided that, for each submodule N of M, there exists an ideal B of R such that $N = \operatorname{ann}_M(B)$. Comultiplication modules have been studied by a number of authors (see, for example, [**3, 4, 5, 8, 10, 14, 15**]). In Theorem 4.3, we characterize when a direct sum of modules is a comultiplication module.

We prove that if R is any ring, then every semisimple R-module is an α-module (Corollary 2.9). In the case where R is a Dedekind domain or a chain ring or the trivial extension $X \ltimes S$ of the injective envelope X of a simple S-module by a Dedekind domain S, then every R-module is an α-module (Corollaries 2.14 and 2.15 and Example 3.15). A ring R is von Neumann regular if and only if the mapping α is an anti-monomorphism where M is the direct sum $\oplus(R/P)$ where P runs through the collection of all prime ideals of R (Theorem 4.7).

2. The mapping α.

Let R be a ring, and let M be an R-module. Recall that we call M an α-module in case the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_R M)$ defined by $\alpha(B) = \operatorname{ann}_M(B)$ is a (lattice) anti-homomorphism. Note the following result.

Lemma 2.1. *The following statements are equivalent for an R-module M.*

(i) *M is an α-module.*
(ii) $\operatorname{ann}_M(B \cap C) \subseteq \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$ *for all ideals B, C of R.*
(iii) $\operatorname{ann}_M(B_1 \cap \cdots \cap B_n) = \operatorname{ann}_M(B_1) + \cdots + \operatorname{ann}_M(B_n)$ *for every positive integer n and ideals B_i $(1 \le i \le n)$ of R.*

Proof. (i) $\Rightarrow$ (ii). Let B and C be any ideals of R. Then

$$\begin{aligned}\operatorname{ann}_M(B \cap C) &= \alpha(B \wedge C) = \alpha(B) \vee \alpha(C)\\ &= \operatorname{ann}_M(B) + \operatorname{ann}_M(C).\end{aligned}$$

(ii) $\Rightarrow$ (iii). Let n be a positive integer, and let B_i $(1 \leq i \leq n)$ be any collection of ideals of R. Clearly, $\mathrm{ann}_M(B_i) \subseteq \mathrm{ann}_M(B_1 \cap \cdots \cap B_n)$ for each $1 \leq i \leq n$. Hence,

$$\mathrm{ann}_M(B_1) + \cdots + \mathrm{ann}_M(B_n) \subseteq \mathrm{ann}_M(B_1 \cap \cdots \cap B_n).$$

However, by (ii) and induction,

$$\mathrm{ann}_M(B_1 \cap \cdots \cap B_n) \subseteq \mathrm{ann}_M(B_1) + \cdots + \mathrm{ann}_M(B_n).$$

This proves (iii).

(iii) $\Rightarrow$ (i). Let B and C be any ideals of R. Then

$$\alpha(B \vee C) = \mathrm{ann}_M(B + C) = \mathrm{ann}_M(B) \cap \mathrm{ann}_M(C) = \alpha(B) \wedge \alpha(C).$$

Moreover, (iii) gives that

$$\alpha(B \wedge C) = \mathrm{ann}_M(B \cap C) = \mathrm{ann}_M(B) + \mathrm{ann}_M(C) = \alpha(B) \vee \alpha(C).$$

Thus, α is a homomorphism and M is an α-module. $\square$

Lemma 2.1 has many consequences, and we give a few of these next.

Corollary 2.2. *Let A be an ideal of a ring R. Then the cyclic R-module R/A is an α-module if and only if $(A :_R B \cap C) = (A :_R B) + (A :_R C)$ for all ideals B and C of R.*

Proof. Let G be any ideal of R. Then

$$\begin{aligned}\mathrm{ann}_{R/A}(G) &= \{r + A : G(r + A) = 0\} \\ &= \{r + A : Gr \subseteq A\} = (A :_R G)/A.\end{aligned}$$

The result follows by Lemma 2.1. $\square$

Corollary 2.3. *Let R be a ring, and let an R-module M be an α-module with $A = \mathrm{ann}_R(M)$. Then $M = \mathrm{ann}_M(B) + \mathrm{ann}_M(C)$ for all ideals B and C of R with $B \cap C \subseteq A$.*

Proof. By Lemma 2.1,

$$M = \mathrm{ann}_M(B \cap C) = \mathrm{ann}_M(B) + \mathrm{ann}_M(C). \qquad \square$$

Recall that a module M is called *hollow* in the case where $M \neq L+N$ for any proper submodules L, N.

Corollary 2.4. *Let R be a ring, and let an R-module M be a hollow α-module with $A = \mathrm{ann}_R(M)$. Let B and C be ideals of R such that $B \cap C \subseteq A$. Then $B \subseteq A$ or $C \subseteq A$.*

Proof. By Corollary 2.3, either $M = \mathrm{ann}_M(B)$ or $M = \mathrm{ann}_M(C)$. Thus, $BM = 0$ and $B \subseteq A$ or $CM = 0$ and $C \subseteq A$. □

Corollary 2.5. *Let M be a module over a ring R such that Rm is an α-module for each element $m \in M$. Then M is an α-module.*

Proof. Let B and C be ideals of R. Let $m \in \mathrm{ann}_M(B \cap C)$. By hypothesis and Lemma 2.1,

$$\begin{aligned} m \in \mathrm{ann}_{Rm}(B \cap C) &= \mathrm{ann}_{Rm}(B) + \mathrm{ann}_{Rm}(C) \\ &\subseteq \mathrm{ann}_M(B) + \mathrm{ann}_M(C). \end{aligned}$$

Thus, $\mathrm{ann}_M(B \cap C) \subseteq \mathrm{ann}_M(B) + \mathrm{ann}_M(C)$. By Lemma 2.1, M is an α-module. □

We next consider some examples of α-modules.

Proposition 2.6. *Let R be a ring, and let an R-module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then M is an α-module if and only if M_i is an α-module for all $i \in I$.*

Proof. Suppose first that M_i is an α-module for all $i \in I$. Let B and C be any ideals of R. Then

$$\begin{aligned} \mathrm{ann}_M(B \cap C) &= \oplus_{i \in I} \left[\mathrm{ann}_{M_i}(B \cap C)\right] \\ &= \oplus_{i \in I} \left[\mathrm{ann}_{M_i}(B) + \mathrm{ann}_{M_i}(C)\right] \\ &= \left[\oplus_{i \in I} \mathrm{ann}_{M_i}(B)\right] + \left[\oplus_{i \in I} \mathrm{ann}_{M_i}(C)\right] \\ &= \mathrm{ann}_M(B) + \mathrm{ann}_M(C). \end{aligned}$$

By Lemma 2.1, M is an α-module.

Conversely, suppose that M is an α-module. Let $i \in I$, and let $K = M_i$. Given any ideals B and C of R we have:

$$\begin{aligned}\operatorname{ann}_K(B \cap C) &= K \cap [\operatorname{ann}_M(B \cap C)] = K \cap [\operatorname{ann}_M(B) + \operatorname{ann}_M(C)] \\ &= [K \cap \operatorname{ann}_M(B)] + [K \cap \operatorname{ann}_M(C)] \\ &= \operatorname{ann}_K(B) + \operatorname{ann}_K(C).\end{aligned}$$

Thus, $\operatorname{ann}_K(B \cap C) = \operatorname{ann}_K(B) + \operatorname{ann}_K(C)$ for all ideals B and C of R. By Lemma 2.1, K is an α-module. It follows that M_i is an α-module for every $i \in I$. □

Let R be any ring. An R-module M is called *prime* in the case when M is non-zero and whenever $rm = 0$ for some $r \in R$, $0 \neq m \in M$ then $rM = 0$. It is well known that the module M is prime if and only if $A = \operatorname{ann}_R(M)$ is a prime ideal of R and the (R/A)-module M is torsion-free. In particular, if P is any prime ideal of R, then the R-module R/P is prime. There is an extensive literature on the topic of prime modules over commutative rings (see, for example, **[6, 9, 11, 12, 13]**).

Proposition 2.7. *Let R be any ring. Then every prime R-module is an α-module.*

Proof. Let M be a prime R-module. Let B and C be ideals of R. Let $m \in \operatorname{ann}_M(B \cap C)$. Then $(B \cap C)m = 0$, and hence, $B(Cm) = 0$. Either $Cm = 0$ or $BM = 0$, in which case $Bm = 0$. In any case, $m \in \operatorname{ann}_R(B) \cup \operatorname{ann}_R(C) \subseteq \operatorname{ann}_R(B) + \operatorname{ann}_R(C)$. By Lemma 2.1, M is an α-module. □

Corollary 2.8. *Let R be any ring. Then every direct sum of prime R-modules is an α-module.*

Proof. By Propositions 2.6 and 2.7. □

Corollary 2.9. *Let R be any ring. Then every semisimple R-module is an α-module.*

Proof. By Corollary 2.8 because every simple R-module is prime. □

Next we consider the case of domains.

Lemma 2.10. *Let R be a domain, and let M be an R-module with torsion submodule T. Then M is an α-module if and only if T is an α-module.*

Proof. Let G be any non-zero ideal of R. Clearly, $\operatorname{ann}_M(G) = \operatorname{ann}_T(G)$. Apply Lemma 2.1. □

Corollary 2.11. *Let R be a domain, and let M be an R-module whose torsion submodule is semisimple. Then M is an α-module.*

Proof. By Lemma 2.10 and Corollary 2.9. □

Next we investigate rings R with the property that every R-module is an α-module.

Theorem 2.12. *The following statements are equivalent for ideals B and C of a ring R.*

(i) $\operatorname{ann}_M(B \cap C) = \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$ *for every R-module M.*
(ii) $\operatorname{ann}_M(B \cap C) = \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$ *for every cyclic R-module M.*
(iii) $R = (B :_R C) + (C :_R B)$.

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii) $\Rightarrow$ (iii). Let M denote the cyclic R-module $R/(B \cap C)$. By hypothesis,

$$R/(B \cap C) = \operatorname{ann}_M(B \cap C) = \operatorname{ann}_M(B) + \operatorname{ann}_M(C).$$

But it is easy to check that

$$\operatorname{ann}_M(B) = \{r + (B \cap C) : Br \subseteq (B \cap C)\} = (C :_R B)/(B \cap C).$$

Similarly, $\operatorname{ann}_M(C) = (B :_R C)/(B \cap C)$. It follows that $R = (B :_R C) + (C :_R B)$.

(iii) $\Rightarrow$ (i). There exist elements $u \in (B :_R C)$ and $v \in (C :_R B)$ such that $1 = u + v$. Let M be any R-module, and let $m \in \operatorname{ann}_M(B \cap C)$. Note that $B(vm) = (vB)m \subseteq (B \cap C)m = 0$ so that $vm \in \operatorname{ann}_M(B)$. Similarly, $um \in \operatorname{ann}_M(C)$. Thus, $m = vm + um \in \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$. We have proved that $\operatorname{ann}_M(B \cap C) \subseteq \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$. Hence, $\operatorname{ann}_M(B \cap C) = \operatorname{ann}_M(B) + \operatorname{ann}_M(C)$. □

Corollary 2.13. *Let R be a ring. Then the following statements are equivalent*:

(i) *Every R-module is an α-module.*
(ii) *Every cyclic R-module is an α-module.*
(iii) $R = (B :_R C) + (C :_R B)$ *for all ideals B and C of R.*

Proof. By Lemma 2.1 and Theorem 2.12. □

Corollary 2.14. *A Noetherian domain R is Dedekind if and only if every R-module is an α-module.*

Proof. By Corollary 2.13 and [**7,** Theorem 25.2]. □

Given a ring R, we shall call an R-module M a *chain module* in case the submodules of M are linearly ordered. That is, M is a chain module if and only if $L \subseteq N$ or $N \subseteq L$ for all submodules L, N of M. The ring R is called a *chain ring* in the case where the R-module R is a chain module. A ring R is called *local* if R contains a unique maximal ideal. Clearly, chain rings are local.

Corollary 2.15. *A ring R is a chain ring if and only if R is local and every R-module is an α-module.*

Proof. Suppose first that R is a chain ring. Then R is a local ring. Let B and C be any ideals of R. Then $B \subseteq C$, and hence $(C :_R B) = R$, or else $C \subseteq B$ and in this case $(B :_R C) = R$. In any case, $R = (B :_R C) + (C :_R B)$. By Corollary 2.13 every R-module is an α-module. Conversely, suppose that R is a local ring such that every R-module is an α-module. Let G and H be ideals of R. By Corollary 2.13, $R = (G :_R H) + (H :_R G)$. But R is a local ring so that $R = (G :_R H)$, and hence $H \subseteq G$, or else $R = (H :_R G)$ and in this case $G \subseteq H$. This proves that R is a chain ring. □

A ring R need not be a chain ring nor a Dedekind domain in order that every R-module be an α-module. For example, we have the following result.

Proposition 2.16. *Let a ring $R = R_1 \oplus \cdots \oplus R_n$ be a direct sum of subrings R_i $(1 \leq i \leq n)$, for some positive integer n. Suppose that, for* *each* $1 \leq i \leq n$*, every* R_i*-module is an* α*-module. Then every* R*-module is an* α*-module.*

Proof. Let B and C be any ideals of R. Then $B = B_1 \oplus \cdots \oplus B_n$ for some ideal B_i of R_i for all $1 \leq i \leq n$. Similarly, $C = C_1 \oplus \cdots \oplus C_n$ for some ideal C_i of R_i for each $1 \leq i \leq n$. This implies that

$$(B :_R C) = (B_1 :_{R_1} C_1) \oplus \cdots \oplus (B_n :_{R_n} C_n),$$

and

$$(C :_R B) = (C_1 :_{R_1} B_1) \oplus \cdots \oplus (C_n :_{R_n} B_n).$$

By Corollary 2.13, $R_i = (B_i :_{R_i} C_i) + (C_i :_{R_i} B_i)$ for all $1 \leq i \leq n$. Thus, $(B :_R C) + (C :_R B) = R_1 \oplus \cdots \oplus R_n = R$. We have proved that $R = (B :_R C) + (C :_R B)$ for all ideals B, C of R, and hence, by Corollary 2.13, every R-module is an α-module. □

3. The mapping β**.** Let R be a ring, and let M be an R-module. Recall that β is the mapping from $\mathcal{L}({}_RM)$ to $\mathcal{L}(R)$ defined by $\beta(N) = \mathrm{ann}_R(N)$ for every submodule N of M. The module M is called a β*-module* in case β is an anti-homomorphism. The proof of the next result is very similar to that of Lemma 2.1 and so is omitted.

Lemma 3.1. *Let* R *be any ring. Then the following statements are equivalent for an* R*-module* M*.*

- (i) M *is a* β*-module.*
- (ii) $\mathrm{ann}_R(L \cap N) \subseteq \mathrm{ann}_R(L) + \mathrm{ann}_R(N)$ *for all submodules* L *and* N *of* M*.*
- (iii) $\mathrm{ann}_R(L_1 \cap \cdots \cap L_n) = \mathrm{ann}_R(L_1) + \cdots + \mathrm{ann}_R(L_n)$ *for every positive integer* n *and submodules* L_i $(1 \leq i \leq n)$ *of* M*.*

Corollary 3.2. *For any ring* R*, every submodule of a* β*-module over* R *is also a* β*-module.*

Proof. By Lemma 3.1. □

Corollary 3.3. *Let* R *be any ring. Then the* R*-module* R *is an* α*-module if and only if it is a* β*-module.*

Proof. By Lemmas 2.1 and 3.1. □

In contrast to Corollary 2.9, we note the following fact.

Corollary 3.4. *Let R be a ring, and let M be any R-module such that there exists a non-zero R-module X with the property that the R-module $X \oplus X$ embeds in M. Then the R-module M is not a β-module.*

Proof. There exist submodules L and N of M such that $L \cap N = 0$ and $L \cong N \cong X$. Thus, $\mathrm{ann}_R(L \cap N) = R \neq \mathrm{ann}_R(X) = \mathrm{ann}_R(L) + \mathrm{ann}_R(N)$. Apply Lemma 3.1. $\square$

We next look at homomorphic images and have the following result.

Theorem 3.5. *The following statements are equivalent for a module M over a general ring R.*

(i) *Every homomorphic image of M is a β-module.*
(ii) *Every submodule of M is a μ-module.*
(iii) *$R = (L :_R N) + (N :_R L)$ for all submodules L and N of M.*

Proof. (i) $\Rightarrow$ (iii). Let L and N be any submodules of M. Then $M/(L \cap N)$ is a β-module. By Lemma 3.1,

$$\begin{aligned} R &= \mathrm{ann}_R((L/(L \cap N)) \cap (N/(L \cap N))) \\ &= \mathrm{ann}_R(L/(L \cap N)) + \mathrm{ann}_R(N/(L \cap N)) \\ &= (N :_R L) + (L :_R N). \end{aligned}$$

(iii) $\Rightarrow$ (i). Let K be any proper submodule of M. Let $\overline{M}$ denote the R-module M/K. Let $\overline{L}$ and $\overline{N}$ be any submodules of $\overline{M}$. Then there exist submodules L and N of M, each containing K, such that $\overline{L} = L/K$ and $\overline{N} = N/K$. Note that

$$\mathrm{ann}_R(\overline{L} \cap \overline{N}) = \mathrm{ann}_R((L \cap N)/K) = (K :_R L \cap N).$$

By hypothesis, $R = (N :_R L) + (L :_R N)$, and hence

$$\begin{aligned} (K :_R L \cap N) &= (K :_R L \cap N)(N :_R L) \\ &\quad + (K :_R L \cap N)(L :_R N) \\ &\subseteq (K :_R L) + (K :_R N) \\ &= \mathrm{ann}_R(\overline{L}) + \mathrm{ann}_R(\overline{N}). \end{aligned}$$

By Lemma 3.1, the R-module $\overline{M}$ is a β-module.

(ii) $\Leftrightarrow$ (iii). By [**17,** Lemma 3.14]. $\square$

Corollary 3.6. *Let R be any ring. If an R-module M is a chain module, then every homomorphic image of M is a β-module. Moreover, the converse holds if R is a local ring.*

Proof. Suppose that M is a chain module. Let L and N be submodules of M. Then $L \subseteq N$ or $N \subseteq L$. Without loss of generality, $L \subseteq N$. Then $L \cap N = L$, and we have the following:

$$\mathrm{ann}_R(L \cap N) = \mathrm{ann}_R(L) = \mathrm{ann}_R(L) + \mathrm{ann}_R(N),$$

because $\mathrm{ann}_R(N) \subseteq \mathrm{ann}_R(L)$. By Proposition 3.5, every homomorphic image of M is a β-module.

Conversely, suppose that R is a local ring and every homomorphic image of M is a β-module. Let L and N be any submodules of M. By Proposition 3.5, $R = (L :_R N) + (N :_R L)$. Because R is a local ring, either $R = (L :_R N)$ and $N \subseteq L$ or $R = (N :_R L)$ and $L \subseteq N$. It follows that M is a chain module. $\square$

We now look at direct sums of modules. Note the following result.

Lemma 3.7. *Let B_1, B_2, C_1 and C_2 be ideals of a ring R such that*

$$R = B_1 + C_2 = B_2 + C_1.$$

Then

$$(B_1 + C_1) \cap (B_2 + C_2) = (B_1 \cap B_2) + (C_1 \cap C_2).$$

Proof. It is clear that $(B_1 \cap B_2) + (C_1 \cap C_2) \subseteq (B_1 + C_1) \cap (B_2 + C_2)$. Next, note that $B_1 + C_1 = (B_1 + C_1)(B_2 + C_1) \subseteq B_1B_2 + C_1$, and hence, $B_1 + C_1 = B_1B_2 + C_1$. Similarly, $B_2 + C_2 = B_2 + C_1C_2$. Therefore, we have by the modular law,

$$\begin{aligned}(B_1 + C_1) \cap (B_2 + C_2) &= (B_1B_2 + C_1) \cap (B_2 + C_1C_2) \\ &= [(B_1B_2 + C_1) \cap B_2] + C_1C_2 \\ &= B_1B_2 + (B_2 \cap C_1) + C_1C_2.\end{aligned}$$

However, $R = B_1 + C_2 = B_2 + C_1$ implies that

$$B_2 \cap C_1 = B_2C_1 = B_2C_1(B_1 + C_2) \subseteq B_1B_2 + C_1C_2.$$

It follows that $(B_1 + C_1) \cap (B_2 + C_2) \subseteq B_1B_2 + C_1C_2 \subseteq (B_1 \cap B_2) + (C_1 \cap C_2)$. $\square$

Theorem 3.8. *Let R be any ring, and let an R-module $M = M_1 \oplus \cdots \oplus M_k$ be the direct sum of submodules M_i $(1 \leq i \leq k)$ for some positive integer k. Then M is a β-module if and only if M_i is a β-module for each $1 \leq i \leq k$ and $R = \mathrm{ann}_R(M_i) + \mathrm{ann}_R(M_j)$ for all integers $1 \leq i < j \leq k$.*

Proof. Suppose first that M is a β-module. By Corollary 3.2, the submodule M_i is a β-module for all $1 \leq i \leq k$. Moreover, for all $1 \leq i < j \leq k$, Lemma 3.1 gives that $R = \mathrm{ann}_R(M_i \cap M_j) = \mathrm{ann}_R(M_i) + \mathrm{ann}_R(M_j)$.

Conversely, suppose the submodules M_i $(1 \leq i \leq k)$ satisfy the stated conditions. By induction on k, it is sufficient to prove the result when $k = 2$. Let L and N be any submodules of M. Let $x \in L$. Then $x = x_1 + x_2$ for some $x_1 \in M_1$ and $x_2 \in M_2$. There exist elements a_1 and a_2 in R such that $1 = a_1 + a_2$, $a_1M_1 = 0$ and $a_2M_2 = 0$. It follows that $x_1 = (1 - a_1)x_1 = a_2x_1 = a_2x \in L \cap M_1$. Similarly, $x_2 \in L \cap M_2$, so that $x \in (L \cap M_1) + (L \cap M_2)$. If $L_1 = L \cap M_1$ and $L_2 = L \cap M_2$, then L_1 is a submodule of M_1, L_2 is a submodule of M_2 and $L = L_1 \oplus L_2$. Similarly, $N = N_1 \oplus N_2$ for some submodules N_1 of M_1 and N_2 of M_2. Let $B_i = \mathrm{ann}_R(L_i)$ $(i = 1, 2)$, and let $C_i = \mathrm{ann}_R(N_i)$ $(i = 1, 2)$. Note that

$$R = \mathrm{ann}_R(M_1) + \mathrm{ann}_R(M_2) \subseteq \mathrm{ann}_R(L_1) + \mathrm{ann}_R(N_2) = B_1 + C_2.$$

Thus, $R = B_1 + C_2$. Similarly, $R = B_2 + C_1$. Next, $L \cap N = (L_1 \cap N_1) \oplus (L_2 \cap N_2)$, and hence, we have by hypothesis and Lemma 3.1,

$$\begin{aligned} \mathrm{ann}_R(L \cap N) &= \mathrm{ann}_R(L_1 \cap N_1) \cap \mathrm{ann}_R(L_2 \cap N_2) \\ &= [\mathrm{ann}_R(L_1) + \mathrm{ann}_R(N_1)] \cap [\mathrm{ann}_R(L_2) + \mathrm{ann}_R(N_2)] \\ &= (B_1 + C_1) \cap (B_2 + C_2). \end{aligned}$$

But Lemma 3.7 now gives

$$(B_1 + C_1) \cap (B_2 + C_2) \subseteq (B_1 \cap B_2) + (C_1 \cap C_2) = \mathrm{ann}_R(L) + \mathrm{ann}_R(N).$$

We have proved that $\operatorname{ann}_R(L \cap N) \subseteq \operatorname{ann}_R(L) + \operatorname{ann}_R(N)$ for all submodules L and N of M. By Lemma 3.1, the R-module M is a β-module. $\square$

Recall that, for any ring R, an R-module U is called *uniform* in the case where U is non-zero and $X \cap Y \neq 0$ for all non-zero submodules X and Y of U. Recall further that a submodule L of an R-module M is *essential* provided $L \cap N \neq 0$ for every non-zero submodule N of M. We shall apply Theorem 3.8 to modules over Dedekind domains. First we note a simple fact.

Lemma 3.9. *Let R be a ring, and let M be an R-module such that M is a β-module. Then every submodule L of M with* $\operatorname{ann}_R(L) = 0$ *is essential in M.*

Proof. Let N be any submodule of M with $L \cap N = 0$. By Lemma 3.1, $R = \operatorname{ann}_R(L \cap N) = \operatorname{ann}_R(L) + \operatorname{ann}_R(N) = \operatorname{ann}_R(N)$. It follows that $N = 0$. Hence, L is an essential submodule of M. $\square$

Proposition 3.10. *Let R be any domain, and let M be a non-zero R-module. Then M is a β-module which is not a torsion R-module if and only if M is a torsion-free uniform module.*

Proof. Suppose that M is a β-module which is not a torsion module. Let m be any element of M which is not a torsion element. Then $Rm \cong R$ so that Rm is a uniform module. Moreover, $\operatorname{ann}_R(Rm) = 0$ implies that Rm is essential in M by Lemma 3.9. Thus, M is a uniform module and is torsion-free. Conversely, suppose that M is a torsion-free uniform module. Then M is definitely not a torsion module. Clearly, $\beta(K) = 0$ for every non-zero submodule K of M, and it easily follows that β is a homomorphism because M is uniform. Thus, M is a β-module. $\square$

The next result characterizes β-modules over a Dedekind domain. Recall that Corollary 2.14 shows that every module over a Dedekind domain is an α-module. If R is any ring, B an ideal of R and M any R-module, then we denote by $\mathcal{A}(M, B)$ the set of all elements m in M such that $B^n m = 0$ for some positive integer n. Note that $\mathcal{A}(M, B)$ is a submodule of M for every ideal B of R.

Theorem 3.11. *Let R be a Dedekind domain which is not a field, and let F denote the field of fractions of R. Then an R-module M is a β-module if and only if either*

(a) *M embeds in the R-module F, or*
(b) *$M \cong E(U)$, the injective envelope of a simple R-module U, or*
(c) *$M \cong (R/P_1^{k_1}) \oplus \cdots \oplus (R/P_n^{k_n})$ for some positive integers n, k_i $(1 \leq i \leq n)$ and distinct maximal ideals P_i $(1 \leq i \leq n)$ of R.*

Proof. Suppose first that M is a β-module. If M is not torsion, then M is a torsion-free uniform R-module (Proposition 3.10), and hence, M embeds in ${}_RF$. Suppose that M is a torsion module. Then M has essential socle, and hence $\operatorname{ann}_R(P) \neq 0$ for some maximal ideal P of R. Let P_i $(i \in I)$ denote the collection of all maximal ideals P of R such that $\operatorname{ann}_R(P) \neq 0$. Let $V_i = \mathcal{A}(M, P_i)$ $(i \in I)$. Note that, in this case, $M = \oplus_{i \in I} V_i$. By Corollary 3.4, the submodule V_i has simple socle U_i (say) for each $i \in I$. Suppose that $\operatorname{ann}_R(V_j) = 0$ for some $j \in I$. By Lemma 3.9, V_j is an essential submodule of M, and hence, $M = V_j$ and M embeds in $\mathrm{E}(U_j)$. But $\operatorname{ann}_R(M) = 0$ now gives that $M \cong \mathrm{E}(U_j)$.

This leaves the case that $\operatorname{ann}_R(V_i) \neq 0$ for all $i \in I$. Suppose that the index set I is infinite. Let $j \in I$. Then the set $I \setminus \{j\}$ is infinite. Let $W = \oplus_{i \in I \setminus \{j\}} V_i$, let S denote the socle of W, and let $A = \operatorname{ann}_R(S)$. Then $A = \cap_{i \in I \setminus \{j\}} P_i$. If $A \neq 0$, then the ring R/A is Artinian, and hence, R/A has only a finite number of maximal ideals. But P_i/A is a maximal ideal of the ring R/A for each $i \in I \setminus \{j\}$, a contradiction. Thus, $A = 0$, and hence, $V_j = 0$ by Lemma 3.9, a contradiction. It follows that the set I is finite. Without loss of generality, we can suppose that $I = \{1, 2, \ldots, n\}$ for some positive integer n.

Let $1 \leq i \leq n$. Note that V_i embeds in $\mathrm{E}(U_i)$. But $\operatorname{ann}_R(V_i) \neq 0$ gives that $V_i \cong R/P_i^{k_i}$ for some positive integer k_i. Thus, $M \cong (R/P_1^{k_1}) \oplus \cdots \oplus (R/P_n^{k_n})$, as required.

Conversely, suppose that M satisfies (a), (b) or (c). If (a) holds, then M is a β-module by Proposition 3.10. If (b) holds, then it is well known that M is a chain module, and hence, M is a β-module by Corollary 3.6. If (c) holds, then the R-module $R/P_i^{k_i}$ is a chain module for each $1 \leq i \leq n$ and hence M is a β-module by Corollary 3.6 and Theorem 3.8. Thus, in any case M is a β-module. $\square$

Corollary 3.12. *Let R be a Dedekind domain, and let M be an R-module such that M is not torsion-free and the mapping $\beta : \mathcal{L}({}_RM) \to \mathcal{L}(R)$ is a homomorphism. Then, the mapping β is a monomorphism.*

Proof. By Theorem 3.11 and [**1,** Theorem 3.9], the above mapping α is a surjection, and by Lemma 1.3, the mapping β is an injection. $\square$

Now we return to consider rings with the property that every module is an α-module. Recall that, if a ring R is a domain, then an R-module M is called *divisible* in the case where $M = cM = \{cm : m \in M\}$ for every non-zero element c of R. Every injective R-module is divisible (see, for example, [**16,** Proposition 2.6]) and clearly every homomorphic image of a divisible module is divisible. We look again at trivial extensions. Note the following result.

Lemma 3.13. *Let S be a domain, let X be a divisible S-module, and let R be the trivial extension $X \ltimes S$. Let B be any ideal of R. Then $B = X \ltimes I$ for some ideal I of S or $B = Y \ltimes 0$ for some submodule Y of X.*

Proof. Suppose that $B \subseteq X \ltimes 0$. Then, it is easy to see that $B = Y \ltimes 0$ for some submodule Y of X. Now suppose that $B \nsubseteq X \ltimes 0$. Then B contains an element (s, x) for some $0 \neq s \in S$, $x \in X$. Next $X \ltimes 0 = (X \ltimes 0)(s, x) \subseteq B$. It is easy to prove that, in this case, there exists an ideal I of S such that $B = X \ltimes I$. $\square$

Theorem 3.14. *Let S be a domain, let X be a divisible S-module, and let $R = X \ltimes S$ be the trivial extension of X by S. Then every R-module is an α-module if and only if:*

(a) *every S-module is an α-module, and*
(b) *every homomorphic image of X is a β-module.*

Proof. Suppose first that R has the property that every R-module is an α-module. Let I and J be any ideals of S. Let $B = X \ltimes I$ and $C = X \ltimes J$. Then B and C are ideals of R. By Corollary 2.13, $R = (B :_R C) + (C :_R B)$. It is easy to check that $B :_R C) = X \ltimes (I :_S J)$ and $(C :_R B) = X \ltimes (J :_S I)$. It follows that $S = (I :_S J) + (J :_S I)$.

We have proved that $S = (I :_S J) + (J :_S I)$ for all ideals I, J of S. By Corollary 2.13, every S-module is an α-module. This proves (a).

Now let Y and Z be any submodules of the S-module X. Let $G = Y \ltimes 0$ and $H = Z \ltimes 0$. Then G and H are ideals of R. Note that $(G :_R H) = X \ltimes (Y :_S Z)$ and $(H :_R G) = X \ltimes (Z :_S Y)$. By Corollary 2.13, $R = (G :_R H) + (H :_R G) = X \ltimes [(Y :_S Z) + (Z :_S Y)]$, and hence, $S = (Y :_S Z) + (Z :_S Y)$. We have proved that $S = (Y :_S Z) + (Z :_S Y)$ for all submodules Y and Z of X. By Theorem 3.5, every homomorphic image of the S-module X is a β-module. This proves (b).

Conversely, suppose that (a) and (b) hold. Let E and F be any ideals of R. If $E \subseteq F$ or $F \subseteq E$, then $R = (E :_R F) + (F :_R E)$. Now suppose that $E \nsubseteq F$ and $F \nsubseteq E$. By Lemma 3.13, either $E+F \subseteq X \ltimes 0$ or $X \ltimes 0 \subseteq E \cap F$. In the first case, $E = U \ltimes 0$ and $F = V \ltimes 0$ for some submodules U, V of the S-module X. Then

$$\begin{aligned}(E :_R F) + (F :_R E) &= [X \ltimes (U :_S V)] + [X \ltimes (V :_S U)]\\ &= X \ltimes S = R,\end{aligned}$$

because $S = (U :_S V) + (V :_S U)$ by (b) and Theorem 3.5. Now suppose that $X \ltimes 0 \subseteq E \cap F$. Then $E = X \ltimes E'$ and $F = X \ltimes F'$ for some ideals E', F' of S. In this case,

$$\begin{aligned}(E :_R F) + (F :_R E) &= [X \ltimes (E' :_S F')] + [X \ltimes (F' :_S E')]\\ &= X \ltimes S = R,\end{aligned}$$

because $S = (E' :_S F') + (F' :_S E')$ by (a) and Corollary 2.13. In any case, $R = (E :_R F) + (F :_R E)$, and this holds for all ideals E and F of R. By Corollary 2.13, every R-module is an α-module. $\square$

Note that, if S is a Dedekind domain, then it is well known that the injective envelope of any simple S-module is a chain module.

Example 3.15. Let S be any Dedekind domain, let U be a simple S-module, and let X denote the injective envelope of U. Let R denote the ring $X \ltimes S$. Then every R-module is an α-module.

Proof. By Corollary 2.14, every S-module is an α-module. The S-module X is a chain module, and hence, every homomorphic image of X is a β-module (Corollary 3.6). Finally, the S-module X is a divisible module, and hence, by Theorem 3.14, every R-module is an α-module. □

Finally, in this section, we briefly consider semisimple modules.

Proposition 3.16. *The following statements are equivalent for a semisimple module M over a ring R.*

(i) *M is a β-module.*
(ii) *M is a μ-module.*
(iii) $R = \mathrm{ann}_R(L) + \mathrm{ann}_R(N)$ *for all submodules L and N of M with $L \cap N = 0$.*

Proof. (i) ⇒ (iii). By Lemma 3.1.

(iii) ⇒ (i). Suppose that (iii) holds. There exist an index set I and simple submodules U_i $(i \in I)$ of M such that $M = \oplus_{i\in I}\, U_i$. Let $P_i = \mathrm{ann}_R(U_i)$ for each $i \in I$. Now let K be any non-zero submodule of M. By the argument used in the sufficiency part of the proof of Theorem 3.8 we obtain that

$$K = \oplus_{i\in I}\,(K \cap U_i) = \oplus_{i\in I'}\, U_i$$

for some non-empty subset I' of I.

Now let L and N be any non-zero submodules of M. There exists a non-empty subset I_1 of I such that $L = \oplus_{i\in I_1}\, U_i$, and there exists a non-empty subset I_2 of I such that $N = \oplus_{i\in I_2}\, U_i$. Clearly, $L \cap N = \oplus_{i\in I_1\cap I_2}\, U_i$, and hence, if $B = \mathrm{ann}_R(L \cap N)$, then $B = \cap_{i\in I_1\cap I_2}\, P_i$. It is also clear that

$$\mathrm{ann}_R(L) = \cap_{i\in I_1}\, P_i = B \cap C,$$

where $C = \cap_{i\in I_1\setminus I_2}\, P_i = \mathrm{ann}_R(\oplus_{i\in I_1\setminus I_2}\, U_i)$. Similarly, $\mathrm{ann}_R(N) = B \cap D$ where $D = \mathrm{ann}_R(\oplus_{i\in I_2\setminus I_1}\, U_i)$. By hypothesis $R = C + D$, and hence,

$$\begin{aligned}\mathrm{ann}_R(L \cap N) = B = B(C + D) &= BC + BD \subseteq (B \cap C) + (B \cap D)\\ &= \mathrm{ann}_R(L) + \mathrm{ann}_R(N).\end{aligned}$$

By Lemma 3.1, M is a β-module.

(ii) ⇔ (iii). By [**17,** Proposition 3.16]. □

An example of a semisimple β-module is given in [**17,** Example 3.17]. On the other hand, if I is an infinite collection of primes in $\mathbb{Z}$, the ring of rational integers, then the $\mathbb{Z}$-module $\oplus_{p\in I}(\mathbb{Z}/\mathbb{Z}p)$ is not a β-module by Proposition 3.16. Note further that, if R is a domain, then, for every non-zero ideal B of R, the R-module B is a β-module by Lemma 3.1, but if R is not Prüfer there exist finitely generated ideals of R which are not μ-modules over R by [**17,** Theorem 3.13]. On the other hand, for any ring R, the R-module R is a μ-module, but if S is a ring and M any non-zero S-module, then the ring $R = (M \oplus M) \ltimes S$ has the property that the R-module R is not a β-module by Corollary 3.4. Thus, although the classes of β-modules and μ-modules coincide for semisimple modules, in general, there is no relationship between them.

4. Anti-monomorphisms and anti-epimorphisms. Let R be a ring and M an R-module. The mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ defined by $\alpha(B) = \mathrm{ann}_M(B)$, for every ideal B of R, can be an anti-homomorphism (that is, M is an α-module) without it being either an anti-monomorphism or an anti-epimorphism. For example, let R be a domain which is not a field, and let M be a non-zero torsion-free R-module. By Corollary 2.11, M is an α-module. For any non-zero ideal B of R, $\alpha(B) = \mathrm{ann}_M(B) = 0$ and $\alpha(0) = \mathrm{ann}_M(0) = M$. Let a be a non-zero element of R which is not a unit. Then $Ra \neq Ra^2$, but $\alpha(Ra) = \alpha(Ra^2)$. Thus, α is not an injection. Note next that M is not a simple R-module. Let K be any proper non-zero submodule of M. Then $K \neq \alpha(C)$ for any ideal C of R. Thus, α is not a surjection.

Next let R be any ring which is not a field, and let U be any simple R-module with $P = \mathrm{ann}_R(U)$. In this case, for any ideal B of R, $\alpha(B) = \mathrm{ann}_U(B) = U$ if $B \subseteq P$ and $\alpha(B) = 0$ otherwise. We have already seen that U is an α-module (Corollary 2.9). However, $\alpha(0) = \alpha(P) = U$ shows that α is not an injection. But α is clearly a surjection. Thus, α is an anti-epimorphism but not an anti-monomorphism. Later, we shall give an example to show that the mapping α can be an anti-monomorphism but not an anti-epimorphism.

Recall that a module M is a comultiplication module for a ring R when the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is a surjection (but not necessarily an anti-epimorphism). We next consider when the direct sum of modules is a comultiplication module. First, we prove two lemmas:

Lemma 4.1. *Let R be a ring, and let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. Suppose that M is a comultiplication module. Then* $\mathrm{ann}_{M_2}(\mathrm{ann}_R(M_1)) = 0$.

Proof. Let $B = \mathrm{ann}_R(M_1)$. By [**1,** Theorem 1.5], we have:

$$M_1 = \mathrm{ann}_M(B) = \mathrm{ann}_{M_1}(B) \oplus \mathrm{ann}_{M_2}(B),$$

and hence $\mathrm{ann}_{M_2}(B) = 0$, as required. $\square$

Lemma 4.2. *Let R be a ring, and let an R-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2 such that* $\mathrm{ann}_{M_1}(\mathrm{ann}_R(M_2)) = 0$ *and M_1 is a comultiplication module. Let a submodule $N = N_1 \oplus N_2$ of M be a direct sum of submodules N_1 of M_1 and N_2 of M_2. Let* $B_i = \mathrm{ann}_R(N_i)$ $(i = 1, 2)$. *Then* $\pi_1(\mathrm{ann}_M(B_1 \cap B_2)) \subseteq N_1$, *where $\pi_1 : M \to M_1$ is the canonical projection.*

Proof. Suppose that $m \in M$ satisfies $(B_1 \cap B_2)m = 0$. There exist elements $m_i \in M_i$ $(i = 1, 2)$ such that $m = m_1 + m_2$. Clearly $(B_1 \cap B_2)m_1 = 0$ and, in this case, $B_2(B_1 m_1) = 0$. But $N_2 \subseteq M_2$ implies that $\mathrm{ann}_R(M_2) \subseteq \mathrm{ann}_R(N_2) = B_2$, which in turn implies that

$$B_1 m_1 \subseteq \mathrm{ann}_{M_1}(B_2) \subseteq \mathrm{ann}_{M_1}(\mathrm{ann}_R(M_2)) = 0.$$

Thus, $m_1 \in \mathrm{ann}_{M_1}(B_1) = N_1$ by [**1,** Theorem 1.5]. $\square$

The following result generalizes [**1,** Theorem 3.1].

Theorem 4.3. *Let R be a ring, and let the R-module $M = \oplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then the R-module M is a comultiplication module if and only if*

(a) *the R-module M_j is a comultiplication module for all $j \in I$,*
(b) *$N = \oplus_{i \in I}(N \cap M_i)$ for every submodule N of M, and*
(c) $\mathrm{ann}_{M_j}(\mathrm{ann}_R(\oplus_{i \neq j} M_i)) = 0$ *for all $j \in I$.*

Proof. Suppose first that the R-module M is a comultiplication module. Then (a) and (b) follow by [**1,** Lemma 2.1] and (c) by Lemma 4.1.

Conversely, suppose that (a), (b) and (c) hold. Let N be any submodule of M, and let $N_i = N \cap M_i$ $(i \in I)$. By (b), $N =$

$\oplus_{i\in I} N_i$. Let $B_i = \mathrm{ann}_R(N_i)$ $(i \in I)$, and let $B = \cap_{i\in I} B_i$. Then $BN = \oplus_{i\in I} BN_i = 0$, so that $N \subseteq \mathrm{ann}_M(B)$. Let $m \in \mathrm{ann}_M(B)$. Let $j \in I$, and let $\pi_j : M \to M_j$ denote the canonical projection. If $C = \cap_{i\neq j} B_i$, then $B = B_j \cap C$ and $C = \mathrm{ann}_R(\oplus_{i\neq j} N_i)$. By (a), M_j is a comultiplication module and, by (c), $\mathrm{ann}_{M_j}(\oplus_{i\neq j} M_i) = 0$. Therefore, we can apply Lemma 4.2 to obtain $\pi_j(m) \in N_j$. It follows that $m \in \oplus_{i\in I} N_i = N$. Hence, $\mathrm{ann}_M(B) \subseteq N$, and we have proved that $N = \mathrm{ann}_M(B)$. It follows that the R-module M is a comultiplication module. □

Corollary 4.4. *Let R be a ring, and let an R-module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ such that $R = \mathrm{ann}_R(M_j) + \mathrm{ann}_R(\oplus_{i\neq j} M_i)$ for all $j \in I$. Then the R-module M is a comultiplication module if and only if the R-module M_i is a comultiplication module for all $i \in I$.*

Proof. It is not difficult to show that the module M satisfies (b) and (c) in Theorem 4.3, and, therefore, the result follows immediately from Theorem 4.3. □

Given a ring R, an R-module M is a comultiplication α-module if and only if the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an anti-epimorphism. Combining Proposition 2.6 and Theorem 4.3 we have the next result without further proof.

Corollary 4.5. *Let R be a ring, and let the R-module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules M_i $(i \in I)$. Then the R-module M is a comultiplication α-module if and only if:*

(a) *the R-module M_j is a comultiplication α-module for all $j \in I$,*
(b) *$N = \oplus_{i\in I} (N \cap M_i)$ for every submodule N of M, and*
(c) *$\mathrm{ann}_{M_j}(\mathrm{ann}_R(\oplus_{i\neq j} M_i)) = 0$ for all $j \in I$.*

Proposition 4.6. *Let R be a ring, and let an R-module $M = \oplus_{i\in I} U_i$ be a direct sum of simple submodules U_i $(i \in I)$ for some non-empty index set I. Let $P_i = \mathrm{ann}_R(U_i)$ $(i \in I)$. Then M is a comultiplication α-module if and only if $R = P_j + (\cap_{i\neq j} P_i)$ for all $j \in I$.*

Proof. By Corollaries 2.9 and 4.4. □

Now we look at a situation when the mapping α is an anti-monomorphism. Let R be any ring, and let B be any ideal of R. Then $\mathcal{V}(B)$ will denote the collection of all prime ideals P of R with $B \subseteq P$. Note that $\mathcal{V}(R)$ is the empty set and $\mathcal{V}(0)$ is the collection of all prime ideals of R. This brings us to the following result.

Theorem 4.7. *Let R be a ring, and let M denote the R-module $\oplus_{P\in\mathcal{V}(0)}(R/P)$. Then the following statements are equivalent*:

(i) *R is von Neumann regular.*
(ii) *The mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an anti-monomorphism.*
(iii) *The mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an injection.*

Proof. (i) $\Rightarrow$ (ii). Because R is a von Neumann regular ring, every prime ideal of R is maximal. Next note that M is an α-module by Corollary 2.9. Let B be any ideal of R. Then $\alpha(B) = \mathrm{ann}_M(B) = \oplus_{P\in\mathcal{V}(B)}(R/P)$. If C is any ideal of R with $\alpha(C) = \alpha(B)$, then $\mathcal{V}(B) = \mathcal{V}(C)$. But this implies

$$B = \cap_{P\in\mathcal{V}(B)} P = \cap_{P\in\mathcal{V}(C)} P = C,$$

by [**2,** Example 18, number 23]. Thus, $\alpha(B) = \alpha(C)$ implies that $B = C$. It follows that α is an anti-monomorphism.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Suppose that the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an injection. Let $a \in R$. Then $\mathcal{V}(Ra) = \mathcal{V}(Ra^2)$, and hence,

$$\alpha(Ra) = \oplus_{P\in\mathcal{V}(Ra)}(R/P) = \oplus_{P\in\mathcal{V}(Ra^2)}(R/P) = \alpha(Ra^2).$$

Because α is an injection, the ideal $Ra = Ra^2$, and hence, $a = aba$ for some $b \in R$. Thus, R is a von Neumann regular ring. $\square$

This allows us to produce examples to show that the mapping α can be an anti-monomorphism without being an anti-epimorphism.

Corollary 4.8. *Let R be a von Neumann regular ring, let P_i $(i \in I)$ denote the collection of all maximal ideals of R, and let $M = \oplus_{i\in I}(R/P_i)$. Then the mapping $\alpha : \mathcal{L}(R) \to \mathcal{L}({}_RM)$ is an anti-monomorphism. Moreover, α is an anti-epimorphism* (*and hence an anti-isomorphism*) *if and only if R is semiprime Artinian.*

Proof. By Theorem 4.7, α is an anti-monomorphism. Now suppose that the R-module M is a comultiplication module. Let $j \in I$. By Proposition 4.6, $R = P_j + (\cap_{i \neq j} P_i)$. But $\cap_{i \in I} P_i = 0$. Thus, $P_j = Re_j$ for some idempotent element e_j of R. We have proved that every maximal ideal of R is generated by an idempotent. Thus, R has no proper essential ideals and hence is semiprime Artinian. Conversely, if R is semiprime Artinian, then α is a surjection by Proposition 4.6. $\square$

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