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MATCHINGS, COVERINGS, AND CASTELNUOVO-MUMFORD REGULARITY

RUSS WOODROOFE

ABSTRACT. We show that the co-chordal cover number of a graph G gives an upper bound for the Castelnuovo-Mumford regularity of the associated edge ideal. Several known combinatorial upper bounds of regularity for edge ideals are then easy consequences of covering results from graph theory, and we derive new upper bounds by looking at additional covering results.

1. Introduction and background. Let G be a graph with vertex set $\{x_1, \dots, x_n\}$, and let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k obtained by associating a variable with each vertex of G. We consider the *edge ideal* of G in R, defined as

$$I(G) = (x_i x_j : \{x_i, x_j\} \text{ an edge of } G).$$

The Castelnuovo-Mumford regularity of an ideal I, denoted by $\operatorname{reg} I$, is one of the main measures of the complexity of I. Several recent papers [**13, 19, 29, 30, 32, 34, 37**] have related the Castelnuovo-Mumford regularity of the edge ideal $I(G)$ with various invariants of the graph G.

The purpose of this paper is to give a new upper bound on $\operatorname{reg}(R/I(G))$, and to show that this new upper bound generalizes several other recently discovered upper bounds.

A graph G is *chordal* if every induced cycle in G has length 3 and is *co-chordal* if the complement graph $\overline{G}$ is chordal. It follows from Fröberg's classification of edge ideals with linear resolutions [**14**] that $\operatorname{reg}(R/I(G)) \leq 1$ if and only if G is co-chordal. (A direct proof using the techniques in Section 3 is also straightforward). The *co-chordal*

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cover number, denoted $\operatorname{cochord} G$, is the minimum number of co-chordal subgraphs required to cover the edges of G.

Our main result is as follows.

Theorem 1. *For any graph G and over any field k, we have $\operatorname{reg}(R/I(G)) \leq \operatorname{cochord} G$.*

We will see the proof to follow almost immediately from a result of Kalai and Meshulam [**22**]. Nevertheless, Theorem 1 provides a fundamental connection between combinatorics and commutative algebra, and it will help us give simple and unified proofs of both known and new upper bounds for the regularity of $R/I(G)$.

A particularly simple condition yielding a co-chordal cover (hence a bound on regularity) is as follows.

Theorem 2. *If G is a graph such that $V(G)$ can be partitioned into an (induced) independent set J_0 together with s cliques $J_1, \dots, J_s$, then $\operatorname{reg}(R/I(G)) \leq s$.*

The following is a recursive version of Theorem 2:

Theorem 3. *If G is a graph such that $J \subseteq V(G)$ induces a clique, then $\operatorname{reg}(R/I(G)) \leq \operatorname{reg}(R/I(G \setminus J)) + 1$, where $G \setminus J$ denotes the induced subgraph on $V(G) \setminus J$.*

In plain language, Theorem 3 says that deleting a clique lowers regularity by at most 1. The author hopes that Theorems 2 and 3 may be helpful to practitioners in the field for quickly finding rough upper estimates of regularity of edge ideals.

The remainder of this paper is organized as follows. In the remainder of this section we review terminology from graph theory. In Section 2, we prove Theorem 1. In Section 3, we introduce the equivalent notion of regularity of a simplicial complex. We then use topological techniques to calculate regularity of several examples, and more generally, to obtain lower bounds. In particular, we give a geometric proof of the well-known fact (Lemma 7) that $\operatorname{reg}(R/I(G))$ is at least the induced matching number of G. In Section 4, we combine Theorem 1 with results from the graph theory literature to prove Theorems 2 and 3. We recover and extend results of [**19, 25**], but show that the results of [**27, 34**] cannot be proved using this technique.

1.1. Terminology and notation from graph theory. All graphs discussed in this paper are simple, with no loops or multiedges. We assume basic familiarity with standard graph theory definitions as in, e.g., [**10, 26**], but review some particular terms we will use.

If $\mathcal{F}$ is a family of graphs, then an $\mathcal{F}$ covering of a graph G is a collection $H_1, \dots, H_s$ of subgraphs of G such that every H_i is in $\mathcal{F}$, and such that $\bigcup E(H_i) = E(G)$. Elsewhere in the literature this notion is sometimes referred to as an $\mathcal{F}$ *edge covering*, to contrast with covers of the vertices. The $\mathcal{F}$ *cover number* is the smallest size of an $\mathcal{F}$ cover. We will mostly be interested in the case where $\mathcal{F}$ is some subfamily of co-chordal graphs.

An *independent set* in a graph G is a subset of pairwise non-adjacent vertices. Similarly, a *clique* is a subset of pairwise adjacent vertices. We do not require cliques to be maximal.

A *matching* in a graph G is a subgraph consisting of pairwise disjoint edges. If the subgraph is an induced subgraph, the matching is an induced matching. The graph consisting of a matching with m edges is denoted as mK_2.

The *independence number* $\alpha(G)$, *clique number* $\omega(G)$ and induced matching number $\operatorname{indmatch} G$ are, respectively, the maximum size of an independent set, clique or induced matching.

A *coloring* of G is a partition of the vertices into (induced) independent sets (*colors*), and the *chromatic number* $\chi(G)$ is the smallest number of colors possible in a coloring of G. A graph G is perfect if $\alpha(H) = \chi(\overline{H})$ for every induced subgraph H of G. It is well-known that the complement of a perfect graph is also perfect.

We denote by P_n the path on n vertices (having edges $\{x_1x_2, x_2x_3, \dots, x_{n-1}, x_n\}$), and by C_n the cycle on n vertices (having the edges of P_n together with x_1x_n).

2. Proof of Theorem 1. As previously mentioned, Theorem 1 is an easy consequence of the following deep result by Kalai and Meshulam [**22**].

Theorem 4 (Kalai and Meshulam [**22**, Theorem 1.2]). *If $I_1, \dots, I_s$ are square-free monomial ideals of a polynomial ring $R = k[x_1, \dots, x_n]$ (for some field k), then*

$$\operatorname{reg}\left(R\middle/(I_1 + \cdots + I_s)\right) \leq \sum_{j=1}^{s} \operatorname{reg}(R/I_j).$$

Remark 5. Theorem 4 was conjectured by Terai [**32**]. Herzog [**21**] later generalized the result to monomial ideals that are not square-free.

Remark 6. Kalai and Meshulam stated [**22**, Theorem 1.2] in terms of $\operatorname{reg}(I_j)$'s, rather than $\operatorname{reg}(R/I_j)$'s. Theorem 4 is equivalent since, by e.g., [**28**, Theorem 1.34], we have $\operatorname{reg} I = \operatorname{reg}(R/I) + 1$.

In the context of edge ideals, Theorem 4 says that if $G_1, \dots, G_s$ are graphs on the same vertex set $\{x_1, \dots, x_n\}$, then

$$\operatorname{reg}\left(R/I\left(\bigcup_{j=1}^{s} G_j\right)\right) \leq \sum_{j=1}^{s} \operatorname{reg}(R/I(G_j)). \tag{1}$$

Proof of Theorem 1. Recall from above that $\operatorname{reg}(R/I(H)) = 1$ if and only if H is co-chordal with at least one edge. The result then follows immediately from (1) by considering the case where each $R/I(G_j)$ has regularity 1.

We comment that (1) can more generally be applied to edge ideals of clutters (i.e., to square-free monomial ideals with degree > 2), but that in this case the set of ideals with linear resolution (that is, smallest possible regularity) is not classified, giving more fragmented results. In this paper we henceforth restrict ourselves to the case of graphs.

3. Lower bounds and simple examples. Before discussing applications, it will be convenient to have lower bounds to compare with the upper bound of Theorem 1. As we will shortly see that $\operatorname{reg}(R/I(H)) \le \operatorname{reg}(R/I(G))$ for every induced subgraph H of G, lower bounds usually come from examples.

We will compute regularity through Hochster's formula (see, e.g., [**28**]), which relates local cohomology of the quotient R/I of a square-free monomial ideal with the simplicial cohomology of the simplicial complex of non-zero square-free monomials in R/I. We refer to [**20**] for basic background on simplicial cohomology, or to [**2**] for a concise reference aimed at combinatorics.

The *Castelnuovo-Mumford regularity* of a simplicial complex Δ over a field k, denoted by $\operatorname{reg}_k \Delta$, is defined to be the maximum i such that the reduced homology $\tilde{H}_{i-1}(\Gamma; k) \neq 0$ for some induced subcomplex Γ of Δ. It is well known to follow from Hochster's formula (together with the Betti number characterization of regularity) that $\operatorname{reg}_k \Delta$ is equal to the Castelnuovo-Mumford regularity of the Stanley-Reisner ring of Δ over k. We remark that complexes with regularity at most d have been referred to as *d-Leray* and have been studied in the context of proving certain Helly-type theorems [**22**].

In the case of the edge ideal of a graph G, let $\operatorname{Ind} G$ denote the *independence complex* of G, consisting of all independent sets of G. In this case our above discussion specializes to the relation:

$$\operatorname{reg}\left(k[x_1, \dots, x_n]\Big/ I(G)\right) = \operatorname{reg}_k(\operatorname{Ind} G). \tag{2}$$

(Note that we write $k[x_1, \dots, x_n]$ rather than R to emphasize the field over which we are working.)

In particular, it follows immediately from the definition of $\operatorname{reg}_k \Delta$ that $\operatorname{reg}_k(\operatorname{Ind} H) \le \operatorname{reg}_k(\operatorname{Ind} G)$ for H an induced subgraph of G. Thus, for example, finding an induced subgraph of G whose independence complex is a d-dimensional sphere would show that $\operatorname{reg}(R/I(G)) = \operatorname{reg}_k(\operatorname{Ind} G) \ge d + 1$.

Such bounds often do not depend on the choice of field k that we work over, and in such cases we will suppress k from our notation.

Recall that an *induced matching* in a graph G is a matching which forms an induced subgraph of G, and that $\operatorname{indmatch} G$ denotes the number of edges in a largest induced matching. Induced matchings have a considerable literature, see e.g., [**1, 5, 6, 11, 16**].

The following is essentially due to Katzman; we will give a short geometric proof.

Lemma 7 (Katzman [**23**, Lemma 2.2]). *For any graph G, we have* $\operatorname{reg}(R/I(G)) \geq \operatorname{indmatch} G$.

Proof. Let $m = \operatorname{indmatch} G$, so that G has mK_2 as an induced subgraph. Notice that if H is the disjoint union of subgraphs H_1 and H_2, then $\operatorname{Ind}(H)$ is the simplicial join $\operatorname{Ind}(H_1) * \operatorname{Ind}(H_2)$. Thus, the independence complex of the disjoint union of m edges is the m-fold join of 0-spheres, hence an $(m-1)$-sphere. (It is the boundary complex of an $(m-1)$-dimensional cross-polytope.) The result follows. $\square$

A more general result follows immediately from the Künneth formula in algebraic topology [**2**, (9.12)].

Lemma 8. *For any field k and simplicial complexes Δ_1 and Δ_2, we have* $\operatorname{reg}_k(\Delta_1 * \Delta_2) = \operatorname{reg}_k \Delta_1 + \operatorname{reg}_k \Delta_2$.

In the context of edge ideal quotients, if G_1 and G_2 are any two graphs, then over any field k, for their disjoint union $G_1 \sqcup G_2$ we have

$$\operatorname{reg}(R/I(G_1 \sqcup G_2)) = \operatorname{reg}(R/I(G_1)) + \operatorname{reg}(R/I(G2)). \tag{3}$$

Thus, Lemma 7 is the special case where we take the disjoint union of graphs with a single edge.

Lemmas 1 and 7 admit the simple combined statement that, for any graph G, we have

$$\operatorname{indmatch} G \leq \operatorname{reg}(R/I(G)) \leq \operatorname{cochord} G. \tag{4}$$

Both inequalities can be strict, as the interested reader can quickly see by examination of C_5 and C_7. Indeed, it follows easily that regularity can be arbitrarily far from both $\operatorname{indmatch} G$ and $\operatorname{cochord} G$:

Proposition 9. *For any nonnegative integers r, s there is a graph G such that*

$$\text{indmatchG} = \operatorname{reg}(R/I(G)) - r \text{ and } \operatorname{cochord} G = \operatorname{reg}(R/I(G)) + s.$$

Proof. Consider r copies of C_5 disjoint union with s copies of C_7. $\square$

Another relevant construction can be found in Lemma 21 and the discussion following.

More generally, Kozlov calculated the homotopy type of the independence complexes of paths and cycles [**24**, Propositions 4.6 and 5.2], from which the following is immediate:

Proposition 10. $\operatorname{reg}(R/I(C_n)) = \operatorname{reg}(R/I(P_n)) = \lfloor (n+1)/3 \rfloor$ *for* $n \geq 3$.

(Regularity of $R/I(P_n)$ was also calculated in [**3**] using purely algebraic methods.)

It is easy to see that the regularity is equal to the lower bound of Lemma 7 in the P_n case, and in the C_n case when $n \not\equiv 2 \pmod 3$; but that $\operatorname{reg}(\operatorname{Ind}(C_{3i+2})) = i+1 = \operatorname{indmatch}(C_{3i+2}) + 1$.

Since the graph formed by two disjoint edges is not co-chordal, we see that cochordal subgraphs of P_n and C_n (for $n \geq 5$) are paths with at most three edges. Thus, regularity is equal to the upper bound of Theorem 1 in the P_n case, and in the C_n case when $n \not\equiv 1 \pmod 3$; but, for $i > 1$, we have $\operatorname{reg}(\operatorname{Ind}(C_{3i+1})) = i = \operatorname{cochord}(C_{3i+1}) - 1$.

By combining Proposition 10 with Lemma 8, we can somewhat improve the induced matching lower bound of Lemma 7:

Corollary 11. *If a graph G has an induced subgraph H which is the disjoint union of edges and cycles*

$$H \cong mK_2 \sqcup \coprod_{j=1}^{n} C_{3i_j+2},$$

then $\operatorname{reg}(R/I(G)) \geq m + n + \sum_{j=1}^{n} i_j$.

4. Applications. We can recover, and in some cases improve, several of the upper bounds for regularity in the combinatorial commutative algebra literature by combining Theorem 1 with covering results from the graph theory literature. Theorem 1 thus seems to capture an essential connection between Castelnuovo-Mumford regularity and pure graph-theoretic invariants.

4.1. Split covers. Although co-chordal covers per se have not been a topic of frequent study, there are many results in the graph theory literature concerning the $\mathcal{F}$-cover number of graphs for various subfamilies of co-chordal graphs. We will review several of these with an eye to regularity.

A *split graph* is a graph H such that $V(H)$ can be partitioned into a clique and an (induced) independent set. It is easy to see that such graphs are both chordal and co-chordal; see e.g., [**26**, Chapter 5] for additional background. Covering the edges of G with split graphs allows us to prove Theorem 2.

Proof of Theorem 2. (Essentially, e.g., [**26**, Lemma 7.5.2]). Let H_i be the subgraph consisting of all edges incident to at least one vertex in J_i. Each H_i can be partitioned as the clique on J_i together with the independent set $V(G) \setminus V(J_i)$. Therefore, each H_i is a split graph. Thus, $H_1, \ldots, H_s$ is a split graph covering, hence a co-chordal covering. The result follows by Theorem 1. $\square$

To help clarify the meaning of the condition in Theorem 2, we notice that when $J_0 = \varnothing$, the sets $J_1, \ldots, J_s$ are exactly an s-coloring of $\overline{G}$.

However, the bound $\operatorname{reg}(R/I(G)) \leq \chi(\overline{G})$ resulting from the $J_0 = \varnothing$ case of Theorem 2 is trivial. Indeed, this bound follows from the inequalities $\chi(\overline{G}) \geq \alpha(G)$ and $\alpha(G) \geq \operatorname{reg}(R/I(G))$. (The latter is immediate by Hochster's formula, as discussed in Section 3, since $\alpha(G) = \dim \operatorname{Ind}(G) + 1$ and $\widetilde{H}_i(\Delta)$ always vanishes above $\dim \Delta$.)

The proof of Theorem 3 is entirely similar:

Proof of Theorem 3. Let H consist of all edges incident to J. Then H is a split graph, with $E(G) = E(H) \cup E(G \setminus J)$, and the result follows from (1). $\square$

We now recall two results of Hà and Van Tuyl, for which we will give new proofs via Theorem 2. The matching number of a graph G, denoted $\nu(G)$, is the size of a maximum matching; that is, the maximum number of pairwise disjoint edges.

Theorem 12 (Hà and Van Tuyl [**19**, Theorem 6.7]). *For any graph* G*, we have* $\operatorname{reg}(R/I(G)) \leq \nu(G)$.

Proof. This is the special case of Theorem 2 where $J_1, \dots, J_s$ is a maximum size family of 2-cliques. □

An easy (stronger) corollary of Theorem 2 is that $\operatorname{reg}(R/I(G))$ is at most the size of a minimum maximal matching. Indeed, we can regard Theorem 2 as it is stated to be a strong generalization of Theorem 12.

We also give a new proof for:

Theorem 13 (Hà and Van Tuyl [**19**, Corollary 6.9]). *If* G *is a chordal graph, then* $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$.

Proof of Theorem 13. Cameron [**5**] observed that a chordal graph G has split cover number (as in Theorem 2) equal to $\operatorname{indmatch} G$; the result follows by (4). □

4.2. Weakly chordal graphs, and techniques for finding co-chordal covers. We can considerably extend Theorem 13 by considering more general covers. A graph G is *weakly chordal* if every induced cycle in both G and $\overline{G}$ has length at most 4. (It is straightforward to show that a chordal graph is weakly chordal.)

Theorem 14. *If* G *is a weakly chordal graph, then* $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$.

Proof. Busch, Dragan and Sritharan [**4**, Proposition 3] show that $\operatorname{indmatch} G = \operatorname{cochord} G$ for any weakly chordal graph G. (Abueida, Busch and Sritharan [**1**, Corollary 1] earlier showed the same result under the additional assumption that G is bipartite.)

The essential technique introduced in [**5**] and further developed in [**1, 4**] is to examine a derived graph G^*, with vertices corresponding to the edges of G, and two edges adjacent unless they form an induced matching in G. Thus, an independent set of G^* corresponds to an induced matching of G. (In graph-theoretic terms, G^* is the square of the line graph of G.)

In a weakly chordal [**4**] (chordal [**5**], chordal bipartite [**1**]) graph, these papers show that

i) G^* is perfect, so that there is a partition of the vertices of G^* into $\alpha(G^*)$ cliques, and

ii) that the subgraph of G corresponding to a maximal clique of G^* is co-chordal.

The equality of $\operatorname{indmatch} G$ and $\operatorname{cochord} G$ follows.

We use a modification of this approach to prove Theorem 16 below.

4.3. Biclique and chain graph covers. Following our terminology from subsection 1.1, the biclique cover number of a graph G is the minimum number of bicliques (complete bipartite graphs) required to cover the edges of G. As a complete bipartite graph $K_{m,n}$ is clearly co-chordal, the biclique cover number is an upper bound for $\operatorname{cochord} G$. More generally, it is straightforward to show that a bipartite graph G is co-chordal if and only $\operatorname{indmatch} G = 1$. Bipartite co-chordal graphs have been called chain graphs.

Recall that a graph is *well-covered* if every maximal independent set has the same cardinality. Kumini showed:

Theorem 15 (Kumini [**25**]). *If G is a well-covered bipartite graph, then* $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$.

We recover Theorem 15 as a corollary of the following chain graph covering result:

Theorem 16. *If G is a well-covered bipartite graph, then* $\operatorname{indmatch} G = \operatorname{cochord} G$.

In order to prove Theorem 16, we will need two lemmas. First, well-covered bipartite graphs have long been known to admit a simple characterization:

Lemma 17 (Ravindra [**31**], Favaron [**12**]; see also Villarreal [**36**]). *If G is a well-covered bipartite graph with no isolated vertices, then G has a perfect matching. Moreover, in every perfect matching M of G, the neighborhood of any edge in M is complete bipartite.*

We will also need the following technical lemma. Two edges are incident if they share a vertex; in particular, we consider an edge to be incident to itself.

Lemma 18. *Let G be a well-covered bipartite graph and M a perfect matching in G. Let M_0 be a subset of M so that no pair of edges in M_0 form an induced matching in G. Then the subgraph H of G consisting of all edges incident to M_0 has* $\operatorname{indmatch} H = 1$ *and is, in particular, co-chordal.*

Proof. Since the neighborhood of any edge in M is complete bipartite, it suffices to show that if e is an edge of H and c_0 an edge of M_0, then e and c_0 do not form a $2K_2$; that is, that there is some edge of G incident to both e and c_0.

If $e \in M_0$, then this is immediate by the hypothesis. Otherwise, $e = \{x, y\}$ where y is in some edge $c_1 = \{y, z\}$ of M_0. By the hypothesis on M, either y or z is in some edge b incident to c_0. If $y \in b$, then we are done. Otherwise, $b = \{z, w\}$ with $w \in c_0$. But then w and x are both neighbors of c_1, hence adjacent by Lemma 17. $\square$

Proof of Theorem 16. Assume, without loss of generality, that G has no isolated vertices, and let M be a perfect matching, as guaranteed to exist by Lemma 17. We construct a new graph M^* with vertices consisting of the edges of M, and with two vertices adjacent unless they form an induced matching in G. Thus, M^* is an induced subgraph of the graph G^* from the discussion following Theorem 14.

Any independent set in M^* still corresponds to an induced matching of G, so that $\alpha(M^*) \leq \operatorname{indmatch} G$. On the other hand, if K^* is a clique in M^*, then Lemma 18 gives the subgraph of all incident edges to be co-chordal. Since every edge in G is incident to at least one edge of M, we get that $\operatorname{cochord} G \leq \chi(\overline{M^*})$.

But Kumini shows [**25**, Discussion 2.8] that the graph obtained from M^* by identifying pairs of vertices v and w with $N[v] = N[w]$ is a comparability graph, hence perfect; so M^* is perfect by, e.g., Diestel [**10**, Lemma 5.5.5]. Hence, we have that $\alpha(M^*) = \chi(\overline{M^*})$, and the result follows. $\square$

We remark that, in Theorem 14, we apply a result from the graph theory literature to prove a new result on regularity while, in Theorem 16, a result from combinatorial commutative algebra guides us to a new min-max result on well-covered bipartite graphs.

4.4. Co-interval covers and boxicity. An *interval graph* is a graph with vertices corresponding to some set of intervals in $\mathbf{R}$ and edges between pairs of intervals that have non-empty intersection. A *co-interval graph* is the complement of an interval graph. Interval graphs are exactly the chordal graphs which can be represented as the incomparability graph of a poset. See [**26**] for general background on such graphs.

The *boxicity* of G, denoted $\operatorname{box} G$, is the co-interval cover number of $\overline{G}$. (The original formulation of boxicity was somewhat different, and the connection with covering is made in [**8**].) Thus, by Theorem 1, we have that $\operatorname{reg}(R/I(G)) \leq \operatorname{box} \overline{G}$.

Since a planar graph G contains no K_5 subgraph, we have that $\operatorname{reg}(R/I(\overline{G})) \leq \dim \operatorname{Ind}(\overline{G})) + 1 = \alpha(\overline{G}) \leq 4$. The literature on boxicity yields a stronger result:

Proposition 19. *If G is a planar graph, then* $\operatorname{reg}(R/I(\overline{G})) \leq 3$. *This upper bound is the best possible.*

Proof. Thomassen [**33**] proves that $\operatorname{box} G \leq 3$. To see that the bound is best possible, notice that the complement of $3K_2$ (that is, the graph consisting of three disjoint edges) is the 1-skeleton of the octahedron, which is well known to be planar. $\square$

By way of contrast, we remark that the proof of Proposition 9 shows that if G is a planar graph, then $\operatorname{reg}(R/I(G))$ may be arbitrarily large.

4.5. Very well covered graphs. In this subsection we present a negative result. A graph is *very well-covered* if it is well-covered and $\alpha(G) = |V|/2$. It is obvious that every well-covered bipartite graph is very well-covered. Mahmoudi et al. [**27**] generalized Theorem 15 to show:

Theorem 20 [**27**]. *If G is a very well-covered graph, then* $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$.

We will demonstrate, however, that the gap between $\operatorname{indmatch} G$ and $\operatorname{cochord} G$ can be arbitrarily large for very well-covered graphs. In particular, the proof via (4) of Theorem 15 cannot be extended to prove Theorem 20.

If G is a graph on n vertices, then let $W(G)$ be the graph on $2n$ vertices obtained by adding a *pendant* (an edge to a new vertex of degree 1) at every vertex of G. This construction has been previously studied in the context of graphs with Cohen-Macaulay edge ideals [**35**], where it has been referred to as *whiskering*; and has been studied in the graph theory literature as a *corona* [**15**]. Because the pendant vertices form a maximal independent set, it is immediate that $W(G)$ is very well-covered.

Lemma 21. *For any graph G, we have* $\operatorname{indmatch} W(G) = \alpha(G)$ *and* $\operatorname{cochord} W(G) = \chi(\overline{G})$.

Proof. For the first equality, we notice that if an induced matching of $W(G)$ contains an edge $\{v, w\}$ of G, then we can get a new induced matching by replacing $\{v, w\}$ with the pendant edge at v. Since a collection of pendant edges forms an induced matching if and only if the corresponding collection of vertices of G is independent, the statement follows.

For the second equality, we first notice that a coloring of $\overline{G}$ partitions the vertices of G into cliques, inducing a covering of $W(G)$ by split graphs (as in Theorem 2). Hence, $\operatorname{cochord} W(G) \leq \chi(\overline{G})$. On the other hand, any co-chordal cover $\{H_i\}$ of $W(G)$ in particular covers the pendant edges, and two pendant edges form an induced matching if the corresponding vertices of G are not connected. Hence, a co-chordal cover induces a covering of the vertices of G by cliques, and thus $\operatorname{cochord} W(G) \geq \chi(\overline{G})$, as desired. ◻

But then, for example, we have $\operatorname{indmatch} W(C_5) = 2$ and $\operatorname{cochord} W(C_5) = 3$. Moreover, it is well-known that the gap between the clique number and chromatic number of $\overline{G}$ can be arbitrarily large, even if $\omega(\overline{G}) = \alpha(G) = 2$. (See, e.g., [**10**, Theorem 5.2.5].) Hence, the gap between $\operatorname{indmatch} W(G)$ and $\operatorname{cochord} W(G)$ can also be arbitrarily large.

Van Tuyl [**34**] has shown an analogue to Theorem 16: that if G is a bipartite graph such that $R/I(G)$ is sequentially Cohen-Macaulay, then $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$. (See his paper [**34**] for definitions and background.) The following example, however, shows that $\operatorname{indmatch} G$ and $\operatorname{reg}(R/I(G))$ may also be strictly less than $\operatorname{cochord} G$ in this situation.

Example 22. Let G be obtained from C_6 by attaching a pendant to vertices x_1, x_2, x_3, x_4. It is easy to see from the conditions given in [**34**] that $\operatorname{Ind} G$ is sequentially Cohen-Macaulay. But, an approach similar to that in Lemma 21 will verify that $\operatorname{indmatch} G = 2$, while $\operatorname{cochord} G = 3$.

4.6. Computational complexity. An immediate consequence of Lemma 21 is that calculating $\operatorname{reg}(R/I(G))$ from the graph G is computationally hard:

Corollary 23. *Given G, calculating* $\operatorname{reg}(R/I(G))$ *is* NP*-hard, even if G is very well covered.*

Proof. One can construct $W(G)$ from G in polynomial time, and $\operatorname{reg}(R/I(W(G))) = \operatorname{indmatch} W(G) = \alpha(G)$. But, checking whether $\alpha(G) \geq C$ is well known to be NP-complete! □

Since computing the independence complex of G is already NP-hard, and as it is hard to imagine finding regularity without computing the independence complex, Corollary 23 is perhaps not too surprising. It might be of more interest to find the computational complexity of computing $\operatorname{reg}(R/I(G))$ from $\operatorname{Ind} G$.

We remark that many of the results we have referenced are from the computer science literature, and efficient algorithms for finding indmatch G and cochord G in special classes of graphs are a main interest of [**1, 4, 5**] and other papers. In particular, given a weakly chordal graph G, we can calculate $\operatorname{reg}(R/I(G)) = \operatorname{indmatch} G$ in polynomial time [**4**, Corollary 8].

In general graphs, however, computing indmatch G or cochord G is NP-hard. It follows from, e.g., the proof of Corollary 23 that determining whether $\operatorname{indmatch} G \geq C$ is NP-complete; while Yannakakis showed [**38**] that determining whether $\operatorname{cochord}(G) \leq C$ is NP-complete. The corresponding problem for split graph covers (as in Theorem 2) is also NP-complete [**7**]. An overview of these and similar hardness results can be found in [**26**, Chapter 7].

4.7. Questions on claw-free graphs. Nevo [**30**] showed that, if G is a ($2K_2$, claw)-free graph, then $\operatorname{reg}(R/I(G)) \leq 2$. Dao, Huneke and Schweig [**9**] have recently given an alternate proof. Can the same be shown using Theorem 1?

Question 24. If G is ($2K_2$, claw)-free, then is $\operatorname{cochord} G \leq 2$?

We notice that a cover by split graphs will not suffice; for example, the Petersen graph P has girth 5. Hence, $\overline{P}$ is ($2K_2$, claw)-free. But it is easy to verify that no two cliques in P satisfy the condition of Theorem 2.

András Gyárfás points out [**18**] that, in [**17**, Problem 5.7], he has asked whether every graph G with $\operatorname{cochord} G = 2$ has $\chi(\overline{G})$ bounded by some function of $\alpha(G)$. We observe that the complement of a graph with girth ≥ 5 is ($2K_2$, claw)-free, with $\alpha(G) = 2$. Since a graph with girth ≥ 5 can have an arbitrarily large chromatic number [**10**, Theorem 5.2.5], a positive answer to Question 24 would imply a negative answer to Gyárfás's question.

If the answer to Question 24 is negative, then the following might still be of interest.

Question 25. If G is claw-free, then does G have a ($2K_2$,claw)-free cover by at most indmatch G subgraphs?

If Question 25 has a positive answer, then a direct application of (1) would then imply that, for a claw-free graph G, we have $\operatorname{reg}(R/I(G)) \leq 2 \cdot \operatorname{indmatch} G$.

After acceptance of this paper, Shahab Haghi and Siamak Yassemi pointed out to me in an email communication (*March,* 2013) *that Question* 25 *has a negative answer for the cyclic graph* C_8*. So far as I am aware, the question remains open as to whether* $\operatorname{reg}(R/I(G)) \leq 2 \cdot \operatorname{indmatch} G$ *for any claw-free graph* G*.*

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Department of Mathematics & Statistics, Mississippi State University, MS 39762
Email address: rwoodroofe@math.msstate.edu