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IMPLICITIZATION OF DE JONQUIÈRES PARAMETRIZATIONS

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ABSTRACT. One introduces a class of projective parameterizations that resemble generalized de Jonquières maps. Any such parametrization defines a birational map $\mathfrak{F}$ of $\mathbf{P}^n$ onto a hypersurface $V(F) \subset \mathbf{P}^{n+1}$ with a strong handle to implicitization. From this side, the theory developed here extends recent work of Benítez and D'Andrea on monoid parameterizations. The paper deals with both the ideal theoretic and effective aspects of the problem. The ring theoretic development gives information on the Castelnuovo-Mumford regularity of the base ideal of $\mathfrak{F}$. From the effective side, we give an explicit formula of $\deg(F)$ involving data from the inverse map of $\mathfrak{F}$ and show how the present parametrization relates to monoid parameterizations.

1. Introduction and notation. Let k denote an arbitrary infinite field which will be assumed to be algebraically closed for the geometric purpose. A rational map $\mathfrak{F} : \mathbf{P}^n \dashrightarrow \mathbf{P}^m$ is defined by $m+1$ forms $\mathbf{f} = \{f_0, \ldots, f_m\} \subset R := k[\mathbf{x}] = k[x_0, \ldots, x_n]$ of the same degree $d \geq 1$, not all null. We often write $\mathfrak{F} = (f_0 : \cdots : f_m)$ to underscore the projective setup.

The image of $\mathfrak{F}$ is the projective subvariety $W \subset \mathbf{P}^m$ whose homogeneous coordinate ring is the k-subalgebra $k[\mathbf{f}] \subset R$ after degree renormalization. Write $S := k[\mathbf{f}] \simeq k[\mathbf{y}]/I(W)$, where $I(W) \subset k[\mathbf{y}] = k[y_0, \ldots, y_m]$ is the homogeneous defining ideal of the image in the embedding $W \subset \mathbf{P}^m$.

We say that $\mathfrak{F}$ is *birational onto the image* if there is a rational map backwards $\mathbf{P}^m \dashrightarrow \mathbf{P}^n$ such that the residue classes $\mathbf{f}' = \{f'_0, \ldots, f'_n\} \subset S$ of its defining coordinates do not simultaneously vanish and satisfy

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the relations

$$
\begin{aligned}
(1)\qquad & (\mathbf{f}'_0(\mathbf{f}) : \cdots : \mathbf{f}'_n(\mathbf{f})) = (x_0 : \cdots : x_n),\\
& (\mathbf{f}_0(\mathbf{f}') : \cdots : \mathbf{f}_m(\mathbf{f}')) \equiv (y_0 : \cdots : y_m) \mod I(W).
\end{aligned}
$$

Let K denote the field of fractions of $S = k[\mathbf{f}]$. Note that the set of coordinates $(f'_0 : \cdots : f'_n)$ defining the "inverse" map is not uniquely defined; any other set $(f''_0 : \cdots : f''_n)$ related to $\mathbf{f}'$ by requiring that it defines the same element of the projective space $\mathbf{P}^n_K = \mathbf{P}^n_k \times_{\mathrm{Spec}(k)} \mathrm{Spec}\,(K)$ will do as well; both tuples are called *representatives* of the rational map (see **[16]** for details). If k is algebraically closed, these relations translate into the usual geometric definition in terms of invertibility of the map on a dense Zariski open set.

A special important case is that of a *Cremona map*, that is, a birational map

$$\mathfrak{G} = (g_0 : \cdots : g_n) : \mathbf{P}^n \dashrightarrow \mathbf{P}^n$$

of $\mathbf{P}^n$ onto itself. We assume, as usual, that the coordinate forms have no proper common factor. In this setting, the common degree $d \geq 1$ of these forms is called the *degree* of $\mathfrak{G}$. Having information about the inverse map, e.g., about its degree, will be quite relevant in the sequel. Thus, for instance, the structural equality

$$(2)\qquad (g_0(g'_0, \ldots, g'_n) : \cdots : g_n(g'_0, \ldots, g'_n)) = (y_0 : \cdots : y_n),$$

involving the inverse map gives a uniquely defined form $D \in R$ such that $g_i(g'_0, \ldots, g'_n) = y_i D$, for every $i = 1, \ldots, n$. We call $D \in k[\mathbf{y}]$ the *target inversion factor* of $\mathfrak{G}$. By symmetry, there is a *source inversion factor* $C \in k[\mathbf{x}]$.

Our basic reference for the above is **[16]**, which contains enough of the introductory material in the form we use here (see also **[6]** for a more general overview).

Now, the problem envisaged in this paper emerges from a particular situation of rational maps, known as *elimination*. Namely, one takes $m = n + 1$ and assumes that $\dim k[\mathbf{f}] = \dim R$ $(= n + 1)$. Therefore, W is a hypersurface defined by an irreducible form $F \in k[\mathbf{y}] = k[y_0, \ldots, y_{n+1}]$. We speak of F informally as the *implicit equation* of $\mathfrak{F}$. Elimination theory in this formulation is the problem of determining F or at least its properties, such as its degree. The set of the given forms defining $\mathfrak{F}$ is called a *parametrization* of F. The theory has an applicable side shown in a very active research area; we refer to some of the related modern work on the subject in the bibliography.

Although the main interest classically focused on *implicitization*, i.e., in deriving the implicit equation F, more recently quite some literature has appeared on the ideal theoretic structure of the parametrization and the algebras naturally involved [**1, 2, 4, 5, 10, 11**]. In this regard, a source of inspiration has been the classical *Sylvester forms*, a slightly imprecise notion to refer to certain generators of the defining ideal of the Rees algebra associated to the base ideal of the rational map $\mathfrak{F}$ (i.e., the ideal generated by the parameterizing forms).

Actually, we go even more special, by dealing with rational maps which, in a sense, are allusive of the classical de Jonquières plane Cremona map. Namely, the class of parametrizations used here are suggestive of the stellar Cremona maps by Pan [**13**], a bona fide generalization of the classical plane de Jonquières maps, and inspired by the results of Benítez and D'Andrea [**1**] on the so-called *monoid parametrizations*.

Precisely, start with a Cremona map $\mathfrak{G} = (g_0 : \cdots : g_n) : \mathbf{P}^n \dashrightarrow \mathbf{P}^n$ as explained above. Let $f, g \in R$ be additional forms of arbitrary degrees $\mathfrak{d} \geq 1$ and $d + \mathfrak{d}$, respectively. We assume throughout that f and g are relatively prime.

Definition 1.1. The rational map $\mathfrak{F} = (g_0 f : \cdots : g_n f : g) : \mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$ will be called a *de Jonquières parametrization.*

Note the easy, though important, fact that $\mathfrak{F}$ is a birational map onto its image $W = V(F)$. This follows immediately from the usual field extension criterion (see, e.g., [**6,** Proposition 1.11]. Moreover, if the inverse of $\mathfrak{G}$ is $\mathfrak{G}^{-1} = (g'_0 : \cdots : g'_n)$, with $g'_i \in k[y_0, \ldots, y_n]$, then $(\overline{g'_0} : \cdots : \overline{g'_n})$ is a representative of the inverse $\mathfrak{F}^{-1}$ of $\mathfrak{F}$, where the bar over an element of $k[\mathbf{y}]$ denotes its class modulo (F); note that this representative of $\mathfrak{F}^{-1}$ does not involve the last variable y_{n+1}.

The Cremona map $\mathfrak{G}$ may be called the *underlying* (or *structural*) Cremona map of $\mathfrak{F}$.

The main results of the paper are stated in Theorem 2.6, Proposition 3.3, Proposition 4.2 and Theorem 4.5.

Let us briefly describe the contents of the next sections.

Section 2 gives the main properties of the base ideal of the parametrization, such as structure of syzygies, free resolution and regularity. Part of the information of this section is crucial for introducing the concept of *syzygetic polynomials* that arise as natural candidates for the implicit equation (often with extraneous factors).

Section 3 deals with the implicit equation F. Here one introduces the basic polynomials that play a role in the nature of F, such as the syzygetic polynomials mentioned before. One heavily draws on the hypothesis that the de Jonquières parametrization is birational, by having the defining parametrization of the inverse map and the inversion factor take control of the situation. This section also examines the details of two main cases of the given de Jonquières parametrization, called the *inclusion case* and the *non-zero-divisor case*, respectively. It is worth pointing out that the first of these two cases covers as a very special case the situation of a monoid parametrization.

In Section 4 one focuses on the so-called "Rees equations" of the parametrization. These are the elements of a minimal set of generators of a presentation ideal (the "Rees ideal") of the Rees algebra of the base ideal of $\mathfrak{F}$, one of which, of course, is F itself. These have been variously studied by several authors, some listed in the references. The idea in this section is based on the method of downgrading that has been used in different sources (e.g., **[2]**, **[9]**, **[11]**). Ours is a modification of this method, hereby called *birational downgrading*, by which we use the forms defining the inverse map rather than the usual procedures in the literature. The main result yields a set of Rees equations candidates for a set of minimal generators, generating an ideal having as a minimal prime component the entire Rees ideal. The sections end with a result giving the precise relation between the Rees ideal of de Jonquières parameterizations and the one of the monoid parameterizations.

2. Syzygetic background. In this section we establish the basic relations of degree 1 of the forms $g_0f, \ldots, g_nf, g$ defining the rational map $\mathcal{F}$ of Definition 1.1. For the next lemma and proposition $(g_0 : \cdots : g_n)$ defines any rational map, not necessarily Cremona.

2.1. A mapping cone. In this part, we state a very general result regarding a certain mapping cone naturally associated to the present data. The construction is completely general and does not require a graded situation. Accordingly, we refresh our data just assuming that $I \subset R := k[x_0, \ldots, x_n]$ is an arbitrary ideal and $f, g \in R$ are given elements.

Lemma 2.1. *If* $\gcd(f, g) = 1$*, then*:

(a) $If : (g) = (I : (g))f$.
(b) *Multiplication by g induces an isomorphism $R/(I : (g))f \simeq (If, g)/If$ of R-modules.*

Proof. (a) The inclusion $If : (g) \supset (I : (g))f$ is obvious regardless of any relative assumption about f, g. Conversely, let $b \in R$ be such that $bg \in If$. Then f divides bg and, since $\gcd(f, g) = 1$, then f divides b. Say, $b = af$, with $a \in R$. Then $(ag)f \in If$; hence, $ag \in I$, i.e., $a \in I : (g)$. Therefore, $b \in (I : (g))f$.

(b) One has $(If, g)/If \simeq (g)/(g) \cap If = (g)/(If : (g))g \simeq R/If : (g)$, where the last isomorphism is multiplication by g^{-1}. Now apply (a). $\square$

Quite generally, a surjective R-module homomorphism $\pi : R^q \twoheadrightarrow I : (g)$ induces a *content map* $c(g) : R^q \to R^{n+1}$. In explicit coordinates, let π be induced by choosing a set of generators $\{c_1, \ldots, c_q\}$ of $I : (g)$, so that $\pi(v_j) = c_j$, where $\{v_1, \ldots, v_q\}$ is the canonical basis of R^q. Given a set $\{g_0, \ldots, g_p\}$ of generators of I, let $\{e_0, \ldots, e_p\}$ denote the canonical basis of R^{p+1}. Write $c_j g = \sum_{i=0}^{p} h_{ij} g_i$, with $h_{ij} \in R$. Then $c(g)(v_j) = \sum_{i=0}^{p} h_{ij} e_i$, for $j = 1, \ldots, q$.

This simple construction will be used in the following result.

Lemma 2.2. *Let $\mathfrak{R}$ and $\mathfrak{S}$ denote finite free resolutions of R/I and $R/(I : (g))f$, respectively. Then multiplication by g lifts to a map $\mathfrak{S} \to \mathfrak{R}$ whose associated mapping cone is a free resolution of $R/(If, g)$. In particular, a syzygy matrix of the generators of (If, g) has the form*

$$\Psi = \begin{pmatrix} \varphi & c(g) \\ \mathbf{0} & -f\pi \end{pmatrix},$$

where φ denotes a syzygy matrix of a given set of generators of I.

Proof. As in Lemma 2.1(b), multiplication by g induces an injective R-module homomorphism $R/(I : (g))f \hookrightarrow R/If$ with image $(If, g)/If$.

This homomorphism lifts to a map of complexes (free resolutions)

$$\begin{array}{lccccccccccc}
\mathfrak{R}: & \cdots & \to & R^{m_1} & \overset{\varphi}{\to} & R^{p+1} & \xrightarrow{f\mathbf{g}} & R & \to & R/If & \to & 0 \\
 & & & \uparrow & & c(g)\uparrow & & \cdot g\uparrow & & \cdot g\uparrow & & \\
\mathfrak{R}: & \cdots & \to & R^{r_1} & \overset{\psi}{\to} & R^{q} & \xrightarrow{f\pi} & R & \to & R/(I:(g))f & \to & 0,
\end{array}$$

where $\mathbf{g} = (g_0 \cdots g_p)$, and $c(g)$ is the above content map. Then the corresponding mapping cone is an R-free resolution of $(R/If)/((If, g)/If) \simeq R/(If, g)$ (see [**7,** Exercise A3.30]). □

2.2. Graded minimality and regularity. We move back to the original graded situation. Namely, set $I = (g_0, \ldots, g_n)$, where the g_i's are forms of degree $d \geq 1$ minimally generating I, and f, g are forms with $\deg(g) = d + \deg(f)$ such that $\gcd(f, g) = 1$. Also, let $\{c_1, \ldots, c_q\}$ be a set of minimal generators of $I : (g)$, with c_j homogeneous of degree C_j.

Let

$$\cdots \to \bigoplus_{j=1}^{m_1} R(-a_{1j}) \overset{\varphi}{\to} \bigoplus_{i=0}^{n+1} R(-d) \overset{\mathbf{g}}{\to} R \to R/I \to 0$$

and

$$\cdots \to \bigoplus_{j=1}^{q_1} R(-C_{1j}) \overset{\psi}{\to} \bigoplus_{j=1}^{q} R(-C_j) \overset{\pi}{\to} R \to R/I : g \to 0$$

stand for minimal graded free resolutions of R/I and $R/I : g$, respectively, from which we immediately derive minimal graded free resolutions of R/If and $R/(I : g)f$:

$$\cdots \bigoplus_{j=1}^{m_1} R(-a_{1j} - \deg(f)) \xrightarrow{\varphi_1 = \varphi} R(-(d + \deg(f)))^{n+1}$$
$$\xrightarrow{f\,\mathbf{g}} R \to R/If \to 0,$$

$$\cdots \to \bigoplus_{j=1}^{q_1} R(-C_{1j} - \deg(f)) \xrightarrow{\psi_1 = \psi} \bigoplus_{j=1}^{q} R(-C_j - \deg(f))$$
$$\xrightarrow{f\,\pi} R \to R/(I : g)f \to 0.$$

Shifting the second of these resolutions by $-(d+\deg(f))$, one obtains a map of complexes, where the vertical homomorphisms are also homogeneous of degree 0

$$\begin{array}{cccccccccc}
\cdots \to & \bigoplus_{j=1}^{m_i} R(-a_{ij}-\mathfrak{d}) & \to \cdots \xrightarrow{\varphi_1} & R(-(d+\mathfrak{d}))^{n+1} & \to & R & \to & R/If & \to \\
 & c_i(g)\uparrow & & c(g)\uparrow & & \cdot g\uparrow & & \cdot g\uparrow & \\
\cdots \to & \bigoplus_{j=1}^{q_i} R(-C_{ij}-(d+2\mathfrak{d})) & \to \cdots \xrightarrow{\psi_1} & \bigoplus_{j=1}^{q} R(-C_j-(d+2\mathfrak{d})) & \to & R(-(d+\mathfrak{d})) & \to & \frac{R}{(I:g)f}(-(d+\mathfrak{d})) & \to
\end{array}$$

where we have written $\mathfrak{d} := \deg(f)$ for editing purposes.

We let $\operatorname{reg}(M)$ denote the Castelnuovo-Mumford regularity of a graded R-module, and let $\operatorname{pd}(M)$ stand for its homological (i.e., projective) dimension.

One has:

Proposition 2.3. *With the above notation, the associated mapping cone $\mathcal{C}_\bullet$ is a graded free resolution of $R/(If,g)$:*

$$\cdots \to \left(\bigoplus_{j=1}^{m_1} R(-a_{1j}-\deg(f))\right) \oplus \left(\bigoplus_{j=1}^{q} R(-C_j-(d+2\deg(f)))\right)$$
$$\xrightarrow{\Psi} R(-(d+\deg(f)))^{n+2} \to R \to R/(If,g) \to 0.$$

Moreover, if $\operatorname{reg}(R/I) < d+\deg(f)-2$*, then this resolution is minimal.*

Proof. Applying the minimality criterion stated in [**7,** Exercise A3.30] it suffices to show that $-a_{ij}-\deg(f) > -C_{ik}-(d+2\deg(f))$ for all i,j,k. Now, on one hand, $a_{ij}-i-1 \leq \operatorname{reg}(R/I)$, for any i,j, and on the other hand, for any i,k, $C_{ik} \geq i-1$, where $C_{0k} = C_k$. Therefore, the condition is fulfilled if the regularity of R/I is bounded as stated. □

Corollary 2.4. *With the above notation, assume that* $\operatorname{pd}(R/(I:g)f) \leq pd(R/I)-1$ *(e.g., if $g \in I$ and I has codimension ≥ 2). Then*

$$\operatorname{pd}(R/(If,g)) \leq \operatorname{pd}(R/I),$$

with equality provided $\operatorname{reg}(R/I) < d+\deg(f)-2$.

Since the preceding regularity bound implies, in particular, the minimality of the above graded free presentation of $R/(If,g)$, thus having a direct impact on the search for a minimal set of bihomogeneous

Rees equations, it is pertinent to understand how this bound reflects on the current data.

Proposition 2.5. *Keeping the previous notation, one has*:

(a) $\operatorname{reg}(R/(If,g)) \leq \max\{\operatorname{reg}(R/I)+\deg(f), \operatorname{reg}(R/(I:g))+d+2\deg(f)-1\}$.

(b) *If, moreover,* $\operatorname{reg}(R/I) \leq d+\deg(f)-2$, *then*

$$\operatorname{reg}(R/(If,g)) = \operatorname{reg}(R/(I:g))+d+2\deg(f)-1.$$

Proof. (a) Computing the regularity in terms of the twists of the graded free resolution $\mathcal{C}_\bullet$ in Proposition 2.3, one finds

$$\begin{aligned}\operatorname{reg}(R/(If,g)) &\leq \max\{\operatorname{reg}(R/If), \operatorname{reg}(R/(I:g)f)+d+\deg(f)-1\}\\ &= \max\{\operatorname{reg}(R/I)+\deg(f), \operatorname{reg}(R/(I:g))\\ &\quad + d+2\deg(f)-1\}.\end{aligned}$$

(b) By the second assertion in Proposition 2.3, the mapping cone is a graded minimal free resolution. Therefore, if $\operatorname{reg}(R/I) < d+\deg(f)-2$, then the maximum in (a) is the second term. $\square$

2.3. Regularity in the case of isolated base points. We keep the notation of the previous subsection. Namely, $I=(g_0,\ldots,g_n)$, where the g_i's are forms of degree $d\geq 1$ minimally generating I, and f,g are nonzero forms such that $\deg(g)=d+\deg(f)$ and $\gcd(f,g)=1$. For any ideal $\mathfrak{a}\subset R$, we denote by $\mathfrak{a}^{\mathrm{sat}}$ its saturation $\mathfrak{a}:(\mathbf{x})^\infty$. If M is a graded R-module, we will set

$$\operatorname{indeg}(M) := \inf\{\mu \mid M_\mu \neq 0\},$$

with the convention that $\operatorname{indeg}(0)=+\infty$, and

$$\operatorname{end}(M) := \sup\{\mu \mid M_\mu \neq 0\},$$

with the convention that $\operatorname{end}(0)=-\infty$.

Theorem 2.6. *Suppose that* $\dim(R/I)\leq 1$. *Then*

(1) $\operatorname{reg}(R/I) = \max\{(n+1)(d-1)-\operatorname{indeg}(I^{\mathrm{sat}}/I), n(d-1)-\operatorname{indeg}((\alpha):I/I)\}$, *where α denotes a maximal regular sequence of d-forms in I.*

(2) *If in addition I is the base ideal of a Cremona map, then* $\operatorname{reg}(R/I) \leq n(d-1)-1$.
(3) $\operatorname{reg}(R/(If,g)) \leq \operatorname{reg}(R/I) + d + 2\deg(f) - 1$.

Proof. (1) We copy *ipsis litteris* the argument in the proof of [**8,** Theorem 1.2], updating the setup. Thus, the ground ring now has dimension $n+1$, the base ideal $I \subset R$ has codimension n (hence, $I^{\mathrm{un}} = I^{\mathrm{sat}}$ and, necessarily, $d \geq 2$). Note that one can always pick a maximal regular sequence of d-forms in I. We thus obtain $(I^{\mathrm{sat}}/I)\check{} = (I^{\mathrm{sat}}/I)((n+1)(d-1))$; hence, $\operatorname{end}(I^{\mathrm{sat}}/I) + \operatorname{indeg}(I^{\mathrm{sat}}/I) = (n+1)(d-1)$.

(2) In the case where I is the base ideal of a Cremona map, $\operatorname{indeg}(I^{\mathrm{sat}}/\ I) \geq d+1$ according to [**14**] (by convention, if I is saturated, one sets $\operatorname{indeg}(I^{\mathrm{sat}}/I) = +\infty$). Therefore, $(n+1)(d-1) - \operatorname{indeg}(I^{\mathrm{sat}}/I) \leq n(d-1)-2$. On the other hand, $(\alpha) \subsetneq I$ since I defines a Cremona map and $d \geq 2$, where α is as in item (1). This gives $\operatorname{indeg}((\alpha) : I/I) \geq 1$ which implies that $n(d-1) - \operatorname{indeg}((\alpha) : I/I) \leq n(d-1)-1$. The assertion then follows from (1).

(3) The following result was proved in [**3**]: if M, N are finitely generated graded modules over R such that $\dim(\operatorname{Tor}_1^R(M,N)) \leq 1$, then $\operatorname{reg}\operatorname{Tor}_0^R(M,N)) \leq \operatorname{reg}(M) + \operatorname{reg}(N)$.

We will apply this assertion with $M = R/If$ and $N = R/(g)$. One has

$$\begin{aligned} \operatorname{Tor}_1^R(R/If, R/g)) &\simeq If \cap (g)/(g)If \simeq (If : g)g/If \cdot g \\ &\simeq (If : g)/If(-\deg(g)) \\ &\simeq (I : g)/I(-\deg(f) - \deg(g)), \end{aligned}$$

whose annihilator is the ideal $I : (I : g)$. Thus, this module has dimension at most one as $\dim R/(I : (I : g)) \leq \dim(R/I) \leq 1$ by assumption. Now, applying the above yields

$$\begin{aligned} \operatorname{reg}(R/(If,g)) &= \operatorname{reg}(\operatorname{Tor}_0^R(R/If, R/g))) \\ &\leq \operatorname{reg}(R/If) + \operatorname{reg}(R/(g)) \\ &= \operatorname{reg}(R/I) + \deg(f) + d + \deg(f) - 1 \\ &= \operatorname{reg}(R/I) + d + 2\deg(f) - 1, \end{aligned}$$

as was to be shown. ∎

3. The search for the implicit equation. We keep the notation of Section 1.

3.1. Monoids and syzygetic polynomials. A form F of the shape $F = G + Hy_{n+1}$, where G, H are forms in $k[y_0, \ldots, y_n]$, is called a *monoid* (cf., **[12]** for generalities on these forms). Thus, this is simply a polynomial of degree 1 in the one variable polynomial ring $B[y_{n+1}]$, with homogeneous coefficients in $B = k[y_0, \ldots, y_n]$. As we will see, a good deal of the results will hereafter involve monoids. We will often say an y_{n+1}-monoid to stress the privileged variable y_{n+1}.

The following gadget will be basic throughout. Consider a syzygy of $J = (If, g)$ as in Lemma 2.2 with nonzero last coordinate. Its polynomial version is a 1-form in $R[y_0, \ldots, y_n, y_{n+1}]$

$$\sum_{i=0}^{n} h_{ij} y_i - f c_j y_{n+1}, \tag{3}$$

where $I : (g) = (\ldots, c_j, \ldots)$ and $c_j g = \sum_{i=0}^{n} h_{ij} g_i$.

Definition 3.1. Suppose that $(g_0 : \cdots : g_n)$ defines a Cremona map, and let $(g'_0 : \cdots : g'_n)$ define its inverse map. The jth *syzygetic polynomial* is the form

$$\sum_{i=0}^{n} h_{ij}(g'_0, \ldots, g'_n) y_i - f(g'_0, \ldots, g'_n) c_j(g'_0, \ldots, g'_n) y_{n+1} \in k[\mathbf{y}] \tag{4}$$
$$:= k[y_0, \ldots, y_n, y_{n+1}],$$

obtained from (3) by evaluating $x_i \mapsto g'_i$, for $i = 0, \ldots, n$.

The main property of such forms is the following.

Lemma 3.2. *The jth syzygetic polynomial as introduced in Definition* 3.1 *belongs to the defining ideal of the Rees algebra of the ideal* (If, g); *in particular, it is a multiple of the implicit equation of the parametrization given by* $\{If, g\}$.

Proof. To see this, recall that, since the rational map defined by the generators of (If, g) is birational onto $V(F)$, one has a k-algebra isomorphism of the Rees algebras

$$\mathcal{R}_R((If,g)) \simeq \mathcal{R}_S(I'), \tag{5}$$

where $I' = (\overline{g'_0}, \ldots, \overline{g'_n})$ and $S = k[\mathbf{y}]/(F)$ (see **[6,** Theorem 2.18, proof of (a) $\Rightarrow$ (b)]). This isomorphism is induced by the identity map of $R[\mathbf{y}] = k[\mathbf{y}][\mathbf{x}] = k[\mathbf{x},\mathbf{y}]$. In terms of the respective defining ideals $\mathcal{J}$ and $\mathcal{K}$ over $k[\mathbf{x},\mathbf{y}]$, we have an equality $\mathcal{J} = \mathcal{K}$. Therefore, by definition of $\mathcal{K}$, the syzygetic polynomial

$$P = P(\mathbf{y}) := \sum_{i=0}^{n} h_i(g'_0, \ldots, g'_n)y_i - f(g'_0, \ldots, g'_n)c_j(g'_0, \ldots, g'_n)y_{n+1}$$

vanishes modulo $Fk[\mathbf{x},\mathbf{y}]$. But, since it is a polynomial in $\mathbf{y}$ only, it necessarily belongs to $(F) \subset k[\mathbf{y}]$, i.e., it is a multiple of the implicit equation F. □

This suggests that a syzygetic polynomial is a fair candidate for the implicit equation F and will coincide with F up to a nonzero field element provided it be irreducible.

3.2. The degree of the implicit equation. In this part we establish a formula for the degree of F in terms of the data introduced so far.

Proposition 3.3. *Suppose that $\mathfrak{G} = (g_0 : \cdots : g_n) : \mathbf{P}^n \dashrightarrow \mathbf{P}^n$ defines a Cremona map of degree d with inverse map $\mathfrak{G}^{-1} = (g'_0 : \cdots : g'_n)$ and target inversion factor $D \subset k[y_0, \ldots, y_n]$. Let $f, g \in R = k[x_0, \ldots, x_n]$ be forms, with $\deg(g) = d + \deg(f)$. Letting $F \subset k[y_0, \ldots, y_n, y_{n+1}]$ denote the implicit equation of the parametrization $(fg_0 : \cdots : fg_n : g) : \mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$, one has:*

(i) *F is the y_{n+1}-monoid*

$$\frac{g(\mathbf{g}') - y_{n+1}f(\mathbf{g}')\,D}{\gcd(g(\mathbf{g}'), f(\mathbf{g}')\,D)}. \tag{6}$$

(ii) $\deg(F) = \deg(g)\deg(\mathfrak{G}^{-1}) - \deg(\gcd(g(\mathbf{g}'), f(\mathbf{g}')\,D)) = \deg(f)\deg(\mathfrak{G}^{-1}) + \deg(D) + 1 - \deg(\gcd(g(\mathbf{g}'), f(\mathbf{g}')\,D))$.

In particular, $\deg(F) \leq \deg(g)\deg(\mathfrak{G}^{-1})$.

Proof. (i) Set $S := k[y_0, \ldots, y_{n+1}]/(F)$ for the homogeneous coordinate ring of the image of $\mathfrak{F}$. Since the associated de Jonquières parametrization defined by the generators of (If, g) is birational, a formula such as (2) implies the vanishing of the 2×2 minors of

$$\begin{pmatrix} (fg_{{}_0})(\mathbf{g}') & \cdots & (fg_{{}_n})(\mathbf{g}') & g(\mathbf{g}') \\ y_0 & \cdots & y_n & y_{n+1} \end{pmatrix}$$

modulo (F). In particular, each of the minors

$$P_i := y_i g(\mathbf{g}') - y_{n+1} f(\mathbf{g}') g_i(\mathbf{g}')$$

fixing the last column, is a multiple of F. On the other hand, using (2) for the Cremona map $\mathfrak{G}$ yields $g_i(\mathbf{g}') = y_i\, D$, for $i = 0, \ldots, n$. Since y_i is not a factor of F, it follows that F divides the nonzero y_{n+1}-monoid $g(\mathbf{g}') - y_{n+1} f(\mathbf{g}')\, D$. Clearly, F is not a factor of $\gcd(g(\mathbf{g}'),\, f(\mathbf{g}')\, D)$ as the latter lives in $k[y_0, \ldots, y_n]$ while F involves effectively the variable y_{n+1}.

Now

$$\frac{g(\mathbf{g}') - y_{n+1} f(\mathbf{g}')\, D}{\gcd(g(\mathbf{g}'), f(\mathbf{g}')\, D)}$$

is an y_{n+1}-monoid with relatively prime components, hence is irreducible. Therefore, it must coincide with F.

(ii) Taking degrees, the first equality in the stated formula follows readily, while the subsequent equality follows from the standing assumption that $\deg(g) = \deg(f) + \deg(\mathfrak{G})$ and from the equality $\deg(\mathfrak{G}) \deg(\mathfrak{G}^{-1}) = \deg(D) + 1$ by definition of the inversion factor D. $\square$

Corollary 3.4. *If* $\gcd(f(\mathbf{g}'), g(\mathbf{g}')) = 1$, *then*

$$\deg(F) = \deg(g) \deg(\mathfrak{G}^{-1}) - \deg(\gcd(g(\mathbf{g}'), D)).$$

In particular, $\deg(f) \deg(\mathfrak{G}^{-1}) + 1 \leq \deg(F) < \deg(g) \deg(\mathfrak{G}^{-1})$.

Proof. The displayed equality follows immediately from the formula in Proposition 3.3, so only the lower bound is the question. For that, writing $\deg(g) = \deg(f) + \deg(\mathfrak{G})$ yields $\deg(F) = \deg(f) \deg(\mathfrak{G}^{-1}) + \deg(\mathfrak{G}) \deg(\mathfrak{G}^{-1}) - \deg(\gcd(g(\mathbf{g}'), D))$. But $\deg(\mathfrak{G}) \deg(\mathfrak{G}^{-1}) = \deg(D)$ $+1$, while obviously $\deg(\gcd(g(\mathbf{g}'), D)) \leq \deg(D)$. $\square$

3.3. The inclusion case. We focus on the case where $g \in I$.

Here one has $I : (g) = R$, which in the notation of subsection 2.1 tells us that $\pi : R \to R$ can be taken to be the identity map and the content map $c(g) : R \to R^{n+1}$ picks up only one additional syzygy. Here the syzygy matrix of the given generators of (If, g) is of the form

$$\Psi = \begin{pmatrix} \varphi & c(g) \\ \mathbf{0} & -f \end{pmatrix}, \tag{7}$$

where φ denotes a syzygy matrix of the given set of generators of I and $c(g)$ stands for the column vector defining the content map.

Thus, there is only one syzygetic polynomial $P := \sum_{i=0}^{n} h_i(\mathbf{g}')y_i - f(\mathbf{g}')y_{n+1}$. Keeping the assumptions of Proposition 3.3, one has:

Proposition 3.5. *Assume that* $\gcd(f(\mathbf{g}'), g(\mathbf{g}')) = 1$. *If* $P \in k[y_0, \ldots, y_n, y_{n+1}]$ *is a syzygetic polynomial, the following conditions are equivalent*:

(i) $g \in I$.
(ii) $\deg(F) = \deg(f)\deg(\mathfrak{G}^{-1}) + 1$ *and* $(F) = (P)$.

Proof. (i) $\Rightarrow$ (ii). Quite generally, when $g \in I$, one has $\deg(P) = \deg(f)\deg(\mathfrak{G}^{-1}) + 1$. On the other hand, by Corollary 3.4, $\deg(F) \geq \deg(f)\deg(\mathfrak{G}^{-1}) + 1$. Since F is a factor of P, we are through.

(ii) $\Rightarrow$ (i). Since $\deg(P) = \deg(f)\deg(\mathfrak{G}^{-1})+1$, confronting with the general shape of P as in Definition 3.1 yields $\deg(c_j(g'_0, \ldots, g'_n)) = 0$ for a generator c_j of the conductor $I : g$. This forces c_j to be invertible, so $g \in I$. □

A special case of Proposition 3.5 is a result of **[1]**:

Corollary 3.6. *If* $\mathfrak{G}$ *is the identity map of* $\mathbf{P}^n$ *and* $\gcd(f, g) = 1$, *then*

$$F = g(y_0, \ldots, y_n) - f(y_0, \ldots, y_n)\, y_{n+1}. \tag{8}$$

In particular, $\deg(F) = \deg(f) + 1$.

Proof. In this case, the inverse is also the identity, so $\gcd(f(\mathbf{g}'), g(\mathbf{g}')) = \gcd(f(\mathbf{y}), g(\mathbf{y})) = 1$. On the other hand, the target inversion factor is 1, so the polynomial (6) is $g(y_0, \ldots, y_n) - f(y_0, \ldots, y_n)\, y_{n+1}$. □

At the other end of the spectrum, so to say, we find as a consequence a more "typical" situation:

Corollary 3.7. *Keeping the notation of Proposition* 3.3, *assume that* $g \in I$. *If* f *is a general form, then* $(F) = (P)$.

Proof. Since f is chosen to be general and g and $\mathbf{g}'$ are fixed once for all, the condition $\gcd(f(\mathbf{g}'), g(\mathbf{g}')) = 1$ is fulfilled. $\square$

If f is not sufficiently general it may happen that P as above is not irreducible as the following example entails.

Example 3.8. Take the maximal minors of the following 4×3 matrix over $R = k[x_0, x_1, x_2, x_3]$

$$\begin{pmatrix} 0 & 0 & -x_1 \\ -x_0 & x_0 - x_1 & x_1 \\ x_0 & 0 & 0 \\ x_2 & -x_3 & x_3 \end{pmatrix}.$$

These 3-forms g_0, g_1, g_2, g_3 define a Cremona map of $\mathbf{P}^3$ with inverse given by the 2-forms

$$-y_0y_3,\ y_0y_2,\ -y_1y_3 - y_2y_3,\ -y_2^2 - y_2y_3$$

([**15,** Section 2.1]). If one takes $g \in I = (g_0, g_1, g_2, g_3)$ and f sufficiently special, but still such that $\gcd(f, g) = 1$, then it is apparent that y_3 will come out as a factor of P, e.g., take $g = x_0g_3 = x_0^3x_3$ (the minor corresponding to the last three rows) and $f = x_0 + x_2$; then $P = -y_3F$, where F is the implicit quadric equation.

Yet another special notable case of $g \in I$ is worth isolating as well, where the data are somewhat twisted around. We recall that a form $g \in R = k[\mathbf{x}]$ is called *homaloidal* if its partial derivatives (the so-called *polar map* of G) define a Cremona map. The ideal generated by the partial derivatives of a form is often called its *gradient ideal.*

Corollary 3.9. $(\operatorname{char}(k) = 0)$. *Let* $g \in R = k[\mathbf{x}]$ *denote a reduced homaloidal form of degree* $d+1$, *let* $I \subset R$ *stand for the gradient ideal* *of g, and let $\{g'_0, \ldots, g'_n\} \subset k[y_0, \ldots, y_n]$ define the inverse map of the polar map of g. If $f = \sum_{i=0}^n \lambda_i x_i$ is a general linear form, then*

$$F = \sum_{i=0}^{n} (y_i - (d+1)\lambda_i \, y_{n+1}) \, g'_i(y_0, \ldots, y_n). \tag{9}$$

In particular, $\deg(F) = \deg(g'_i) + 1$.

The polynomial (9) might be called the *general Eulerian equation* of a polar Cremona map.

Remark 3.10. We note that, under the hypothesis that $g \in I$, there is an inclusion $J \subset I$. This triggers a natural injection $\mathcal{R}(J) \subset \mathcal{R}(I)$ of Rees algebras. Thus, in principle, this would give information about the defining Rees equations of J out of these of the base ideal I. However, setting up explicit presentations requires moving around variables, so the ultimate computational advantage is not so clear. Also note that, if, moreover, I is saturated and f is sufficiently general, then $J = I \cap (f, g)$ and $J : I = (f, g)$ (to see the last equality, note it is obvious if $\operatorname{ht} I \geq 3$ since $\{f, g\}$ is a regular sequence, and if $\operatorname{ht} I = 2$ we just need that no minimal prime of I be a minimal prime of (f, g), which is the case if f is sufficiently general). One may ask how implicitization may profit from this simple situation of linkage in a *coarse* sense.

3.4. The non-zero-divisor case. Assume that g is a non-zero-divisor on R/I. In this situation, $I : (g) = I$; hence, the map π in Lemma 2.2 boils down to the structural surjection $\varphi : R^{n+1} \to I$. Accordingly, the content map $c(g)$ reduces to g times the identity map of R^{n+1}. Therefore, a presentation matrix of $J = (If, g)$ now has the form

$$\Psi = \begin{pmatrix} \varphi & g \cdot \mathbf{1}_{n+1} \\ \mathbf{0} & -f\mathbf{g} \end{pmatrix},$$

where φ is a syzygy matrix of I and $\mathbf{g} = (g_0 \cdots g_n)$.

Proposition 3.11. *Let $\mathfrak{G} = (g_0 : \cdots : g_n) : \mathbf{P}^n \dashrightarrow \mathbf{P}^n$ stand for a Cremona map of degree d with base ideal $I = (g_0, \ldots, g_n)$, and let $f, g \in R$ be given as before. Suppose that g is a non-zero-divisor on*

R/I. Then the implicit equation F is a factor of

$$P := g(g'_0, \ldots, g'_n) - f(g'_0, \ldots, g'_n)\, D\, y_{n+1},$$

where $g'_0, \ldots, g'_n \subset k[y_0, \ldots, y_n]$ define the inverse $\mathfrak{G}^{-1}$ to $\mathfrak{G}$ and D is the target inversion factor. In particular, one has

$$\deg(F) \leq \deg(f)\deg(\mathfrak{G}^{-1}) + \deg(D) + 1.$$

Moreover, the following conditions are equivalent:

(a) $(P) = (F)$,
(b) $\gcd(g(\mathbf{g}'), f(\mathbf{g}')D) = 1$,
(c) $\deg(F) = \deg(f)\deg(\mathfrak{G}^{-1}) + \deg(D) + 1$.

Proof. Drawing on the above format of Ψ, consider the 1-form corresponding to a Koszul syzygy as above

$$Q_i(\mathbf{x}, \mathbf{y}) := g y_i - f g_i y_{n+1}, \quad i \in \{0, \ldots, n\},$$

and take the corresponding syzygetic $\mathbf{y}$-polynomial

$$P_i := g(g'_0, \ldots, g'_n) y_i - f(g'_0, \ldots, g'_n)\, g_i(g'_0, \ldots, g'_n)\, y_{n+1}.$$

Note that P_i is the numerator in the expression of F as obtained in the proof of Proposition 3.3 (i).

Clearly, then, (a) through (c) are equivalent assertions. $\square$

Remark 3.12. One wonders what is a more precise choice of f, g that guarantees the irreducibility of the form P in the above proposition. Note that all Koszul-like syzygies of $J = (If, g)$ give rise to the same polynomial P, so there is not much elbow room from this angle.

4. The search for Rees equations.

4.1. The birational downgrading method. For the results of this section, we recall a form of the so-called downgrading map in the context of birational maps. Versions of this notion have been considered before in different contexts ([**2**], [**9**], [**11**]).

Let $\mathbf{x} := \{x_0, \ldots, x_n\}$ and $\mathbf{y} := \{y_0, \ldots, y_n\}$ be two sets of mutually independent variables over k. Given a bihomogeneous polynomial $Q = Q(\mathbf{x}, \mathbf{y}) \in k[\mathbf{x}, \mathbf{y}]$ of bidegree (p, q) $(p \geq 1)$, choose bihomogeneous polynomials $Q_i(\mathbf{x}, \mathbf{y})$, $0 \leq i \leq n$, such that $Q = \sum_{i=0}^{n} x_i\, Q_i(\mathbf{x}, \mathbf{y})$, called an $\mathbf{x}$-*framing* of Q. In addition, fix a sequence of forms of the same degree $\mathfrak{H} := \{\mathfrak{h}_0, \ldots, \mathfrak{h}_n\} \subset k[\mathbf{y}]$.

The polynomial $\sum_{i=0}^{n} \mathfrak{h}_i\, Q_i(\mathbf{x}, \mathbf{y})$ is called an $\mathfrak{H}$-*downgraded polynomial* of Q. We use the notation $D_{\mathfrak{H}}(Q)$ for an $\mathfrak{H}$-downgraded polynomial even though it is not well-defined since the $\mathbf{x}$-*framing* is only stable modulo the trivial (Koszul) relations of $\mathbf{x}$. We will also allow for a harmless flat extension such as $k[\mathbf{x}, \mathbf{y}] \subset k[\mathbf{x}, \mathbf{y}, \mathbf{z}]$, where $\mathbf{z}$ is an additional set of variables.

This general notion will be applied to forms in $k[\mathbf{y}, y_{n+1}]$ while $\mathfrak{H} \subset k[\mathbf{y}]$ is the set of forms defining the inverse of a Cremona map $\mathfrak{F} : \mathbf{P}^n \dashrightarrow \mathbf{P}^n$–in which case, we talk informally about a *birational downgrading.* The common downgrading is typically the case where the Cremona map is the identity map.

As in subsection 3.1, we stick to the notation $\mathcal{R}_R(J) \simeq R[\mathbf{y}]/\mathcal{J}$ for the Rees algebra of an ideal $J \subset R := k[\mathbf{x}]$ even if $\mathbf{x}$ and $\mathbf{y}$ have different cardinalities.

Lemma 4.1. *Let* $\mathbf{g} = \{g_0, \ldots, g_n\} \subset R$ *be forms of fixed degree defining a Cremona map* $\mathfrak{G}$ *of* $\mathbf{P}^n$*, not necessarily without a proper common divisor. Let* $g = g_{n+1} \in R$ *stand for an additional form, of the same degree. Write* $\mathcal{J} \subset R[\mathbf{y}, y_{n+1}]$ *for the presentation of the Rees algebra of the ideal* $J = (\mathbf{g}, g)$ *based on these generators. Let* $\mathfrak{H} := \{g'_0, \ldots, g'_n\} \subset k[\mathbf{y}]$ *denote the set of defining forms of the inverse map to* $\mathfrak{G}$*, and let* $D_{\mathfrak{H}}$ *denote the corresponding birational downgrading. Then*

$$Q = \sum_{r \geq 0} Q_r(\mathbf{x}, \mathbf{y}) y_{n+1}^r \in \mathcal{J} \Longrightarrow D_{\mathfrak{H}}(Q) = \sum_{r \geq 0} D_{\mathfrak{H}}(Q_r) y_{n+1}^r \in \mathcal{J}.$$

Proof. Since the rational map $\mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$ is birational onto its image and $\mathfrak{H}$ (modulo the implicit equation) defines its inverse, by a similar token as in (5), one has an isomorphism

$$\mathcal{R}_R((\mathbf{g}, g)) \simeq \mathcal{R}_S((\mathfrak{H})), \tag{10}$$

where $S = k[\mathbf{y}, y_{n+1}]/(F)$ is the homogeneous coordinate ring of the image of $\mathfrak{F}$. One proceeds as in the argument of Lemma 3.2, with the obvious adaptation, namely, instead of evaluating fully by $x_i \mapsto g'_i$, one only evaluates the variables in a frame. Since fully evaluating either Q or its downgraded partner $D_{\mathfrak{H}}(Q)$ gives a form vanishing on $\mathfrak{H}$ by the isomorphism (10), one has $D_{\mathfrak{H}}(Q) \in \mathcal{J}$ as stated. $\square$

For the subsequent results, we need an iterated version of the framing-downgrading gadget $D_{\mathfrak{H}}(Q)$. Namely, one sets

$$D^0_{\mathfrak{H}}(Q) = Q, \qquad D^{(\ell)}_{\mathfrak{H}}(Q) := D_{\mathfrak{H}}(D^{(\ell-1)}_{\mathfrak{H}}(Q)), \quad \text{for all } \ell \geq 1. \tag{11}$$

We say that $D^{(\ell)}_{\mathfrak{H}}(Q)$ is *fully downgraded* when it eventually lands in $k[y_0, \ldots, y_{n+1}]$, that is, when $\ell = \deg_{\mathbf{x}}(Q)$.

We now apply to the original setup of the base ideal $(If, g) \subset R$, where $I = (g_0, \ldots, g_n)$ is the base ideal of the Cremona map $\mathfrak{G}$, and $\gcd(f, g) = 1$. As before, let $g'_0, \ldots, g'_n$ have $\gcd 1$, defining the inverse map to $\mathfrak{G}$. Accordingly, we take $\mathfrak{H} = \{g'_0, \ldots, g'_n\}$. Note that our previous syzygetic polynomials are among the fully downgraded $D^{(\ell)}_{\mathfrak{H}}(Q)$, for Q a syzygy of J with nonzero last coordinate.

Proposition 4.2. *The defining ideal of the Rees algebra of the ideal $J = (If, g)$ is a minimal prime of the ideal*

$$\mathfrak{D} := \left(\mathcal{I},\ \{D^{(\ell)}_{g'_0,\ldots,g'_n}(Q),\ 0 \leq \ell \leq \deg_{\mathbf{x}}(Q)\}\right),$$

where $\mathcal{I}$ stands for the defining ideal of $\mathcal{R}_R(I)$ and $Q \in k[\mathbf{x}, \mathbf{y}]$ runs through the biforms corresponding to the syzygies of J with nonzero last coordinate.

Proof. By Lemma 4.1, $\mathfrak{D}$ is contained in the presentation ideal of $\mathcal{R}_R(J)$ which has codimension $n+1$ and is a prime ideal. Therefore, it suffices to show that $\mathfrak{D}$ has codimension $n+1$ as well. But $\mathcal{I}$ is a prime ideal of codimension n and, moreover, is contained in the ideal

$(\mathbf{x})k[\mathbf{x},\mathbf{y},y_{n+1}]$ because I is generated by algebraically independent elements over k. Since the fully downgraded elements of $\mathfrak{D}$ belong to $k[\mathbf{y},y_{n+1}]$, this ideal has codimension at least one more. $\square$

Concerning the problem of determining a set of generators of the presentation ideal of $\mathcal{R}_R(J)$, it is not enough to assume that f is a general form in order that the (uniquely determined) fully downgraded be irreducible, as we have seen in the non-zero-divisor case. But, even when no Koszul relation is a minimal syzygy generator, taking f general may not help, as the following simple example indicates.

Example 4.3. Let $I=(x_0x_1,x_0x_2,x_1x_2)$ define the standard quadratic plane Cremona map. Let $f=\lambda_0x_0+\lambda_1x_1+\lambda_2x_2$ be a general form (at least $\lambda_0\lambda_1\lambda_2\neq 0$). Take $g=x_0^2x_1-x_2^3$, for example. The conductor $I:(g)$ is generated by $\{x_0,x_1\}$. Accordingly, a set of minimally generating syzygies of $J=(If,g)$ consists of two linear syzygies coming from I and two additional syzygies corresponding to x_0,x_1. The syzygetic polynomial out of any of the two last syzygies has degree 5 and has a so-called *extraneous* factor of degree 1.

Question 4.4. *Suppose that f is a general form. Does a set of generators of the presentation ideal of $\mathcal{R}_R(J)$ consist of those of $\mathcal{I}$ plus the downgraded polynomials*

$$\{D^{(\ell)}_{g'_0,\dots,g'_n}(Q),\ 0\leq \ell\leq \deg_{\mathbf{x}}(Q)\}$$

divided by the corresponding extraneous factors?

The question lacks any precision since one would have to define "extraneous factor." In any case, the ideal $\mathfrak{D}$ has a central place in this approach, which could be called the *downgraded Rees ideal.*

4.2. The method of the associated monoid parametrization. In this subsection we will take a slightly different approach to get to the presentation ideal of the Rees algebra of $J=(If,g)\subset k[\mathbf{x}]$ defining a de Jonquières parametrization $\mathfrak{F}:\mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}$, with underlying Cremona map $\mathfrak{G}:\mathbf{P}^n\dashrightarrow\mathbf{P}^n$. As in the earlier notation, $I=(g_0,\dots,g_n)\subset k[\mathbf{x}]$, while the inverse map $\mathfrak{G}^{-1}$ is defined by certain forms $g'_0,\dots,g'_n\in k[\mathbf{y}]=k[y_0,\dots,y_n]$.

By Proposition 3.3 (i), F is an y_{n+1}-monoid, say, $F = F_\delta - y_{n+1}F_{\delta-1}$, where $\delta = \deg(F)$ and $F_\delta, F_{\delta-1} \in k[\mathbf{y}]$ are forms of degrees $\delta, \delta-1$, respectively, such that $\gcd(F_\delta, F_{\delta-1}) = 1$.

Set $h_\delta := F_\delta(\mathbf{x})$ and $h_{\delta-1} := F_{\delta-1}(\mathbf{x})$, so $h_\delta, h_{\delta-1} \in k[\mathbf{x}]$ are forms of degrees $\delta, \delta-1$, respectively. Consider the standard monoid parametrization of $\mathrm{Im}\,(\mathfrak{F})$ defined by $h_\delta, h_{\delta-1}$, namely:

$$\mathfrak{M} := (h_{\delta-1}x_0 : \cdots : h_{\delta-1}x_n : -h_\delta) : \mathbf{P}^n \dashrightarrow \mathbf{P}^{n+1}. \tag{12}$$

Write $K := (h_{\delta-1}x_0, \ldots, h_{\delta-1}x_n, h_\delta) \subset k[\mathbf{x}]$ for the base ideal of $\mathfrak{M}$.

Next is the main result of this part.

Theorem 4.5. *With the above notation, one has*:

(a) $\mathfrak{F}$ *and* $\mathfrak{M}$ *have the same implicit equation.*

(b) $\mathfrak{F} = \mathfrak{G} \circ \mathfrak{M}$.

(c) *Let*

$$\mathcal{R}(J) \simeq k[\mathbf{x}, \mathbf{y}, y_{n+1}]/\mathcal{I}_{\mathfrak{F}}$$

and

$$\mathcal{R}(K) \simeq k[\mathbf{x}, \mathbf{y}, y_{n+1}]/\mathcal{I}_{\mathfrak{M}}$$

be presentations of the two Rees algebras based on the given generators. Then

$$\mathcal{I}_{\mathfrak{F}} = \mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) : C^\infty$$

and

$$\mathcal{I}_{\mathfrak{M}} = \mathcal{I}_{\mathfrak{F}}(\mathfrak{G}^{-1}) : D^\infty,$$

with

$$\mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) := \{\mathfrak{h}(g_0, \ldots, g_n; \mathbf{y}, y_{n+1}) \mid \mathfrak{h}(\mathbf{x}; \mathbf{y}, y_{n+1}) \in \mathcal{I}_{\mathfrak{M}}\}$$

and

$$\mathcal{I}_{\mathfrak{F}}(\mathfrak{G}^{-1}) := \{\mathfrak{h}(g'_0(\mathbf{x}), \ldots, g'_n(\mathbf{x}); \mathbf{y}, y_{n+1}) \mid \mathfrak{h}(\mathbf{x}; \mathbf{y}, y_{n+1}) \in \mathcal{I}_{\mathfrak{F}}\}$$

where $C \in k[\mathbf{x}]$ *and* $D \in k[\mathbf{y}]$ *are, respectively, the source inversion factor and the target inversion factor of* $\mathfrak{G}$.

Proof. (a) This follows immediately from Corollary 3.6 and the definition of the forms $h_\delta, h_{\delta-1}$.

(b) This is pretty much tautological as the inverse of $\mathfrak{M}$ is the restriction to $V(F) \subset \mathbf{P}^{n+1}$ of the map $(y_0 : \cdots : y_n)$, a special case of a de Jonquières parametrization, but otherwise very well known (see, e.g., [**12**]). Then, obviously $\mathfrak{M}^{-1} \circ \mathfrak{F} = \mathfrak{G}$, as required.

(c) We first show the inclusions $\mathcal{I}_{\mathfrak{F}}(\mathfrak{G}^{-1}) \subset \mathcal{I}_{\mathfrak{M}}$ and $\mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) \subset \mathcal{I}_{\mathfrak{F}}$.

For the first of these, let $\mathfrak{h}(\mathbf{x}; \mathbf{y}, y_{n+1}) \in \mathcal{I}_{\mathfrak{F}}$ be a bihomogeneous element. By definition, one has $\mathfrak{h}(\mathbf{x}; fg_0, \ldots, fg_n, g) = 0$, while we wish to show that $\mathfrak{h}(\mathbf{g}'(\mathbf{x}); h_{a-1}x_0, \ldots, h_{a-1}x_n, h_a) = 0$. For this, let D denote the target inversion factor of $\mathfrak{G}$. Recall that

$$h_\delta = \frac{g(\mathbf{g}'(\mathbf{x}))}{\deg(f)} \quad \text{and} \quad h_{\delta-1} = \frac{f(\mathbf{g}'(\mathbf{x}))\, D(\mathbf{x})}{\deg(f)},$$

where $\deg(f) = \gcd(g(\mathbf{g}'(\mathbf{x})), f(\mathbf{g}'(\mathbf{x}))\, D(\mathbf{x}))$. Since $\mathfrak{h}$ is bihomogeneous, we can pull out a power of $\deg(f)$ as a factor; hence, the assertion is equivalent to showing that

$$\mathfrak{h}(\mathbf{g}'(\mathbf{x}); f(\mathbf{g}'(\mathbf{x}))D(\mathbf{x})x_0, \ldots, f(\mathbf{g}'(\mathbf{x}))D(\mathbf{x})x_n, g(\mathbf{g}'(\mathbf{x}))) = 0. \tag{13}$$

By definition, $D(\mathbf{x})x_i = g_i(\mathbf{g}'(\mathbf{x}))$, for all i. Therefore, (13) is equivalent to the vanishing of

$$\begin{aligned}\mathfrak{h}(\mathbf{g}'(\mathbf{x}); f(\mathbf{g}'(\mathbf{x}))g_0(\mathbf{g}'(x)), \ldots, f(\mathbf{g}'(\mathbf{x}))g_n(\mathbf{g}'(x)), g(\mathbf{g}'(\mathbf{x})))\\ = \mathfrak{h}(\mathbf{x}; fg_0, \ldots, fg_n, g)(\mathbf{g}'(\mathbf{x})).\end{aligned}$$

The rightmost polynomial is the result of evaluating the null polynomial, so itself is null.

To argue for the second inclusion above, likewise let $\mathfrak{h}(\mathbf{x}; \mathbf{y}, y_{n+1}) \in \mathcal{I}_{\mathfrak{M}}$ be a bihomogeneous element. By definition, $\mathfrak{h}(\mathbf{x}; h_{\delta-1}\mathbf{x}, h_\delta) = 0$, whereas one wishes to show that

$$H := \mathfrak{h}(\mathbf{g}(\mathbf{x}); f(\mathbf{x})\mathbf{g}(\mathbf{x}), g(\mathbf{x})) = 0.$$

For this, we first prove that substituting $\mathbf{g}'(\mathbf{x})$ for $\mathbf{x}$ in H gives zero; namely, by a similar token as above, using the characteristic property of the target factor D, there are suitable integers s, r such that:

$$H(\mathbf{g}'(\mathbf{x})) = \mathfrak{h}(\mathbf{g}(\mathbf{g}'(\mathbf{x})); f(\mathbf{g}'(\mathbf{x}))g_0(\mathbf{g}'(\mathbf{x})), \ldots, f(\mathbf{g}'(\mathbf{x}))g_n(\mathbf{g}'(\mathbf{x})), g(\mathbf{g}'(\mathbf{x})))$$

$$= \deg(f)^r D^s \mathfrak{h}\left(\mathbf{x}; \frac{f(\mathbf{g}'(\mathbf{x}))}{\deg(f)}\mathbf{g}(\mathbf{g}'(\mathbf{x})), \frac{g(\mathbf{g}'(\mathbf{x}))}{\deg(f)}\right)$$
$$= \deg(f)^r D^s \mathfrak{h}(\mathbf{x}; h_{\delta-1}\mathbf{x}, h_\delta) = 0.$$

Consider now the source inversion factor $C \in k[\mathbf{x}]$, whose characteristic property is that $g_i'(\mathbf{g}) = Cx_i$, for all i. Then, for a suitable exponent t, one has

$$\begin{aligned} C^t\, H = C^t\, \mathfrak{h}(\mathbf{g}(\mathbf{x}); f(\mathbf{x})\mathbf{g}(\mathbf{x}), g(\mathbf{x})) &= (\mathfrak{h}(\mathbf{g}(\mathbf{x}); f(\mathbf{x})\mathbf{g}(\mathbf{x}), g(\mathbf{x})))(\mathbf{g}'(\mathbf{g}(\mathbf{x}))) \\ = ((\mathfrak{h}(\mathbf{g}(\mathbf{x}); f(\mathbf{x})\mathbf{g}(\mathbf{x}), g(\mathbf{x})))\, (\mathbf{g}'(\mathbf{x})))\, (\mathbf{g}(\mathbf{x})) &= H(\mathbf{g}'(\mathbf{x}))(\mathbf{g}(\mathbf{x})) = 0, \end{aligned}$$

which proves the assertion.

To complete the proof of the theorem, we show the equality $\mathcal{I}_{\mathfrak{F}} = \mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) : C^\infty$, the other equality being proved in the same fashion. Since $\mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) \subset \mathcal{I}_{\mathfrak{F}}$ and the ideal $\mathcal{I}_{\mathfrak{F}}$ is prime, the inclusion $\mathcal{I}_{\mathfrak{F}} \supset \mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) : C^\infty$ is clear. Conversely, let $\mathfrak{h}(\mathbf{x};\mathbf{y}) \in \mathcal{I}_{\mathfrak{F}}$. By what we have proved above, $\mathfrak{h}(\mathbf{g}'(\mathbf{x});\mathbf{y}) \in \mathcal{I}_{\mathfrak{M}}$, and hence $\mathfrak{h}(\mathbf{g}'(\mathbf{g}(\mathbf{x}));\mathbf{y}) \in \mathcal{I}_{\mathfrak{M}}(\mathfrak{G})$. Again, $\mathbf{g}'(\mathbf{g}(\mathbf{x})) = C\mathbf{x}$ and $\mathfrak{h}(\mathbf{x};\mathbf{y})$ is bihomogeneous. Therefore, for a suitable exponent u, one has $C^u\mathfrak{h}(\mathbf{x};\mathbf{y}) = \mathfrak{h}(\mathbf{g}'(\mathbf{g}(\mathbf{x}));\mathbf{y}) \in \mathcal{I}_{\mathfrak{M}}(\mathfrak{G})$, which says that $\mathfrak{h}(\mathbf{x};\mathbf{y}) \in \mathcal{I}_{\mathfrak{M}}(\mathfrak{G}) : C^\infty$. This proves the other inclusion. $\square$

Remark 4.6. The result of Proposition 4.2 and the one of Theorem 4.5 (c) give different approaches to describe the presentation ideal $\mathcal{I}_{\mathfrak{F}}$ in an explicit way. The first has the advantage of stressing a mechanical way to get the downgraded Rees ideal $\mathcal{D}$; unfortunately, the final step may depend on the knowledge of the primary decomposition of $\mathcal{D}$. The second has the advantage of starting with the simpler ideal $\mathcal{I}_{\mathfrak{M}}$ but is dependent on knowing the implicit equation beforehand and a source inversion factor (the latter being equivalent, in practice, to be able to get an inverse map explicitly). This ideal has also been described in [**1,** Theorem 3.1] in a sort of "reverse" downgrading process starting with the equation F.

It might be appropriate comparing the two procedures for computational as well as theoretical purposes.

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