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EXTREMAL REES ALGEBRAS

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Dedicated to Jürgen Herzog for his numerous contributions to Commutative Algebra on the occasion of his 70th birthday.

ABSTRACT. We study almost complete intersection ideals whose Rees algebras are extremal in the sense that some of their fundamental metrics–depth or relation type–have maximal or minimal values in the class. The focus is on those ideals that lead to *almost* Cohen-Macaulay algebras, and our treatment is wholly concentrated on the nonlinear relations of the algebras. Several classes of such algebras are presented, some of a combinatorial origin. We offer a different prism to look at these questions with accompanying techniques. The main results are effective methods to calculate the invariants of these algebras.

1. Introduction.

Our goal is the study of the defining equations of the Rees algebras $\mathbf{R}[It]$ of classes of almost complete intersection ideals when one of its important metrics, especially depth or reduction number, attains an extreme value in the class. We are going to show that such algebras occur frequently and develop novel means to identify them. As a consequence, interesting properties of such algebras have been discovered. We argue that several questions, while often placed in the general context of Rees algebra theory, may be viewed as subproblems in this more narrowly defined environment.

Let $\mathbf{R}$ be a Cohen-Macaulay local ring of dimension d, or a polynomial ring $\mathbf{R} = k[x_1, \ldots, x_d]$ for k a field. By an *almost complete intersection*

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we mean an ideal $I = (a_1, \ldots, a_g, a_{g+1})$ of codimension g where the subideal $J = (a_1, \ldots, a_g)$ is a complete intersection and $a_{g+1} \notin J$. By the *equations* of I, it is meant a free presentation of the Rees algebra $\mathbf{R}[It]$ of I,

$$(1)\quad 0 \longrightarrow \mathbf{L} \longrightarrow \mathbf{B} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_{g+1}] \xrightarrow{\psi} \mathbf{R}[It] \longrightarrow 0, \quad \mathbf{T}_i \longmapsto f_i t.$$

More precisely, $\mathbf{L}$ is the defining ideal of the Rees algebra of I, but we refer to it simply as the *ideal of equations* of I. We are particularly interested in establishing the properties of $\mathbf{L}$ when $\mathbf{R}[It]$ is Cohen-Macaulay or has almost maximal depth. This broader view requires a change of focus from $\mathbf{L}$ to one of its quotients. We are going to study some classes of ideals whose Rees algebras have these properties. They tend to occur in classes where the reduction number $\mathrm{red}_J(I)$ attains an extremal value.

We first set up the framework to deal with properties of $\mathbf{L}$ by a standard decomposition. We keep the notation of above, $I = (J, a)$. The presentation ideal $\mathbf{L}$ of $\mathbf{R}[It]$ is a graded ideal $\mathbf{L} = L_1 + L_2 + \cdots$, where L_1 are linear forms in the $\mathbf{T}_i$ defined by a matrix ϕ of the syzygies of I, $L_1 = [\mathbf{T}_1, \ldots, \mathbf{T}_{g+1}] \cdot \phi$. Our basic prism is given by the exact sequence

$$0 \longrightarrow \mathbf{L}/(L_1) \longrightarrow \mathbf{B}/(L_1) \longrightarrow \mathbf{R}[It] \longrightarrow 0.$$

Here $\mathbf{B}/(L_1)$ is a presentation of the symmetric algebra of I and $\mathbf{S} = \mathrm{Sym}\,(I)$ is a Cohen-Macaulay ring under very broad conditions, including when I is an ideal of finite colength. The emphasis here will be entirely on $T = \mathbf{L}/(L_1)$, which we call the *module of nonlinear* relations of I. The usefulness arises because of the fact exhibited in the exact sequence

$$(2)\qquad 0 \longrightarrow T \longrightarrow \mathbf{S} \longrightarrow \mathbf{R}[It] \longrightarrow 0.$$

• (Proposition 2.5). *T is a Cohen-Macaulay $\mathbf{S}$-module if and only if* $\mathrm{depth}\,\mathbf{R}[It] \geq d$.

In general, we say that a commutative Noetherian ring R is *almost Cohen-Macaulay* (aCM for short) if $\mathrm{grade}\,(\mathfrak{m}) \geq \mathrm{height}\,(\mathfrak{m}) - 1$ for every maximal ideal $\mathfrak{m}$ of R. We note that $\mathbf{L}$ carries a very different kind of information than T does. The advantage lies in the flexibility of treating Cohen-Macaulay modules versus Cohen-Macaulay ideals: the means to test for Cohen-Macaulayness in modules are more plentiful than in ideals. An elementary example lies in the proof of:

- (Theorem 2.10). Suppose that $\mathbf{R}$ is a Cohen-Macaulay local ring and I is an $\mathfrak{m}$-primary almost complete intersection such that $\mathbf{S} = \mathrm{Sym}\,(I)$ is reduced. Then $\mathbf{R}[It]$ is almost Cohen-Macaulay.

We shall now discuss our more technical results. Throughout, $(\mathbf{R}, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d (where we include rings of polynomials and the ideals are homogeneous), and I is an almost complete intersection $I = (J, a)$ of finite colength.

One technique we bring in to the treatment of equations of Rees algebras is the theory of the Sally module. It gives a very direct relationship between Cohen-Macaulayness of T and of the Sally module $S_J(I)$ of I relative to J. $S_J(I)$ also gives a quick connection between the Castelnuovo regularity and the relation type of $\mathbf{R}[It]$ and those of $S_J(I)$. A criterion of Huckaba ([**21**]) (of which we give a quick proof for completeness) gives a method to test the Cohen-Macaulayness of T in terms of the values of the first Hilbert coefficient $e_1(I)$ of I (Theorem 3.7). It is particularly well-suited for the case when I is generated by homogeneous polynomials defining a birational mapping for then the value of $e_1(I)$ is known.

Our approach to the estimation for $\nu(T)$, the minimum number of generators of T, passes through the determination of an effective formula for $\deg \mathbf{S}$, the multiplicity of $\mathbf{S}$:

- (Theorem 3.20). If I is generated by forms of degree n, then

$$\deg \mathbf{S} = \sum_{j=0}^{d-1} n^j + \lambda(I/J).$$

The summation accounts for $\deg \mathbf{R}[It]$, according to [**16**], so $\deg T = \lambda(I/J)$. This is a number that will control the number of generators of T, and therefore of $\mathbf{L}$, whenever $\mathbf{R}[It]$ is almost Cohen-Macaulay. It achieves the goal of finding estimates for the number of generators of $\mathbf{L}$ and of its *relation type*, that is

$$\mathrm{reltype}\,(I) = \inf\{n \mid \mathbf{L} = (L_1, L_2, \ldots, L_n)\}.$$

Two other metrics of interest, widely studied for homogeneous ideals but not limited to them, are the following. One seeks to bound the *saturation exponent* of $\mathbf{L}/(L_1)$ (which was introduced in [**19**] and has a simple ring-theoretic explanation as the index of nilpotency of $\mathbf{S}$),

$$\operatorname{sdeg}(I) = \inf\{s \mid \mathfrak{m}^s\mathbf{L} \subset (L_1)\},$$

and the other is the degree of the special fiber $\mathcal{F}(I)$ of $\mathbf{R}[It]$, also called the *elimination degree* of I,

$$\operatorname{edeg}(I) = \deg \mathcal{F}(I) = \inf\{s \mid L_s \not\subset \mathfrak{m}\mathbf{B}\}.$$

While $\operatorname{retype}(I)$ is the most critical of these numbers, the other two are significant because they are often found linked to the syzygies of I. Our notion of extremality will cover the supremum or infimum values of these degrees in a given class of ideals but also their relationship to the cohomology of $\mathbf{R}[It]$ as expressed by the depth of the algebra.

Two classes of almost Cohen-Macaulay algebras arise from certain homogeneous ideals. First, we show that binary ideals with one linear syzygy have this property. This has been proved by several authors. We offer a very short proof using the technology of the Sally module (Proposition 4.6). It runs for a few lines and gives no details of the projective resolution of that algebra besides the fact that it has the appropriate length. We include it because we have found no similar technique in the literature. The proof structure, a simple combinatorial obstruction to the aCM property, is used repeatedly to examine the occurrence of the property amongst ideals generated by quadrics in 4-space.

A different class of algebras is that associated to monomials. These ideals have the form $I = (x^\alpha, y^\beta, z^\gamma, x^a y^b z^c)$. We showed that

- (Proposition 4.12). The ideals (x^n, y^n, z^n, xyz), $n \geq 3$, and $(x^n, y^n, z^n, w^n, xyzw)$, $n \geq 4$, have almost Cohen-Macaulay Rees algebras.

We expect these statements are still valid in higher dimensions. Our proofs were computer-assisted as we used Macaulay 2 ([**11**]) to derive deeper heuristics.

2. Approximation complexes and almost complete intersections. A main source of extremal Rees algebras lie in the construction of approximation complexes. We quickly recall them and some of their main properties.

2.1. The $\mathcal{Z}$-complex. These are complexes derived from Koszul complexes and arise as follows (for details, see [**13, 14**], [**29**, Chapter 4]). Let $\mathbf{R}$ be a commutative ring, F a free $\mathbf{R}$-module of rank n with a basis $\{e_1, \ldots, e_n\}$, and $\varphi : F \to \mathbf{R}$ a homomorphism. The exterior algebra $\bigwedge F$ of F can be endowed with a differential

$$\partial_\varphi : \bigwedge^r F \longrightarrow \bigwedge^{r-1} F,$$

$$\partial_\varphi(v_1 \wedge v_2 \wedge \cdots \wedge v_r) = \sum_{i=1}^{r} (-1)^i \varphi(v_i)(v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_r).$$

The complex $\mathbf{K}(\varphi) = \{\bigwedge F, \partial_\varphi\}$ is called the *Koszul complex* of φ. Another notation for it is: If $\mathbf{x} = \{\varphi(e_1), \ldots, \varphi(e_n)\}$, denote the Koszul complex by $\mathbf{K}(\mathbf{x})$.

Let $\mathbf{S} = S(F) = \mathrm{Sym}\,(F) = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$, and consider the exterior algebra of $F \otimes_{\mathbf{R}} S(F)$. It can be viewed as a Koszul complex obtained from $\{\bigwedge F, \partial_\varphi\}$ by change of scalars $\mathbf{R} \to \mathbf{S}$, and another complex defined by the $\mathbf{S}$-homomorphism

$$\psi : F \otimes_{\mathbf{R}} S(F) \longrightarrow S(F), \quad \psi(e_i) = \mathbf{T}_i.$$

The two differentials ∂_φ and ∂_ψ satisfy

$$\partial_\varphi \partial_\psi + \partial_\psi \partial_\varphi = 0,$$

which leads directly to the construction of several complexes.

Definition 2.1. Let $\mathbf{Z}$, $\mathbf{B}$ and $\mathbf{H}$ be the modules of cycles, boundaries and the homology of $\mathbf{K}(\varphi)$.

- The $\mathcal{Z}$-complex of φ is $\mathcal{Z} = \{\mathbf{Z} \otimes_{\mathbf{R}} \mathbf{S}, \partial\}$

$$0 \longrightarrow Z_n \otimes \mathbf{S}[-n] \longrightarrow \cdots \longrightarrow Z_1 \otimes \mathbf{S}[-1] \longrightarrow \mathbf{S} \longrightarrow 0,$$

where ∂ is the differential induced by ∂_ϕ.

- The $\mathcal{B}$-complex of φ is the subcomplex of $\mathcal{Z}$

$$0 \longrightarrow B_n \otimes \mathbf{S}[-n] \longrightarrow \cdots \longrightarrow B_1 \otimes \mathbf{S}[-1] \longrightarrow \mathbf{S} \longrightarrow 0.$$

- The $\mathcal{M}$-complex of φ is $\mathcal{M} = \{\mathbf{H} \otimes_{\mathbf{R}} \mathbf{S}, \partial\}$

$$0 \longrightarrow \mathrm{H}_n \otimes \mathbf{S}[-n] \longrightarrow \cdots \longrightarrow \mathrm{H}_1 \otimes \mathbf{S}[-1] \longrightarrow \mathrm{H}_0 \otimes \mathbf{S} \longrightarrow 0,$$

where ∂ is the differential induced by ∂_ψ.

These are complexes of graded modules over the polynomial ring $\mathbf{S} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_n]$.

Proposition 2.2. *Let $I = \varphi(F)$. Then*:

(i) *The homology of $\mathcal{Z}$ and of $\mathcal{M}$ depend only on I*;

(ii) $\mathcal{H}_0(\mathcal{Z}) = \mathrm{Sym}\,(I)$.

Acyclicity. The homology of the Koszul complex $\mathbf{K}(\varphi)$ is not fully independent of I; for instance, it depends on the number of generators. An interest here is the ideals whose $\mathcal{Z}$ complexes are acyclic.

We recall a broad setting that gives rise to almost complete intersections with Cohen-Macaulay symmetric algebras. For a systematic examination of the notions here we refer to [**14**]. It is centered on one approximation complex associated to an ideal, the so-called $\mathcal{Z}$-complex. A significant interest for us is the following.

Theorem 2.3 [**14**, Theorem 10.1]. *Let $\mathbf{R}$ be a Cohen-Macaulay local ring, and let I be an ideal of positive height. Assume*:

(a) $\nu(I_{\mathfrak{p}}) \leq \mathrm{height}\,\mathfrak{p} + 1$ *for every* $\mathfrak{p} \supset I$;

(b) $\mathrm{depth}\,(\mathrm{H}_i)_{\mathfrak{p}} \geq \mathrm{height}\,\mathfrak{p} - \nu(I_{\mathfrak{p}}) + 1$ *for every* $\mathfrak{p} \supset I$ *and every* $0 \leq i \leq \nu(I_{\mathfrak{p}}) - \mathrm{height}\, I_{\mathfrak{p}}$.

Then

(i) *The complex $\mathcal{Z}$ is acyclic.*

(ii) $\mathrm{Sym}\,(I)$ *is a Cohen-Macaulay ring.*

Corollary 2.4. *Let* $\mathbf{R}$ *be a Cohen-Macaulay local ring of dimension* $d \geq 1$*, and let* I *be an almost complete intersection. The complex* $\mathcal{Z}$ *is acyclic and* $\mathrm{Sym}\,(I)$ *is a Cohen-Macaulay algebra in the following cases*:

(i) (See also [**26**].) I *is* $\mathfrak{m}$*-primary. In this case* $\mathrm{Sym}\,(I)$ *has Cohen-Macaulay type* $d-1$*.*

(ii) $\text{height}\, I = d-1$*. Furthermore, if* I *is generically a complete intersection then* I *is of linear type.*

(iii) $\text{height}\, I = d-2$ *and* $\text{depth}\, \mathbf{R}/I \geq 1$*. Furthermore, if* $\nu(I_{\mathfrak{p}}) \leq \text{height}\, \mathfrak{p}$ *for* $I \subset \mathfrak{p}$*, then* I *is of linear type.*

A different class of ideals with Cohen-Macaulay symmetric algebras is treated in [**22**].

2.2. The canonical presentation. Let $\mathbf{R}$ be a Cohen-Macaulay local domain of dimension $d \geq 1$, and let I be an almost complete intersection as in Corollary 2.4. The ideal of equations $\mathbf{L}$ can be studied in two stages: (L_1) and $\mathbf{L}/(L_1) = T$:

$$0 \longrightarrow T \longrightarrow \mathbf{S} = \mathbf{B}/(L_1) = \mathrm{Sym}\,(I) \longrightarrow \mathbf{R}[It] \longrightarrow 0. \tag{3}$$

We will argue that this exact sequence is very useful. Note that $\mathrm{Sym}\,(I)$ and $\mathbf{R}[It]$ have dimension $d+1$, and that T is the $\mathbf{R}$-torsion submodule of $\mathbf{S}$. Let us give some of its properties.

Proposition 2.5. *Let* I *be an ideal as above.*

(i) (L_1) *is a Cohen-Macaulay ideal of* $\mathbf{B}$*.*

(ii) T *is a Cohen-Macaulay* $\mathbf{S}$*-module if and only if depth* $\mathbf{R}[It] \geq d$*.*

(iii) *If* I *is* $\mathfrak{m}$*-primary, then* $\mathcal{N} = T \cap \mathfrak{m}\mathbf{S}$ *is the nil radical of* $\mathbf{S}$ *and* $\mathcal{N}^s = 0$ *if and only if* $\mathfrak{m}^s T = 0$*. This is equivalent to saying that* $\mathrm{sdeg}\,(I)$ *is the index of nilpotency of* $\mathrm{Sym}\,(I)$*.*

(iv) $T = \mathcal{N} + \mathcal{F}$*, where* $\mathcal{F}$ *is a lift in* $\mathbf{S}$ *of the relations in* $\mathbf{S}/\mathfrak{m}\mathbf{S}$ *of the special fiber ring* $\mathcal{F}(I) = \mathbf{R}[It] \otimes \mathbf{R}/\mathfrak{m}$*. In particular if* $\mathcal{F}(I)$ *is a hypersurface ring,* $T = (f, \mathcal{N})$*.*

Proof. (i) comes from Corollary 2.4.

(ii) In the defining sequence of T,

$$0 \longrightarrow (L_1) \longrightarrow \mathbf{L} \longrightarrow T \longrightarrow 0,$$

since (L_1) is a Cohen-Macaulay ideal of codimension g, as a $\mathbf{B}$-module, we have $\operatorname{depth}(L_1) = d+2$, while $\operatorname{depth}\mathbf{L} = 1+\operatorname{depth}\mathbf{R}[It]$. It follows that $\operatorname{depth} T = \min\{d+1, 1+\operatorname{depth}\mathbf{R}[It]\}$. Since T is a module of Krull dimension $d+1$, it is a Cohen-Macaulay module if and only if $\operatorname{depth}\mathbf{R}[It] \geq d$.

(iii) $\mathfrak{m}\mathbf{S}$ and T are both minimal primes and, for large n, $\mathfrak{m}^n T = 0$. Thus, T and $\mathfrak{m}\mathbf{S}$ are the only minimal primes of $\mathbf{S}$, $\mathcal{N} = \mathfrak{m}\mathbf{S} \cap T$. To argue the equality of the two indices of nilpotency, let n be such that $\mathfrak{m}^n T = 0$. The ideal $\mathfrak{m}^n\mathbf{S} + T$ has positive codimension, so it contains regular elements since $\mathbf{S}$ is Cohen-Macaulay. Therefore, to show

$$\mathfrak{m}^s T = 0 \Longleftrightarrow \mathcal{N}^s = 0,$$

it is enough to multiply both expressions by $\mathfrak{m}^n\mathbf{S} + T$. The verification is immediate.

(iv) Tensoring the sequence (3) by $\mathbf{R}/\mathfrak{m}$ gives the exact sequence

$$0 \longrightarrow \mathfrak{m}\mathbf{S} \cap T/\mathfrak{m}T = \mathcal{N}/\mathfrak{m}T \longrightarrow T/\mathfrak{m}T \longrightarrow \mathbf{S}/\mathfrak{m}\mathbf{S} \longrightarrow \mathcal{F}(I) \longrightarrow 0.$$

By Nakayama's lemma, we may ignore $\mathfrak{m}T$ and recover T as asserted. □

The main intuition derived from Proposition 2.5 is that, whatever methods are developed to study the equations of $\mathbf{R}[It]$ when this algebra is Cohen-Macaulay, should apply in case they are almost Cohen-Macaulay.

Remark 2.6. If I is not $\mathfrak{m}$-primary but still satisfies one of the other conditions of Corollary 2.4, the nilradical $\mathcal{N}$ of $\mathbf{S}$ is given by $T \cap N_0\mathbf{S}$, where N_0 is the intersection of the minimal primes $\mathfrak{p}$ for which $I_{\mathfrak{p}}$ is not of linear type.

2.3. Reduced symmetric algebras.

Proposition 2.7. *Let $\mathbf{R}$ be a Gorenstein local domain of dimension d, and let J be a parameter ideal. If J contains two minimal* *generators in* $\mathfrak{m}^2$*, then the Rees algebra of* $I = J\colon\mathfrak{m}$ *is Cohen-Macaulay and* $\mathbf{L} = (L_1, \mathbf{f})$ *for some quadratic form* $\mathbf{f}$*.*

Proof. The equality $I^2 = JI$ comes from [**4**]. The Cohen-Macaulayess of $\mathbf{R}[It]$ is a general argument (in [**4**] and probably elsewhere). Let $\mathbf{f}$ denote the quadratic form

$$\mathbf{f} = \mathbf{T}_{d+1}^2 + \text{lower terms.}$$

Let us show that $\mathfrak{m}T = 0$. Reduction modulo $\mathbf{f}$ can be used to present any element in $\mathbf{L}$ as

$$F = \mathbf{T}_{d+1} \cdot A + B \in \mathbf{L},$$

where A and B are forms in $\mathbf{T}_1, \ldots, \mathbf{T}_d$. Since $I = J : \mathfrak{m}$, any element in $\mathfrak{m}\mathbf{T}_{d+1}$ is equivalent, modulo L_1, to a linear form in the other variables. Consequently,

$$\mathfrak{m}F \subset (L_1, \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_d]) \cap \mathbf{L} \subset (L_1),$$

as desired.

By Proposition 2.5, we have the exact sequence

$$0 \longrightarrow T \longrightarrow \mathbf{S}/\mathfrak{m}\mathbf{S} \longrightarrow \mathcal{F}(I) \longrightarrow 0.$$

But T is a maximal Cohen-Macaulay $\mathbf{S}$-module, and so it is also a maximal Cohen-Macaulay $\mathbf{S}/\mathfrak{m}\mathbf{S}$-module as well. It follows that T is generated by a monic polynomial that divides the image of $\mathbf{f}$ in $\mathbf{S}/\mathfrak{m}\mathbf{S}$. It is now clear that $T = (\mathbf{f})\mathbf{S}$. □

Corollary 2.8. *For the ideals above* $\mathrm{Sym}\,(I)$ *is reduced.*

We now discuss a generalization, but since we are still developing the examples, we are somewhat informal.

Corollary 2.9. *Suppose the syzygies of* I *are contained in* $\mathfrak{m}^s\mathbf{B}$ *and that* $\mathfrak{m}^s\mathbf{L} \subset (L_1)$*. We have the exact sequence*

$$0 \longrightarrow T \longrightarrow \mathbf{S}/\mathfrak{m}^s\mathbf{S} \longrightarrow \mathbf{R}[It] \otimes \mathbf{R}/\mathfrak{m}^s = \mathcal{F}_s(I) \longrightarrow 0. \tag{4}$$

If $\mathbf{R}[It]$ *is almost Cohen-Macaulay,* T *is a Cohen-Macaulay module that is an ideal of the polynomial ring* $\mathbf{C} = \mathbf{R}/(\mathfrak{m}^s)[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$*, a ring of multiplicity* $\binom{s+d-1}{d}$*. Therefore, we have that* $\nu(T) \leq \binom{s+d-1}{d}$*.*

Note that also here $\mathcal{F}_s(I)$ is Cohen-Macaulay. We wonder whether $\mathcal{F}(I)$ is Cohen-Macaulay.

Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring and I an almost complete intersection as in Corollary 2.4. We examine the following surprising fact.

Theorem 2.10. *Suppose that* $\mathbf{R}$ *is a Cohen-Macaulay local ring and* I *is an* $\mathfrak{m}$*-primary almost complete intersection such that* $\mathbf{S} = \mathrm{Sym}\,(I)$ *is reduced. Then* $\mathbf{R}[It]$ *is an almost Cohen-Macaulay algebra.*

Proof. Since $0 = \mathcal{N} = T \cap \mathfrak{m}\mathbf{S}$, on one hand from (3) we have that T satisfies the S_2 condition of Serre, that is,

$$\operatorname{depth} T_P \geq \inf\{2, \dim T_P\}$$

for every prime ideal P of $\mathbf{S}$. On the other hand, from (4), T is an ideal of the polynomial ring $\mathbf{S}/\mathfrak{m}\mathbf{S}$. It follows that $T = (\mathbf{f})\mathbf{S}$, and consequently, $\operatorname{depth} \mathbf{R}[It] \geq d$. $\square$

Example 2.11. If $\mathbf{R} = \mathbf{Q}[x, y]/(y^4 - x^3)$, $J = (x)$ and $I = J : (x, y) = (x, y^3)$, $\operatorname{depth} \mathbf{R}[It] = 1$.

There are a number of immediate observations.

Corollary 2.12. *If* I *is an ideal as in Theorem* 2.10*, then the special fiber ring* $\mathcal{F}(I)$ *is Cohen-Macaulay.*

Remark 2.13. If I is an almost complete intersection as in (2.4) and its radical is a regular prime ideal P, that is $\mathbf{R}/P$ is regular local ring, the same assertions will apply if $\mathrm{Sym}\,(I)$ is reduced.

3. Almost Cohen-Macaulay algebras. We begin our treatment of the properties of an ideal when its Rees algebra $\mathbf{A} = \mathbf{R}[It]$ is almost Cohen-Macaulay. We first describe a large class of examples.

3.1. Direct links of Gorenstein ideals. We briefly outline a broad class of extremal Rees algebras. Let $(\mathbf{R}, \mathfrak{m})$ be a Gorenstein local ring of dimension $d \geq 1$. A natural source of almost complete intersections in $\mathbf{R}$ are direct links of Gorenstein ideals. That is, let K be a Gorenstein ideal of $\mathbf{R}$ of codimension s, that is, $\mathbf{R}/K$ is a Gorenstein ring of dimension $d - s$. If $J = (a_1, \ldots, a_s) \subset K$ is a complete intersection of codimension s, $J \neq K$, $I = J : K$ is an almost complete intersection, $I = (J, a)$. Depending on K, sometimes these ideals come endowed with very good properties. Let us recall one of them.

Proposition 3.1. *Let $(\mathbf{R}, \mathfrak{m})$ be a Noetherian local ring of dimension d.*

(i) [**4**, Theorem 2.1]. *Suppose $\mathbf{R}$ is a Cohen-Macaulay local ring, let $\mathfrak{p}$ be a prime ideal of codimension s such that $\mathbf{R}_\mathfrak{p}$ is a Gorenstein ring and let J be a complete intersection of codimension s contained in $\mathfrak{p}$. Then, for $I = J : \mathfrak{p}$, we have $I^2 = JI$ in the following two cases*:

(a) $\mathbf{R}_\mathfrak{p}$ *is not a regular local ring*;

(b) *if $\mathbf{R}_\mathfrak{p}$ is a regular local ring two of the elements a_i belong to $\mathfrak{p}^{(2)}$.*

(ii) [**3**, Theorem 3.7]. *Suppose J is an irreducible $\mathfrak{m}$-primary ideal. Then*

(a) *either there exists a minimal set of generators $\{x_1, \ldots, x_d\}$ of $\mathfrak{m}$ such that $J = (x_1, \ldots, x_{d-1}, {x_d}^r)$, or*

(b) *$I^2 = JI$ for $I = J : \mathfrak{m}$.*

The following criterion is a global version of Corollary 3.13.

Proposition 3.2. *Let $\mathbf{R}$ be a Gorenstein local ring and $I = (J, a)$ an almost complete intersection* (*when we write $I = (J, a)$ we always mean that J is a reduction*). *If I is an unmixed ideal* (*height unmixed*) *then* $\operatorname{red}_J(I) \leq 1$ *if and only if $J : a = I_1(\phi)$.*

Proof. Since the ideal JI is also unmixed, to check the equality $J : a = I_1(\phi)$ we only need to check at the minimal primes of I (or, of J, as they are the same). Now Corollary 3.13 applies. ◻

If, in Proposition 3.1, $\mathbf{R}$ is a Gorenstein local ring and I is a Cohen-Macaulay ideal, their associated graded rings are Cohen-Macaulay, while the Rees algebras are also Cohen-Macaulay if $\dim \mathbf{R} \geq 2$.

Theorem 3.3. *Let* $\mathbf{R}$ *be a Gorenstein local ring and* I *a Cohen-Macaulay ideal that is an almost complete intersection. If* $\mathrm{red}_J(I) \leq 1$*, then in the canonical representation*

$$0 \longrightarrow T \longrightarrow \mathbf{S} \longrightarrow \mathbf{R}[It] \longrightarrow 0,$$

(i) *If* $\dim \mathbf{R} \geq 2$ $\mathbf{R}[It]$ *is Cohen-Macaulay.*

(ii) T *is a Cohen-Macaulay module over* $\mathbf{S}/(I_1(\phi))\mathbf{S}$*, in particular*

$$\nu(T) \leq \deg \mathbf{R}/I_1(\phi).$$

Example 3.4. Let $\mathbf{R} = k[x_1, \ldots, x_d]$, k an algebraically closed field, and let $\mathfrak{p}$ be a homogeneous prime ideal of codimension $d-1$. Suppose $J = (a_1, \ldots, a_{d-1})$ is a complete intersection of codimension $d-1$ with at least two generators in $\mathfrak{p}^2$. Since $\mathbf{R}/\mathfrak{p}$ is regular, $I = J : \mathfrak{p}$ is an almost complete intersection and $I^2 = JI$. Since $\mathfrak{p}$ is a complete intersection, say $\mathfrak{p} = (x_1 - c_1x_d, x_2 - c_2x_d, \ldots, x_{d-1} - c_{d-1}x_d)$, $c_i \in k$, we write the matrix equation $J = \mathbf{A} \cdot \mathfrak{p}$, where $\mathbf{A}$ is a square matrix of size $d-1$. This is the setting where the Northcott ideals occur, and therefore $I = (J, \det \mathbf{A})$.

By Theorem 3.3 (ii), $\nu(T) \leq \deg(\mathbf{R}/\mathfrak{p}) = 1$. Thus, $\mathbf{L}$ is generated by the syzygies of I (which are well-understood) plus a quadratic equation.

3.2. Metrics of aCM Rees algebras. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d, and let I be an almost complete intersection of finite colength. We assume that $I = (J, a)$, where J is a minimal reduction of I. These assumptions will hold for the remainder of the section. We emphasize that they apply to the case when $\mathbf{R}$ is a polynomial ring over a field and I is a homogeneous ideal.

In the next statement we highlight information about the equations of I that are a direct consequence of the aCM hypothesis. In the next segments we begin to obtain the required data in an explicit form. As for notation, $\mathbf{B} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$ and, for a graded $\mathbf{B}$-module A, $\deg(A)$ denotes the multiplicity relative to the maximal homogeneous ideal $\mathcal{M}$ of $\mathbf{B}$, $\deg(A) = \deg(\mathrm{gr}_{\mathcal{M}}(A))$. In actual computations $\mathcal{M}$ can be replaced by a reduction. For instance, if E is a graded $\mathbf{R}$-module and $A = E \otimes_{\mathbf{R}} \mathbf{B}$, picking a reduction J for $\mathfrak{m}$ gives the reduction $\mathcal{N} = (J, \mathbf{T}_1, \ldots, \mathbf{T}_{d+1})$ of $\mathcal{M}$. It will follow that $\deg(A) = \deg(E)$.

Theorem 3.5. *If the algebra $\mathbf{R}[It]$ is almost Cohen-Macaulay, in the canonical sequence*

$$0 \longrightarrow T \longrightarrow \mathbf{S} \longrightarrow \mathbf{R}[It] \longrightarrow 0,$$

(i) $\mathrm{reg}(\mathbf{R}[It]) = \mathrm{red}_J(I) + 1$.

(ii) $\nu(T) \leq \deg(\mathbf{S}) - \deg(\mathbf{R}[It])$.

Proof. (i) follows from Corollary 3.9. As for (ii), since T is a Cohen-Macaulay module, $\nu(T) \leq \deg(T)$. □

The goal is to find $\deg(T)$, $\deg(\mathbf{R}[It])$ and $\deg(\mathbf{S})$ in terms of more direct metrics of I. This will be answered in Theorem 3.17.

Cohen-Macaulayness of the Sally module. Fortunately there is a simple criterion to test whether $\mathbf{R}[It]$ is an aCM algebra: It is so if and only if it satisfies the Huckaba test:

$$e_1(I) = \sum_{j\geq 1} \lambda(I^j/JI^{j-1}).$$

Needless to say, this is exceedingly effective if you already know $e_1(I)$, in particular there is no need to determine the equations of $\mathbf{R}[It]$ for the purpose.

Let $\mathbf{R}$ be a Noetherian ring, I an ideal and J a reduction of I. The Sally module of I relative to J, $S_J(I)$, is defined by the exact sequence of $\mathbf{R}[Jt]$-modules

$$0 \longrightarrow I\mathbf{R}[Jt] \longrightarrow I\mathbf{R}[It] \longrightarrow S_J(I) = \bigoplus_{j\geq 2} I^j/IJ^{j-1} \longrightarrow 0.$$

The definition applies more broadly to other filtrations. We refer the reader to [**30**, page 101] for a discussion. Of course, this module depends on the chosen reduction J, but its Hilbert function and its depth are independent of J. There are extensions of this construction to more general reductions-and we employ one below.

If $\mathbf{R}$ is a Cohen-Macaulay local ring and I is $\mathfrak{m}$-primary with a minimal reduction, $S_J(I)$ plays a role in mediating among properties of $\mathbf{R}[It]$.

Proposition 3.6. *Suppose* $\mathbf{R}$ *is a Cohen-Macaulay local ring of dimension* d. *Then*:

(i) *If* $S_J(I) = 0$ *then* $\mathrm{gr}_I(\mathbf{R})$ *is Cohen-Macaulay.*

(ii) *If* $S_J(I) \neq 0$, *then* $\dim S_J(I) = d$.

Some of the key properties of the Sally module are in display in the next result ([**21**, Theorem 3.1]). It converts the property of $\mathbf{R}[It]$ being almost Cohen-Macaulay into the property of $S_J(I)$ being Cohen-Macaulay.

Theorem 3.7 (Huckaba theorem). *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Cohen-Macaulay local ring of dimension* $d \geq 1$ *and* J *a parameter ideal. Let* $\mathcal{A} = \{I_n, n \geq 0\}$ *be a filtration of* $\mathfrak{m}$*-primary ideals such that* $J \subset I_1$ *and* $\mathbf{B} = \mathbf{R}[I_n t^n, n \geq 0]$ *is* $\mathbf{A} = \mathbf{R}[Jt]$*-finite. Define the Sally module* $S_{\mathbf{B}/\mathbf{A}}$ *of* $\mathbf{B}$ *relative to* $\mathbf{A}$ *by the exact sequence*

$$0 \longrightarrow I_1\mathbf{A} \longrightarrow I_1\mathbf{B} \longrightarrow S_{\mathbf{B}/\mathbf{A}} \longrightarrow 0.$$

Suppose $S_{\mathbf{B}/\mathbf{A}} \neq 0$. *Then*:

(i) $e_0(S_{\mathbf{B}/\mathbf{A}}) = e_1(\mathbf{B}) - \lambda(I_1/J) \leq \sum_{j\geq 2} \lambda(I_j/JI_{j-1})$.

(ii) *The following conditions are equivalent*:

(a) $S_{\mathbf{B}/\mathbf{A}}$ *is Cohen-Macaulay*;

(b) $\operatorname{depth} \mathrm{gr}_{\mathcal{A}}(\mathbf{R}) \geq d - 1$;

(c) $e_1(\mathbf{B}) = \sum_{j\geq 1} \lambda(I_j/JI_{j-1})$;

(d) *If* $I_N = I^n$, $\mathbf{R}[It]$ *is almost Cohen-Macaulay.*

Proof. If $J = (\mathbf{x}) = (x_1, \ldots, x_d)$, $S_{\mathbf{B}/\mathbf{A}}$ is a finite module over the ring $\mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_d]$, $\mathbf{T}_i \to x_i t$. Note that

$$\lambda(S_{\mathbf{B}/\mathbf{A}}/\mathbf{x}S_{\mathbf{B}/\mathbf{A}}) = \sum_{j\geq 2} \lambda(I_j/JI_{j-1}),$$

which shows the first assertion.

For the equivalencies, first note that equality means that the first Euler characteristic $\chi_1(\mathbf{x}; S_{\mathbf{B}/\mathbf{A}})$ vanishes which, by Serre's theorem ([**1**, Theorem 4.7.10]), says that $S_{\mathbf{B}/\mathbf{A}}$ is Cohen-Macaulay. The final assertion comes from the formula for the multiplicity of $S_{\mathbf{B}/\mathbf{A}}$ in terms of $e_1(\mathbf{B})$ ([**30**, Theorem 2.5]). □

Castelnuovo regularity. The Sally module also encodes information about the Castelnuovo regularity $\operatorname{reg}(\mathbf{R}[It])$ of the Rees algebra. The following proposition and its corollary are extracted from the literature [**20, 28**] or proved directly by adding the exact sequence that defines $S_J(I)$ (note that $I\mathbf{R}[Jt]$ is a maximal Cohen-Macaulay module) to the canonical sequences relating $\mathbf{R}[It])$ to $\operatorname{gr}_I(\mathbf{R})$ and $\mathbf{R}$ via $I\mathbf{R}[It]$ (see [**28**, Section 3]).

Proposition 3.8. *Let $\mathbf{R}$ be a Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal and J a minimal reduction. Then*:

$$\operatorname{reg}(\mathbf{R}[It]) = \operatorname{reg}(S_J(I)).$$

In particular,

$$\operatorname{reltype}(I) \leq \operatorname{reg}(S_J(I)).$$

Corollary 3.9. *If I is an almost complete intersection and $\mathbf{R}[It]$ is almost Cohen-Macaulay, then*

$$\operatorname{reltype}(I) = \operatorname{red}_J(I) + 1.$$

The Sally fiber of an ideal. To help analyze the problem, we single out an extra structure. Let $(\mathbf{R}, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d > 0$, I an $\mathfrak{m}$-primary ideal and J one of its minimal reductions.

Definition 3.10 (Sally fiber). The *Sally fiber* of I is the graded module

$$F(I) = \bigoplus_{j\geq 1} I^j/JI^{j-1}.$$

$F(I)$ is an Artinian $\mathbf{R}[Jt]$-module whose last non-vanishing component is I^r/JI^r, $r = \mathrm{red}_J(I)$. The equality $e_1(I) = \lambda(F(I))$ is the condition for the almost Cohen-Macaulayness of $\mathbf{R}[It]$. We note that $F(I)$ is the fiber of $S_J(I)$ extended by the term I/J. To obtain additional control over $F(I)$, we are going to endow it with additional structures in cases of interest.

Suppose $\mathbf{R}$ is a Gorenstein local ring, $I = (J, a)$. The modules $F_j = I^j/JI^{j-1}$ are cyclic modules over the Artinian Gorenstein ring $\mathbf{A} = \mathbf{R}/J : a$. We turn $F(I)$ into a graded module over the polynomial ring $\mathbf{A}[s]$ by defining

$$a^j \in F_j \longmapsto s \cdot a^j = a^{j+1} \in F_{j+1}.$$

This is clearly well defined and has $s^r \cdot F(I) = 0$. Several of the properties of the F_n's arise from this representation; for instance, the length of F_j is non-increasing. Thus, $F(I)$ is a graded module over the Artinian Gorenstein ring $\mathbf{B} = \mathbf{A}[s, s^r = 0]$.

Remark 3.11. The variation of the values of F_j is connected to the degrees of the generators of $\mathbf{L}$. For convenience, we set $I = (J, a)$ and $\mathbf{B} = \mathbf{R}[u, \mathbf{T}_1, \ldots , \mathbf{T}_d]$, with u corresponding to a. For example:

(i) Suppose that, for some s, $\mathbf{f}_s = \lambda(F_s) = 1$. This means that we have d equations of the form

$$\mathbf{h}_i = x_i u^s + g_i \in \mathbf{L}_s,$$

where $g_i \in (\mathbf{T}_1, \ldots , \mathbf{T}_d)\mathbf{B}_{s-1}$. Eliminating the x_i, we derive a nonvanishing monic equation in $\mathbf{L}$ of degree $d \cdot s$. Thus, $\mathrm{red}_J(I) \leq ds - 1$.

(ii) A more delicate observation is that, whenever $\mathbf{f}_s > \mathbf{f}_{s+1}$, then there are *fresh* equations in $\mathbf{L}_{s+1}$. Let us explain why this happens: $\mathbf{f}_s = \lambda(JI^{s-1} : I^s)$, that is, the ideal L_s contains elements of the form

$$c \cdot u^s + \mathbf{g}, \qquad c \in JI^{s-1} : I^s, \qquad \mathbf{g} \in (\mathbf{T}_1, \dots, \mathbf{T}_d)\mathbf{B}_{s-1}.$$

Since $\mathbf{f}_{s+1} < \mathbf{f}_s$, $JI^s : I^{s+1}$ properly contains $JI^{s-1} : I^s$, which means that we must have elements in L_{s+1}

$$d \cdot u^{s+1} + \mathbf{g},$$

with $d \notin JI^{s-1} : I^s$ and $\mathbf{g} \in (\mathbf{T}_1, \dots, \mathbf{T}_d)\mathbf{B}_s$. Such elements cannot belong to $L_s \cdot \mathbf{B}_1$, so they are fresh generators.

The converse also holds.

A toolbox. We first give a simplified version of [**4**, Proposition 2.2]. Suppose $\mathbf{R}$ is a Gorenstein local ring of dimension d. Consider the two exact sequences:

$$0 \longrightarrow J/JI = (\mathbf{R}/I)^d \longrightarrow \mathbf{R}/JI \longrightarrow \mathbf{R}/J \longrightarrow 0$$

and the syzygetic sequence

$$0 \longrightarrow \delta(I) \longrightarrow \mathrm{H}_1(I) \longrightarrow (\mathbf{R}/I)^{d+1} \longrightarrow I/I^2 \longrightarrow 0.$$

The first gives

$$\lambda(\mathbf{R}/JI) = d \cdot \lambda(\mathbf{R}/I) + \lambda(\mathbf{R}/J),$$

the other

$$\lambda(\mathbf{R}/I^2) = (d+2)\lambda(\mathbf{R}/I) - \lambda(\mathrm{H}_1(I)) + \lambda(\delta(I)).$$

Thus,

$$\lambda(I^2/JI) = \lambda(I/J) - \lambda(\delta(I)),$$

since $\mathrm{H}_1(I)$ is the canonical module of $\mathbf{R}/I$. Taking into account the syzygetic formula in [**19**], we finally have:

Proposition 3.12. *Let* $(\mathbf{R}, \mathfrak{m})$ *be a Gorenstein local ring of dimension* $d > 0$*,* $J = (a_1, \ldots, a_d)$ *a parameter ideal and* $I = (J, a)$ *and* $a \in \mathfrak{m}$*. Then*:

$$\begin{aligned}\lambda(I^2/JI) &= \lambda(I/J) - \lambda(\mathbf{R}/I_1(\phi)) \\ &= \lambda(\mathbf{R}/J : a) - \lambda(\mathbf{R}/I_1(\phi)) \\ &= \lambda(\mathbf{R}/J : a) - \lambda(\mathrm{Hom}\,(\mathbf{R}/I_1(\phi), \mathbf{R}/J : a) \\ &= \lambda(\mathbf{R}/J : a) - \lambda((J : a) : I_1(\phi))/J : a) \\ &= \lambda(\mathbf{R}/(J : a) : I_1(\phi)).\end{aligned}$$

Note that, in dualizing $\mathbf{R}/I_1(\phi)$, we made use of the fact that $\mathbf{R}/J : a$ is a Gorenstein ring.

Corollary 3.13. $I^2 = JI$ *if and only if* $J : a = I_1(\phi)$*. In this case, if* $d > 1$ *the algebra* $\mathbf{R}[It]$ *is Cohen-Macaulay.*

Corollary 3.14. *If* $\mathbf{R}[It]$ *is an* aCM *algebra and* $\mathrm{red}_J(I) = 2$*, then* $e_1(I) = 2 \cdot \lambda(I/J) - \lambda(\mathbf{R}/I_1(\phi))$.

Remark 3.15. We could enhance these observations considerably if formulas for $\lambda(JI^2 : I^3)$ were to be developed. More precisely, how do the syzygies of I affect $JI^2 : I^3$?

3.3. Multiplicities and number of relations. To benefit from Theorem 3.5, we need to have effective formulas for $\deg(\mathbf{S})$ and $\deg(\mathbf{R}[It])$. We are going to develop them now.

Proposition 3.16. *Let* $\mathbf{R} = k[x_1, \ldots, x_d]$ *and* I *be an almost complete intersection as above,* $I = (f_1, \ldots, f_d, f_{d+1}) = (J, f_{d+1})$ *generated by forms of degree* n*. Then* $\deg(\mathbf{R}[It]) = \sum_{j=0}^{d-1} n^j$.

Proof. After an elementary observation, we make use of one of the beautiful multiplicity formulas of [**16**]. Set $A = \mathbf{R}[It]$, $A_0 = \mathbf{R}[Jt]$, $\mathcal{M} = (\mathfrak{m}, It)A$ and $\mathcal{M}_0 = (\mathfrak{m}, Jt)A_0$. Then

$$\deg(\mathrm{gr}_{\mathcal{M}_0}(A_0)) = \deg(\mathrm{gr}_{\mathcal{M}_0}(A)) = \deg(\mathrm{gr}_{\mathcal{M}}(A)),$$

the first equality is because $A_0 \to A$ is a finite rational extension, and the second is because $(\mathfrak{m}, Jt)A$ is a reduction of $(\mathfrak{m}, It)A$. Now we use [**16**, Corollary 1.5] that gives $\deg(A_0)$. $\square$

The multiplicity of the symmetric algebra. We shall now prove one of our main results, a formula for $\deg S(I)$ for ideals generated by forms of the same degree. Let $\mathbf{R} = k[x_1, \ldots, x_d]$, $I = (\mathbf{f}) = (f_1, \ldots, f_d, f_{d+1})$ be an almost complete intersection generated by forms of degree n. At some point we assume, harmlessly, that $J = (f_1, \ldots, f_d)$ is a complete intersection. There will be a slight change of notation in the rest of this section. We set $\mathbf{B} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$ and $\mathbf{S} = \mathrm{Sym}\,(I)$.

Theorem 3.17 (Degree formula). $\deg \mathbf{S} = \sum_{j=0}^{d} n^j - \lambda(\mathbf{R}/I)$.

Proof. Let $\mathbf{K}(\mathbf{f}) = \bigwedge \mathbf{R}^{d+1}(-n)$ be the Koszul complex associated to $\mathbf{f}$,

$$0 \longrightarrow K_{d+1} \longrightarrow K_d \longrightarrow \cdots \longrightarrow K_2 \longrightarrow K_1 \longrightarrow K_0 \longrightarrow 0,$$

and consider the associated $\mathcal{Z}$-complex

$$\begin{aligned} 0 \longrightarrow Z_d \otimes \mathbf{B}(-d) \longrightarrow Z_{d-1} \otimes \mathbf{B}(-d+1) \longrightarrow \cdots \\ \longrightarrow Z_2 \otimes \mathbf{B}(-2) \longrightarrow Z_1 \otimes \mathbf{B}(-1) \xrightarrow{\psi} \mathbf{B} \longrightarrow 0. \end{aligned}$$

This complex is acyclic with $\mathcal{H}_0(\mathcal{Z}) = \mathbf{S} = \mathrm{Sym}\,(I)$. Now we introduce another complex obtained by replacing $Z_1 \otimes \mathbf{B}(-1)$ by $B_1 \otimes \mathbf{B}(-1)$, where B_1 is the module of 1-boundaries of $\mathbf{K}(\mathbf{f})$, followed by the restriction of ψ to $B_1 \otimes \mathbf{B}(-1)$.

This defines another acyclic complex, $\mathcal{Z}^*$, actually the $\mathcal{B}$-complex of $\mathbf{f}$, and we set $\mathcal{H}_0(\mathcal{Z}^*) = \mathbf{S}^*$. The relationship between $\mathbf{S}$ and $\mathbf{S}^*$ is given in the following observation:

Lemma 3.18. $\deg \mathbf{S}^* = \deg \mathbf{S} + \lambda(\mathbf{R}/I)$.

Proof. Consider the natural mapping between $\mathcal{Z}$ and $\mathcal{Z}^*$:

$$\begin{tikzcd}
0 \arrow[r] & Z_d \otimes \mathbf{B}(-d) \arrow[r] \arrow[d, "\phi_d"'] & \cdots \arrow[r] & Z_2 \otimes \mathbf{B}(-2) \arrow[r] \arrow[d, "\phi_2"'] & B_1 \otimes \mathbf{B}(-1) \arrow[r] \arrow[d] & \mathbf{B} \arrow[r] \arrow[d] & \mathbf{S}^* \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_d \otimes \mathbf{B}(-d) \arrow[r] & \cdots \arrow[r] & Z_2 \otimes \mathbf{B}(-2) \arrow[r] & Z_1 \otimes \mathbf{B}(-1) \arrow[r] & \mathbf{B} \arrow[r] & \mathbf{S} \arrow[r] & 0.
\end{tikzcd}$$

The maps $\phi_2, \dots, \phi_d$ are isomorphisms while the other maps are defined above. They induce the short exact sequence of modules of dimension $d+1$,

$$0 \longrightarrow (Z_1/B_1) \otimes \mathbf{B}(-1) \longrightarrow \mathbf{S}^* \longrightarrow \mathbf{S} \longrightarrow 0.$$

Note that $Z_1/B_1 = \mathrm{H}_1(\mathbf{K}(\mathbf{f}))$ is the canonical module of $\mathbf{R}/I$, and therefore has the same length as $\mathbf{R}/I$. Finally, by the additivity formula for the multiplicities ([**8**, Lemma 13.2]),

$$\deg \mathbf{S}^* = \deg \mathbf{S} + \lambda(Z_1/B_1),$$

as desired. $\square$

We now give our main calculation of multiplicities.

Lemma 3.19. $\deg \mathbf{S}^* = \sum_{j=0}^{d} n^j$.

Proof. We note that the $\mathcal{Z}^*$-complex is homogeneous for the total degree [as required for the computation of multiplicities] provided the Z_i's and B_1 have the same degree. We can conveniently write B_i for Z_i, $i \geq 2$. This is clearly the case since they are generated in degree n. This is not the case for Z_1. However, when $\mathbf{f}$ is a regular sequence, all the Z_i are equigenerated, an observation we shall make use of below.

Since the modules of $\mathcal{Z}^*$ are homogeneous, we have that the Hilbert series of $\mathbf{S}^*$ is given as

$$H_{\mathbf{S}^*}(\mathbf{t}) = \frac{\sum_{i=0}^{d}(-1)^i h_{B_i}(\mathbf{t})\mathbf{t}^i}{(1-\mathbf{t})^{2d+1}} = \frac{h(\mathbf{t})}{(1-\mathbf{t})^{2d+1}},$$

where $h_{B_i}(\mathbf{t})$ are the h-polynomials of the B_i. More precisely, each of the terms of $\mathcal{Z}^*$ is a $\mathbf{B}$-module of the form $A \otimes \mathbf{B}(-r)$ where A

is generated in a same degree. Such modules are isomorphic to their associated graded modules.

The multiplicity of $\mathbf{S}^*$ is given by the standard formula

$$\deg \mathbf{S}^* = (-1)^d \frac{h^{(d)}(1)}{d!}.$$

We now indicate how the $h_{B_i}(\mathbf{t})$ are assembled. Let us illustrate the case when $d = 4$ and $i = 1$. B_1 has a free resolution of the strand of the Koszul complex

$$0 \longrightarrow \mathbf{R}(-3n) \longrightarrow \mathbf{R}^5(-2n) \longrightarrow \mathbf{R}^{10}(-n) \longrightarrow \mathbf{R}^{10} \longrightarrow B_1 \longrightarrow 0,$$

so that

$$h_{B_1}(\mathbf{t}) = 10 - 10\mathbf{t}^n + 5\mathbf{t}^{2n} - \mathbf{t}^{3n},$$

and similarly for all B_i.

We are now ready to make our key observation. Consider a complete intersection P generated by $d+1$ forms of degree n in a polynomial ring of dimension $d+1$, and set $\mathbf{S}^{**} = \mathrm{Sym}\,(P)$. The corresponding approximation complex now has $B_1 = Z_1$. The approach above would, for the new Z_i, give the same h-polynomials of the B_i in the case of an almost complete intersection (but in dimension d). This means that the Hilbert series of $\mathbf{S}^{**}$ is given by

$$H_{\mathbf{S}^{**}}(\mathbf{t}) = \frac{h(\mathbf{t})}{(1-\mathbf{t})^{2d+2}}.$$

It follows that $\deg \mathbf{S}^*$ can be computed as the degree of the symmetric algebra generated by a regular sequence of $d+1$ forms of degree n, a result that is given in [**16**]. Thus,

$$\deg \mathbf{S}^* = \deg \mathbf{S}^{**} = \sum_{j=0}^{d} n^j,$$

and the calculation of $\deg \mathbf{S}$ is complete. $\square$

We will now write Theorem 3.17 in a more convenient formulation for applications.

Theorem 3.20. *Let* $\mathbf{R} = k[x_1, \ldots, x_d]$ *and* $I = (f_1, \ldots, f_d, f_{d+1})$ *be an ideal of forms of degree* n*. If* $J = (f_1, \ldots, f_d)$ *is a complete intersection, then*

$$\deg \mathbf{S} = \sum_{j=0}^{d-1} n^j + \lambda(\mathbf{R}/J : I).$$

Proof. The degree formula gives

$$\begin{aligned}\deg \mathbf{S} &= \sum_{j=0}^{d-1} n^j + [n^d - \lambda(R/I)] = \sum_{j=0}^{d-1} n^j + [\lambda(\mathbf{R}/J) - \lambda(\mathbf{R}/I)] \\ &= \sum_{j=0}^{d-1} n^j + \lambda(I/J) = \sum_{j=0}^{d-1} n^j + \lambda(\mathbf{R}/J : I). \qquad \square\end{aligned}$$

Corollary 3.21. *Let* $I = (J, a)$ *be an ideal of finite colength as above. Then the module of nonlinear relations satisfies* $\deg(T) = \lambda(I/J)$*. In particular, if* $\mathbf{R}[It]$ *is almost Cohen-Macaulay,* T *can be generated by* $\lambda(I/J)$ *elements.*

Proof. From the sequence of modules of the same dimension

$$0 \longrightarrow T \longrightarrow \mathbf{S} \longrightarrow \mathbf{R}[It] \longrightarrow 0,$$

we have

$$\deg(T) = \deg \mathbf{S} - \deg \mathbf{R}[It] = \lambda(I/J). \qquad \square$$

The last assertion of this corollary can also be obtained from [**24**, Theorem 4.1].

The Cohen-Macaulay type of the module of nonlinear relations. We recall the terminology of Cohen-Macaulay type of a module. Set $\mathbf{B} = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_{d+1}]$. If E is a finitely generated $\mathbf{B}$-module of codimension r, we say that $\operatorname{Ext}^r_{\mathbf{B}}(E, \mathbf{B})$ is its canonical module. It is the first non vanishing $\mathrm{Ext}^i_{\mathbf{B}}(E,\mathbf{B})$ module, denoted by ω_E. The minimal number of the generators of ω_E is called the *Cohen-Macaulay type* of E and is denoted by $\mathrm{type}\,(E)$. When E is graded and Cohen-Macaulay, it gives the last Betti number of a projective resolution of E. It can be expressed in different ways, for example, for the module of nonlinear relations $\omega_T = \mathrm{Ext}^d_{\mathbf{B}}(T,\mathbf{B}) = \mathrm{Hom}_{\mathbf{S}}(T,\omega_{\mathbf{S}})$.

Proposition 3.22. *Let* $\mathbf{R}$ *be a Gorenstein local ring of dimension* $d \geq 2$ *and* $I = (J,a)$ *be an ideal of finite colength as above. If* $\mathbf{R}[It]$ *is an* aCM *algebra and* $\omega_{R[It]}$ *is Cohen-Macaulay, then the type of the module* T *of nonlinear relations satisfies*

$$\mathrm{type}\,(T) \leq \mathrm{type}\,(S_J(I)) + d - 1,$$

where $S_J(I)$ *is the Sally module.*

Proof. We set $\mathcal{R} = \mathbf{R}[It]$ and $\mathcal{R}_0 = \mathbf{R}[Jt]$. First apply $\mathrm{Hom}_{\mathbf{B}}(\cdot,\mathbf{B})$ to the basic presentation

$$0 \longrightarrow T \longrightarrow \mathbf{S} \longrightarrow \mathcal{R} \longrightarrow 0,$$

to obtain the cohomology sequence

$$0 \longrightarrow \omega_{\mathcal{R}} \longrightarrow \omega_{\mathbf{S}} \longrightarrow \omega_T \longrightarrow \mathrm{Ext}^{d+1}_{\mathbf{B}}(\mathcal{R},\mathbf{B}) \longrightarrow 0. \tag{5}$$

Now apply the same functor to the exact sequence of $\mathbf{B}$-modules

$$0 \longrightarrow I \cdot \mathcal{R}[-1] \longrightarrow \mathcal{R} \longrightarrow \mathbf{R} \longrightarrow 0$$

to obtain the exact sequence

$$\begin{aligned}0 \longrightarrow \omega_{\mathcal{R}} \stackrel{\theta}{\longrightarrow} \omega_{I\mathcal{R}[-1]} \longrightarrow \mathrm{Ext}^{d+1}_{\mathbf{B}}(\mathbf{R},\mathbf{B})&\\ = \mathbf{R} \longrightarrow \mathrm{Ext}^{d+1}_{\mathbf{B}}(\mathcal{R},\mathbf{B}) \longrightarrow \mathrm{Ext}^{d+1}_{\mathbf{B}}(I\mathcal{R}[-1],\mathbf{B}) \longrightarrow 0.&\end{aligned}$$

Since $\omega_{\mathcal{R}}$ is Cohen-Macaulay and $\dim \mathbf{R} \geq 2$, the cokernel of θ is either $\mathbf{R}$ or an $\mathfrak{m}$-primary ideal that satisfies the condition S_2 of Serre. The only choice is $\mathrm{coker}\,(\theta) = \mathbf{R}$. Therefore,

$$\mathrm{Ext}^{d+1}_{\mathbf{B}}(\mathcal{R},\mathbf{B}) \simeq \mathrm{Ext}^{d+1}_{\mathbf{B}}(I\mathcal{R}[-1],\mathbf{B}).$$

Now we approach the module $\operatorname{Ext}_{\mathbf{B}}^{d+1}(I\mathcal{R}, \mathbf{B})$ from a different direction. We note that $\mathcal{R}$, but not $\mathbf{S}$ and T, is also a finitely generated $\mathbf{B}_0 = \mathbf{R}[\mathbf{T}_1, \ldots, \mathbf{T}_d]$-module as it is annihilated by a monic polynomial $\mathbf{f}$ in $\mathbf{T}_{d+1}$ with coefficients in $\mathbf{B}_0$. By Rees's theorem, we have that, for all i, $\operatorname{Ext}_{\mathbf{B}}^{i}(\mathcal{R}, \mathbf{B}) = \operatorname{Ext}_{\mathbf{B}/(\mathbf{f})}^{i-1}(\mathcal{R}, \mathbf{B}/(\mathbf{f}))$, and a similar observation applies to $I \cdot \mathcal{R}$.

Next consider the finite, flat morphism $\mathbf{B}_0 \to \mathbf{B}/(\mathbf{f})$. For any $\mathbf{B}/(\mathbf{f})$-module E with a projective resolution $\mathbf{P}$, we have that $\mathbf{P}$ is a projective $\mathbf{B}_0$-resolution of E. This means that the isomorphism of complexes

$$\begin{aligned}\operatorname{Hom}_{\mathbf{B}_0}(\mathbf{P}, \mathbf{B}_0) &\simeq \operatorname{Hom}_{\mathbf{B}/(\mathbf{f})}(\mathbf{P}, \operatorname{Hom}_{\mathbf{B}_0}(\mathbf{B}/(\mathbf{f}), \mathbf{B}_0)) \\ &= \operatorname{Hom}_{\mathbf{B}/(\mathbf{f})}(\mathbf{P}, \mathbf{B}/(\mathbf{f}))\end{aligned}$$

gives isomorphisms for all i

$$\operatorname{Ext}_{\mathbf{B}_0}^{i}(E, \mathbf{B}_0) \simeq \operatorname{Ext}_{\mathbf{B}/(\mathbf{f})}^{i}(E, \mathbf{B}/(\mathbf{f})).$$

Thus,

$$\operatorname{Ext}_{\mathbf{B}}^{i}(\mathcal{R}, \mathbf{B}) \simeq \operatorname{Ext}_{\mathbf{B}_0}^{i-1}(\mathcal{R}, \mathbf{B}_0).$$

In particular, $\omega_{\mathcal{R}} = \operatorname{Ext}_{\mathbf{B}_0}^{d-1}(\mathcal{R}, \mathbf{B}_0)$.

Finally, apply $\operatorname{Hom}_{\mathbf{B}_0}(\cdot, \mathbf{B}_0)$ to the exact sequence of $\mathbf{B}_0$-modules and examine its cohomology sequence.

$$0 \longrightarrow I \cdot \mathcal{R}_0 \longrightarrow I \cdot \mathcal{R} \longrightarrow S_J(I) \longrightarrow 0$$

is then

$$\begin{aligned}0 \longrightarrow \omega_{I\mathcal{R}} \longrightarrow \omega_{I\mathcal{R}_0} \longrightarrow \omega_{S_J(I)} \longrightarrow &\operatorname{Ext}_{\mathbf{B}_0}^{d}(\mathcal{R}, \mathbf{B}_0) \\ &= \operatorname{Ext}_{\mathbf{B}}^{d+1}(\mathcal{R}, \mathbf{B}) \longrightarrow 0.\end{aligned}$$

Taking this into (5) and the $\operatorname{type}(\mathbf{S}) = d - 1$ gives the desired estimate. $\square$

Remark 3.23. A class of ideals with $\omega_{\mathcal{R}}$ Cohen-Macaulay is discussed in Corollary 4.2 (b).

4. Distinguished aCM algebras. This section treats several classes of Rees algebras which are almost Cohen-Macaulay.

4.1. Equi-homogeneous acis. We shall now treat an important class of extremal Rees algebras. Let $\mathbf{R} = k[x_1, \ldots, x_d]$, and let $I = (a_1, \ldots, a_d, a_{d+1})$ be an ideal of finite colength, that is, $\mathfrak{m}$-primary. We further assume that the first d generators form a regular sequence and $a_{d+1} \notin (a_1, \ldots, a_d)$. If $\deg a_i = n$, the integral closure of $J = (a_1, \ldots, a_d)$ is the ideal $\mathfrak{m}^n$, in particular, J is a minimal reduction of I. The integer $\operatorname{edeg}(I) = \operatorname{red}_J(I) + 1$ is called the *elimination degree* of I. The study of the equations of I, that is, of $\mathbf{R}[It]$, depends on a comparison between the metrics of $\mathbf{R}[It]$ to those of $\mathbf{R}[\mathfrak{m}^n t]$, which are well known.

Proposition 4.1 [**19**, Condition (B)]. *The following conditions are equivalent*:

(i) Φ *is a birational mapping, that is, the natural embedding* $\mathcal{F}(I) \hookrightarrow \mathcal{F}(\mathfrak{m}^n)$ *induces an isomorphism of quotient fields*;

(ii) $\operatorname{red}_J(I) = n^{d-1} - 1$;

(iii) $e_1(I) = ((d-1)/2)(n^d - n^{d-1})$;

(iv) $\mathbf{R}[It]$ *satisfies the* R_1 *condition of Serre.*

Condition (B). As a convenience of exposition, we refer to the equivalence conditions in Proposition 4.1 simply as Condition (B).

Corollary 4.2. *For an ideal* I *that satisfies Condition* (B), *the following hold*:

(i) *The algebra* $\mathbf{R}[It]$ *is not Cohen-Macaulay except when* $I = (x_1, x_2)^2$.

(ii) *The canonical module of* $\mathbf{R}[It]$ *is Cohen-Macaulay.*

Proof. (i) follows from the condition of Goto-Shimoda [**10**] that the reduction number of a Cohen-Macaulay Rees algebra $\mathbf{R}[It]$ must satisfy $\operatorname{red}_J(I) \le \dim \mathbf{R} - 1$, which in the case $n^{d-1} - 1 \le d - 1$ is only met if $d = n = 2$.

(ii) The embedding $\mathbf{R}[It] \hookrightarrow \mathbf{R}[\mathfrak{m}^n t]$ being an isomorphism in codimension one, the corresponding canonical modules are isomorphic.

The canonical module of a Veronese subring such as $\mathbf{R}[\mathfrak{m}^n t]$ is well known (see [**15**], [**17**, page 187]; see also [**2**, Proposition 2.2]).

Binary ideals. These are the ideals of $\mathbf{R} = k[x, y]$ generated by 3 forms of degree n. Many of their Rees algebras are almost Cohen-Macaulay. We will showcase the technology of the Sally module in treating a much-studied class of ideals. First we discuss a simple case (see also [**18**]).

Proposition 4.3. *Let ϕ be a 3×2-matrix of quadratic forms in $\mathbf{R}$, and let I be the ideal given by its 2×2 minors. Then $\mathbf{R}[It]$ is almost Cohen-Macaulay.*

Proof. These ideals have reduction number 1 or 3. In the first case, all of its Sally modules vanish and $\mathbf{R}[It]$ is Cohen-Macaulay.

In the other case, I satisfies Condition (B) and $e_1(I) = \binom{4}{2} = 6$. A simple calculation shows that $\lambda(\mathbf{R}/I) = 12$, so that $\lambda(I/J) = 16 - 12 = 4$. To apply Theorem 3.7, we need to verify the equation

$$f_1 + f_2 + f_3 = 6. \tag{6}$$

We already have $f_1 = 4$. To calculate f_2, we need to take $\lambda(R/I_1(\phi))$ in Corollary 3.12. $I_1(\phi)$ is an ideal generated by 2 generators or $I_1(\phi) = (x, y)^2$. But, in the first case, the Sylvester resultant of the linear equations of $\mathbf{R}[It]$ would be a quadratic polynomial, that is, I would have reduction number 1, which would contradict the assumption. Thus, by Corollary 3.12, $f_2 = 4 - \lambda(\mathbf{R}/I_1(\phi)) = 1$. Since $f_2 \geq f_3 > 0$, we have $f_3 = 1$, and equation (6) is satisfied. $\square$

We have examined higher degree examples of ideals of this type which are/are not almost Cohen-Macaulay. Quite a lot is known about the following ideals. $\mathbf{R} = k[x, y]$ and I is a codimension 2 ideal given by that 2×2 minors of a 3×2 matrix with homogeneous entries of degrees 1 and $n - 1$.

Theorem 4.4. *If $I_1(\phi) = (x, y)$ then:*

(i) $\deg \mathcal{F}(I) = n$*, that is, I satisfies Condition* (B).

(ii) $\mathbf{R}[It]$ *is almost Cohen-Macaulay.*

(iii) *The equations of* $\mathbf{L}$ *are given by a straightforward algorithm.*

Proof. The proof of (i) is in [**6**] and in other sources ([**5, 23**]; see also [**7**, Theorem 2.2] for a broader statement in any characteristic and [**27**, Theorem 4.1] in characteristic zero), and of (ii) in [**23**, Theorem 4.4], while (iii) was conjecturally given in [**18**, Conjecture 4.8] and proved in [**6**]. We give a combinatorial proof of (ii) in Proposition 4.6. $\square$

We note that $\deg(\mathbf{S}) = 2n$, since $\mathbf{S}$ is a complete intersection defined by two forms of (total) degrees 2 and n, while $\mathbf{R}[Jt]$ is defined by one equation of degree $n+1$. Thus, $\nu(T) \leq 2n - (n+1) = n-1$, which is the number of generators given in the algorithm.

We point out a property of module T. We recall that an $\mathbf{A}$-module is an *Ulrich* module if it is a maximal Cohen-Macaulay module with $\deg M = \nu(M)$ [**12**].

Corollary 4.5. *T is an Ulrich $\mathbf{S}$-module.*

Considerable numerical information in Theorem 4.4 will follow from:

Proposition 4.6. *If $\deg\alpha = 1$ and $\deg\beta = n-1$, then $\lambda(F_j) = n-j$. In particular, $\mathbf{R}[It]$ is almost Cohen-Macaulay.*

Proof. Note that, since the ideal satisfies Condition (B), $F_{n-1} \neq 0$. On the other hand, $\mathbf{L}$ contains fresh generators in all degrees $j \leq n$. This means that, for $f_j = \lambda(F_j)$,

$$f_j > f_{j+1} > 0, \quad j < n.$$

Since $f_1 = n-1$, the decreasing sequence of integers

$$n-1 = f_1 > f_2 > \cdots > f_{n-2} > f_{n-1} > 0$$

implies that $f_j = n-j$. Finally, applying Theorem 3.7, we have that $\mathbf{R}[It]$ is an aCM algebra since $\sum_j f_j = e_1(I) = \binom{n}{2}$. $\square$

Quadrics. Here we explore sporadic classes of aCM algebras defined by quadrics in $k[x_1, x_2, x_3, x_4]$.

First, we use Proposition 3.12 to look at other cases of quadrics. For $d = 3$, $n = 2$, $\operatorname{edeg}(I) = 2$ or 4. In the first case, $J : a = I_1(\phi)$. In addition, $J : a \neq \mathfrak{m}$ since the socle degree of $\mathbf{R}/J$ is 3. Then $\mathbf{R}[It]$ is Cohen-Macaulay. If $\operatorname{edeg}(I) = 4$ we must have (and conversely) $\lambda(\mathbf{R}/J : a) = 2$ and $I_1(\phi) = \mathfrak{m}$. Then $\mathbf{R}[It]$ is almost Cohen-Macaulay.

Next we treat almost complete intersections of finite colength generated by quadrics of $\mathbf{R} = k[x_1, x_2, x_3, x_4]$. Sometimes we denote the variables by $x, y, \dots,$ or use these symbols to denote (independent) linear forms. For notation, we use $J = (a_1, a_2, a_3, a_4)$ and $I = (J, a)$.

Our goal is to address the following:

Question 4.7. Let I be an almost complete intersection generated by 5 quadrics of x_1, x_2, x_3, x_4. If I satisfies Condition (B), in which cases is $\mathbf{R}[It]$ an almost Cohen-Macaulay algebra? In this case, what are the generators of its module of nonlinear relations?

In order to make use of Theorem 3.7, our main tools are Corollary 3.12 and [**19**, Theorem 2.2]. They make extensive use of the syzygies of I. The question forks into three cases, but our analysis is complete in only one of them.

The Hilbert functions of quaternary quadrics. We make a quick classification of the Hilbert functions of the ideals $I = (J, a)$. Since $I/J \simeq \mathbf{R}/J : a$ and J is a complete intersection, the problem is equivalent to determining the Hilbert functions of $\mathbf{R}/J : a$, with $J : a$ a Gorenstein ideal. The Hilbert function $H(\mathbf{R}/I)$ of $\mathbf{R}/I$ is $H(\mathbf{R}/J) - H(\mathbf{R}/J : a)$. We will need the Hilbert function of the corresponding canonical module in order to make use of [**19**, Proposition 3.7], giving information about $L_2/\mathbf{B}_1L_1$.

We shall refer to the sequences $(f_1, f_2, f_3, \dots,)$, $f_i = \lambda(I^i/JI^{i-1})$, as the **f**-sequence of (I, J). We recall that these sequences are monotonic and that, if I satisfies Condition (B), $\sum_{i\geq 1} f_i = e_1(I) = 12$.

Proposition 4.8. *Let* $\mathbf{R} = k[x_1, x_2, x_3, x_4]$ *and* $I = (J, a)$ *be an almost complete intersection generated by* 5 *quadrics, where* J *is a complete intersection. Then* $L = \lambda(\mathbf{R}/J : a) \leq 6$, *and the possible Hilbert functions of* $(\mathbf{R}/J : a)$ *are*:

$$\begin{aligned}
L &= 6 : (1,4,1), \quad (1,2,2,1)^*, \ (1,1,1,1,1,1)^* \\
L &= 5 : (1,3,1), \quad (1,1,1,1,1)^* \\
L &= 4 : (1,2,1), \quad (1,1,1,1)^* \\
L &= 3 : (1,1,1)^* \\
L &= 2 : (1,1)^{**} \\
L &= 1 : (1)^{**}.
\end{aligned}$$

If I *satisfies Condition* (B), *the corresponding Hilbert function is one of the unmarked sequences above.*

Proof. Since $\lambda(\mathfrak{m}^2/I) \geq 5$ and $\lambda(\mathfrak{m}^2/J) = 11$, $L = \lambda(\mathbf{R}/J : a) \leq 6$. Because the Hilbert function of $R/(J : a)$ is symmetric and $L \leq 6$, the list includes all viable Hilbert functions.

Let us first rule out those marked with a*, while those marked with a** cannot satisfy Condition (B). In each of these, $J : a$ contains at least 2 linearly independent linear forms, which we denote by x, y, so that $J : a/(x, y)$ is a Gorenstein ideal of the regular ring $\mathbf{R}/(x, y)$. It follows that $(J : a)/(x, y)$ is a complete intersection. In the case of $(1, 2, 2, 1)$, $J : a = (x, y, \alpha, \beta)$, where α is a form of degree 2 and β a form of degree 3, since $\lambda(\mathbf{R}/J : a) = 6$. Since $J \subset J : a$, all the generators of J must be contained in (x, y, α), which is impossible by the Krull theorem. Those strings with at least three 1's are also excluded since $J : a$ would have the form (x, y, z, w^s), $s \geq 3$, and the argument above applies. The case $(1, 1)$, $J : a = (x, y, z, w^2)$ means that $I_1(\phi) = J : a$, or $J : a = \mathfrak{m}$. In the first case, by Corollary 3.13, $I^2 = JI$. In the second case, $I_1(\phi) = \mathfrak{m}$. This will imply that $\lambda(I^2/JI) = 2 - 1 = 1$, and therefore I will not satisfy Condition (B) (we need the summation to total 12). $\square$

Hilbert function $(1, 4, 1)$**.** If $R/J : a$ has Hilbert function $(1, 4, 1)$, $J : a \subset \mathfrak{m}^2$, but we cannot have equality since $\mathfrak{m}^2$ is not a Gorenstein ideal. We also have $I_1(\phi) \subset \mathfrak{m}^2$. If they are not equal, $I_1(\phi) = J : a$, which by Corollary 3.13 would mean that $\mathrm{red}_J(I) = 1$.

Theorem 4.9. *Suppose I satisfies Condition* (B) *and* $I_1(\phi) \subset \mathfrak{m}^2$. *Then* $\mathbf{R}[It]$ *is almost Cohen-Macaulay.*

Proof. The assumption $I_1(\phi) \subset \mathfrak{m}^2$ means that the Hilbert function of $J : a$ is $(1,4,1)$ and vice-versa. Note also that, by assumption, $\lambda(I^7/JI^6) \neq 0$. Since $\lambda(I/J) = \lambda(\mathbf{R}/J : a) = 6$, it suffices to show that $\lambda(I^2/JI) = 1$. From $\lambda(\mathfrak{m}^2/I) = 5$, the module $\mathfrak{m}^2/I$ is of length 5 minimally generated by 5 elements. Therefore, $\mathfrak{m}^3 \subset I$, actually $\mathfrak{m}^3 = \mathfrak{m}I$.

There is an isomorphism $\mathbf{h} : \mathbf{R}/J : a \simeq I/J$, $r \mapsto ra$. It moves the socle of $\mathbf{R}/J : a$ into the socle of I/J. If $a \notin J : a$, then $\mathfrak{m}^2 = (J : a, a)$ and a gives the socle of $\mathbf{R}/J : a$; thus, it is mapped by $\mathbf{h}$ into the socle of I/J, that is, $\mathfrak{m} \cdot a^2 \in J$. Thus, $\mathfrak{m} \cdot a^2 \in \mathfrak{m}^3 J \subset JI$. On the other hand, if $a \in J : a$, then, since $a^2 \in J$, we have $a^2 \in \mathfrak{m}^2 J$ and $\mathfrak{m} \cdot a^2 \in \mathfrak{m}^3 J \subset JI$.

An example is $J = (x^2, y^2, z^2, w^2)$, $a = xy + xz + xw + yz$. □

Hilbert function $(1,3,1)$**.** Our discussion about this case is very sparse.

• For these Hilbert functions, $J : a = (x, P)$, where P is a Gorenstein ideal in a regular local ring of dimension 3, and therefore is given by the Pfaffians of a skew-symmetric matrix, necessarily 5×5. Since $J \subset J : a$, L must contain forms of degree 2. In addition, P is given by five 2-forms (and $(x, \mathfrak{m}^2)/(x, P)$ is the socle of $\mathbf{R}/J : a$).

• If I satisfies Condition (B), $\mathbf{R}[It]$ is almost Cohen-Macaulay if and only if $\lambda(I^2/JI) = 2$ and $\lambda(I^3/JI^2) = 1$. The first equality, by Proposition 3.12, requires $\lambda(\mathbf{R}/I_1(\phi)) = 3$ which gives that $I_1(\phi)$ contains the socle of $J : a$ and another independent linear form. In all, it means that $I_1(\phi) = (x, y, (z, w)^2)$. On the other hand, $\lambda(I^2/JI) = 2$ means that $JI : I^2 = (x, y, z, w^2)$ (after more label changes).

• An example is $J = (x^2, y^2, z^2, w^2)$ with $a = xy + yz + zw + wx + yw$. The ideal $I = (J, a)$ satisfies Condition (B).

Hilbert function $(1,2,1)$**.** We do not have the full analysis of this case either.

• An example is $J = (x^2, y^2, z^2, w^2)$, and $a = xy + yz + xw + zw$.

The ideal $I = (J, a)$ satisfies Condition (B). The expected $\mathbf{f}$-sequence of such ideals is $(4, 3, 1, 1, 1, 1, 1)$.

• If I satisfies Condition (B), then $I_1(\phi) = \mathfrak{m}$. We know that $I_1(\phi) \neq J : a$, so $I_1(\phi) \supset \mathfrak{m}^2$, that is, $I_1(\phi) = (x, y, \mathfrak{m}^2)$, $(x, y, z, \mathfrak{m}^2)$ or $\mathfrak{m}$. Let us exclude the first two cases.

$(x, y, \mathfrak{m}^2)$. This leads to two equations

$$\begin{aligned} xa &= xb + yc \\ ya &= xd + ye, \end{aligned}$$

with $b, c, d, e \in J$. But this gives the equation $(a - b)(a - e) - dc = 0$, and $\mathrm{red}_J(I) \leq 1$.

$(x, y, z, \mathfrak{m}^2)$. Then the Hilbert function of $\mathbf{R}/I_1(\phi)$ is $(1, 1)$. According to [**19**, Proposition 3.7], L_2 has a form of bidegree $(1, 2)$, with coefficients in $I_1(\phi)$, that is, in (x, y, z). This gives three forms with coefficients in this ideal, two in degree 1, so by elimination we get a monic equation of degree 4.

We summarize the main points of these observations into a normal form assertion.

Proposition 4.1. *Let I be an ideal that satisfies Condition* (B)*, and the Hilbert function of $\mathbf{R}/J : a$ is* $(1, 2, 1)$*. Then, up to a change of variables to $\{x, y, z, w\}$, I is a Northcott ideal, that is, there is a 4×4-matrix $\mathbf{A}$,*

$$\mathbf{A} = \left[\begin{array}{c} \mathbf{B} \\ \hline \mathbf{C} \end{array} \right],$$

where $\mathbf{B}$ is a 2×4-matrix whose entries are linear forms and $\mathbf{C}$ is a matrix with scalar entries and $\mathbf{V} = [x, y, \alpha, \beta]$, where α, β are quadratic forms in z, w such that

$$I = (\mathbf{V} \cdot \mathbf{A}, \det \mathbf{A}).$$

Proof. There are two independent linear forms in $J : a$ which we denote by x, y. We observe that $(J : a)/(x, y)$ is a Gorenstein ideal in a polynomial ring of dimension 2, so it is a complete intersection:

$J : a = (x, y, \alpha, \beta)$, with α and β forms of degree 2 (as $\lambda(\mathbf{R}/J : a) = 4$), from which we remove the terms in x, y, that is, we may assume $\alpha, \beta \in (z, w)^2$.

Since $J \subset J : a$, we have a matrix $\mathbf{A}$,

$$J = [x, y, \alpha, \beta] \cdot \mathbf{A} = \mathbf{V} \cdot \mathbf{A}.$$

By duality, $I = J : (J : a)$, which by the Northcott theorem [**25**] gives

$$I = (J, \det \mathbf{A}).$$

Note that a gets, possibly, replaced by $\det \mathbf{A}$. The statement about the degrees of the entries of $\mathbf{A}$ is clear. $\square$

Example 4.11. Let

$$\mathbf{A} = \begin{bmatrix} x+y & z+w & x-w & z \\ z & y+w & x-z & y \\ 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{v} = [\, x, y, z^2 + zw + w^2, z^2 - w^2 \,].$$

This ideal satisfies Condition (B), but $\mathbf{R}[It]$ is not aCM. This is unfortunate but opens the question of when such ideals satisfy Condition (B). The $\mathbf{f}$-sequence here is $(4, 3, 3, 1, 1, 1, 1)$.

The degrees of L. We examine how the Hilbert function of $\mathbf{R}/J : a$ organizes the generators of $\mathbf{L}$. We denote the presentations variables by $u, \mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4$, with u corresponding to a.

• $(1, 4, 1)$. We know (Theorem 4.9) that $JI : I^2 = \mathfrak{m}$. This means that we have forms

$$\begin{aligned} \mathbf{h}_1 &= xu^2 + \cdots \\ \mathbf{h}_2 &= yu^2 + \cdots \\ \mathbf{h}_3 &= zu^2 + \cdots \\ \mathbf{h}_4 &= wu^2 + \cdots \end{aligned}$$

with the $(\cdots)$ in $(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4)\mathbf{B}_1$. The corresponding resultant, of degree 8, is nonzero.

• $(1,2,1)$. There are two forms of degree 1 in $\mathbf{L}$,

$$\begin{aligned}\mathbf{f}_1 &= xu + \cdots\\ \mathbf{f}_2 &= yu + \cdots .\end{aligned}$$

The forms in $L_2/\mathbf{B}_1L_1$ have coefficients in $\mathfrak{m}^2$. This will follow from $I_1(\phi) = \mathfrak{m}$. We need a way to generate two forms of degree 3. Since we expect $JI : I^2 = \mathfrak{m}$, this would mean the presence of two forms in L_3,

$$\begin{aligned}\mathbf{h}_1^* &= zu^3 + \cdots\\ \mathbf{h}_2^* &= wu^3 + \cdots ,\end{aligned}$$

which, together with $\mathbf{f}_1$ and $\mathbf{f}_2$, would give the nonzero degree 8 resultant.

• $(1,3,1)$. There is a form $\mathbf{f}_1 = xu + \cdots \in L_1$ and two forms in L_2

$$\begin{aligned}\mathbf{h}_1 &= yu^2 + \cdots\\ \mathbf{h}_2 &= zu^2 + \cdots ,\end{aligned}$$

predicted by [**19**, Proposition 3.7], if $I_1(\phi) = (x, y, z, w^2)$. (There are indications that this is always the case.) We need a cubic equation $\mathbf{h}_3^* = wu^3 + \cdots$ to launch the nonzero resultant of degree 8.

For all quaternary quadrics with $\mathbf{R}[It]$ almost Cohen-Macaulay, Corollary 3.21 says that $\nu(T) \leq \lambda(\mathbf{R}/J : a)$. Let us compare to the actual number of generators in the examples discussed above:

$$\begin{bmatrix} \nu(T) & & \lambda(I/J)\\ 5 & (1,4,1) & 6\\ 4 & (1,3,1) & 5\\ 4 & (1,2,1) & 4 \end{bmatrix}.$$

We note that, in the last case, T is an Ulrich module.

4.2. Monomial ideals. Monomial ideals of finite colength which are almost complete intersections have a very simple description. We examine a narrow class of them. Let $\mathbf{R} = k[x, y, z]$ be a polynomial ring over an infinite field, and let J and I be $\mathbf{R}$-ideals such that

$$J = (x^a,\ y^b,\ z^c) \subset (J,\ x^\alpha y^\beta z^\gamma) = I.$$

This is the general form of almost complete intersections of $\mathbf{R}$ generated by monomials. Perhaps the most interesting cases are those where

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}<1.$$

This inequality ensures that J is not a reduction of I. Let

$$Q=(x^a-z^c,\ y^b-z^c,\ x^\alpha y^\beta z^\gamma),$$

and suppose that $a>3\alpha$, $b>3\beta$, $c>3\gamma$. Note that $I=(Q,\ z^c)$. Then Q is a minimal reduction of I and the reduction number $\mathrm{red}_Q(I)\le 2$. We label these ideals $I(a,b,c,\alpha,\beta,\gamma)$.

We will examine in detail the cases $a=b=c=n\ge 3$ and $\alpha=\beta=\gamma=1$. We want to argue that $\mathbf{R}[It]$ is almost Cohen-Macaulay. To benefit from the monomial generators in using *Macaulay 2*, we set $I=(xyz,x^n,y^n,z^n)$. Setting $\mathbf{B}=\mathbf{R}[u,\mathbf{T}_1,\mathbf{T}_2,\mathbf{T}_3]$, we claim that

$$\begin{aligned}\mathbf{L}=(z^{n-1}u-xy\mathbf{T}_3,y^{n-1}u-xz\mathbf{T}_2,x^{n-1}u-yz\mathbf{T}_1,\\ z^n\mathbf{T}_2-y^n\mathbf{T}_3,z^n\mathbf{T}_1-x^n\mathbf{T}_3,y^n\mathbf{T}_1-x^n\mathbf{T}_2,\\ y^{n-2}z^{n-2}u^2-x^2\mathbf{T}_2\mathbf{T}_3,x^{n-2}z^{n-2}u^2-y^2\mathbf{T}_1\mathbf{T}_3,\\ x^{n-2}y^{n-2}u^2-z^2\mathbf{T}_1\mathbf{T}_2,x^{n-3}y^{n-3}z^{n-3}u^3-\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3).\end{aligned}$$

We also want to show that these ideals define an almost Cohen-Macaulay Rees algebra.

There is a natural specialization argument. Let X, Y and Z be new indeterminates, and let $\mathbf{B}_0=\mathbf{B}[X,Y,Z]$. In this ring, define the ideal $\mathbf{L}_0$ obtained by replacing in the list above of generators of $\mathbf{L}$, x^{n-3} by X and accordingly x^{n-2} by xX, and so on; carry out similar actions on the other variables:

$$\begin{aligned}\mathbf{L}_0=(z^2Zu-xy\mathbf{T}_3,y^2Yu-xz\mathbf{T}_2,x^2Xu-yz\mathbf{T}_1,\\ z^3Z\mathbf{T}_2-y^3Y\mathbf{T}_3,z^3Z\mathbf{T}_1-x^3X\mathbf{T}_3,y^3Y\mathbf{T}_1-x^3X\mathbf{T}_2,\\ yzYZu^2-x^2\mathbf{T}_2\mathbf{T}_3,xzXZu^2-y^2\mathbf{T}_1\mathbf{T}_3,\\ xyXYu^2-z^2\mathbf{T}_1\mathbf{T}_2,XYZu^3-\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3).\end{aligned}$$

Invoking *Macaulay 2* gives a (non-minimal) projective resolution

$$0\longrightarrow\mathbf{B}_0^4\xrightarrow{\phi_4}\mathbf{B}_0^{17}\xrightarrow{\phi_3}\mathbf{B}_0^{22}\xrightarrow{\phi_2}\mathbf{B}_0^{10}\xrightarrow{\phi_1}\mathbf{B}_0\longrightarrow\mathbf{B}_0/\mathbf{L}_0\longrightarrow 0.$$

We claim that the specialization $X \to x^{n-3}$, $Y \to y^{n-3}$, $Z \to z^{n-3}$ gives a projective resolution of $\mathbf{L}$.

• Call $\mathbf{L}'$ the result of the specialization in $\mathbf{B}$. We argue that $\mathbf{L}^{\mathfrak{p}} = \mathbf{L}$.

• Inspection of the Fitting ideal F of ϕ_4 shows that it contains $(x^3, y^3, z^3, u^3, \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3)$. From standard theory, the radicals of the Fitting ideals of ϕ_2 and ϕ_2 contain $\mathbf{L}_0$, and therefore the radicals of the Fitting ideals of these mappings after specialization will contain the ideal (L_1) of $\mathbf{B}$, as $L_1 \subset \mathbf{L}'$.

• Because (L_1) has codimension 3, by the acyclicity theorem ([**1**, 1.4.13]), the complex gives a projective resolution of $\mathbf{L}'$. Furthermore, as proj. $\dim \mathbf{B}/\mathbf{L}' \leq 4$, $\mathbf{L}'$ has no associated primes of codimension ≥ 5. Meanwhile, the Fitting ideal of ϕ_4 having codimension ≥ 5 forbids the existence of associated primes of codimension 4. Thus, $\mathbf{L}'$ is unmixed.

• Finally, in $(L_1) \subset \mathbf{L}'$, as $\mathbf{L}'$ is unmixed, its associated primes are minimal primes of (L_1), but by Proposition 2.5 (iii), there are just two such, $\mathfrak{m}\mathbf{B}$ and $\mathbf{L}$. Since $\mathbf{L}' \not\subset \mathfrak{m}\mathbf{B}$, $\mathbf{L}$ is its unique associated prime. Localizing at $\mathbf{L}$ gives the equality of $\mathbf{L}'$ and $\mathbf{L}$ since $\mathbf{L}$ is a primary component of (L_1).

Let us sum up this discussion:

Proposition 4.12. *The Rees algebra of* $I(n,n,n,1,1,1)$, $n \geq 3$, *is almost Cohen-Macaulay.*

Corollary 4.13. $e_1(I(n,n,n,1,1,1)) = 3(n+1)$.

Proof. This follows easily since $e_0(I) = 3n^2$, the colengths of the monomial ideals I and $I_1(\phi)$ directly calculated and $\mathrm{red}_J(I) = 2$ so that

$$\begin{aligned} e_1(I) &= \lambda(I/J) + \lambda(I^2/JI) \\ &= \lambda(I/J) + [\lambda(I/J) - \lambda(\mathbf{R}/I_1(\phi))] \\ &= (3n-1) + 4. \qquad \square \end{aligned}$$

Remark 4.14. We have also experimented with other cases beyond those with xyz and in higher dimension as well.

• In $\dim \mathbf{R} = 4$, the ideal $I = I(n,n,n,n,1,1,1,1) = (x_1^n, x_2^n, x_3^n, x_4^n, x_1x_2x_3x_4)$, $n \geq 4$, has a Rees algebra $\mathbf{R}[It]$ which is almost Cohen-Macaulay.

• The argument used was a copy of the previous case, but we needed to make an adjustment in the last step to estimate the codimension of the last Fitting ideal F of the corresponding mapping ϕ_5. This is a large matrix, so it would not be possible to find the codimension of F by looking at all its maximal minors. Instead, one argues as follows. Because I is $\mathfrak{m}$-primary, $\mathfrak{m} \subset \sqrt{F}$, so we can drop the entries in ϕ_5 in $\mathfrak{m}$. Inspection will give $u^{16} \in F$, so dropping all u's gives additional minors in $\mathbf{T}_1, \ldots, \mathbf{T}_4$, for height $(F) \geq 6$. This suffices to show that $\mathbf{L} = \mathbf{L}'$.

Conjecture 4.15. *Let I be a monomial ideal of $k[x_1, \ldots, x_d]$. If I is an almost complete intersection of finite colength, its Rees algebra $\mathbf{R}[It]$ is almost Cohen-Macaulay.*

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