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REGULARITY BOUNDS FOR BINOMIAL EDGE IDEALS

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Dedicated to Professor Jürgen Herzog on the occasion of his 70th birthday

ABSTRACT. We show that the Castelnuovo-Mumford regularity of the binomial edge ideal of a graph is bounded below by the length of its longest induced path and bounded above by the number of its vertices.

1. Introduction. Let G be a simple graph on the vertex set $[n] = \{1, 2, \dots, n\}$. The *binomial edge ideal* J_G of G, introduced by Herzog et al. [**4**] and Ohtani [**8**], is the ideal in the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ over a field K, defined by

$$J_G = (x_i y_j - x_j y_i : \{i, j\} \text{ is an edge of } G).$$

From an algebraic viewpoint, it is of interest to study relations between algebraic properties of J_G and combinatorial properties of G. In this note, we prove the following simple combinatorial bounds for the regularity of binomial edge ideals.

Theorem 1.1. *Let G be a simple graph on $[n]$, and let ℓ be the length of the longest induced path of G. Then*

$$\ell + 1 \le \operatorname{reg}(J_G) \le n.$$

2. A lower bound. In this section, we prove the lower bound in Theorem 1.1. Throughout the paper, we will use the standard terminologies of graph theory in [**2**].

We consider the $\mathbf{N}^n$-grading of S defined by $\deg x_i = \deg y_i = \mathbf{e}_i$, where $\mathbf{e}_i$ is the ith unit vector of $\mathbf{N}^n$. Binomial edge ideals are $\mathbf{N}^n$-graded by definition. For an $\mathbf{N}^n$-graded S-module M and $\mathbf{a} \in \mathbf{N}^n$,

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we write $M_{\mathbf{a}}$ for the graded component of M of degree $\mathbf{a}$ and write $\beta_{i,\mathbf{a}}(M) = \dim_K \mathrm{Tor}_i(M,K)_{\mathbf{a}}$ for the $\mathbf{N}^n$*-graded Betti numbers* of M. Also, for $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{N}^n$, let $\mathrm{supp}\,(\mathbf{a}) = \{i \in [n] : a_i \neq 0\}$ and $|\mathbf{a}| = a_1 + \cdots + a_n$. Then the $\mathbf{N}$*-graded Betti numbers* of M are $\beta_{i,j}(M) = \sum_{\mathbf{a} \in \mathbf{N}^n, |\mathbf{a}|=j} \beta_{i,\mathbf{a}}(M)$ and the (*Castelnuovo-Mumford*) *regularity* of M is

$$\mathrm{reg}\,(M) = \max\{j : \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}.$$

For a simple graph G on the vertex set $[n]$ and for a subset $W \subset [n]$, we write G_W for the induced subgraph of G on W. For convenience, we consider that G_W has the vertex set $[n]$ and regard J_{G_W} as an ideal of S.

Lemma 2.1. *Let G be a simple graph on $[n]$, and let $W \subset [n]$. Then, for any $\mathbf{a} \in \mathbf{N}^n$ with $\mathrm{supp}\,(\mathbf{a}) \subset W$, one has*

$$\beta_{i,\mathbf{a}}(J_G) = \beta_{i,\mathbf{a}}(J_{G_W}) \quad \text{for all } i.$$

Proof. Let

$$\mathcal{F} : 0 \longrightarrow \bigoplus_{\mathbf{a} \in \mathbf{N}^n} S^{\beta_{p,\mathbf{a}}(J_G)}(-\mathbf{a}) \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{a} \in \mathbf{N}^n} S^{\beta_{0,\mathbf{a}}(J_G)}(-\mathbf{a}) \xrightarrow{\phi} S$$

be the $\mathbf{N}^n$-graded minimal free resolution of S/J_G, where p is the projective dimension of J_G. Consider its subcomplex

$$\mathcal{F}' : 0 \longrightarrow \bigoplus_{\substack{\mathbf{a} \in \mathbf{N}^n \\ \mathrm{supp}\,(\mathbf{a}) \subset W}} S^{\beta_{p,\mathbf{a}}(J_G)}(-\mathbf{a}) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{\substack{\mathbf{a} \in \mathbf{N}^n \\ \mathrm{supp}\,(\mathbf{a}) \subset W}} S^{\beta_{0,\mathbf{a}}(J_G)}(-\mathbf{a}) \xrightarrow{\phi'} S.$$

We claim that $\mathcal{F}'$ is the minimal free resolution of S/J_{G_W}. It is clear that $\mathrm{coker}\,\phi' = S/J_{G_W}$. Hence, what we must prove is that $\mathcal{F}'$ is acyclic. To prove this, it is enough to show that the multigraded component $\mathcal{F}'_{\mathbf{a}}$ is acyclic for any $\mathbf{a} \in \mathbf{N}^n$ with $\mathrm{supp}\,(\mathbf{a}) \subset W$.

Let $\mathbf{a} \in \mathbf{N}^n$ with $\operatorname{supp}(\mathbf{a}) \subset W$. Since, for any $\mathbf{b} \in \mathbf{N}^n$, $S(-\mathbf{b})_{\mathbf{a}}$ is non-zero if and only if $\mathbf{a} - \mathbf{b}$ is non-negative, we have

$$\mathcal{F}_{\mathbf{a}} = \mathcal{F}'_{\mathbf{a}},$$

which implies that $\mathcal{F}'_{\mathbf{a}}$ is acyclic since $\mathcal{F}$ is a minimal free resolution. □

Corollary 2.2. *With the same notation as in Lemma* 2.1*, one has* $\beta_{i,j}(J_G) \geq \beta_{i,j}(J_{G_W})$ *for all* i, j*.*

Corollary 2.3. *Let* G *be a simple graph on* $[n]$*, and let* ℓ *be the length of the longest induced path of* G*. Then* $\operatorname{reg}(J_G) \geq \ell + 1$*.*

Proof. Observe that the binomial edge ideal of a path of length ℓ is a complete intersection having ℓ generators of degree 2 and has the regularity $\ell + 1$. Then the statement follows from Corollary 2.2. □

3. An upper bound. In this section, we prove the upper bound in Theorem 1.1.

We consider the $\mathbf{N}^{2n}$-grading of S defined by $\deg x_i = \mathbf{e}_i$ and $\deg y_i = \mathbf{e}_{i+n}$. Binomial edge ideals are not $\mathbf{N}^{2n}$-graded but monomial ideals in S are $\mathbf{N}^{2n}$-graded. To simplify the notation, we identify the multidegree $(\mathbf{a}, \mathbf{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbf{N}^{2n}$ and the monomial $x^{\mathbf{a}} y^{\mathbf{b}} = x_1^{a_1} \cdots x_n^{a_n} y_1^{b_1} \cdots y_n^{b_n}$, and, for an $\mathbf{N}^{2n}$-graded S-module M, write

$$\beta_{i,x^{\mathbf{a}}y^{\mathbf{b}}}(M) = \beta_{i,(\mathbf{a},\mathbf{b})}(M).$$

Also, we write

$$P(M,t) = \sum_{k=0}^{2n} \sum_{(\mathbf{a},\mathbf{b}) \in \mathbf{N}^{2n}} \beta_{k,(\mathbf{a},\mathbf{b})}(M) x^{\mathbf{a}} y^{\mathbf{b}} t^k$$

for the ($\mathbf{N}^{2n}$-graded) *Poincaré series* of M.

Lemma 3.1. *Let* $m_1, \ldots, m_g$ *be monomials in* S *and* $I = (m_1, \ldots, m_g)$*. Then*

$$P(S/I, t) \leq 1 + \sum_{m_j \notin (m_1, \ldots, m_{j-1})} P\big(S/\big((m_1, \ldots, m_{j-1}) : m_j\big), t\big) m_j t,$$

where the inequality is coefficient-wise.

Proof. The assertion follows from the short exact sequence

$$0 \longrightarrow S/\big((m_1,\dots,m_{j-1}):m_j\big) \xrightarrow{\times m_j} S/(m_1,\dots,m_{j-1}) \longrightarrow S/(m_1,\dots,m_j) \longrightarrow 0$$

for $j=2,3,\dots,g$, by mapping cone construction (cf. [**9**, Construction 27.3]). ◻

We now consider binomial edge ideals. In the rest of this section, we fix a simple graph G on $[n]$. We say that a path

$$P : s = v_0 \longrightarrow v_1 \longrightarrow \cdots \longrightarrow v_r = t$$

of G is *admissible* if $s < t$ and, for $k = 1,2,\dots,r-1$, one has either $v_k < s$ or $v_k > t$. The vertices s and t are called the *ends* of P and the vertices $v_1,\dots,v_{r-1}$ are called the *inner vertices* of P.

For an admissible path $P : s = i_0 \to i_1 \to \cdots \to i_r = t$, we define the monomial

$$m_P = \Big(\prod_{v_k<s} y_{v_k}\Big)\Big(\prod_{v_k>t} x_{v_k}\Big) x_s y_t.$$

Let $\mathcal{P}(G)$ be the set of all admissible paths of G, and let $>_{\mathrm{lex}}$ be the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. For an ideal $I \subset S$, let $\mathrm{in}_{>_{\mathrm{lex}}}(I)$ be the initial ideal of I with respect to $>_{\mathrm{lex}}$. The following result is due to Herzog et al. [**4**, Theorem 2.1] and Ohtani [**8**, Theorem 3.2].

Lemma 3.2. $\mathrm{in}_{>_{\mathrm{lex}}}(J_G) = (m_P : P \in \mathcal{P}(G))$.

Note that our definition of the admissibility is different from that in [**4**]. In particular, the generators in Lemma 3.2 may not be minimal.

The next property is a key lemma to prove the main result.

Lemma 3.3. *Let $P : s = v_0 \to \cdots \to v_r = t$ be an admissible path and $1 \le k \le r-1$.*

(i) *If $v_k < s$, then there is an $\ell > k$ such that $P' : v_k \to v_{k+1} \to \cdots \to v_\ell$ is an admissible path of G and $m_{P'}$ divides $x_{v_k} m_P$.*

(ii) *If $v_k > t$, then there is an $\ell < k$ such that $P' : v_\ell \to v_{\ell+1} \to \cdots \to v_k$ is an admissible path of G and $m_{P'}$ divides $y_{v_k} m_P$.*

Proof. We prove (i) (the proof for (ii) is similar). Let $\ell > k$ be the smallest integer satisfying $i_k < i_\ell \leq t$. Then the path $P' : v_k \to v_{k+1} \to \cdots \to v_\ell$ satisfies the desired condition. ◻

We call a path P' satisfying condition (i) or (ii) in Lemma 3.3 a *wedge* of P at v_k.

From now on, we fix an ordering

$$P_1, P_2, \ldots, P_g$$

of the admissible paths of G, where $g = \#\mathcal{P}(G)$, such that if the length of P_i is smaller than that of P_j, then $i < j$. To simplify the notation, we write

$$m_k = m_{P_k},$$

for $k = 1, 2, \ldots, g$. Then $\mathrm{in}_{>_{\mathrm{lex}}}(J_G) = (m_1, \ldots, m_g)$. By the choice of the ordering, if P_i is a wedge of P_j, then $i < j$. This fact immediately implies the following property.

Lemma 3.4. *Let $1 < j \leq g$, and let s and t be the ends of P_j with $s < t$. For any inner vertex v of P_j, one has $x_v \in (m_1, \ldots, m_{j-1}) : m_j$ if $v < s$ and $y_v \in (m_1, \ldots, m_{j-1}) : m_j$ if $v > t$.*

For a monomial $w \in S$, let

$$\mathrm{mult}\,(w) = \{k \in [n] : x_k y_k \text{ divides } w\}.$$

Note that, for a squarefree monomial $w \in S$, one has $\deg w \leq n + \#\mathrm{mult}\,(w)$. Since the regularity does not decrease under taking initial ideals (see e.g., [**9**, Theorem 22.9]), the next statement proves the remaining part of Theorem 1.1.

Proposition 3.5. *For any monomial $w \in S$ and an integer $p > 0$, one has*

$$\beta_{p,w}\left(S/\mathrm{in}_{>_{\mathrm{lex}}}(J_G)\right) = 0 \quad \text{if } \#\mathrm{mult}\,(w) \geq p.$$

In particular, $\mathrm{reg}\,(\mathrm{in}_{>_{\mathrm{lex}}}(J_G)) \leq n$.

Proof. The second statement follows from the first statement together with the fact that the multigraded Betti numbers of a squarefree monomial ideal is concentrated in squarefree degrees. Thus, we have proved the first statement.

We first introduce some notations. Let $\mathcal{M} = \{m_1, m_2, \dots, m_g\}$. We say that a subset $F = \{m_{i_1}, m_{i_2}, \dots, m_{i_k}\} \subset \mathcal{M}$, where $i_1 < \cdots < i_k$, is a *Lyubeznik subset* of $\mathcal{M}$ (of size k) if, for $j = 1, 2, \dots, k$, any monomial m_ℓ with $\ell < i_j$ does not divide $\mathrm{lcm}\,(m_{i_j}, m_{i_{j+1}}, \dots, m_{i_k})$. We prove the assertion by the following two claims.

Claim 1. Let $F = \{m_{i_1}, \dots, m_{i_k}\}$, where $i_1 < \cdots < i_k$, be a Lyubeznik subset of $\mathcal{M}$. Then

(i) $\mathrm{mult}\,(\mathrm{lcm}\,(F))$ contains no inner vertices of P_{i_1}.

(ii) if $\mathrm{mult}\,(\mathrm{lcm}\,(F))$ contains no inner vertices of P_{i_j} for $j = 2, 3, \dots, k$, then $\#\mathrm{mult}\,(\mathrm{lcm}\,(F)) \le k - 1$.

Claim 2. Let $F = \{m_{i_1}, \dots, m_{i_k}\}$, where $i_1 < \cdots < i_k$, be a Lyubeznik subset of $\mathcal{M}$ and w a monomial of S. Let $p > 0$ be an integer. Suppose

(a) $\beta_{p,w}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k})) \neq 0$, and

(b) $\mathrm{mult}\,(w \cdot \mathrm{lcm}\,(F))$ contains no inner vertices of P_{i_δ} for $\delta = 2, 3, \dots, k$.

Then there is a Lyubeznik subset $\widetilde{F} = \{m_{j_1}, \dots, m_{j_\ell}\}$, where $j_1 < \cdots < j_\ell$, of $\mathcal{M}$ and a monomial $\widetilde{w}$ such that

(a′) $\beta_{p-1,\widetilde{w}}(S/((m_1, \dots, m_{j_1-1}) : m_{j_1} \cdots m_{j_\ell})) \neq 0$,

(b′) $\mathrm{mult}\,(\widetilde{w} \cdot \mathrm{lcm}\,(\widetilde{F}))$ contains no inner vertices of P_{j_δ} for $\delta = 2, 3, \dots, \ell$, and

(c′) $\#\mathrm{mult}\,(\widetilde{w} \cdot \mathrm{lcm}\,(\widetilde{F})) - \#\widetilde{F} = \#\mathrm{mult}\,(w \cdot \mathrm{lcm}\,(F)) - \#F - 1$.

We first show that these claims prove the desired statement. Let $u \in S$ be a monomial such that $\beta_{p,u}(S/\mathrm{in}_{>_{\mathrm{lex}}}(J_G)) \neq 0$ with $p > 0$. We show that there is a Lyubeznik subset F such that

$$\#\mathrm{mult}\,(u) = \#\mathrm{mult}\,(\mathrm{lcm}\,(F)) - \#F + p \tag{1}$$

and F satisfies the assumption of Claim 1 (ii). Note that this proves the desired statement by Claim 1 (ii).

Recall $\mathrm{in}_{>_{\mathrm{lex}}}(J_G) = (m_1, \dots, m_g)$. By Lemma 3.1, there is a Lyubeznik subset $\{m_j\}$ of size 1 such that $\beta_{p-1,u/m_j}(S/((m_1, \dots, m_{j-1}) : m_j)) \neq 0$. If $p = 1$, then $u = m_j$ and the set $\{m_j\}$ has the desired property (1). Suppose $p > 1$. Then the pair of the Lyubeznik set $\{m_j\}$ and a monomial u/m_j satisfies assumptions (a) and (b) of Claim 2. Thus, by applying Claim 2 repeatedly, one obtains a Lyubeznik subset $F = \{m_{i_1}, \dots, m_{i_k}\}$ and a monomial w such that

- $\beta_{0,w}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k})) \neq 0$, and
- $\#\mathrm{mult}\,(w \cdot \mathrm{lcm}\,(F)) - \#F = \#\mathrm{mult}\,(u) - p$.

The first condition says $w = x^{\mathbf{0}} y^{\mathbf{0}}$, where $\mathbf{0} = (0, \dots, 0)$, and the second condition proves that F satisfies the desired property (1).

In the rest, we prove Claims 1 and 2.

Proof of Claim 1. (i) Suppose to the contrary that there is an inner vertex v of P_{i_1} which belongs to mult (lcm (F)). Let P_j be a wedge of P_{i_1} at v. Then $j < i_1$ and m_j divides lcm $(m_{i_1}, \dots, m_{i_k})$ by Lemma 3.3. This contradicts the definition of Lyubeznik sets.

(ii) Let $s_1, t_1, s_2, t_2, \dots, s_k, t_k$ be the ends of $P_{i_1}, \dots, P_{i_k}$, where $s_j < t_j$ for all j. By (i) and the assumption, mult (lcm (F)) contains no inner vertices of P_{i_j} for all j. Hence

$$\#\mathrm{mult}\,(\mathrm{lcm}\,(F)) \leq \#\mathrm{mult}\,(x_{s_1} y_{t_1} x_{s_2} y_{t_2} \cdots x_{s_k} y_{t_k}) \leq k - 1,$$

where the last inequality follows from $s_1 < t_1, \dots, s_k < t_k$. ◻

Proof of Claim 2. We consider two cases.

Case 1: Suppose that mult $(w \cdot \mathrm{lcm}\,(F))$ contains an inner vertex v of P_{i_1}. Consider the case that x_v divides m_{i_1} (the case that y_v divides m_{i_1} is similar). Since y_v does not divide lcm (F) by Claim 1 (i), y_v divides w. Then, as $y_v \in (m_1, \dots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k}$ by Lemma 3.4, we have $\beta_{p,w}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k})) \neq 0$ if and only if $\beta_{p-1,w/y_v}(S/((m_1, \dots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k})) \neq 0$. Then the pair of the set $\widetilde{F} = F$ and the monomial $\widetilde{w} = w/y_v$ satisfies (a′), (b′) and (c′), as desired.

Case 2: Suppose that mult $(w \cdot \mathrm{lcm}\,(F))$ contains no inner vertices of P_{i_1}. For $j = 1, 2, \dots, i_1 - 1$, let

$$\overline{m}_j = \frac{m_j}{\gcd(m_j, m_{i_1} \cdots m_{i_k})}.$$

Then we have

$$(\overline{m}_1, \ldots, \overline{m}_{i_1-1}) = (m_1, \ldots, m_{i_1-1}) : m_{i_1} \cdots m_{i_k}.$$

By Lemma 3.1 and (a), there is a $1 \leq i_0 < i_1$ such that $\overline{m}_{i_0} \notin (\overline{m}_1, \ldots, \overline{m}_{i_0-1})$ and

$$\beta_{p-1, w/\overline{m}_{i_0}} \big(S/\big((\overline{m}_1, \ldots, \overline{m}_{i_0-1}) : \overline{m}_{i_0}\big)\big) \neq 0. \tag{2}$$

Let $\widetilde{w} = w/\overline{m}_{i_0}$ and $\widetilde{F} = \{m_{i_0}, m_{i_1} \ldots, m_{i_k}\}$. Since, for $\ell < i_0$, $\overline{m}_\ell$ divides $\overline{m}_{i_0}$ if and only if m_ℓ divides $\mathrm{lcm}\,(m_{i_0}, m_{i_1}, \ldots, m_{i_k})$, $\widetilde{F}$ is a Lyubeznik subset. Also, since

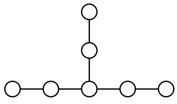
$$(\overline{m}_1, \ldots, \overline{m}_{i_0-1}) : \overline{m}_{i_0} = (m_1, \ldots, m_{i_0-1}) : m_{i_0} m_{i_1} \cdots m_{i_k},$$

(2) and the fact $w \cdot \mathrm{lcm}\,(F) = \widetilde{w} \cdot \mathrm{lcm}\,(\widetilde{F})$ say that the pair $\widetilde{F}$ and $\widetilde{w}$ satisfies (a′), (b′) and (c′) as desired. □

Remark 3.6. Although we use conditions that appear in Lyubeznik resolutions [**5**], Lyubeznik resolutions themselves seem not to prove Proposition 3.5.

Remark 3.7. Madani and Kiani [**7**, Theorem 3.2] gave a better upper bound when G is closed. They proved that, if G is closed, then $\mathrm{reg}\, I_G$ is bounded above by the number of maximal cliques of G plus one, which is smaller than or equal to the number of the vertices of G by Dirac's theorem on chordal graphs.

Example 3.8. Both inequalities in Theorem 1.1 could be strict. Indeed, the regularity of the binomial edge ideal of the following graph is 6. However, the graph has 7 vertices and the length of its longest induced path is 4.



Remark 3.9. A similar bound holds for the depth of S/J_G. Let K_n be the complete graph on $[n]$. If G is a connected graph on $[n]$, then J_{K_n} is an associated prime of S/J_G by [**4**, Corollary 3.9] and $\dim S/J_{K_n} = n+1$. This fact implies $\operatorname{depth}(S/J_G) \leq n+1$ (see [**1**, Proposition 1.2.13]).

We end this note with the following conjecture.

Conjecture 3.10. *Let G be a graph on $[n]$. If $\operatorname{reg}(J_G) = n$, then G is a path of length $n-1$.*

We verify Conjecture 3.10 for graphs with at most 9 vertices in characteristic 0 and 2 by using Macaulay2 [**3**]. For this computation, we use the list of graphs with at most 9 vertices in [**7**].

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