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MONOIDS OF MODULES OVER RINGS OF INFINITE COHEN-MACAULAY TYPE

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ABSTRACT. Given a one-dimensional analytically unramified local ring $(R, \mathfrak{m})$, let $\mathfrak{C}(R)$ denote the monoid of isomorphism classes of maximal Cohen-Macaulay R-modules (together with $[0]$) with operation given by $[M]+[N] = [M \oplus N]$. If R is complete, then the Krull-Remak-Schmidt property holds; i.e., direct-sum decompositions of finitely generated R-modules are unique. If R is not complete, then properties of the monoid $\mathfrak{C}(R)$ measure how far R is from having the Krull-Remak-Schmidt property. Using a list of ranks of indecomposable maximal Cohen-Macaulay modules over the $\mathfrak{m}$-adic completion of R, we give a description of the monoid $\mathfrak{C}(R)$ when R has infinite Cohen-Macaulay type. Under certain hypotheses we show that, for arbitrary integers s and t both greater than one, there exists a maximal Cohen-Macaulay R-module M such that $M \cong L_1 \oplus \cdots \oplus L_s$ and $M \cong N_1 \oplus \cdots \oplus N_t$ for indecomposable maximal Cohen-Macaulay R-modules L_i and N_j.

1. Introduction. Let R be a commutative ring, and let $\mathcal{C}$ be a class of R-modules closed under isomorphism, finite direct sums and direct summands. We say the *Krull-Remak-Schmidt property* holds for the class $\mathcal{C}$ if, whenever $M_1 \oplus M_2 \oplus \cdots \oplus M_s \cong N_1 \oplus N_2 \oplus \cdots \oplus N_t$ for indecomposable modules M_i, $N_j \in \mathcal{C}$, then

(1) $t = s$, and

(2) there exists a permutation σ of the set $\{1, \ldots, s\}$ such that $M_i \cong N_{\sigma(i)}$ for each $i \in \{1, \ldots, s\}$.

Over a complete local ring, the Krull-Remak-Schmidt property holds for the class of finitely generated modules (see [**16**, Theorem 5.20]). Many authors, including Evans [**6**, Section 1] and Wiegand [**18**, Sections 3 and 4], have produced examples of noncomplete local rings for which direct-sum decompositions of finitely generated modules are

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nonunique. One way to study direct-sum decompositions over a local ring R is to consider the monoid of isomorphism classes of finitely generated R-modules with operation given by $[M]+[N]=[M\oplus N]$. In particular, this monoid is free if and only if the Krull-Remak-Schmidt property holds for the class of finitely generated R-modules. Moreover, certain invariants of the monoid measure nonuniqueness of direct-sum decompositions over R. This approach has been used frequently; see for example [**1, 2, 7**].

We restrict our attention to one-dimensional analytically unramified local rings $(R,\mathfrak{m})$ and to the class of maximal Cohen-Macaulay R-modules. Given a list of ranks of indecomposable modules over the $\mathfrak{m}$-adic completion of R, we construct the monoid $\mathfrak{C}(R)$ of isomorphism classes of maximal Cohen-Macaulay R-modules (together with $[0]$). In [**2**], Baeth and Luckas describe the monoid $\mathfrak{C}(R)$ when R has finite Cohen-Macaulay type. Our goal in this paper is to study the monoid $\mathfrak{C}(R)$ when R has infinite Cohen-Macaulay type.

In Section 2, we recall several results about ranks of indecomposable maximal Cohen-Macaulay modules. In Section 3, we describe the monoid $\mathfrak{C}(R)$ as a Diophantine monoid and, in Sections 4 and 5, we give properties of $\mathfrak{C}(R)$. In particular, in Section 5, we consider the elasticity of $\mathfrak{C}(R)$, an invariant that measures how far $\mathfrak{C}(R)$ is from being free. Under additional hypotheses, we prove that for an arbitrary pair of integers s and t, both greater than one, there exists a maximal Cohen-Macaulay R-module $M_{s,t}$ such that

$$M_{s,t}\cong L_1\oplus L_2\oplus\cdots\oplus L_s\cong N_1\oplus N_2\oplus\cdots\oplus N_t$$

for indecomposable maximal Cohen-Macaulay R-modules L_i and N_j. In fact, a stronger result is shown in Theorem 5.4. We conclude, in Section 6, with examples of local integral domains $(R,\mathfrak{m})$ whose $\mathfrak{m}$-adic completions have exactly two minimal prime ideals. These examples illustrate properties studied in earlier sections. Throughout the paper, we also make remarks comparing our results obtained when R has infinite Cohen-Macaulay type with the corresponding results in [**2**] obtained when R has finite Cohen-Macaulay type.

2. Ranks of indecomposable modules. Recall that a commutative ring S is *local* if S is a Noetherian ring with exactly one maximal ideal. A local ring $(S, \mathfrak{n})$ is *analytically unramified* if the $\mathfrak{n}$-adic completion of S is reduced. Throughout the paper, we assume that $(R, \mathfrak{m})$ is a one-dimensional analytically unramified local ring with minimal prime ideals $P_1, \ldots, P_s$. (Note that in this context a finitely generated R-module is maximal Cohen-Macaulay if and only if it is non-zero and torsion-free.)

We say $(R, \mathfrak{m})$ has *finite Cohen-Macaulay type* if there exist only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay R-modules. Otherwise, we say R has *infinite Cohen-Macaulay type*. For a maximal Cohen-Macaulay R-module M, the *rank of M at the minimal prime ideal P_i*, denoted $\mathrm{rank}_{P_i}(M)$, is the dimension of the vector space M_{P_i} over the field R_{P_i}. The *rank* of M is the s-tuple $(r_1, \ldots, r_s)$, where $r_i = \mathrm{rank}_{P_i}(M)$.

When R has infinite Cohen-Macaulay type, we do not have a complete description of the s-tuples that occur as the ranks of indecomposable maximal Cohen-Macaulay R-modules. However, we do know some ranks that always occur. For example, from the work of Wiegand [**17**, Section 2], it is known that for every positive integer r, there exists an indecomposable maximal Cohen-Macaulay R-module of constant rank $(r, \ldots, r)$ (see also [**12**, Theorem 1.4]). The following result from Saccon's Ph.D. thesis describes the ranks that occur for indecomposable maximal Cohen-Macaulay R-modules when there is at least one minimal prime ideal P such that the ring R/P has infinite Cohen-Macaulay type.

Theorem 2.1 [**15**, Theorem 3.4.1]. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with minimal prime ideals $P_1, \ldots, P_s$. Assume R/P_{i_0} has infinite Cohen-Macaulay type for some $i_0 \in \{1, \ldots, s\}$. Let $(r_1, \ldots, r_s)$ be a non-zero s-tuple of nonnegative integers with $r_i \leq 2r_{i_0}$ for all $i \in \{1, \ldots, s\}$.*

(1) *There exists an indecomposable maximal Cohen-Macaulay R-module of rank $(r_1, \ldots, r_s)$.*

(2) *If the residue field k is infinite, then the set of isomorphism classes of indecomposable maximal Cohen-Macaulay R-modules of rank $(r_1, \ldots, r_s)$ has cardinality $|k|$.*

As we will see in Sections 3 and 5, the information provided by Theorem 2.1 is enough to describe direct-sum decompositions of maximal

Cohen-Macaulay R-modules when the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has at least one minimal prime ideal Q such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.

Remark 2.2. Theorem 2.1 generalizes the main result in Crabbe-Saccon (see [**5**, Main Theorem]). Although more information about the matrices describing the monoid $\mathfrak{C}(R)$ could be gleaned from Theorem 2.1, the theorem in [**5**] is sufficient for most of the results in Section 3. The full strength of Theorem 2.1 is required to prove results about the elasticity of the monoid $\mathfrak{C}(R)$ in Section 5.

Remark 2.3. When R has finite Cohen-Macaulay type, the ranks that occur for indecomposable maximal Cohen-Macaulay R-modules are completely determined and listed in [**3**, Main Theorem]. The list is dramatically different from the list of ranks provided in Theorem 2.1. If R is a one-dimensional analytically unramified local ring of finite Cohen-Macaulay type, then R has at most three minimal prime ideals, and the rank of every indecomposable maximal Cohen-Macaulay R-module occurs in the following list: 1, 2, 3, $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$, $(0,1,1)$, $(1,0,1)$, $(1,1,0)$, $(1,1,1)$ or $(2,1,1)$. (The lack of symmetry in the last possibility presumes a fixed ordering of the minimal prime ideals. The point is that one cannot have both an indecomposable module of rank $(2,1,1)$ and an indecomposable module of rank $(1,2,1)$.)

3. The monoid $\mathfrak{C}(R)$. Let $(R,\mathfrak{m},k)$ be a one-dimensional analytically unramified local ring. We denote the isomorphism class of an R-module M by $[M]$. The goal of this section is to describe the monoid $\mathfrak{C}(R)$ of isomorphism classes of maximal Cohen-Macaulay R-modules (together with $[0]$) with operation given by $[M]+[N]=[M\oplus N]$. In order to study $\mathfrak{C}(R)$, it is useful to pass to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R and to consider $\mathfrak{C}(R)$ as a submonoid of $\mathfrak{C}(\widehat{R})$.

We first recall some terminology about monoids; we refer the reader to Geroldinger and Halter-Koch [**8**] and Halter-Koch [**10**] for details. Let $\mathbf{N}_0$ denote the set of nonnegative integers. For a (possibly infinite) index set Ω, let $\mathbf{N}_0^{(\Omega)}$ denote the direct sum of $|\Omega|$ copies of $\mathbf{N}_0$, indexed by the elements of Ω.

3.1. Krull monoids. A *monoid* H is a commutative cancellative (additive) semigroup with identity 0. We further assume that 0 is the only invertible element (i.e., if $x + y = 0$, then $x = y = 0$). For x, $y \in H$, we say $x \leq y$ if $x + z = y$ for some $z \in H$. A submonoid K of a monoid H is *full* if, for every x, $y \in K$ with $x = y + z$ for some $z \in H$, we have $z \in K$. A monoid H is *free* provided $H \cong \mathbf{N}_0^{(\Omega)}$ for some (possibly infinite) index set Ω. A non-zero element x of a monoid H is an *atom* (or is *irreducible*) if x cannot be written as the sum of two non-zero elements of H. We assume all monoids are *atomic*; that is, every non-zero element can be expressed as a sum of irreducible elements.

A monoid homomorphism $\varphi: H_1 \to H_2$ is a *divisor homomorphism* if $\varphi(x) \leq \varphi(y)$ in H_2 implies $x \leq y$ in H_1. A monoid H is a *Krull monoid* provided there exists a divisor homomorphism from H to a free monoid $\mathbf{N}_0^{(\Omega)}$ for some index set Ω. A *divisor theory* is a divisor homomorphism $\varphi: H \to \mathbf{N}_0^{(\Omega)}$ such that every element of $\mathbf{N}_0^{(\Omega)}$ is the greatest lower bound (in the product partial ordering) of some finite set of elements in $\varphi(H)$. Every Krull monoid admits a divisor theory (see [**10**, Theorem 23.4]).

The *quotient group* $\mathcal{Q}(H)$ of a monoid H is the group of formal differences

$$\mathcal{Q}(H) := \{x - y \mid x, y \in H\}.$$

Given a divisor theory $\varphi: H \to \mathbf{N}_0^{(\Omega)}$, there is an induced quotient homomorphism $\mathcal{Q}(\varphi): \mathcal{Q}(H) \to \mathcal{Q}(\mathbf{N}_0^{(\Omega)})$, where $\mathcal{Q}(\mathbf{N}_0^{(\Omega)}) \cong \mathbf{Z}^{(\Omega)}$. The *divisor class group* $\mathcal{C}l(H)$ is the cokernel of the map $\mathcal{Q}(\varphi)$. The divisor class group of H depends only on H, and not on the divisor theory for H (see [**4**, page 76]).

For a Krull monoid H, the following conditions are equivalent:

(1) H is free.

(2) Every non-zero element of H has a unique representation as a sum of atoms (up to a permutation).

(3) $\mathcal{C}l(H) = 0$.

Thus, the divisor class group $\mathcal{C}l(H)$ is a useful invariant that describes factorizations in the monoid H and measures how far H is from being free.

3.2. The monoid $\mathfrak{C}(R)$ as a Diophantine monoid. Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with $\mathfrak{m}$-adic completion $\widehat{R}$. Let $\mathfrak{C}(R)$ denote the set of isomorphism classes of maximal Cohen-Macaulay R-modules, together with $[0]$. We give $\mathfrak{C}(R)$ the structure of a commutative (additive) semigroup by defining

$$[M] + [N] := [M \oplus N].$$

Observe that, over a local ring, cancellation holds for finitely generated modules; that is, if $M \oplus A \cong M \oplus B$, then $A \cong B$ (see [**6**, Proposition 1]). Thus $\mathfrak{C}(R)$ is a monoid as defined in subsection 3.1. Since the class of maximal Cohen-Macaulay R-modules is closed under direct summands, finite direct sums, and isomorphism, the monoid $\mathfrak{C}(R)$ carries information about direct-sum decompositions over R, for example, whether the Krull-Remak-Schmidt property holds for the class of maximal Cohen-Macaulay R-modules, and, if it does not, how badly it fails (see Section 5). In this subsection, we describe $\mathfrak{C}(R)$ as a Diophantine monoid, i.e., as the set of nonnegative integer solutions of a system of homogeneous linear equations with integer coefficients. Even though the term "Diophantine monoid" usually refers to the monoid of solutions of a system of finitely many equations in finitely many variables, we use this terminology for the monoid $\mathfrak{C}(R)$ regardless of whether it is finitely generated.

Consider the monoid $\mathfrak{C}(\widehat{R})$ of isomorphism classes of maximal Cohen-Macaulay $\widehat{R}$-modules (together with $[0]$). Since the Krull-Remak-Schmidt property holds over complete local rings, we have $\mathfrak{C}(\widehat{R}) \cong \mathbf{N}_0^{(\Lambda)}$, where Λ is the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. The natural map from the category of finitely generated R-modules to the category of finitely generated $\widehat{R}$-modules, sending M to $M \otimes_R \widehat{R}$, induces a monoid homomorphism $\mathfrak{C}(R) \to \mathfrak{C}(\widehat{R})$, sending $[M]$ to $[M \otimes_R \widehat{R}]$. This monoid homomorphism is injective (see [**9**, Proposition 2.5.8]) and a divisor homomorphism (see [**18**, page 544]). As $\mathfrak{C}(\widehat{R}) \cong \mathbf{N}_0^{(\Lambda)}$, we see that $\mathfrak{C}(R)$ is a Krull monoid, and we can consider $\mathfrak{C}(R)$ as a full submonoid of $\mathbf{N}_0^{(\Lambda)}$.

The following definitions are useful in understanding how $\mathfrak{C}(R)$ sits inside $\mathfrak{C}(\widehat{R})$. Given a local ring $(S, \mathfrak{n})$ with $\mathfrak{n}$-adic completion $\widehat{S}$, we say a finitely generated $\widehat{S}$-module M is *extended* if $M \cong N \otimes_S \widehat{S}$ for some (necessarily finitely generated) S-module N. We say M is *minimally* *extended* if $M \neq 0$ is extended, and no non-zero proper direct summand of M is extended. We record the following lemma for later use. For a finitely generated S-module N, we identify $N \otimes_S \widehat{S}$ with the completion $\widehat{N}$ of N.

Lemma 3.1. *Let $(S, \mathfrak{n})$ be a local ring, and let $\widehat{S}$ denote its $\mathfrak{n}$-adic completion. Let M be an extended $\widehat{S}$-module with $M \cong N \otimes_S \widehat{S}$ for an S-module N. Then M is minimally extended if and only if N is an indecomposable S-module.*

Proof. Suppose M is minimally extended and $N \cong N_1 \oplus N_2$ for S-modules N_1 and N_2. Then $M \cong \widehat{N}_1 \oplus \widehat{N}_2$ as $\widehat{S}$-modules. Since M is minimally extended, either $\widehat{N}_1 = 0$ or $\widehat{N}_2 = 0$, and hence either $N_1 = 0$ or $N_2 = 0$. Thus N is indecomposable.

Now suppose N is indecomposable as an S-module and L is a direct summand of M. If L is extended, then $L \cong \widehat{P}$ for an S-module P. Since $\widehat{P}$ is a direct summand of $\widehat{N}$, P is a direct summand of N by [**19**, Proposition 1.2]. Since N is indecomposable, either $P = 0$ or $P = N$. It follows that $L \cong 0$ or $L \cong M$, and hence M is minimally extended. □

Given a list of all the indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules as well as their ranks, we can determine $\mathfrak{C}(R)$ using the following proposition, which is an immediate consequence of a result of Levy and Odenthal [**14**, Theorem 6.2].

Proposition 3.2. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Let M be a finitely generated $\widehat{R}$-module. Then M is extended from an R-module if and only if $\operatorname{rank}_P(M) = \operatorname{rank}_Q(M)$ whenever P and Q are minimal prime ideals of $\widehat{R}$ lying over the same minimal prime ideal of R.*

The description of the monoid $\mathfrak{C}(R)$ depends on the *splitting number* q of R defined by

$$q := |\operatorname{MinSpec}(\widehat{R})| - |\operatorname{MinSpec}(R)|,$$

where $\operatorname{MinSpec}(R)$ and $\operatorname{MinSpec}(\widehat{R})$ denote the set of minimal prime ideals of R and $\widehat{R}$, respectively. If t_i is the number of minimal prime ideals of $\widehat{R}$ lying over the minimal prime ideal P_i of R, then $\sum_{i=1}^{s} t_i$ is the number of minimal prime ideals of $\widehat{R}$, and

$$q = (t_1 - 1) + (t_2 - 1) + \cdots + (t_s - 1) = \sum_{i=1}^{s} t_i - s.$$

The following proposition describes the monoid $\mathfrak{C}(R)$ when $q = 0$.

Proposition 3.3. *If $(R, \mathfrak{m}, k)$ is a one-dimensional analytically unramified local ring with splitting number $q = 0$, then $\mathfrak{C}(R)$ is a free monoid.*

Proof. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Since $q = 0$, there is a one-to-one correspondence between the minimal prime ideals of R and the minimal prime ideals of $\widehat{R}$. By Proposition 3.2, every finitely generated $\widehat{R}$-module is extended from an R-module, and thus $\mathfrak{C}(R) \cong \mathfrak{C}(\widehat{R})$. Since $\widehat{R}$ is a complete local ring, $\mathfrak{C}(\widehat{R})$ is free and so is $\mathfrak{C}(R)$. $\square$

In the following setup, used throughout Section 3, we assume $q \geq 1$.

General Setup 3.4. Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with minimal prime ideals $P_1, \ldots, P_s$, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. For each $i \in \{1, \ldots, s\}$, let $Q_{i,1}, \ldots, Q_{i,t_i}$ denote the minimal prime ideals of $\widehat{R}$ lying over the minimal prime ideal P_i of R.

Let q be the splitting number of R, and assume $q \geq 1$. Observe that there is at least one index $i \in \{1, \ldots, s\}$ such that $t_i \geq 2$. Let $p \in \{1, \ldots, s\}$ be the number of minimal prime ideals P_i of R with $t_i \geq 2$. After renumbering (if necessary), we may assume that $P_1, \ldots, P_p$ are the minimal prime ideals of R with $t_i \geq 2$, and $P_{p+1}, \ldots, P_s$ are the minimal prime ideals of R with $t_i = 1$.

The full submonoid $\mathfrak{C}(R) \subseteq \mathfrak{C}(\widehat{R}) \cong \mathbf{N}_0^{(\Lambda)}$ can be described as follows (see [**7**, page 9]).

Construction 3.5. Let Λ denote the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. For an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module M, let

$$(r_{1,1}, \ldots, r_{1,t_1}, \ldots, r_{p,1}, \ldots, r_{p,t_p}, r_{p+1,1}, \ldots, r_{s,1})$$

denote its rank, where $r_{i,j} = \operatorname{rank}_{Q_{i,j}}(M)$ is the rank of M at $Q_{i,j}$.

Set $\mathcal{A}(R)$ to be the $q \times |\Lambda|$ matrix with entries in $\mathbf{Z}$, where the column indexed by the isomorphism class $[M] \in \Lambda$ is the transpose of the vector

$$[r_{1,1}-r_{1,2} \ \cdots \ r_{1,1}-r_{1,t_1} \quad \cdots \ r_{p,1}-r_{p,2} \ \cdots \ r_{p,1}-r_{p,t_p}].$$

By Proposition 3.2, we have $\mathfrak{C}(R) \cong \ker(\mathcal{A}(R)) \cap \mathbf{N}_0^{(\Lambda)}$.

3.3. Towards a description of the matrix $\mathcal{A}(R)$. The goal of this subsection is to give a description of the matrix $\mathcal{A}(R)$ and thus of $\mathfrak{C}(R)$.

3.3.1. The matrix $\mathcal{T}$. We first introduce a matrix $\mathcal{T}$ with entries in the set $\{0, 1, -1\}$, and then show that this matrix always occurs as a submatrix of the matrix $\mathcal{A}(R)$.

Construction 3.6. Let $\{t_1, \ldots, t_p\}$ be a (finite) sequence of integers with $t_i \geq 2$ for all i, and set $q = \sum_{j=1}^{p} t_j - p$. Fix $i \in \{1, \ldots, p\}$. Let A_i be the set of $(t_i - 1) \times 1$ column vectors all of whose entries are either 0 or 1, and let B_i be the set of $(t_i - 1) \times 1$ column vectors all of whose entries are either 0 or -1. Set $C_i := A_i \cup B_i$, and note that $|C_i| = 2^{t_i} - 1$. Now consider all the $q \times 1$ column vectors of the form

$$\left[\begin{array}{c} T_1 \\ \hline \vdots \\ \hline T_p \end{array}\right], \tag{3.1}$$

where $T_i \in C_i$ for each $i \in \{1, \ldots, p\}$. Let $\mathcal{T}$ be the $q \times \prod_{i=1}^{p}(2^{t_i} - 1)$ matrix formed from the $\prod_{i=1}^{p}(2^{t_i} - 1)$ column vectors of the form (3.1). (The order in which the columns appear in $\mathcal{T}$ does not matter.)

Proposition 3.7. *Let $(R, \mathfrak{m}, k)$ and $\widehat{R}$ be as in General Setup* 3.4. *For each column $\boldsymbol{\alpha}$ of $\mathcal{T}$, there exist nonnegative integers $r_{i,j}$ and an* *indecomposable maximal Cohen-Macaulay $\widehat{R}$-module $M_{\boldsymbol{\alpha}}$ of rank*

$$(r_{1,1},\dots,r_{1,t_1},\dots,r_{p,1},\dots,r_{p,t_p},r_{p+1,1},\dots,r_{s,1})$$

such that

$$\boldsymbol{\alpha} = [r_{1,1}-r_{1,2}\ \cdots\ r_{1,1}-r_{1,t_1}\ \cdots\ r_{p,1}-r_{p,2}\ \cdots\ r_{p,1}-r_{p,t_p}]^T. \tag{3.2}$$

Proof. Let $\boldsymbol{\alpha} = [a_{1,2}\ \cdots\ a_{1,t_1}\ \cdots\ a_{p,2}\ \cdots\ a_{p,t_p}]^T$ be a column of $\mathcal{T}$. For each $i \in \{1,\dots,p\}$ and for each $j \in \{2,\dots,t_i\}$, define

$$r_{i,1} := \begin{cases} 1 & \text{if } a_{i,j} \in \{0,1\} \text{ for all } j \in \{2,\dots,t_i\}, \\ 0 & \text{if } a_{i,j} = -1 \text{ for some } j \in \{2,\dots,t_i\}, \end{cases}$$

and

$$r_{i,j} := \begin{cases} 1 - a_{i,j} & \text{if } r_{i,1} = 1, \\ -a_{i,j} & \text{if } r_{i,1} = 0. \end{cases}$$

If $1 \le p \le s-1$, then set $r_{i,1} := 1$ for $i \in \{p+1,\dots,s\}$.

Consider the tuple

$$\underline{r} := (r_{1,1},\dots,r_{1,t_1},\dots,r_{p,1},\dots,r_{p,t_p},r_{p+1,1},\dots,r_{s,1}),$$

and note that $\underline{r}$ satisfies (3.2). Also, $\underline{r} \neq 0$ and $r_{i,j} \in \{0,1\}$ for all i and j. Define $V := \{(i,j) \mid r_{i,j} \neq 0\}$. Then the module $M_{\boldsymbol{\alpha}} := \widehat{R}/\cap_{(i,j)\in V} Q_{i,j}$ is an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $\underline{r}$. ◻

Remark 3.8. We emphasize that, if S is a one-dimensional local ring with minimal prime ideals $Q_1,\dots,Q_t$ and $I \subseteq \{1,\dots,t\}$, then $S/\cap_{i\in I} Q_i$ is an indecomposable maximal Cohen-Macaulay S-module of rank $(r_1,\dots,r_t)$, where $r_i = 1$ if $i \in I$, and $r_i = 0$ if $i \notin I$. (Modules of ranks consisting of zeros and ones always exist regardless of whether R has finite Cohen-Macaulay type or infinite Cohen-Macaulay type.) By Proposition 3.7, the matrix $\mathcal{A}(R)$ contains, as a submatrix, the matrix $\mathcal{T}$ from Construction 3.6.

3.3.2. More on the matrix $\mathcal{A}(R)$. The following more specific setup requires at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.

Setup 3.9. Let $(R, \mathfrak{m}, k)$ and $\widehat{R}$ be as in General Setup 3.4; that is, $(R, \mathfrak{m}, k)$ is a one-dimensional analytically unramified local ring with splitting number $q \geq 1$, and $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R. Let $P_1, \ldots, P_s$ be the minimal prime ideals of R, and for each $i \in \{1, \ldots, s\}$, let $Q_{i,1}, \ldots, Q_{i,t_i}$ denote the minimal prime ideals of $\widehat{R}$ lying over P_i. Assume that there is at least one minimal prime ideal $Q_{i,j}$ of $\widehat{R}$ such that $\widehat{R}/Q_{i,j}$ has infinite Cohen-Macaulay type.

For each minimal prime ideal P_i of R, let $u_i \in \{0, 1, \ldots, t_i\}$ denote the number of minimal prime ideals $Q_{i,j}$ of $\widehat{R}$ (lying over P_i) such that $\widehat{R}/Q_{i,j}$ has infinite Cohen-Macaulay type. Let $p \in \{1, \ldots, s\}$ denote the number of minimal prime ideals P_i of R with $t_i \geq 2$, and let $l \in \{0, 1, \ldots, p\}$ denote the number of minimal prime ideals P_i of R with $t_i \geq 2$ and $u_i \geq 1$. Let $m \in \{0, 1, \ldots, s-p\}$ denote the number of minimal prime ideals P_i of R with $t_i = 1$ and $u_i = 1$.

After renumbering (if necessary), we may assume that:

(i) $P_1, \ldots, P_l$ are the minimal prime ideals of R with $t_i \geq 2$ and $u_i \geq 1$; $Q_{1,1}, \ldots, Q_{1,u_1}, \ldots, Q_{l,1}, \ldots, Q_{l,u_l}$ are the minimal prime ideals of $\widehat{R}$ (lying over $P_1, \ldots, P_l$, respectively) such that $\widehat{R}/Q_{i,j}$ has infinite Cohen-Macaulay type, and $Q_{1,u_1+1}, \ldots, Q_{1,t_1}, \ldots, Q_{l,u_l+1}, \ldots, Q_{l,t_l}$ are the minimal prime ideals of $\widehat{R}$ (lying over $P_1, \ldots, P_l$, respectively) such that $\widehat{R}/Q_{i,j}$ has finite Cohen-Macaulay type;

(ii) $P_{l+1}, \ldots, P_p$ are the minimal prime ideals of R with $t_i \geq 2$ and $u_i = 0$ (that is, the branches $\widehat{R}/Q_{i,j}$ have finite Cohen-Macaulay type for all $i \in \{l+1, \ldots, p\}$ and for all $j \in \{1, \ldots, t_i\}$);

(iii) $P_{p+1}, \ldots, P_{p+m}$ are the minimal prime ideals of R with $t_i = 1$ and $u_i = 1$ (that is, the branches $\widehat{R}/Q_{i,1}$ have infinite Cohen-Macaulay type for all $i \in \{p+1, \ldots, p+m\}$), and

(iv) $P_{p+m+1}, \ldots, P_s$ are the minimal prime ideals of R with $t_i = 1$ and $u_i = 0$ (that is, the branches $\widehat{R}/Q_{i,1}$ have finite Cohen-Macaulay type for all $i \in \{p+m+1, \ldots, s\}$).

In the following three propositions, we give descriptions of the matrix $\mathcal{A}(R)$ introduced in Construction 3.5. The next proposition describes $\mathcal{A}(R)$ when there is an index $i_0 \in \{p+1, \ldots, p+m\}$ such that $t_{i_0} = 1$ and $u_{i_0} = 1$; that is, there is exactly one minimal prime ideal $Q_{i_0,1}$ of

$\widehat{R}$ lying over the minimal prime ideal P_{i_0} of R, and $\widehat{R}/Q_{i_0,1}$ has infinite Cohen-Macaulay type.

Proposition 3.10. *Let* $(R, \mathfrak{m}, k)$ *and* $\widehat{R}$ *be as in Setup* 3.9*. Assume* $1 \le m \le s-p$*, and fix an index* $i_0 \in \{p+1, \ldots, p+m\}$*. For every vector* $\boldsymbol{\alpha} \in \mathbf{Z}^{(q)}$*,* $\mathcal{A}(R)$ *contains* $|k|\aleph_0$ *columns coinciding with* $\boldsymbol{\alpha}$*. In particular,* $\mathcal{A}(R)$ *contains* $|k|\aleph_0$ *copies of an enumeration of* $\mathbf{Z}^{(q)}$*.*

Proof. Let $\boldsymbol{\alpha} = [a_{1,2} \ \cdots \ a_{1,t_1} \ \cdots \ a_{p,2} \ \cdots \ a_{p,t_p}]^T \in \mathbf{Z}^{(q)}$. Choose an integer a such that $a > |a_{i,j}|$ for all i and j. Set $r_{i,1} := a$ for each $i \in \{1, \ldots, p\}$, and $r_{i,j} := a - a_{i,j}$ and for each $i \in \{1, \ldots, p\}$ and for each $j \in \{2, \ldots, t_i\}$. Choose an integer b such that $b > r_{i,j}$ for all $i \in \{1, \ldots, p\}$ and for all $j \in \{1, \ldots, t_i\}$, and set $r_{i,1} := b$ for each $i \in \{p+1, \ldots, s\}$.

Consider the non-zero tuple

$$\underline{r} := (r_{1,1}, \ldots, r_{1,t_1}, \ldots, r_{p,1}, \ldots, r_{p,t_p}, r_{p+1,1}, \ldots, r_{s,1}).$$

By construction, each $r_{i,j}$ is a nonnegative integer and $r_{i_0,1} \ge r_{i,j}$ for all i and j. Since $\widehat{R}/Q_{i_0,1}$ has infinite Cohen-Macaulay type, Theorem 2.1 implies the existence of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module $M_{\boldsymbol{\alpha}}$ of rank $\underline{r}$. The column of $\mathcal{A}(R)$ indexed by $[M_{\boldsymbol{\alpha}}]$ is $\boldsymbol{\alpha}$. Theorem 2.1 also guarantees that, for each $n \in \mathbf{N}_0$, there exists an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module $M_{\boldsymbol{\alpha},n}$ of rank $\underline{r} + (n, \ldots, n)$; the column of $\mathcal{A}(R)$ indexed by $[M_{\boldsymbol{\alpha},n}]$ is also $\boldsymbol{\alpha}$. Thus, $\boldsymbol{\alpha}$ occurs at least $\aleph_0$ times as a column of $\mathcal{A}(R)$.

If the residue field k is finite, then $\mathcal{A}(R)$ has $\aleph_0 = |k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$. If the residue field k is infinite, then Theorem 2.1 guarantees the existence of $|k|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of the same rank $\underline{r} + (n, \ldots, n)$. Thus, $\mathcal{A}(R)$ has $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$, and hence $\mathcal{A}(R)$ contains $|k|\aleph_0$ copies of an enumeration of $\mathbf{Z}^{(q)}$. $\square$

The next proposition describes $\mathcal{A}(R)$ when there is an index $i_0 \in \{1, \ldots, l\}$ such that $t_{i_0} \ge 2$ and $u_{i_0} \ge 1$; that is, there are at least two minimal prime ideals $Q_{i_0,1}$ and $Q_{i_0,2}$ of $\widehat{R}$ lying over the minimal prime ideal P_{i_0} of R, and $\widehat{R}/Q_{i_0,1}$ has infinite Cohen-Macaulay type.

Set $q_0 = 0$, and for $i \in \{1, \ldots, p\}$, define

$$q_i := \sum_{j=1}^{i} (t_j - 1). \tag{3.3}$$

Proposition 3.11. *Let $(R, \mathfrak{m}, k)$ and $\widehat{R}$ be as in Setup* 3.9*. Assume $1 \le l \le p \le s$, and fix an index $i_0 \in \{1, \ldots, l\}$. For every vector $\boldsymbol{\alpha} \in \mathbf{Z}^{(q_{i_0-1})} \oplus \mathbf{N}_0^{(t_{i_0}-1)} \oplus \mathbf{Z}^{(q-q_{i_0})}$, $\mathcal{A}(R)$ contains $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$.*

Proof. Let $\boldsymbol{\alpha} = [a_{1,2} \cdots a_{1,t_1} \cdots a_{i_0,2} \cdots a_{i_0,t_{i_0}} \cdots a_{p,2} \cdots a_{p,t_p}]^T \in \mathbf{Z}^{(q_{i_0-1})} \oplus \mathbf{N}_0^{(t_{i_0}-1)} \oplus \mathbf{Z}^{(q-q_{i_0})}$. Choose an integer a such that $a > |a_{i,j}|$ for all i and j.

If $s = 1$, then $i_0 = 1$, $t_{i_0} = t_1 \ge 2$ and $\boldsymbol{\alpha} \in \mathbf{N}_0^{(t_1-1)}$. Set $r_{1,1} := a$ and $r_{1,j} := a - a_{1,j}$ for each $j \in \{2, \ldots, t_1\}$. Observe that $r_{1,1} \ge r_{1,j}$ for all j since $a_{1,j} \in \mathbf{N}_0$.

If $s \ge 2$, then set $r_{i,1} := a$ for each $i \in \{1, \ldots, s\} \setminus \{i_0\}$, and $r_{i,j} := a - a_{i,j}$ for each $i \in \{1, \ldots, p\} \setminus \{i_0\}$ and for each $j \in \{2, \ldots, t_i\}$. Choose an integer b such that $b > r_{i,j}$ for all $i \ne i_0$ and $j \in \{1, \ldots, t_i\}$. Set $r_{i_0,1} := b$ and $r_{i_0,j} := b - a_{i_0,j}$ for each $j \in \{2, \ldots, t_{i_0}\}$. Observe that $r_{i_0,1} \ge r_{i_0,j}$ for all j since $a_{i_0,j} \in \mathbf{N}_0$.

Consider the non-zero tuple

$$\underline{r} := (r_{1,1}, \ldots, r_{1,t_1}, \ldots, r_{p,1}, \ldots, r_{p,t_p}, r_{p+1,1}, \ldots, r_{s,1}).$$

By construction, each $r_{i,j}$ is a nonnegative integer and $r_{i_0,1} \ge r_{i,j}$ for all i and j. As in the proof of Proposition 3.10, we conclude that $\mathcal{A}(R)$ contains $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$. $\square$

The next proposition describes $\mathcal{A}(R)$ when there is an index $i_0 \in \{1, \ldots, l\}$ such that $t_{i_0} \ge u_{i_0} \ge 2$; that is, there are at least two minimal prime ideals $Q_{i_0,1}$ and $Q_{i_0,2}$ of $\widehat{R}$ lying over the minimal prime ideal P_{i_0} of R, and both $\widehat{R}/Q_{i_0,1}$ and $\widehat{R}/Q_{i_0,2}$ have infinite Cohen-Macaulay type. In this case, we gain additional information about some of the columns of $\mathcal{A}(R)$.

Let $i \in \{1, \ldots, l\}$. If $u_i \geq 2$, then for $n \in \{2, \ldots, u_i\}$, define $Z_{i,n} \subseteq \mathbf{Z}^{(t_i-1)}$ by

$$(3.4) \quad Z_{i,n} := \Big\{ [\, z_{i,2} \cdots z_{i,t_i} \,]^T \mid z_{i,n} < 0 \text{ and } z_{i,n} \leq z_{i,j} \text{ for } j = 2, \ldots, t_i \Big\}.$$

Proposition 3.12. *Let $(R, \mathfrak{m}, k)$ and $\widehat{R}$ be as in Setup 3.9. Assume $1 \leq l \leq p \leq s$; in addition, assume there is an index $i_0 \in \{1, \ldots, l\}$ such that $u_{i_0} \geq 2$. For every $j \in \{2, \ldots, u_{i_0}\}$ and for every vector $\boldsymbol{\alpha} \in \mathbf{Z}^{(q_{i_0}-1)} \oplus Z_{i_0,j} \oplus \mathbf{Z}^{(q-q_{i_0})}$, $\mathcal{A}(R)$ contains $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$.*

Proof. Fix $j_0 \in \{2, \ldots, u_{i_0}\}$. Let

$$\boldsymbol{\alpha} = [a_{1,2} \ \cdots \ a_{1,t_1} \ \cdots \ a_{i_0,2} \ \cdots \ a_{i_0,t_{i_0}} \ \cdots \ a_{p,2} \ \cdots \ a_{p,t_p}]^T$$

be an element of $\mathbf{Z}^{(q_{i_0}-1)} \oplus Z_{i_0,j_0} \oplus \mathbf{Z}^{(q-q_{i_0})}$. Choose an integer a such that $a > |a_{i,j}|$ for all i and j.

If $s = 1$, then $i_0 = 1$, $t_{i_0} = t_1 \geq 2$ and $\boldsymbol{\alpha} \in Z_{1,j_0}$. Set $r_{1,1} := a$ and $r_{1,j} := a - a_{1,j}$ for each $j \in \{2, \ldots, t_1\}$. Since $\boldsymbol{\alpha} \in Z_{1,j_0}$, we have $a_{1,j_0} < 0$ and $a_{1,j} - a_{1,j_0} \in \mathbf{N}_0$. Thus $r_{1,j_0} \geq r_{1,j}$ for all $j \in \{1, \ldots, t_1\}$.

If $s \geq 2$, then set $r_{i,1} := a$ for each $i \in \{1, \ldots, s\} \setminus \{i_0\}$, and $r_{i,j} := a - a_{i,j}$ for each $i \in \{1, \ldots, p\} \setminus \{i_0\}$ and for each $j \in \{2, \ldots, t_i\}$. Choose an integer b such that $b > r_{i,j}$ for all $i \neq i_0$ and for all $j \in \{1, \ldots, t_i\}$, and $b > a_{i_0,j} - a_{i_0,j_0}$ for all $j \in \{2, \ldots, t_{i_0}\}$. Set $r_{i_0,1} := b + a_{i_0,j_0}$ and $r_{i_0,j} := b - (a_{i_0,j} - a_{i_0,j_0})$ for each $j \in \{2, \ldots, t_{i_0}\}$. Since $\boldsymbol{\alpha} \in \mathbf{Z}^{(q_{i_0}-1)} \oplus Z_{i_0,j_0} \oplus \mathbf{Z}^{(q-q_{i_0})}$, we have $a_{i_0,j_0} < 0$ and $a_{i_0,j} - a_{i_0,j_0} \in \mathbf{N}_0$. Thus $r_{i_0,j_0} \geq r_{i_0,j}$ for all $j \in \{1, \ldots, t_{i_0}\}$.

Consider the non-zero tuple

$$\underline{r} := (r_{1,1}, \ldots, r_{1,t_1}, \ldots, r_{p,1}, \ldots, r_{p,t_p}, r_{p+1,1}, \ldots, r_{s,1}).$$

By construction, each $r_{i,j}$ is a nonnegative integer and $r_{i_0,j_0} \geq r_{i,j}$ for all i and j. Since $\widehat{R}/Q_{i_0,j_0}$ has infinite Cohen-Macaulay type, as before we conclude that $\mathcal{A}(R)$ contains $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$. $\square$

Corollary 3.13. *Let $(R, \mathfrak{m}, k)$ and $\widehat{R}$ be as in Setup 3.9. Assume either $q = 1$ or there is an index $i_0 \in \{1, \ldots, s\}$ such that $\widehat{R}/Q_{i_0,j}$ has* *infinite Cohen-Macaulay type for all minimal prime ideals $Q_{i_0,j}$ of $\widehat{R}$ lying over the minimal prime ideal P_{i_0} of R. For every vector $\boldsymbol{\alpha} \in \mathbf{Z}^{(q)}$, $\mathcal{A}(R)$ contains $|k|\aleph_0$ columns coinciding with $\boldsymbol{\alpha}$. In particular, $\mathcal{A}(R)$ contains $|k|\aleph_0$ copies of an enumeration of $\mathbf{Z}^{(q)}$.*

Proof. Suppose $q = 1$, and without loss of generality, assume $\widehat{R}/Q_{1,1}$ has infinite Cohen-Macaulay type. Let a be a non-zero element in $\mathbf{Z}$. Theorem 2.1 guarantees the existence of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(2a, a, 0, \dots, 0)$ if $a > 0$ and of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(-a, -2a, 0, \dots, 0)$ if $a < 0$. Moreover, there is an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of constant rank $(1, \dots, 1)$. From Construction 3.5, we see that 0 and a appear as entries of $\mathcal{A}(R)$. In addition, as seen in the proof of Proposition 3.10, Theorem 2.1 guarantees that every $a \in \mathbf{Z}$ appears $|k|\aleph_0$ times as an entry of $\mathcal{A}(R)$. Hence $\mathcal{A}(R)$ contains $|k|\aleph_0$ copies of an enumeration of $\mathbf{Z}$.

Now suppose that there is an index $i_0 \in \{1, \dots, s\}$ such that $\widehat{R}/Q_{i_0,j}$ has infinite Cohen-Macaulay type for all minimal prime ideals $Q_{i_0,j}$ of $\widehat{R}$ lying over the minimal prime ideal P_{i_0} of R. If $i_0 \in \{p+1, \dots, p+m\}$, then Proposition 3.10 guarantees that every vector in $\mathbf{Z}^{(q)}$ appears $|k|\aleph_0$ times as a column of $\mathcal{A}(R)$. If $i_0 \in \{1, \dots, l\}$, then $u_{i_0} = t_{i_0} \geq 2$. Let $\boldsymbol{\alpha} = [a_{1,2} \ \cdots \ a_{1,t_1} \ \cdots \ a_{i_0,2} \ \cdots \ a_{i_0,t_{i_0}} \ \cdots \ a_{p,2} \cdots \ a_{p,t_p}]^T \in \mathbf{Z}^{(q)}$. If $a_{i_0,j} \geq 0$ for all $j \in \{2, \dots, t_{i_0}\}$, then Proposition 3.11 guarantees that $\boldsymbol{\alpha} \in \mathbf{Z}^{(q_{i_0-1})} \oplus \mathbf{N}_0^{(t_{i_0}-1)} \oplus \mathbf{Z}^{(q-q_{i_0})}$ appears $|k|\aleph_0$ times as a column of $\mathcal{A}(R)$. If $a_{i_0,j} < 0$ for an index $j \in \{2, \dots, t_{i_0}\}$, then we can choose j_0 such that $a_{i_0,j_0} \leq a_{i_0,j}$ for all $j \in \{2, \dots, t_{i_0}\}$, and so $\boldsymbol{\alpha} \in \mathbf{Z}^{(q_{i_0-1})} \oplus Z_{i_0,j_0} \oplus \mathbf{Z}^{(q-q_{i_0})}$. Since $j_0 \in \{2, \dots, t_{i_0}\} = \{2, \dots, u_{i_0}\}$, Proposition 3.12 guarantees that $\boldsymbol{\alpha}$ appears $|k|\aleph_0$ times as a column of $\mathcal{A}(R)$. Hence, $\mathcal{A}(R)$ contains $|k|\aleph_0$ copies of an enumeration of $\mathbf{Z}^{(q)}$. $\square$

3.3.3. A characterization of the monoid $\mathfrak{C}(R)$. We now summarize the results obtained previously and give a description of the matrix $\mathcal{A}(R)$, and hence of the monoid $\mathfrak{C}(R)$, when the $\mathfrak{m}$-adic completion $\widehat{R}$ has at least one minimal prime ideal Q such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type. Recall the definitions of q_i in (3.3) and $Z_{i,n}$ in (3.4).

Notation 3.14. With the notation introduced in Setup 3.9, set $I_1 := \{1, \ldots, l\}$ if $l \neq 0$, and set $I_2 := \{p+1, \ldots, p+m\}$ if $m \neq 0$. Note that $I_1 \cup I_2 \neq \varnothing$ since, by assumption, there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.

Let W_i be a $q \times \aleph_0$ matrix whose columns are an enumeration of $\mathbf{Z}^{(q_{i-1})} \oplus \mathbf{N}_0^{(t_i - 1)} \oplus \mathbf{Z}^{(q-q_i)}$ if $i \in I_1$ and an enumeration of $\mathbf{Z}^{(q)}$ if $i \in I_2$ (the order of the enumeration does not matter). If the residue field k is finite, then let $\mathcal{W}_i := W_i$; otherwise, let $\mathcal{W}_i$ be the $q \times |k|\aleph_0$ matrix consisting of $|k|$ copies of W_i arranged "horizontally." If $I_1 \neq \varnothing$, then for each $i \in I_1$ with $u_i \geq 2$ and for each $j \in \{2, \ldots, u_i\}$, let $V_{i,j}$ be a $q \times \aleph_0$ matrix whose columns are an enumeration of $\mathbf{Z}^{(q_{i-1})} \oplus Z_{i,j} \oplus \mathbf{Z}^{(q-q_i)}$ (the order of the enumeration does not matter). If the residue field k is finite, then let $\mathcal{V}_{i,j} := V_{i,j}$; otherwise, let $\mathcal{V}_{i,j}$ be the $q \times |k|\aleph_0$ matrix consisting of $|k|$ copies of $V_{i,j}$ arranged "horizontally." Set

$$\mathcal{W} := [\, \mathcal{W}_1 \quad \cdots \quad \mathcal{W}_l \quad \mathcal{W}_{p+1} \quad \cdots \quad \mathcal{W}_{p+m} \,]$$

and

$$\mathcal{V} := [\, \mathcal{V}_{1,2} \quad \cdots \quad \mathcal{V}_{1,u_1} \quad \cdots \quad \mathcal{V}_{l,2} \quad \cdots \quad \mathcal{V}_{l,u_l} \,] .$$

Note that, if $I_1 = \varnothing$, then $\mathcal{W} := [\mathcal{W}_{p+1} \ \cdots \ \mathcal{W}_{p+m}]$, and if $I_2 = \varnothing$, then $\mathcal{W} := [\mathcal{W}_1 \ \cdots \ \mathcal{W}_l]$.

Theorem 3.15. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with splitting number q. Let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R, and let Λ denote the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. Assume that there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.*

(1) *If $q = 0$, then $\mathfrak{C}(R) \cong \mathfrak{C}(\widehat{R}) \cong \mathbf{N}_0^{(\Lambda)}$.*

(2) *If $q \geq 1$, then $\mathfrak{C}(R) \cong \operatorname{Ker}(\mathcal{A}(R)) \cap \mathbf{N}_0^{(\Lambda)}$. The matrix $\mathcal{A}(R)$ is given by*

$$\mathcal{A}(R) = [\mathcal{T} \mid \mathcal{W} \mid \mathcal{V} \mid \mathcal{U}] ,$$

where $\mathcal{T}$ is described in Construction 3.6, *$\mathcal{W}$ and $\mathcal{V}$ are defined in Notation* 3.14, *and $\mathcal{U}$ is an integer matrix with q rows* (*and possibly infinitely many columns*).

Furthermore, if either $q = 1$ or there is a minimal prime ideal P of R such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type for all minimal prime ideals Q of $\widehat{R}$ lying over P, then $\mathcal{A}(R)$ contains $|k|\aleph_0$ copies of an enumeration of $\mathbf{Z}^{(q)}$.

Remark 3.16. When R has finite Cohen-Macaulay type, a characterization of the monoid $\mathfrak{C}(R)$ is given in [**2**, Proposition 3.3]. In this case, the monoid $\mathfrak{C}(R)$ is either free or isomorphic to $\operatorname{Ker}(\mathcal{A}(R)) \cap \mathbf{N}_0^{(n)}$, where n is the number of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. The matrix $\mathcal{A}(R)$ is either a $1 \times n$ matrix with entries in the set $\{0, 1, -1\}$, or a $2 \times n$ matrix with columns in the set $\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

4. The divisor class group of $\mathfrak{C}(R)$. As observed in subsection 3.2, the monoid homomorphism $\mathfrak{C}(R) \to \mathfrak{C}(\widehat{R})$, mapping $[M]$ to $[M \otimes_R \widehat{R}]$, is an injective divisor homomorphism. In this section, we study when $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is a divisor theory and compute the divisor class group $\mathcal{C}l(\mathfrak{C}(R))$ in terms of the splitting number q of R.

We need the following lemma, which generalizes Lemma 2.1 in [**7**]. First note that, for a fixed positive integer q and an index set Ω, a $q \times |\Omega|$ integer matrix $\mathcal{D}$ can be regarded as a homomorphism $\mathcal{D} \colon \mathbf{Z}^{(\Omega)} \to \mathbf{Z}^{(q)}$.

Lemma 4.1. *Fix an integer $q \geq 1$, and let I_q denote the $q \times q$ identity matrix. Let Ω be an index set, and let $\mathcal{D}$ be a $q \times |\Omega|$ integer matrix whose columns are indexed by Ω. Assume $\mathcal{D} = [D_1 \mid D_2]$, where D_1 is the $q \times (2q + 2)$ integer matrix*

$$\left[\begin{array}{c|c|cc} & & 1 & -1 \\ I_q & -I_q & \vdots & \vdots \\ & & 1 & -1 \end{array}\right],$$

and D_2 is an arbitrary integer matrix with q rows (and possibly infinitely many columns). Let $H := \operatorname{Ker}(\mathcal{D}) \cap \mathbf{N}_0^{(\Omega)}$.

(1) *The map $\mathcal{D} \colon \mathbf{Z}^{(\Omega)} \to \mathbf{Z}^{(q)}$ is surjective.*

(2) *The natural inclusion $H \hookrightarrow \mathbf{N}_0^{(\Omega)}$ is a divisor theory.*

(3) $\operatorname{Ker}(\mathcal{D}) = \mathcal{Q}(H)$.

(4) $\mathcal{C}l(H) \cong \mathbf{Z}^{(q)}$.

(5) *If every vector in* $\mathbf{Z}^{(q)}$ *occurs at least once as a column of* $\mathcal{D}$*, then every divisor class in* $\mathcal{C}l(H)$ *contains an atom of* $\mathbf{N}_0^{(\Omega)}$.

Proof. For each $i \in \{1, \ldots, q\}$, the standard unit vector $\mathbf{e}_i \in \mathbf{Z}^{(q)}$ occurs as a column of $\mathcal{D}$; thus, the map $\mathcal{D}\colon \mathbf{Z}^{(\Omega)} \to \mathbf{Z}^{(q)}$ is surjective.

To prove (2) first note that H is full in $\mathbf{N}_0^{(\Omega)}$ and hence $H \hookrightarrow \mathbf{N}_0^{(\Omega)}$ is a divisor homomorphism. Now let β be an arbitrary element of $\mathbf{N}_0^{(\Omega)}$. Write $-\mathcal{D}\beta = [d_1 \ \cdots \ d_q]^T$, where $d_i \in \mathbf{Z}$. Define

$$\mathsf{M} := \max\{0, d_1, \ldots, d_q\} \text{ and } \mathsf{m} := \min\{0, d_1, \ldots, d_q\}.$$

For $i \in \{1, \ldots, 2q+2\}$, let $\boldsymbol{\varepsilon}_i$ denote the unit vector in $\mathbf{N}_0^{(\Omega)}$ with support $\{i\}$. Define $\beta_1 := \sum_{i=1}^{q}(d_i - \mathsf{m})\boldsymbol{\varepsilon}_i - \mathsf{m}\boldsymbol{\varepsilon}_{2q+2}$, and note that $\beta_1 \in \mathbf{N}_0^{(\Omega)}$. Similarly define $\beta_2 := \sum_{i=1}^{q}(\mathsf{M} - d_i)\boldsymbol{\varepsilon}_{q+i} + \mathsf{M}\boldsymbol{\varepsilon}_{2q+1} \in \mathbf{N}_0^{(\Omega)}$. Then $\beta + \beta_1$ and $\beta + \beta_2$ are both in H. By construction β is the greatest lower bound of the set $\{\beta + \beta_1, \beta + \beta_2\} \subseteq H$. Hence $H \hookrightarrow \mathbf{N}_0^{(\Omega)}$ is a divisor theory.

In (3) note that the inclusion $\mathcal{Q}(H) \subseteq \operatorname{Ker}(\mathcal{D})$ is clear. To show the reverse inclusion, let $\alpha \in \operatorname{Ker}(\mathcal{D})$, and write $\alpha = \beta - \gamma$ for some β, $\gamma \in \mathbf{N}_0^{(\Omega)}$. Proceed as in the proof of (2) to find $\beta_1 \in \mathbf{N}_0^{(\Omega)}$ such that $\beta + \beta_1 \in H$. Then $\gamma + \beta_1 = \beta + \beta_1 - \alpha \in \mathbf{N}_0^{(\Omega)} \cap \operatorname{Ker}(\mathcal{D}) = H$. Thus, $\alpha = (\beta + \beta_1) - (\gamma + \beta_1) \in \mathcal{Q}(H)$, proving (3).

Property (4) follows from (1), (2) and (3). In fact the natural inclusion $H \subseteq \mathbf{N}_0^{(\Omega)}$ induces the inclusion $\mathcal{Q}(H) \subseteq \mathbf{Z}^{(\Omega)}$, and thus $\mathcal{C}l(H) = \mathbf{Z}^{(\Omega)}/\mathcal{Q}(H) = \mathbf{Z}^{(\Omega)}/\operatorname{Ker}(\mathcal{D}) \cong \mathbf{Z}^{(q)}$.

To prove (5), let $\alpha \in \mathcal{C}l(H) \cong \mathbf{Z}^{(q)}$. By the additional hypothesis, there is at least one column index $\omega \in \Omega$ such that the ω-th column of $\mathcal{D}$ coincides with α. Thus the unit vector $\boldsymbol{\varepsilon}_\omega \in \mathbf{N}_0^{(\Omega)}$ with support $\{\omega\}$ is an atom of $\mathbf{N}_0^{(\Omega)}$ in the divisor class α. ◻

The next proposition is useful for describing certain invariant properties of the monoid $\mathfrak{C}(R)$ (see Section 5).

Proposition 4.2. *Let* $(R, \mathfrak{m}, k)$ *be a one-dimensional analytically unramified local ring with splitting number* q*. Let* $\widehat{R}$ *denote the* $\mathfrak{m}$*-adic* *completion of R, and let Λ denote the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. Assume there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type. Let $\mathfrak{C}(R) \cong \operatorname{Ker}(\mathcal{A}(R)) \cap \mathbf{N}_0^{(\Lambda)}$ as in Construction* 3.5.

(1) *The natural inclusion $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is a divisor theory.*

(2) *The divisor class group $\mathcal{C}l(\mathfrak{C}(R))$ of $\mathfrak{C}(R)$ is*:

$$\mathcal{C}l(\mathfrak{C}(R)) \cong \begin{cases} 0 & \text{if } q = 0, \\ \mathbf{Z}^{(q)} & \text{if } q \geq 1. \end{cases}$$

(3) *If $q \geq 1$, and if each vector in $\mathbf{Z}^{(q)}$ occurs as a column of the matrix $\mathcal{A}(R)$, then every divisor class in $\mathcal{C}l(\mathfrak{C}(R))$ contains an atom of $\mathbf{N}_0^{(\Lambda)}$.*

Proof. If $q = 0$, then $\mathfrak{C}(R) \cong \mathfrak{C}(\widehat{R})$ is free by Proposition 3.3. Hence $\mathcal{C}l(\mathfrak{C}(R)) = 0$, and the inclusion $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is a divisor theory.

Now assume $q = 1$, and let s denote the number of minimal prime ideals of R. Without loss of generality, assume there are exactly two minimal prime ideals $Q_{1,1}$ and $Q_{1,2}$ of $\widehat{R}$ lying over the minimal prime ideal P_1 of R. Since the indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules $M_1 = \widehat{R}/Q_{1,1}$ and $M_2 = \widehat{R}/Q_{1,2}$ have ranks $(1, 0, \ldots, 0)$ and $(0, 1, 0, \ldots, 0)$, respectively, the columns of $\mathcal{A}(R)$ indexed by $[M_1]$ and $[M_2]$ have entries 1 and -1, respectively.

If $s = 1$, then by Theorem 2.1 there are indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules N_1 and N_2 of ranks $(2, 1)$ and $(1, 2)$, respectively. Thus, the columns of $\mathcal{A}(R)$ indexed by $[N_1]$ and $[N_2]$ have entries 1 and -1, respectively.

If $s \geq 2$, then the indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules $M_{1,1} = \widehat{R}/(Q_{1,1} \cap Q_{2,1})$ and $M_{1,2} = \widehat{R}/(Q_{1,2} \cap Q_{2,1})$ have ranks $(1, 0, 1, 0 \ldots, 0)$ and $(0, 1, 1, 0 \ldots, 0)$, respectively (recall that $Q_{2,1}$ denotes the minimal prime ideal of $\widehat{R}$ lying over the minimal prime ideal P_2 of R). Thus, the columns of $\mathcal{A}(R)$ indexed by $[M_{1,1}]$ and $[M_{1,2}]$ have entries 1 and -1, respectively.

When $q = 1$, we conclude that each element in the set $\{-1, 1\}$ appears at least twice as an entry in the matrix $\mathcal{A}(R)$, and so $\mathcal{A}(R)$ contains the matrix D_1 from the previous lemma. When $q \geq 2$, the matrix $\mathcal{A}(R)$ contains the matrix $\mathcal{T}$ defined in Construction 3.6, and hence the matrix D_1. Therefore, when $q \geq 1$, the proposition follows from Lemma 4.1. $\square$

Remark 4.3. If $q = 1$, then the proof of Proposition 4.2 shows that the natural inclusion $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is always a divisor theory provided $s \geq 2$ (regardless of whether R has finite Cohen-Macaulay type or infinite Cohen-Macaulay type). If $q = 1$ and $s = 1$, then $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is a divisor theory provided there is a minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type, but need not be if R has finite Cohen-Macaulay type. We refer the reader to [**1**, page 937] for an example of an integral domain R with $q = 1$ (and $s = 1$) such that $\mathfrak{C}(R)$ is free of rank different than the rank of $\mathfrak{C}(\widehat{R})$. In this case, the inclusion $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is not a divisor theory.

If $q = 0$ or $q \geq 2$, then the natural inclusion $\mathfrak{C}(R) \hookrightarrow \mathfrak{C}(\widehat{R})$ is always a divisor theory, regardless of whether R has finite Cohen-Macaulay type or infinite Cohen-Macaulay type (see the proof of Proposition 4.2).

5. Non-unique factorization in $\mathfrak{C}(R)$.

We use the results of Sections 3 and 4, along with the list of ranks provided in Section 2, to show how badly the Krull-Remak-Schmidt property fails for the class of maximal Cohen-Macaulay modules over rings of infinite Cohen-Macaulay type. We recall the following definitions, and we refer the reader to [**8, 10**] for details.

Let H be a monoid, and let h be a non-zero element of H. The *set of lengths* of h is $\mathsf{L}(h) := \{n \mid h = a_1 + \cdots + a_n \text{ for atoms } a_i \in H\}$. The *elasticity* of h is $\rho(h) := \sup \mathsf{L}(h)/\inf \mathsf{L}(h)$. Let $\mathsf{R}(H)$ denote the set of elasticities of non-zero elements of H. The *elasticity* of H is $\rho(H) := \sup \mathsf{R}(H)$. We say H is *fully elastic* if $\mathsf{R}(H) = \mathbf{Q} \cap [1, \rho(H)]$ (or $\mathsf{R}(H) = \mathbf{Q} \cap [1, \infty)$ when $\rho(H) = \infty$). The elasticity of H and the set of elasticities of non-zero elements in H give a coarse measure of how far the monoid H is from being free.

5.1. Infinite elasticity.

Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with splitting number q, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. We analyze the elasticity of the monoid $\mathfrak{C}(R)$ when there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.

The elasticity of $\mathfrak{C}(R)$ depends only upon q. When $q \geq 1$, we use Theorem 2.1 to choose indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of nonconstant rank, and to construct minimally extended $\widehat{R}$-modules that are completions of indecomposable R-modules. Direct sums of such indecomposable R-modules show failure of uniqueness in the direct-sum behavior of maximal Cohen-Macaulay R-modules.

Theorem 5.1. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with splitting number q, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Assume that there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.*

(1) *If $q = 0$, then $\mathfrak{C}(R)$ is free, and the Krull-Remak-Schmidt property holds for the class of maximal Cohen-Macaulay R-modules.*

(2) *If $q \geq 1$, then $\rho(\mathfrak{C}(R)) = \infty$.*

Proof. Let $P_1, \dots, P_s$ denote the minimal prime ideals of R. For each $i \in \{1, \dots, s\}$, let $Q_{i,1}, \dots, Q_{i,t_i}$ denote the minimal prime ideals of $\widehat{R}$ lying over the minimal prime ideal P_i of R. Without loss of generality, assume that $\widehat{R}/Q_{1,1}$ has infinite Cohen-Macaulay type.

Assume $q = 0$. Then $\mathfrak{C}(R) \cong \mathfrak{C}(\widehat{R}) \cong \mathbf{N}_0^{(\Lambda)}$, where Λ denotes the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. Hence $\mathfrak{C}(R)$ is free with elasticity $\rho(\mathfrak{C}(R)) = 1$ (cf. Proposition 3.3). Now assume $q \geq 1$. Without loss of generality, we may assume either $t_1 = 1$ and $t_2 \geq 2$, or $t_1 \geq 2$. By Remark 3.8, there exist indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules A and B of ranks $(0, 1, 0, \dots, 0)$ and $(1, 0, 1, \dots, 1)$, respectively. Fix positive integers m and n with $m > n$. Theorem 2.1 guarantees the existence of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules $C_{m,n}$ and $D_{m,n}$ of ranks $(m+n, m, m+n, \dots, m+n)$ and $(m, m+n, m, \dots, m)$, respectively. Consider the following $\widehat{R}$-modules:

$$A \oplus B, \quad C_{m,n} \oplus D_{m,n}, \quad A^{(n)} \oplus C_{m,n}, \quad B^{(n)} \oplus D_{m,n}.$$

By Proposition 3.2, these modules are extended, and hence there exist maximal Cohen-Macaulay R-modules X, Y, Z and W such that

$\widehat{X} \cong A \oplus B$, $\widehat{Y} \cong C_{m,n} \oplus D_{m,n}$, $\widehat{Z} \cong A^{(n)} \oplus C_{m,n}$, and $\widehat{W} \cong B^{(n)} \oplus D_{m,n}$.

Since $\widehat{X}$, $\widehat{Y}$, $\widehat{Z}$ and $\widehat{W}$ are minimally extended, Lemma 3.1 guarantees that X, Y, Z and W are indecomposable R-modules. Moreover, X, Y, Z and W are non-isomorphic R-modules. By faithfully flat descent of isomorphism (see [**9**, Proposition 2.5.8]), we have the following isomorphism of R-modules:

$$X^{(n)} \oplus Y \cong Z \oplus W.$$

We obtain a module that can be decomposed as the direct sum of $n+1$ indecomposable modules and as the direct sum of two indecomposable modules. Thus, $\mathfrak{C}(R)$ has elements with elasticity at least $(n+1)/2$. Since n is an arbitrary positive integer, we conclude that if $q \geq 1$, then $\rho(\mathfrak{C}(R)) = \infty$. $\square$

Remark 5.2. When R has finite Cohen-Macaulay type, the elasticity of $\mathfrak{C}(R)$ is computed in [**2**, Theorem 3.4]. In this case, $\rho(\mathfrak{C}(R)) \in \{1, 3/2\}$.

5.2. Full elasticity. We recall the following result on sets of lengths of Krull monoids with infinite divisor class group.

Theorem 5.3 [**11**, Theorem 1]. *Let H be a Krull monoid with infinite divisor class group G (and divisor theory $H \to \mathbf{N}_0^{(\Omega)}$). Assume every divisor class in $\mathcal{C}l(H)$ contains an atom of $\mathbf{N}_0^{(\Omega)}$. For every nonempty finite subset $L \subseteq \{2, 3, \dots\}$, there exists $h \in H$ such that $\mathsf{L}(h) = L$.*

We apply Theorem 5.3 to the monoid $\mathfrak{C}(R)$ and, as a consequence, we obtain that $\mathfrak{C}(R)$ is fully elastic.

Theorem 5.4. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with splitting number $q \geq 1$, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Assume that there is at least one minimal prime ideal Q of $\widehat{R}$ such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type.*

In addition, assume either $q = 1$ or there is a minimal prime ideal P of R such that $\widehat{R}/Q$ has infinite Cohen-Macaulay type for all minimal prime ideals Q of $\widehat{R}$ lying over P. Given an arbitrary nonempty finite set $L \subseteq \{2, 3, \dots\}$, there exists a maximal Cohen-Macaulay R-module M such that M is the direct sum of l indecomposable maximal Cohen-Macaulay R-modules if and only if $l \in L$.

Proof. Theorem 3.15 guarantees that the matrix $\mathcal{A}(R)$ contains at least one copy of an enumeration of $\mathbf{Z}^{(q)}$. By Proposition 4.2, the divisor class group of $\mathfrak{C}(R)$ is $\mathbf{Z}^{(q)}$, and every divisor class in $\mathcal{C}l(\mathfrak{C}(R))$ contains an atom of $\mathbf{N}_0^{(\Lambda)}$. Now apply Theorem 5.3. □

Corollary 5.5. *Under the same hypotheses as in Theorem* 5.4*, the monoid $\mathfrak{C}(R)$ has infinite elasticity and, in addition, is fully elastic.*

Proof. By Theorem 5.1, $\rho(\mathfrak{C}(R)) = \infty$. Given $p \in \mathbf{Q} \cap [1, \infty)$, write $p = a/b$ for positive integers a and b. Apply Theorem 5.4 to $L = \{2a, 2b\} \subseteq \{2, 3, \dots\}$. □

5.3. A lower bound on the elasticity of $\mathfrak{C}(R)$. If we know how the minimal prime ideals of $\widehat{R}$ lie over the minimal prime ideals of R, then we can compute a lower bound on the elasticity of the monoid $\mathfrak{C}(R)$. This lower bound does not depend upon whether R has finite Cohen-Macaulay type or infinite Cohen-Macaulay type. We emphasize that the lower bound is obtained without explicitly constructing indecomposable maximal Cohen-Macaulay R-modules of a specific rank. This result generalizes Proposition 3.5 in [**2**].

Proposition 5.6. *Let $(R, \mathfrak{m}, k)$ be a one-dimensional analytically unramified local ring with splitting number $q \geq 1$, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Let $P_1, \dots, P_s$ denote the minimal prime ideals of R. For each $i \in \{1, \dots, s\}$, let t_i denote the number of minimal prime ideals of $\widehat{R}$ lying over the minimal prime ideal P_i of R, and let p denote the number of minimal prime ideals of R with $t_i \geq 2$. Then $\rho(\mathfrak{C}(R)) \geq (q + p)/2p$.*

Proof. Order the minimal prime ideals $P_1, \dots, P_s$ of R so that $t_i \geq 2$ for all $i \in \{1, \dots, p\}$. Let $Q_{i,1}, \dots, Q_{i,t_i}$ denote the minimal prime ideals of $\widehat{R}$ lying over the minimal prime ideal P_i of R. For every $i \in \{1, \dots, p\}$ and for every $j \in \{1, \dots, t_i\}$, let

$$A_{i,j} := \frac{\widehat{R}}{Q_{i,j}} \quad \text{and} \quad B_{i,j} := \frac{\widehat{R}}{\left(\bigcap_{(u,v)\neq(i,j)} Q_{u,v}\right)}.$$

Now let

$$A_i := \bigoplus_{j=1}^{t_i} A_{i,j}, \quad B_i := \bigoplus_{j=1}^{t_i} B_{i,j}, \quad \text{and} \quad C_{i,j} := A_{i,j} \oplus B_{i,j}.$$

Since A_i, B_i and $C_{i,j}$ are minimally extended $\widehat{R}$-modules, Proposition 3.2 and Lemma 3.1 guarantee the existence of indecomposable maximal Cohen-Macaulay R-modules X_i, Y_i and $W_{i,j}$ such that

$$\widehat{X}_i \cong A_i, \quad \widehat{Y}_i \cong B_i, \quad \text{and} \quad \widehat{W_{i,j}} \cong C_{i,j}.$$

By faithfully flat descent of isomorphism, we have the following isomorphism of R-modules:

$$\left(\bigoplus_{i=1}^{p} X_i\right) \bigoplus \left(\bigoplus_{i=1}^{p} Y_i\right) \cong \bigoplus_{i=1}^{p} \left(\bigoplus_{j=1}^{t_i} W_{i,j}\right).$$

Thus, we obtain a module that can be expressed as the direct sum of $2p$ indecomposable modules and as the direct sum of $t_1 + \cdots + t_p = q + p$ indecomposable modules. We conclude that $\rho(\mathfrak{C}(R)) \geq (q+p)/2p$. $\square$

From Proposition 5.6, we see that if $q > p$, then $\mathfrak{C}(R)$ has elasticity greater than one.

6. Examples. In this final section, we provide examples of one-dimensional local integral domains and direct-sum decompositions of maximal Cohen-Macaulay modules in order to illustrate how the Krull-Remak-Schmidt property fails over these rings. The following result proves the existence of local integral domains whose completions are the rings we will consider.

Proposition 6.1 [**13**, Theorem 1]. *Let S be a complete local ring. Then S is the completion of a local integral domain if and only if the following conditions hold.*

(1) *The prime ring π of S is an integral domain, and S is a torsion-free π-module.*

(2) *Either S is a field or* $\operatorname{depth}(S) \geq 1$.

In the following examples, we consider equicharacteristic Cohen-Macaulay local rings of dimension one. Thus, the conditions of Proposition 6.1 are automatically satisfied. In particular, we consider local integral domains $(R, \mathfrak{m})$ whose $\mathfrak{m}$-adic completions have two minimal prime ideals (that is, $q = 1$).

We record the following proposition which shows that, given an indecomposable maximal Cohen-Macaulay module, minimally generated by t elements, of rank $(r_1, \dots, r_s)$, one can construct an indecomposable maximal Cohen-Macaulay module of rank $(t - r_1, \dots, t - r_s)$. Recall that a ring S is a *hypersurface* if $S = T/(f)$, where $(T, \mathfrak{m}_T)$ is a regular local ring and $f \in \mathfrak{m}_T$, $f \neq 0$.

Proposition 6.2. *Let S be a hypersurface. If there exists an indecomposable maximal Cohen-Macaulay S-module, minimally generated by t elements, of rank $(r_1, \dots, r_s)$, then there exists an indecomposable maximal Cohen-Macaulay S-module of rank $(t - r_1, \dots, t - r_s)$.*

Proof. We refer to [**20**, Chapter 7]. Let M be an indecomposable non-free maximal Cohen-Macaulay S-module of rank $(r_1, \dots, r_s)$. Since M has no free summand, M is the cokernel of a reduced matrix factorization (φ, ψ) by Corollary 7.6. Moreover, M has a periodic free resolution of period 2 by Proposition 7.2. Thus, the first syzygy $\operatorname{Syz}^1_S(M)$ of M is the cokernel of (ψ, φ), and so $\operatorname{Syz}^1_S(M)$ has no non-zero free summand. By Proposition 7.7, $\operatorname{Syz}^1_S(M)$ is an indecomposable maximal Cohen-Macaulay S-module. Using the additive property of rank on the short exact sequence $0 \to \operatorname{Syz}^1_S(M) \to S^{(t)} \to M \to 0$, we have that $\operatorname{rank}(\operatorname{Syz}^1_S(M)) = (t - r_1, \dots, t - r_s)$. $\square$

Example 6.3. Let $(R, \mathfrak{m})$ be a one-dimensional local integral domain whose $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the (D_5)-singularity:

$$\widehat{R} \cong \frac{\mathbf{C}[\![x,y]\!]}{((x^2-y^3)y)}.$$

Note that $\widehat{R}$ has *finite* Cohen-Macaulay type. The minimal prime ideals of $\widehat{R}$ are $Q_{1,1}=(x^2-y^3)/((x^2-y^3)y)$ and $Q_{1,2}=(y)/((x^2-y^3)y)$, both lying over the minimal prime ideal $P_1=(0)$ of R. In [**1**, Theorem 4.2], there is a classification of the ranks of the indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules (together with the number of non-isomorphic indecomposable modules for each rank). Up to isomorphism, there are four indecomposable $\widehat{R}$-modules of constant rank $(1,1)$, one indecomposable $\widehat{R}$-module of rank $(0,1)$, one indecomposable $\widehat{R}$-module of rank $(1,2)$ and two indecomposable $\widehat{R}$-modules of rank $(1,0)$.

By [**2**, Theorem 3.4], we conclude that $\rho(\mathfrak{C}(R))=1$, but $\mathfrak{C}(R)$ is not free. In particular, we have:

$$\mathfrak{C}(R) \cong \operatorname{Ker}\left([\,0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad -1 \quad -1\,]\right) \cap \mathbf{N}_0^{(8)},$$

and $\mathcal{C}l(\mathfrak{C}(R)) \cong \mathbf{Z}$.

Example 6.4. Let $(R,\mathfrak{m})$ be a one-dimensional local integral domain whose $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the ring:

$$\widehat{R} \cong \frac{\mathbf{C}[\![x,y]\!]}{((x^3-y^7)y)}.$$

The minimal prime ideals of $\widehat{R}$ are $Q_{1,1}=(x^3-y^7)/((x^3-y^7)y)$ and $Q_{1,2}=(y)/((x^3-y^7)y)$. Note that $\widehat{R}/Q_{1,1} \cong \mathbf{C}[\![x,y]\!]/(x^3-y^7)$ has *infinite* Cohen-Macaulay type, and $\widehat{R}/Q_{1,2} \cong \mathbf{C}[\![x]\!]$ has *finite* Cohen-Macaulay type. For each positive integer m, Theorem 1.4 in [**12**] guarantees the existence of $|\mathbf{C}|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of constant rank (m,m). Moreover, for each positive integer m, Theorem 2.1 guarantees the existence of $|\mathbf{C}|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of rank $(m,0)$. If t is the minimal number of generators of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(m,0)$, then Proposition 6.2 implies the existence of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(t-m,t)$. Thus, since there exist $|\mathbf{C}|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules $\{M_\gamma\}_{\gamma\in\mathbf{C}}$ of rank $(m,0)$, there exist $|\mathbf{C}|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of rank $(t_{M_\gamma}-m, t_{M_\gamma})$, where t_{M_γ} is the minimal number of generators of M_γ. Using Construction 3.5, let

$$\mathcal{A}(R) := [0 \ \cdots \ 0 \ \ 1 \ \cdots \ 1 \ \ 2 \ \cdots \ 2 \ \cdots \ -1 \ \cdots \ -1 \ \ -2 \ \cdots \ -2 \ \cdots],$$

where each integer appears $|\mathbf{C}|\aleph_0 = |\mathbf{C}|$ times as an entry of the matrix $\mathcal{A}(R)$. Then

$$\mathfrak{C}(R) \cong \mathrm{Ker}\,(\mathcal{A}(R)) \cap \mathbf{N}_0^{(\mathbf{C})}.$$

By Proposition 4.2, $\mathcal{C}l(\mathfrak{C}(R)) \cong \mathbf{Z}$, as in the previous example. However, unlike in the finite Cohen-Macaulay case, $\rho(\mathfrak{C}(R)) = \infty$ and $\mathfrak{C}(R)$ is fully elastic. Moreover, by Theorem 5.4, for an arbitrary nonempty finite set $L \subseteq \{2,3,\dots\}$, there exists a maximal Cohen-Macaulay R-module M such that M is the direct sum of l indecomposable maximal Cohen-Macaulay R-modules if and only if $l \in L$.

We conclude with an example in which $\widehat{R}$ has infinite Cohen-Macaulay type and $\widehat{R}/Q$ has finite Cohen-Macaulay type for all minimal prime ideals Q of $\widehat{R}$.

Example 6.5. Let $(R,\mathfrak{m})$ be a one-dimensional local integral domain whose $\mathfrak{m}$-adic completion $\widehat{R}$ is isomorphic to the ring:

$$\widehat{R} \cong \frac{\mathbf{C}[\![x,y]\!]}{((x^3-y^4)x)}.$$

The minimal prime ideals of $\widehat{R}$ are $Q_{1,1} = (x^3-y^4)/((x^3-y^4)x)$ and $Q_{1,2} = (x)/((x^3-y^4)x)$. Note that $\widehat{R}$ has *infinite* Cohen-Macaulay type, but both $\widehat{R}/Q_{1,1} \cong \mathbf{C}[\![x,y]\!]/(x^3-y^4)$ and $\widehat{R}/Q_{1,2} \cong \mathbf{C}[\![y]\!]$ have *finite* Cohen-Macaulay type. For rings of this type, we do not have a complete list of ranks of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. As in the previous example, for each positive integer m, there exist $|\mathbf{C}|$ pairwise non-isomorphic indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of constant rank (m,m). From Saccon's Ph.D. thesis [**15**, page 67], the ordered pairs $(m+1,m)$ and $(m+2,m)$, where $m \geq 0$, do occur as ranks of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. As before, if s (respectively, t) is the minimal number of generators of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(m+1, m)$ (respectively, $(m+2, m)$), then Proposition 6.2 implies the existence of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module of rank $(s-m-1, s-m)$ (respectively, $(t-m-2, t-m)$).

Without further investigation of the ranks of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules, we only know the existence of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of ranks (m, n) with $m - n \in \{-2, -1, 0, 1, 2\}$. Using Construction 3.5, let

$$A_1 := [0 \ \cdots \ 0 \quad 1 \ \cdots \ 1 \quad 2 \ \cdots \ 2 \quad -1 \ \cdots \ -1 \quad -2 \ \cdots \ -2],$$

where 0 appears $|\mathbf{C}|\aleph_0 = |\mathbf{C}|$ times as an entry of the matrix A_1 and each element in $\{-2, -1, 1, 2\}$ appears $\aleph_0$ times as an entry of A_1. Let $\mathcal{A}(R) = [A_1 \mid A_2]$, where A_2 is an integer matrix. Then

$$\mathfrak{C}(R) \cong \operatorname{Ker}(\mathcal{A}(R)) \cap \mathbf{N}_0^{(\Lambda)},$$

where Λ is the set of isomorphism classes of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules. We do not know whether there is an upper or lower bound on the entries of the matrix $\mathcal{A}(R)$; however, we can show that $\rho(\mathfrak{C}(R)) \geq 3/2$ as follows.

Let A, B and C be indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules of rank $(1,0)$, $(0,1)$ and $(2,0)$, respectively. Proposition 6.2 guarantees the existence of an indecomposable maximal Cohen-Macaulay $\widehat{R}$-module D_s of rank $(s-2, s)$ for an integer $s \geq 3$. Consider the following $\widehat{R}$-modules:

$$A \oplus B, \quad C \oplus D_s, \quad A^{(2)} \oplus D_s, \quad B^{(2)} \oplus C.$$

Proceeding as in Section 5, we observe that there exist non-isomorphic indecomposable maximal Cohen-Macaulay R-modules X, Y, Z and W such that

$$\widehat{X} \cong A \oplus B, \ \widehat{Y} \cong C \oplus D_s, \ \widehat{Z} \cong A^{(2)} \oplus D_s, \text{ and } \widehat{W} \cong B^{(2)} \oplus C.$$

By faithfully flat descent of isomorphism, we have the following isomorphism of R-modules:

$$X^{(2)} \oplus Y \cong Z \oplus W.$$

We obtain a module whose elasticity is greater than or equal to 3/2, and thus $\rho(\mathfrak{C}(R)) \geq 3/2$. Note that this lower bound on the elasticity $\rho(\mathfrak{C}(R))$ is greater than the one given in Proposition 5.6. In order to gain more information about the monoid $\mathfrak{C}(R)$, we need to study the tuples that occur as ranks of indecomposable maximal Cohen-Macaulay $\widehat{R}$-modules.

We conclude with the following open question, whose answer would be sufficient to determine the monoid $\mathfrak{C}(R)$ up to isomorphism.

Question. Which ranks occur for indecomposable maximal Cohen-Macaulay R-modules, when R is a one-dimensional analytically unramified local ring of infinite Cohen-Macaulay type with R/P of finite Cohen-Macaulay type for all minimal prime ideals P of R?

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