JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 2, Summer 2012

ZERO DIVISOR GRAPHS FOR MODULES OVER COMMUTATIVE RINGS

M. BEHBOODI

ABSTRACT. In this article, we give several generalizations of the concept of zero-divisor elements in a commutative ring with identity to modules. Then, for each R-module M, we associate three undirected (simple) graphs $\Gamma^*({}_RM) \subseteq \Gamma({}_RM) \subseteq \Gamma_*({}_RM)$ which, for $M = R$, all coincide with the zero-divisor graph of R. The main objective of this paper is to study the interplay of module-theoretic properties of M with graph-theoretic properties of these graphs.

0. Introduction. Let R be a commutative ring with identity and $Z(R)$ its set of zero divisors. In [**6**], Anderson and Livingston associated an undirected (simple) graph $\Gamma(R)$ to R with vertices $Z(R)^* := Z(R) \setminus \{0\}$ and with two distinct vertices x and y adjacent if $xy = 0$, and then studied the relationship between the properties of $\Gamma(R)$ and R. This graph is defined somewhat differently from the graph introduced by Beck [**8**], who took the set of vertices to be all of R. Recently, this subject has received a good deal of attention from several authors assigning a graph to a ring or a group and then studying the algebraic properties of these objects via their associated graphs; see, for instance, [**1–8, 13, 14, 18, 19, 23**]. Moreover, Redmond in [**20**] has considered the zero-divisor graph for arbitrary rings (see also [**1**]). In the present article, we introduce and study several generalizations of zero-divisor graphs to modules M over a commutative ring R which, for $M = R$, all coincide with $\Gamma(R)$ (the zero-divisor graph of R). Our main objective is to establish connections between module theoretic properties with the properties of associated graphs.

Throughout, all rings are commutative with identity elements, and all modules are unitary. The symbol $\subseteq$ denotes containment, and $\subset$ denotes proper containment for sets. If N is a submodule (respectively,

2010 AMS *Mathematics subject classification.* Primary 13A99, 3C05, 13C99.
Keywords and phrases. Zero-divisor graph, prime module, multiplication-like modules, virtually divisible modules.
Received by the editors on April 10, 2008, and in revised form on September 4, 2008.
DOI:10.1216/JCA-2012-4-2-175

proper submodule) of M, we write $N \leq M$ (respectively, $N < M$). We denote by $\mathrm{soc}\,(M)$ and $\dim\,(R)$ the socle of M and the classical Krull dimension of R, respectively.

Let R be a commutative ring and M an R-module. For $x \in M$, we denote the annihilator of the factor module M/Rx by I_x (i.e., $I_x := \{r \in R \mid rM \subseteq Rx\}$). It is clear that, for each $x \in M$, $\mathrm{Ann}\,(M) \subseteq I_x$ and $Rx = M$ if and only if $I_x = R$. In particular, if $M = R$, then for each $x \in R$, $I_x = \mathrm{Ann}\,(R/Rx) = Rx$. This means that $I_xI_y = 0$, if and only if $xy = 0$. Therefore, an element $x \in R$ is a zero-divisor in R if and only if $I_xI_yR = 0$ for some nonzero $y \in R$. We shall use these facts to give several generalizations of the concept of zero-divisor elements in a commutative ring to modules. An element x of M is called a:

– *weak zero-divisor*, if either $x = 0$ or $I_xI_yM = 0$ for some nonzero $y \in M$ with $I_y \subset R$;

– *zero-divisor*, if either $x = 0$ or $0 \neq I_x$ and $I_xI_yM = 0$ for some nonzero $y \in M$ with $0 \neq I_y \subset R$;

– *strong zero-divisor*, if either $x = 0$ or $\mathrm{Ann}\,(M) \subset I_x$ and $I_xI_yM = 0$ for some nonzero $y \in M$ with $\mathrm{Ann}\,(M) \subset I_y \subset R$.

For any R-module M, we denote $Z_*(M)$, $Z(M)$ and $Z^*(M)$, respectively, for the set of weak zero-divisors, zero-divisors and strong zero-divisors of M. We note that $Z^*(M) \subseteq Z(M) \subseteq Z_*(M)$ and, these facts are clear when $M = R$, all of the above concepts coincide with the set of zero-divisor elements of R. Now, for an R-module M, we let $\widetilde{Z}_*(M) := Z_*(M) \setminus \{0\}$, $\widetilde{Z}(M) := Z(M) \setminus \{0\}$ and $\widetilde{Z}^*(M) := Z^*(M) \setminus \{0\}$. Then we associate three (simple) graphs $\Gamma_*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma^*({}_RM)$ to M with vertices $\widetilde{Z}_*(M)$, $\widetilde{Z}(M)$ and $\widetilde{Z}^*(M)$, respectively, and the vertices x and y are adjacent if and only if $I_xI_yM = 0$. It is clear that we have $\Gamma^*({}_RM) \subseteq \Gamma({}_RM) \subseteq \Gamma_*({}_RM)$ as the induced subgraphs.

Recall that a graph Γ is connected if there is a path between any two distinct vertices. For distinct vertices x and y of Γ, let $d(x, y)$ be the length of the shortest path from x to y ($d(x, y) = \infty$ if there is no such path). The diameter of Γ is $\mathrm{diam}\,(\Gamma) = \sup\{d(x, y) \mid x$ and y are distinct vertices of $\Gamma\}$. The girth of Γ, denoted by $g(\Gamma)$, is defined as the length of the shortest cycle in Γ ($g(\Gamma) = \infty$ if G contains no cycles).

This article consists of four sections. In Section 1, we show that, for any module M, either $\Gamma^*({}_RM) = \Gamma({}_RM)$ or $\Gamma({}_RM) = \Gamma_*({}_RM)$ and, also, $\Gamma_*({}_RM)$ is always connected with $\operatorname{diam}(\Gamma({}_RM)) \leq 3$. Moreover, if $\Gamma_*({}_RM)$ contains a cycle, then $g(\Gamma_*({}_RM)) \leq 4$. In Section 2, we shall be able to characterize all R-modules M for which $\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM)$. In fact, we show that multiplication-like modules (defined later) are the only modules with this property. In Section 3, we shall study R-modules M for which $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M} := M \setminus \{0\}$ and $\Gamma({}_RM)$ is the empty graph. We also investigate R-modules M for which $\Gamma^*({}_RM)$ is the empty graph and $\Gamma({}_RM)$ is a complete graph with vertices $\widetilde{M}$. In particular, prime modules M for which $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ are characterized. In the final section, we show that, for an R-module M, $\Gamma_*({}_RM)$ is finite if and only if either M is a finite module or a prime multiplication-like module. In particular, if $1 \leq |\Gamma_*({}_RM)| < \infty$, then M is finite and not a simple module.

1. Various zero-divisor graphs for modules. We begin this section with the following definition (given an R-module M, and given $x \in M$, we will denote $\operatorname{Ann}(M/Rx)$ by I_x).

Definition 1.1. Let M be an R-module. An element x of M is called a:

– *weak zero-divisor*, if either $x = 0$ or $I_xI_yM = 0$ for some nonzero $y \in M$ with $I_y \subset R$.

– *zero-divisor*, if either $x = 0$ or $0 \neq I_x$ and $I_xI_yM = 0$ for some nonzero $y \in M$ with $0 \neq I_y \subset R$.

– *strong zero-divisor*, if either $x = 0$ or $\operatorname{Ann}(M) \subset I_x$ and $I_xI_yM = 0$ for some nonzero $y \in M$ with $\operatorname{Ann}(M) \subset I_y \subset R$.

For any module M we denote $Z_*(M)$, $Z(M)$ and $Z^*(M)$, respectively, for the set of weak zero-divisor, zero-divisor and strong zero-divisor elements of M. It is clear that

$$Z^*(M) \subseteq Z(M) \subseteq Z_*(M).$$

The following evident proposition shows that, for $M = R$, all of the above concepts coincide with the set of zero-divisor elements of R.

Proposition 1.2. *Let R be a ring and $x \in R$. Then the following are equivalent.*

(1) *x is a zero divisor element in R;*

(2) *x is a weak zero divisor element in ${}_RR$;*

(3) *x is a zero divisor element in ${}_RR$;*

(4) *x is a strong zero divisor element in ${}_RR$.*

Now, for an R-module M, we let $\widetilde{Z}_*(M) := Z_*(M) \setminus \{0\}$, $\widetilde{Z}(M) := Z(M) \setminus \{0\}$ and $\widetilde{Z}^*(M) := Z^*(M) \setminus \{0\}$. Then we associate three undirected (simple) graphs $\Gamma_*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma^*({}_RM)$ to M with vertices $\widetilde{Z}_*(M)$, $\widetilde{Z}(M)$ and $\widetilde{Z}^*(M)$, respectively, and for which the vertices x and y are adjacent if and only if $I_xI_yM = 0$. It is clear that we have $\Gamma^*({}_RM) \subseteq \Gamma({}_RM) \subseteq \Gamma_*({}_RM)$ as induced subgraphs.

Let $\Gamma(R)$ be the zero-divisor graph of a ring R. By Proposition 1.2, we have the following corollary.

Corollary 1.3. *Let R be a ring. Then*

$$\Gamma^*({}_RR) = \Gamma({}_RR) = \Gamma_*({}_RR) = \Gamma(R).$$

Proposition 1.4. *Let M be an R-module with $I = \mathrm{Ann}\,(M)$. Then*

$$\Gamma_*({}_RM) = \Gamma_*({}_{R/I}M) \quad \text{and} \quad \Gamma^*({}_RM) = \Gamma^*({}_{R/I}M).$$

Proof. Let $x \in \widetilde{Z}_*(M)$. Then there exists a $0 \neq y \in M$ such that $I_xI_yM = 0$. It is clear that $I \subseteq I_x \cap I_y$, $\mathrm{Ann}_{R/I}(M/Rx) = I_x/I$, $\mathrm{Ann}_{R/I}(M/Ry) = I_y/I$ and $(I_x/I)(I_y/I)M = 0$. It follows that $x \in Z_*({}_RM)$ if and only if $x \in Z_*({}_{R/I}M)$, and the vertices x and y are adjacent in $\Gamma_*({}_RM)$ if and only if x and y are adjacent in $\Gamma_*({}_{R/I}M)$. Therefore, $\Gamma_*({}_RM) = \Gamma_*({}_{R/I}M)$. Similarly, we can show that $\Gamma^*({}_RM) = \Gamma^*({}_{R/I}M)$. $\square$

The proofs of Corollaries 1.5 and 1.6 follow immediately from Proposition 1.4. In fact, the next corollary shows that, for any R-module M, either $\Gamma({}_RM) = \Gamma^*({}_RM)$ or $\Gamma({}_RM) = \Gamma_*({}_RM)$.

Corollary 1.5. *Let M be an R-module.* (a) *If M is a faithful module, then* $\Gamma({}_RM) = \Gamma^*({}_RM)$.

(b) *If M is not a faithful module, then* $\Gamma({}_RM) = \Gamma_*({}_RM)$.

Corollary 1.6. *Let $M = Rm$ be a cyclic R-module. Then*

$$\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM) = \Gamma(R/I),$$

where $I = \mathrm{Ann}\,(m)$.

In [**6**, Theorem 2.3], it is shown that, for any commutative ring R, $\Gamma(R)$ is connected and $\mathrm{diam}\,(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 7$ (and by [**19**], $g(\Gamma(R)) \leq 4$). In the next theorem, we extend these results to modules. We show that, for every R-module M, $\Gamma_*(M)$ is a connected graph and has diameter less than or equal to 3. Moreover, if $\Gamma_*({}_RM)$ contains a cycle, then $g(\Gamma_*({}_RM)) \leq 4$.

We need the following lemma.

Lemma 1.7. *Let M be an R-module and $x, y \in \widetilde{Z}_*(M)$. If $x — y$ is a path in $\Gamma_*(M)$, then for each $0 \neq r \in R$, either $ry = 0$ or $x — ry$ is also a path in $\Gamma_*(M)$.*

Proof. Let $x, y \in \widetilde{Z}_*(M)$ and $r \in R$. Assume that $x — y$ is a path in $\Gamma_*(M)$ and $ry \neq 0$. Then $I_xI_yM = 0$. It is clear that $I_{ry} \subseteq I_y$ and so $I_xI_{ry}M \subseteq I_xI_yM = 0$. Thus, $x — ry$ is also a path in $\Gamma_*(M)$. □

The proof of the following theorem is in many ways similar to the proof of the corresponding argument in [**6**].

Theorem 1.8. *Let M be an R-module. Then $\Gamma_*({}_RM)$ is a connected graph and* $\mathrm{diam}\,(\Gamma_*({}_RM)) \leq 3$. *Moreover, if $\Gamma_*({}_RM)$ contains a cycle, then $g(\Gamma_*({}_RM)) \leq 4$.*

Proof. Let $x, y \in \widetilde{Z}_*(M)$ be distinct. If $I_xI_yM = 0$, then $d(x, y) = 1$. So suppose that $I_xI_yM \neq 0$. Since $I_xI_yM \subseteq Rx \cap Ry$, this implies that $Rx \cap Ry \neq 0$. If $(I_x)^2M = (I_y)^2M = 0$, then for each $0 \neq z \in Rx \cap Ry$,

$I_z \subseteq I_x \cap I_y$. So $x — z — y$ is a path of length 2; thus, $d(x, y) = 2$. If $(I_x)^2M = 0$ and $(I_y)^2M \neq 0$, then there is a $b \in \widetilde{Z}_*(M) \setminus \{x, y\}$ such that $I_bI_yM = 0$. If $I_bI_xM = 0$, then $x — b — y$ is a path of length 2. Let $I_bI_xM \neq 0$. Then, for each $0 \neq c \in Rb \cap Rx$, $I_c \subseteq I_b \cap I_x$, so that $x — c — y$ is a path of length 2. In either case, $d(x, y) = 2$. A similar argument holds if $(I_y)^2M = 0$ and $(I_x)^2M \neq 0$. Thus, we may assume that $I_xI_yM, (I_x)^2M$ and $(I_y)^2M$ are all nonzero. Hence, there are $a, b \in \widetilde{Z}_*(M) \setminus \{x, y\}$ with $I_aI_xM = I_bI_yM = 0$. If $I_a = I_b$, then $x — a — y$ is a path of length 2. Thus we may assume that $I_a \neq I_b$. If $I_aI_bM = 0$, then $x — a — b — y$ is a path of length 3, and hence $d(x, y) \leq 3$. If $I_aI_bM \neq 0$, then $Ra \cap Rb \neq 0$ and, for every $0 \neq d \in Ra \cap Rb$, $x — d — y$ is a path of length 2; thus, $d(x, y) = 2$. Hence $d(x, y) \leq 3$, and thus $\operatorname{diam}(\Gamma({}_RM)) \leq 3$.

Now let $(x_1, \dots, x_n)$ be a cycle with length n. Note that $n \geq 3$. Define $x_0 := x_n$ and $x_{n+1} := x_1 = x$. If there is an $i \in \{1, 2, n\}$ such that $Rx_i \cap \{x_{i-1}, x_{i+1}\} \neq \varnothing$, then letting

$$l(i) = \begin{cases} (x, x_2, x_n) & \text{if } i = 1, \\ (x, x_2, x_3) & \text{if } i = 2, \\ (x, x_{n-1}, x_n) & \text{if } i = n. \end{cases}$$

Now, by Lemma 1.7, it is easy to see that $l(i)$ is a cycle for $i = 1, 2, n$. Hence, $g(\Gamma_*({}_RM)) \leq 3$. Henceforth, assume $Rx_i \cap \{x_{i-1}, x_{i+1}\} = \varnothing$ for all $i \in \{1, 2, n\}$. Thus, the proof will now break into two cases:

Case 1. Suppose $Rx_i \subseteq \{x_{i-1}, x_i, x_{i+1}, 0\}$ for all $i \in \{1, 2, n\}$. Then we must have $Rx_i = \{x_i, 0\}$ for all $i \in \{1, 2, n\}$. Consequently, $Rx_2 \cap Rx_n = 0$. In this case $I_{x_2}I_{x_n}M \subseteq Rx_2 \cap Rx_n = 0$, and so (x, x_2, x_n) is a cycle with length 3, and thus $g(\Gamma_*({}_RM)) \leq 3$.

Case 2. Suppose there is an $i \in \{1, 2, n\}$ such that $Rx_i \nsubseteq \{x_{i-1}, x_i, x_{i+1}, 0\}$. Pick $y \in Rx_i \setminus \{x_{i-1}, x_i, x_{i+1}, 0\}$. Define

$$l(i) = \begin{cases} (x, x_2, y, x_n) & \text{if } i = 1, \\ (x, x_2, x_3, y) & \text{if } i = 2, \\ (x, y, x_{n-1}, x_n) & \text{if } i = n. \end{cases}$$

By Lemma 1.7, it is straightforward to verify that $l(i, y)$ is a cycle for all $i = 1, 2, n$, and hence $g(\Gamma_*({}_RM)) \leq 4$. $\square$

In view of the above result, we state the following conjecture:

Conjecture 1.9. *Let M be an R-module. Then $\Gamma^*({}_RM)$ is also connected and* $\operatorname{diam}(\Gamma^*({}_RM)) \leq 3$. *Moreover, if $\Gamma^*({}_RM)$ contains a cycle, then $g(\Gamma^*({}_RM)) \leq 4$.*

Let Γ be a graph with vertices V and, let $\varnothing \neq A$, $B \subseteq V$. Then $A \leftrightsquigarrow B$ means that, for each $a \in A$, $b \in B$, $a — b$ is a path in Γ. Also, for each nonzero R-module M, we shall denote the set of all nonzero elements of M by $\widetilde{M}$ (i.e., $\widetilde{M} = M \setminus \{0\}$).

Lemma 1.10. *Let M be an R-module, and let $M = M_1 \oplus M_2$ where M_1, M_2 are nonzero R-submodules of M. Then $\widetilde{M_1}, \widetilde{M_2} \subseteq \widetilde{Z}_*(M)$ and $\widetilde{M_1} \leftrightsquigarrow \widetilde{M_2}$ in $\Gamma_*(M)$. Moreover, if $0 \neq x, \in \widetilde{Z}_*(M_1)$, then $(x, 0) \in \widetilde{Z}_*(M)$ and, if the vertices x and y are adjacent in $\Gamma_*(M_1)$, then $(x, 0)$, $(y, 0)$ are adjacent in $\Gamma_*({}_RM)$.*

Proof. Let $0 \neq x \in M_1$ and $0 \neq y \in M_2$. It is clear that

$$I_x = \operatorname{Ann}\left(\frac{M}{Rx}\right) = \operatorname{Ann}\left(\frac{M_1 \oplus M_2}{Rx \oplus (0)}\right) = \operatorname{Ann}\left(\frac{M_1}{Rx} \oplus M_2\right)$$
$$I_y = \operatorname{Ann}\left(\frac{M}{Ry}\right) = \operatorname{Ann}\left(\frac{M_1 \oplus M_2}{(0) \oplus Ry}\right) = \operatorname{Ann}\left(M_1 \oplus \frac{M_2}{Ry}\right).$$

It follows that $I_x \subseteq \operatorname{Ann}(M_2)$ and $I_y \subseteq \operatorname{Ann}(M_1)$. Thus, $I_xI_yM = 0$, i.e., $x — y$ is a path in $\Gamma_*(M)$. Therefore, $\widetilde{M_1}, \widetilde{M_2} \subseteq \widetilde{Z}_*(M)$ and $\widetilde{M_1} \leftrightsquigarrow \widetilde{M_2}$ in $\Gamma_*(M)$.

Now let $x \in \widetilde{Z}(M_1)$. Then there exists a $0 \neq y \in M_1$ such that $I_xI_yM_1 = 0$, where $I_x = \operatorname{Ann}(M_1/Rx)$ and $I_y = \operatorname{Ann}(M_1/Ry)$. Clearly,

$$I_{(x,0)} = \operatorname{Ann}\left(\frac{M_1 \oplus M_2}{R(x,0)}\right) = \operatorname{Ann}\left(\frac{M_1}{Rx} \oplus M_2\right),$$
$$J_{(y,0)} = \operatorname{Ann}\left(\frac{M_1 \oplus M_2}{R(y,0)}\right) = \operatorname{Ann}\left(\frac{M_1}{Ry} \oplus M_2\right).$$

It follows that $I_{(x,0)} \subseteq I_x$, $J_{(y,0)} \subseteq I_y$ and $I_{(x,0)}M_2 = J_{(y,0)}M_2 = 0$. Thus, $I_{(x,0)}J_{(y,0)}M = 0$, i.e., $(x, 0)$ and $(y, 0)$ are adjacent in $\Gamma_*({}_RM)$. $\square$

Theorem 1.11. *Let $M = M_1 \oplus M_2$ where M_1, M_2 are nonzero R-modules. If $\Gamma_*(M_1) \neq \varnothing$, then $\Gamma_*(M_1) \cong G$, where G is induced subgraph of $\Gamma_*({}_RM)$ with vertices $\{(x,0) \in \widetilde{Z}_*(M) \mid x \in \widetilde{Z}_*(M_1)\}$.*

Proof. The proof follows directly from Lemma 1.10. □

Proposition 1.12. *Let M be a non-simple semisimple R-module. Then $\Gamma_*({}_RM)$ is a connected graph with vertices $\widetilde{M}$. Moreover, if Rx and Ry are two distinct simple R-submodules of M, then $x — y$ is a path in $\Gamma_*(M)$.*

Proof. Since every proper submodule of a semisimple module M is a direct summand of M, this follows from Lemma 1.10. □

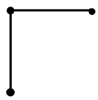
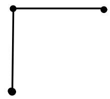
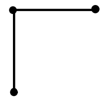
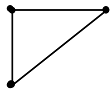
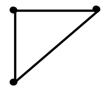
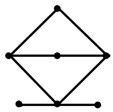
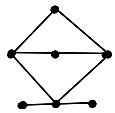
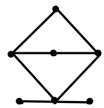
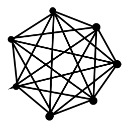
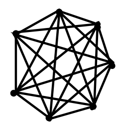
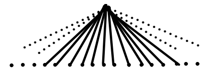
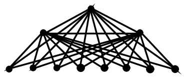
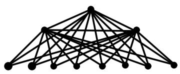
Corollary 1.13. *Let M be a non-simple homogeneous semisimple R-module. Then $\Gamma^*({}_RM) = \varnothing$ and $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$.*

Proof. Since every cyclic submodule of a homogeneous semisimple module M is simple, by Proposition 1.12, $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$. Also $\Gamma^*({}_RM)$ is the empty graph since $\mathrm{Ann}\,(M)$ is a maximal ideal. □

We conclude this section with the following example.

Example 1.14. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}_2 \oplus \mathbf{Z}_4$. Then M has eight elements and the nonzero elements of M are: $m_1 = (1,0)$, $m_2 = (0,1)$, $m_3 = (0,2)$, $m_4 = (0,3)$, $m_5 = (1,1)$, $m_6 = (1,2)$ and $m_7 = (1,3)$. It is easy to check that $I_{m_1} = I_{m_6} = 4\mathbf{Z} = \mathrm{Ann}\,(M)$ and $I_{m_2} = I_{m_3} = I_{m_4} = I_{m_5} = I_{m_7} = 2\mathbf{Z}$. Thus, by Definition 1.1, $\widetilde{Z}_*(M) = \widetilde{Z}(M) = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\}$ and $\widetilde{Z}^*(M) = \{m_2, m_3, m_4, m_5, m_7\}$. Since $I_{m_i} I_{m_j} M = 4\mathbf{Z}(\mathbf{Z}_2 \oplus \mathbf{Z}_4) = 0$ for all $1 \leq i,\ j \leq 7$, we conclude that $\Gamma^*({}_RM)$ is complete with five vertices: m_2, m_3, m_4, m_5 and m_7, but $\Gamma_*({}_RM)$ and $\Gamma^*({}_RM)$ are complete with seven vertices: m_1, m_2, m_3, m_4, m_5, m_6 and m_7.

TABLE 1. Graphs of several **Z**-modules.

$\mathbb{Z}$-modules	$\Gamma^*(M)$	$\Gamma(M)$	$\Gamma_*(M)$
$M = \mathbb{Z}_6$			
$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\emptyset$		
$M = \mathbb{Z}_{12}$			
$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\emptyset$		
$M = \mathbb{Z}_{p^\infty}$ or $\mathbb{Q}$	$\emptyset$	$\emptyset$	$\Gamma_*(M)$ is a complete graph with vertices $\tilde{M}$
$M = \mathbb{Z} \oplus \mathbb{Z}_2$	$\emptyset$	$\emptyset$	
$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3$	$\emptyset$		
$M = \mathbb{Z}_p$ or $\mathbb{Z}$	$\emptyset$	$\emptyset$	$\emptyset$
$M = \bigoplus_{i=1}^n \mathbb{Z},\ n \geq 2$	$\emptyset$	$\emptyset$	$\Gamma_*(M)$ is a complete graph with vertices $\tilde{M}$

Example 1.15. Table 1 shows the zero divisor graphs for several **Z**-modules.

2. Multiplication-like modules. Let R be a ring and M an R-module. If $M \neq 0$ and $\mathrm{Ann}\,(M) = \mathrm{Ann}\,(N)$ for all nonzero submodules N of M, then M is called a prime module. It is immediate that $\mathrm{Ann}\,(M)$ is a prime ideal, and it is called the affiliated prime of $_RM$ (see [**9–12, 15**] for some known results about prime modules). Also, an R-module M is called a multiplication module if each submodule of M is of the form IM, where I is an ideal of R, i.e., for each $0 \neq m \in M$ there is an ideal I of R such that $Rm = IM$ (see [**15**] for some known results about multiplication modules). Let N be a nonzero submodule of a multiplication module M. It is clear that, for each nonzero submodule N of M, $N = \mathrm{Ann}\,(M/N)M$ and so $\mathrm{Ann}\,(M) \subset \mathrm{Ann}\,(M/N)$ (i.e., there is an $r \in R \backslash \mathrm{Ann}\,(M)$ with $rM \subseteq N$). Therefore, we have the following definition.

Definition 2.1. Let M be an R-module. We say that M is a *multiplication-like module* if, for each nonzero submodule N of M, $\mathrm{Ann}\,(M) \subset \mathrm{Ann}\,(M/N)$.

Multiplication-like modules have been considered in [**17, 21**]. For example, in [**17**, Corollary 1.6] it is shown that, if R is a commutative ring and M is a multiplication-like R-module, then M is prime if and only if $\mathrm{Ann}\,(M)$ is a prime ideal.

Clearly, every multiplication module is multiplication-like, but hitherto we have not found any example where M is multiplication-like and M is not a multiplication module. Therefore, we have the following question.

Question 2.2. Let M be a multiplication-like R-module. Is M a multiplication R-module?

Later in this section, we shall be able to characterize all modules M for which $\Gamma^*(_RM) = \Gamma(_RM) = \Gamma_*(_RM)$. In fact, we show that multiplication-like modules are the only modules with this property.

Lemma 2.3. *Let M be an R-module. Then M is multiplication-like if and only if, for each $0 \neq m \in M$,* $\mathrm{Ann}\,(M) \subset I_m$.

Proof. The necessity is clear. Conversely, suppose that, for each $0 \neq m \in M$, $\mathrm{Ann}\,(M) \subset I_m$. Let N be a nonzero submodule of M. For each $0 \neq x \in N$, there exists an ideal I_x such that $0 \neq I_x M \subseteq Rx$. Let $I = \sum_{0 \neq x \in N} I_x$. Then $0 \neq IM \subseteq N$. It follows that M is a multiplication-like module. $\square$

For an R-module M, we define:

$$\underline{\mathcal{A}}(M) := \{x \in M \mid \mathrm{Ann}\,(M) \subset I_x\}$$
$$\underline{\mathcal{A}}(M) := \{x \in M \mid \mathrm{Ann}\,(M) = I_x\}.$$

It is clear that $M = \overline{\mathcal{A}}(M) \cup \underline{\mathcal{A}}(M)$ and $\overline{\mathcal{A}}(M) \cap \underline{\mathcal{A}}(M) = \varnothing$. By Lemma 2.1, M is multiplication-like if and only if $\overline{\mathcal{A}}(M) = \widetilde{M}$ if and only if $\underline{\mathcal{A}}(M) = \{0\}$.

Now we are in a position to characterize those R-modules for which the zero-divisor graphs $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$ coincide.

Theorem 2.4. *Let M be an R-module. Then $\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM)$ if and only if M is a multiplication-like module.*

Proof. ($\Rightarrow$). Suppose $\Gamma^*({}_RM) = \Gamma_*({}_RM)$. If x is a nonzero element in $\underline{\mathcal{A}}(M)$, then $I_x = \mathrm{Ann}\,(M)$. Therefore, $I_x I_y M = 0$ for each $0 \neq y \in M$. It follows that x is a vertex in $\Gamma_*({}_RM)$, so that x is a vertex in $\Gamma^*({}_RM)$. Thus, $\mathrm{Ann}\,(M) \subset I_x$, a contradiction. Thus, $\underline{\mathcal{A}}(M) = \{0\}$. Hence, by Lemma 2.3, M is a multiplication-like module.

($\Leftarrow$). Suppose that M is a multiplication-like module. It is clear that, if $\Gamma_*({}_RM) = \varnothing$, then $\Gamma^*({}_RM) = \varnothing$. Assume that $\Gamma_*({}_RM) \neq \varnothing$, and fix a vertex $x \in \Gamma_*({}_RM)$. Hence, there is a $0 \neq y \in M$ such that $I_x I_y M = 0$. For every $0 \neq m \in M$, $\mathrm{Ann}\,(M) \subset I_m$, so that x is a vertex in $\Gamma^*({}_RM)$. Thus, $\widetilde{Z}_*(M) \subseteq \widetilde{Z}^*(M)$. Now let $x — y$ be a path in $\Gamma_*({}_RM)$. Then $I_x I_y M = 0$ and so $x — y$ is a path in $\Gamma^*({}_RM)$. Thus, $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$. $\square$

Since every multiplication module is multiplication-like, we have the following corollary.

Corollary 2.5. *Let M be a multiplication R-module. Then*

$$\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM).$$

It is clear that, for a commutative ring R, $\Gamma(R)$ is the empty graph if and only if R is an integral domain. Here, we generalize this fact to modules. In fact, we characterize all R-modules M for which $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$ are the empty graph.

Theorem 2.6. *Let M be an R-module. Then the following statements are equivalent.*

(1) $\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM) = \varnothing$ *(i.e.,* $\Gamma_*({}_RM) = \varnothing$*).*

(2) *M is a prime multiplication-like module.*

(3) *M is a multiplication-like module for which* $\mathrm{Ann}\,(M)$ *is a prime ideal.*

Proof. (1) $\Rightarrow$ (2). Let $\Gamma_*({}_RM) = \varnothing$. By Theorem 2.2, M is a multiplication-like module. Assume that M is not a prime module, i.e., $\mathrm{Ann}\,(M)$ is not a prime ideal (see [**17**, Corollary 1.6]). Therefore, $IJM = 0$ for some ideals $I, J \supsetneqq \mathrm{Ann}\,(M)$. Since $IM \neq 0$ and $JM \neq 0$, so that there exist $0 \neq x \in IM$ and $0 \neq y \in JM$. Then $I_xM \subseteq Rx \subseteq IM$ and $I_yM \subseteq Ry \subseteq JM$. Thus, $I_yI_xM \subseteq I_yIM = II_yM \subseteq IJM = 0$. It follows that $x, y \in \widetilde{Z}_*(M)$, a contradiction. Thus M is a prime multiplication-like module.

(2) $\Rightarrow$ (1). Suppose M is a prime multiplication-like module. Then, for every $0 \neq x \in M$, $\mathrm{Ann}\,(M) \subset I_x$. It follows that $I_xI_yM \neq 0$ for each $0 \neq x, y \in M$. So, $\Gamma_*({}_RM) = \varnothing$.

(2) $\Leftrightarrow$ (3) is by [**17**, Corollary 1.6]. $\square$

An R-module M is called indecomposable if $M \neq 0$ and M cannot be written as a direct sum of nonzero submodules. It is clear that every prime ring R is indecomposable as an R-module. Now we generalize this fact to multiplication-like modules.

Proposition 2.7. *Every prime multiplication-like R-module is an indecomposable module.*

Proof. Let M be a prime multiplication-like R-module. By Theorem 2.6, $\Gamma_*({}_RM) = \varnothing$. If $M = M_1 \oplus M_2$, where M_1 and M_2 are nonzero R-modules, then by Lemma 1.10, $\Gamma_*({}_RM) \neq \varnothing$, a contradiction. Thus M is an indecomposable module. $\square$

Proposition 2.8. *Let M be a prime multiplication-like R-module with* $\operatorname{soc}(M) \neq 0$. *Then M is a simple module.*

Proof. Let M be a prime multiplication-like module with $\operatorname{soc}(M) \neq 0$. By [**11**, Corollary 1.9], every prime module with nonzero socle is a homogeneous semisimple module. Thus, by Proposition 2.7, M must be a simple module. $\square$

Corollary 2.9. *Let M be an Artinian prime multiplication-like module. Then M is a simple module.*

Proof. This immediately follows from Proposition 2.8. $\square$

We recall that, if U and M are $R-$modules, then we say U is $M-$injective if, for every submodule N of M, each homomorphism $N \to U$ can be extended to $M \to U$, and an R-module M is called co-semisimple if every simple module is $M-$injective (see, for example [**22**], for definition and characterization). Every semisimple module is of course co-semisimple. In [**11**, Corollary 1.9], the authors proved that a co-semisimple module M over a commutative ring R is prime if and only if M is a homogeneous semisimple module. Thus, we have the following proposition.

Proposition 2.10. *Every prime multiplication-like co-semisimple module is simple.*

3. Virtually divisible modules. Let R be an integral domain. We recall the definition of a divisible R-module M. An R module

M is called divisible if $rM = M$, for all $0 \neq r \in R$. For example, every injective module is divisible. Over $\mathbf{Z}$, or more generally over any principal ideal domain, the divisible modules are exactly the injective modules. Over other domains, however, divisible modules need not be injective (see [**16**, Exercise 6F]).

Let M be a nonzero divisible R-module. It is clear that, for each proper submodule N of M, we have $\mathrm{Ann}\,(M/N) = \mathrm{Ann}\,(M) = 0$. It thus follows that, if R is not a field, then $\Gamma({}_RM) = \varnothing$ and $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ (see Table 1, where M is the $\mathbf{Z}$-module $\mathbf{Z}_{p^\infty}$ or $\mathbf{Q}$). Now let R be a commutative ring (not necessarily a domain) and M a homogeneous semisimple R-module. It is clear that $\mathrm{Ann}\,(M)$ is a maximal ideal and so, for each proper submodule N of M, we have $\mathrm{Ann}\,(M/N) = \mathrm{Ann}\,(M)$. Thus, if R is not a field and M is not a simple module, then, for each $0 \neq x \in M$, $I_x = \mathrm{Ann}\,(M)$. It follows that $\Gamma^*({}_RM)$ is the empty graph and $\Gamma({}_RM)$ is a complete graph with vertices $\widetilde{M}$ (see Table 1, where M is the $\mathbf{Z}$-module $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$).

In this section, we shall study R-modules M for which $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ and $\Gamma({}_RM) = \varnothing$. Also, we study R-modules M for which $\Gamma({}_RM)$ is a complete graph with vertices $\widetilde{M}$ and $\Gamma^*({}_RM) = \varnothing$. In particular, prime modules M for which $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ are characterized.

Definition 3.1. Let M be a nonzero R-module. We say that M is a *virtually divisible module* if $\mathrm{Ann}\,(M/N) = \mathrm{Ann}\,(M)$ for each proper submodule N of M. Also, M is called a *weakly virtually divisible module* if $\mathrm{Ann}\,(M/Rm) = \mathrm{Ann}\,(M)$ for each proper cyclic submodule Rm of M (i.e., $I_x = \mathrm{Ann}\,(M)$ for each $0 \neq x \in M$ which $Rx \neq M$).

Example 3.2. (i) Let R be an integral domain. It is clear that every divisible R-module is virtually divisible but the converse is not true (for example, every non-simple homogeneous semisimple $\mathbf{Z}$-module M is virtually divisible but it is not a divisible $\mathbf{Z}$-module).

(ii) Let R be any ring. Then every homogeneous semisimple R-module is virtually divisible (see Proposition 3.3).

(iii) It is clear that every virtually divisible R-module is weakly virtually divisible but the converse is not true (for example, $M = \mathbf{Z} \oplus \mathbf{Z}$

is a weakly virtually divisible **Z**-module but it is not a virtually divisible **Z**-module).

(iv) Let R be an integral domain. Then every non-cyclic free R-module F is a weakly virtually divisible R-module (see Proposition 3.9 and Theorem 3.11).

Proposition 3.3. *Let M be an R-module with $\mathcal{P} = \mathrm{Ann}\,(M)$. Then M is virtually divisible if and only if $\mathcal{P}$ is a prime ideal and M is a divisible $R/\mathcal{P}$-module.*

Proof. Suppose M is virtually divisible. Let $ab \in \mathcal{P}$, where $a, b \in R$. Assume $aM \neq 0$; then aM is a nonzero submodule of M. If $aM \neq M$, then $\mathrm{Ann}\,(M/aM) = \mathrm{Ann}\,(M) = \mathcal{P}$ (since M is virtually divisible) and so $a \in \mathrm{Ann}\,(M/aM) = \mathrm{Ann}\,(M)$, a contradiction. Thus, $aM = M$ and so $bM = baM = 0$. It follows that $b \in \mathrm{Ann}\,(M) = \mathcal{P}$. Therefore, $\mathcal{P}$ is a prime ideal. Now, let $0 \neq r \in R \setminus \mathcal{P}$. Then $rM \neq 0$. If $rM \neq M$, then $r \in \mathrm{Ann}\,(M/rM) = \mathrm{Ann}\,(M) = \mathcal{P}$, a contradiction. Thus, $rM = M$ i.e., $(r+\mathcal{P})M = M$ and so M is divisible as a $R/\mathcal{P}$-module.The converse is clear. $\square$

In the following theorem there are several equivalent statements for a virtually divisible module.

Theorem 3.4. *Let M be an R-module. Then the following are equivalent.*

(1) *M is virtually divisible.*

(2) $\mathcal{P} = \mathrm{Ann}\,(M)$ *is a prime ideal and M is a divisible $R/\mathcal{P}$-module.*

(3) *Each direct summand of M is a virtually divisible module.*

(4) *For each $a \in R$, we have $aM = M$ or $aM = 0$.*

(5) *For each ideal I of R, we have $IM = M$ or $IM = 0$.*

Proof. The equivalence of (1) and (2) is from Proposition 3.3, and the equivalence of (4) and (5) is clear.

(2) $\Rightarrow$ (3). Let N be a direct summand of M. Then $M = N \oplus K$, for some $K \leq M$. If $N = (0)$, then we are through. Let $N \neq (0)$. Since

$\mathcal{P} = \mathrm{Ann}\,(M)$ is a prime ideal and M is a divisible $R/\mathcal{P}$-module, the factor module M/K is also a divisible $R/\mathcal{P}$-module. Now by (1) $\Leftrightarrow$ (2), $\mathrm{Ann}\,(N) = \mathrm{Ann}\,(M) = \mathcal{P}$, and N is a divisible $R/\mathcal{P}$-module (since, $M/K \cong N$). Thus, N is a virtually divisible R-module.

(3) $\Rightarrow$ (1) is evident.

(2) $\Rightarrow$ (4). Let $a \in R$ and $aM \neq 0$. Then $a \notin \mathrm{Ann}\,(M) = \mathcal{P}$. Since M is a divisible $R/\mathcal{P}$-module, $(a + \mathcal{P})M = M$, i.e., $aM = M$.

(4) $\Rightarrow$ (2). Let $a, b \in R$ and $abM = 0$. If $bM \neq 0$, then by our hypothesis $bM = M$ and so $aM = 0$. Thus, $\mathcal{P} = \mathrm{Ann}\,(M)$ is a prime ideal. Now let $r \in R \setminus \mathcal{P}$. Then $rM = M$, and so $(r + \mathcal{P})M = M$. Thus, M is a divisible $R/\mathcal{P}$-module. □

Next, we determine virtually divisible modules over one-dimensional domains.

Corollary 3.5. *Let R be an integral domain with* $\dim\,(R) = 1$*, and let M be an R-module. Then M is a virtually divisible R-module if and only if one of the following statements hold.*

(1) *M is a homogeneous semisimple module.*

(2) *M is a divisible module.*

Proof. ($\Rightarrow$). Let M be a virtually divisible R-module. By Proposition 3.3, $\mathcal{P} = \mathrm{Ann}\,(M)$ is a prime ideal and M is a divisible $R/\mathcal{P}$-module. If $\mathcal{P} = 0$, then M is a divisible R-module but, if $\mathcal{P} \neq 0$ then $\mathcal{P}$ is a maximal ideal and so M is a homogeneous semisimple module.

($\Leftarrow$). This immediately follows from Theorem 3.4. □

Remark 3.6. Let R be an integral domain which is not a field. Then every divisible R-module M has no maximal submodule, for otherwise if M is a divisible R-module with a maximal submodule N, then $\mathrm{Ann}\,(M/N) = \mathcal{P}$ is a maximal ideal of R. This means that $M = \mathcal{P}M \subseteq N$, a contradiction. In particular, if R is a one-dimensional Noetherian integral domain, then an R-module M is divisible if and only if it has no maximal submodule (see [**11**, Corollary 3.3]).

The following proposition shows that, if M is a finitely generated module, then homogeneous semisimplicity and virtually divisibility of M coincide.

Proposition 3.7. *Let R be a ring, and let M be a finitely generated R-module. Then M is virtually divisible if and only if M is a homogeneous semisimple module.*

Proof. Let M be a finitely generated virtually divisible R-module. Then by Proposition 3.3, $\mathcal{P} = \mathrm{Ann}\,(M)$ is a prime ideal of R and M is a divisible $R/\mathcal{P}$-module. If $\mathcal{P}$ is not a maximal ideal of R, then $R/\mathcal{P}$ is an integral domain which is not a field. By Remark 3.6, M as an $R/\mathcal{P}$-module has no maximal submodule; this is a contradiction (since M is a finitely generated $R/\mathcal{P}$-module). Therefore, $\mathcal{P}$ is a maximal ideal of R, and so M is a homogeneous semisimple module. $\square$

Proposition 3.8. *Let M be a weakly virtually divisible R-module which is not cyclic. Then:*

(1) *$\Gamma^*({}_RM)$ is the empty graph and $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$.*

(2) *If M is faithful, then $\Gamma({}_RM)$ is also the empty graph.*

(3) *If M is not faithful, then $\Gamma({}_RM)$ is a complete graph with vertices $\widetilde{M}$.*

Proof. Let M be a weakly virtually divisible module which is not cyclic. Then $I_x = \mathrm{Ann}\,(M)$, for all $x \in M$. Now by Definition 1.1, $\Gamma^*({}_RM)$ is the empty graph and $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$. If M is faithful, then by Corollary 1.5, $\Gamma^*({}_RM) = \Gamma({}_RM)$ and so $\Gamma({}_RM)$ is the empty graph. Now let $\mathrm{Ann}\,(M) \neq 0$. Then by Corollary 1.5, $\Gamma({}_RM) = \Gamma_*({}_RM)$ and so $\Gamma({}_RM)$ is a complete graph with vertices $\widetilde{M}$. $\square$

Let R be a ring which is not a field, and let M be a homogeneous semisimple R-module. Then $\mathrm{Ann}\,(M)$ is a maximal ideal and so $\mathrm{Ann}\,(M) \neq 0$. Therefore, M is a non-faithful virtually divisible R-module and so we have the following corollary.

Corollary 3.9. *Let R be a ring which is not a field. Let M be a homogeneous semisimple R-module. Then*:

(1) *if M is a simple module, then $\Gamma^*({}_RM) = \Gamma({}_RM) = \Gamma_*({}_RM) = \varnothing$.*

(2) *If M is not a simple module, then $\Gamma^*({}_RM) = \varnothing$ and $\Gamma({}_RM) = \Gamma_*({}_RM)$ are complete graphs with vertices $\widetilde{M}$.*

Next, we shall determine the structure of the all zero divisor graphs for free modules over an integral domain R.

Proposition 3.10. *Let R be an integral domain and M a free R-module. Then*:

(1) *$M \cong R$ if and only if $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$ are the empty graph.*

(2) *$M \ncong R$ if and only if $\Gamma^*({}_RM)$ and $\Gamma({}_RM)$ are the empty graph and $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$.*

Proof. If M and R are isomorphic, it is clear that all the graphs in question are empty. For the converse, let $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$ be the empty graphs. By Theorem 2.6, M is a prime multiplication-like module. Now by Proposition 2.7, M is an indecomposable module and so $M \cong R$.

To see (2), let $M = \oplus_{\lambda\in\Lambda} R$ where Λ is an index set with $|\Lambda| \geq 2$. Let $0 \neq x = (x_\lambda)_{\lambda\in\Lambda} \in M$ where $x_\lambda \in R$ for each $\lambda \in \Lambda$. Thus, $x_\mu \neq 0$ for some $\mu \in \Lambda$ and also $I_xM = \oplus_{\lambda\in\Lambda} I_x \subseteq Rx = R(x_\lambda)_{\lambda\in\Lambda}$. If $I_x \neq 0$ and $0 \neq a \in I_x$, then we put $y_\mu = 0$ and $y_\lambda = a$ for each $\lambda \neq \mu$. Then $(y_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda} I_x$, and so there exists a $t \in R$ such that $(y_\lambda)_{\lambda\in\Lambda} = t(x_\lambda)_{\lambda\in\Lambda}$. It follows that $0 = y_\mu = tx_\mu$ and $a = tx_\lambda$ for each $\lambda \neq \mu$. Since R is a domain and $x_\mu \neq 0$, $t = 0$; hence, $a = 0$, a contradiction. Thus, $I_x = 0$ for each $0 \neq x \in M$. This implies that $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$, and since M is a faithful R-module, by Definition 1.1, $\Gamma^*({}_RM) = \Gamma({}_RM) = \varnothing$. The converse is clear. □

Theorem 3.11. *Let M be a prime R-module. Then $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ if and only if M is a non-simple weakly virtually divisible module.*

Proof. Suppose $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$. Then, for distinct $x, y \in \widetilde{M}$, I_x and I_y are two ideals of R such that $I_xI_yM = 0$. Since $\mathrm{Ann}\,(M)$ is a prime ideal, either $I_xM = 0$ or $I_yM = 0$, i.e., for each $0 \neq x, y \in M$, either $I_x = \mathrm{Ann}\,(M)$ or $I_y = \mathrm{Ann}\,(M)$. We claim that $\mathrm{Ann}\,(M) = I_x$, for each $x \in M$, for otherwise, $\mathrm{Ann}\,(M) \subset I_{x_0}$ for some $0 \neq x_0 \in M$. We will show that $Rx_0 = \{0, x_0\}$. Let $rx_0 \neq x_0$ where $r \in R$. Since $I_{x_0}M \subseteq Rx_0$ and $rI_{x_0}M \subseteq Rrx_0$, we have $rI_{x_0} \subseteq I_{rx_0} = \mathrm{Ann}\,(M)$. Since $\mathrm{Ann}\,(M)$ is a prime ideal of R and $I_{x_0} \nsubseteq Ann(M)$, $rM = 0$ and so $rx_0 = 0$. Thus, $Rx_0 = \{0, x_0\}$, and hence Rx_0 is a simple submodule of M. Therefore, $\mathrm{soc}\,(M) \neq 0$. Since M is a prime module, by [**11**, Corollary 1.9], M is a homogeneous semisimple module, i.e., $\mathrm{Ann}\,(M)$ is a maximal ideal of R. It follows that $I_x = \mathrm{Ann}\,(M)$, a contradiction. Therefore, $\mathrm{Ann}\,(M) = I_x$, for each $x \in M$ i.e., M is a weakly virtually divisible module. Since M is a prime module, M is a nonzero module and so $\widetilde{M} \neq \varnothing$ and hence $\Gamma_*({}_RM)$ is a non-empty graph. Therefore, M is not a simple module. The converse is clear. $\square$

Theorem 3.12. *Let M be a nonzero R-module for which* $\mathrm{Ann}\,(M)$ *is a prime ideal and* $\mathrm{soc}\,(M) = 0$. *Then $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$ if and only if M is a non-simple weakly virtually divisible module.*

Proof. Suppose $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$. Then, for distinct $x, y \in M \setminus \{0\}$, I_x and I_y are two ideals of R such that $I_xI_yM = 0$. Since $\mathrm{Ann}\,(M)$ is a prime ideal, either $I_xM = 0$ or $I_yM = 0$ i.e., for each $0 \neq x, y \in M$, either $I_x = \mathrm{Ann}\,(M)$ or $I_y = \mathrm{Ann}\,(M)$. Exactly as in the proof of Theorem 3.11, if I_{x_0} does not equal $\mathrm{Ann}\,(M)$, we have that $Rx_0 = \{0, x_0\}$ which is a simple submodule of M and so $\mathrm{soc}\,(M) \neq 0$, a contradiction. Therefore, $\mathrm{Ann}\,(M) = I_x$, for each $x \in M$, i.e., M is a weakly virtually divisible module. Since M is nonzero, $\Gamma_*({}_RM)$ is a non-empty graph. Therefore, M is not a simple module. The converse is clear. $\square$

Next, we give some properties of the zero-divisor graphs for a certain class of modules. In fact, the following theorem more or less summarizes the overall situation for a module whose annihilator is prime.

Theorem 3.13. *Let M be an R-module for which $Ann(M)$ is a prime ideal.*

(a) *If M is multiplication-like, then $\Gamma^*({}_RM)$, $\Gamma({}_RM)$ and $\Gamma_*({}_RM)$ are the empty graph.*

(b) *Assume that M is not a multiplication-like module. Then*:

(1) *if M is a faithful weakly virtually divisible module (i.e., $\underline{\mathcal{A}}(M) = M$), then $\Gamma_*({}_RM)$ is a complete graph with vertices $\widetilde{M}$. Moreover, $\Gamma^*({}_RM)$ and $\Gamma({}_RM)$ are the empty graph.*

(2) *if M is faithful and it is not a weakly virtually divisible module (i.e., $\underline{\mathcal{A}}(M) \neq \widetilde{M}$), then $\Gamma_*({}_RM)$ is a connected graph with vertices $\widetilde{M}$ such that $\underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \overline{\mathcal{A}}(M)$. Moreover, $\Gamma^*(M)$ and $\Gamma({}_RM)$ are the empty graph.*

(3) *if M is a non-faithful weakly virtually divisible module (i.e., $\mathrm{Ann}\,(M) \neq 0$ and $\underline{\mathcal{A}}(M) = M$), then $\Gamma(M) = \Gamma_*({}_RM)$ and this graph is complete with vertices $\widetilde{M}$. Moreover, $\Gamma^*(M)$ is the empty graph.*

(4) *If M is not faithful and it is not a weakly virtually divisible module (i.e., $\mathrm{Ann}\,(M) \neq 0$ and $\underline{\mathcal{A}}(M) \neq M$), then $\Gamma(M) = \Gamma_*({}_RM)$, and this graph is complete with vertices $\widetilde{M}$ such that*

$$\underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \overline{\mathcal{A}}(M).$$

Moreover, $\Gamma^*(M)$ is the empty graph.

Proof. Part (a) is just Theorem 2.6. Parts (1), (2) and (3) of (b) follow from Proposition 3.8. For the last implication, assume M is not faithful and $\underline{\mathcal{A}}(M) \neq M$. By Corollary 1.5, $\Gamma(M) = \Gamma_*(M)$. Since M is not multiplication-like, $\overline{\mathcal{A}}(M) \neq \varnothing$ and $\{0\} \subset \underline{\mathcal{A}}(M) \subset M$ (see the comments before Theorem 2.4). Let $x \in \underline{\mathcal{A}}(M)$ and $y \in \overline{\mathcal{A}}(M)$. Then $0 \neq \mathrm{Ann}\,(M) = I_x$ and $0 \neq \mathrm{Ann}\,(M) \subset I_y$. Hence, $I_xI_yM = 0$, i.e., x and y are adjacent in $\Gamma_*(M)$. If $x_1, x_2 \in \underline{\mathcal{A}}(M)$, then $I_{x_1}I_{x_2}M = 0$. Thus, x_1 and x_2 are also adjacent in $\Gamma_*({}_RM)$). If $z \in \overline{\mathcal{A}}(M) \cap \widetilde{Z}_*(M)$, then $I_zI_tM = 0$, for some $t \in \overline{\mathcal{A}}(M)$. Since $\mathrm{Ann}\,(M)$ is a prime ideal, so that $I_tM = 0$ (i.e., $I_t = \mathrm{Ann}\,(M)$), a contradiction. Thus, $\widetilde{Z}_*(M) \cap \overline{\mathcal{A}}(M) = \varnothing$. Hence, $\Gamma({}_RM) = \Gamma_*({}_RM)$ is a connected graph with vertices $\widetilde{M}$, in which at least $\underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \underline{\mathcal{A}}(M) \setminus \{0\} \leftrightsquigarrow \overline{\mathcal{A}}(M)$. □

Corollary 3.14. *Let M be an R-module for which* $\mathrm{Ann}\,(M)$ *is a prime ideal. Then* $\Gamma_*({}_RM)$ *is connected and* $\mathrm{diam}\,\Gamma_*({}_RM) \leq 2$*. Moreover, if M is a finite module with $|M| \geq 4$, then $\Gamma_*({}_RM)$ contains a cycle and $g(\Gamma_*({}_RM)) \leq 3$.*

Proof. The proof is immediate from Theorem 3.13. □

4. On the zero-divisor graphs of finite modules. Let R be a ring. For an R-module M we let $\Gamma = \Gamma_*({}_RM)$, $\Gamma({}_RM)$ or $\Gamma^*({}_RM)$. Of course, Γ may be infinite (i.e., a module may have an infinite number of weak zero-divisors, zero-divisors or strong zero-divisors). But Γ is probably of most interest when it is finite, for then one can draw Γ. In [**6**], Anderson and Livingston prove that $\Gamma(R)$ is finite if and only if either R is finite or an integral domain. In particular, if $1 \leq \Gamma(R) < \infty$ then R is finite and not a field. In this section, we give a generalization of this fact to modules.

Theorem 4.1. *Let M be an R-module. Then $\Gamma_*({}_RM)$ is finite if and only if either M is finite or a prime multiplication-like module. In particular, if $1 \leq |\Gamma_*({}_RM)| < \infty$, then M is finite and not a simple module.*

Proof. ($\Rightarrow$). Suppose that $\Gamma_*({}_RM)$ is finite. We proceed by cases.

Case 1. $\underline{\mathcal{A}}(M) \neq \{0\}$. Then there exists a $0 \neq x \in M$ such that $I_x = \mathrm{Ann}\,(M)$. Thus, $I_xI_yM = 0$ for all $0 \neq y \in M$, i.e., $\widetilde{Z}_*(M) = \widetilde{M}$ and so M is finite.

Case 2. $\underline{\mathcal{A}}(M) = \{0\}$. Then, by the comments before Theorem 2.4, M is multiplication-like.

Subcase 1. $\Gamma_*({}_RM) = \varnothing$. Then by Theorem 2.6, M is prime.

Subcase 2. $\Gamma_*({}_RM) \neq \varnothing$ (and finite). If M is finite, there is nothing to prove.

Suppose M is infinite. Since M is multiplication-like, for each nonzero submodule N of M, $\mathrm{Ann}\,(M/N) \neq 0$ (in particular, $I_x \neq 0$ for all $0 \neq x \in M$). Since $\Gamma_*({}_RM) \neq \varnothing$, $I_xI_yM = 0$ for some $x, y \in \widetilde{M}$. By Lemma 1.7, for each $0 \neq r \in R$, either $ry = 0$ or $x — ry$ is also a path in $\Gamma_*({}_RM)$. It follows that $Rx \subseteq Z_*(M)$. Thus, Rx is finite. Since $0 \neq I_xM \subseteq Rx$, I_xM is also finite. Let $a \in I_x$ such that $0 \neq aM$. Then aM is finite, and there exists an ideal J of R such that $0 \neq JM \subseteq aM$. If M is not finite, then there is an element $m_0 \in M$ with $T := \{m \in M \mid am_0 = am\}$ infinite. It follows that $N := \{m \in M \mid am = 0\}$ is a nonzero submodule of M and N is infinite. Since M is virtually multiplication, there is an ideal I of R such that $0 \neq IM \subseteq N$. Now let $0 \neq jm_1 \in JM$. Then $I_{jm_1}M \subseteq Rjm_1 \subseteq JM$ and so, for each $0 \neq m \in N$, $I_mI_{jm_1}M \subseteq I_mJM \subseteq I_maM = aI_mM \subseteq aN = 0$. Therefore, $N \subseteq Z_*(M) = \widetilde{Z}_*(M) \cup \{0\}$, a contradiction.

($\Leftarrow$). If M is finite, there is nothing to do, and if M is prime and multiplication-like, Theorem 3.13 does the job. $\square$

We conclude this paper with the following conjecture.

Conjecture 4.2. *Let M be an R-module. If $\Gamma^*({}_RM)$ is finite and nonempty, then M is finite.*

Acknowledgments. The author would like to thank the anonymous referee for a careful check of the details and for helpful comments that improved this paper.

REFERENCES

1. S. Akbari and H.R. Maimani, *Zero-divisor graphs of non-commutative rings*, J. Algebra **296** (2006), 462–479.

2. S. Akbari, H.R. Maimani and S. Yassemi, *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra **270** (2003), 169–180.

3. S. Akbari and A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra **274** (2004), 847–855.

4. D.D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra **159** (1993), 500–514.

5. D.F. Anderson, A. Frazier, A. Lauve and P.S. Livingston, *The zero-divisor graph of a commutative ring*, II, Lect. Notes Pure Appl. Math. **220**, Dekker, New York, 2001.

6. D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.

7. M. Axtell, J. Coykendall and J. Stickles, *Zero-divisor graphs of polynomials and power series over commutative rings*, Comm. Algebra **33** (2005), 2043–2050.

8. I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

9. M. Behboodi, *A generalization of the classical Krull dimension for modules*, J. Algebra **305** (2006), 1128–1148.

10. M. Behboodi and H. Koohy, *On minimal prime ideals*, Far East J. Math. Sci. **6** (2002), 83–90.

11. M. Behboodi, O.A.S. Karamzadeh and H. Koohy, *Modules whose certain ideals are prime*, Vietnam J. Math. **32** (2004), 303–317.

12. J. Dauns, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156–181.

13. F. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum **65** (2002), 206–214.

14. F. DeMeyer and K. Schneider, *Automorphisms and zero-divisor graphs of commutative rings*, Internat. J. Commutative Rings **1** (2002), 93–106.

15. Z.A. El-Bast and P.F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), 755–779.

16. K.R. Goodearl and R.B. Warfild, *An introduction to non-commutative Noetherian rings*, London Math. Soc. Student Texts **16**, Cambridge University Press, Cambridge, 1989.

17. A. Haghany and M.R. Vedadi, *Endoprime modules*, Acta Math. Hungar. **106** (2005), 89–99.

18. R. Levy and J. Shapiro, *The zero-divisor graph of von Neumann regular rings*, Comm. Algebra **30** (2002), 745–750.

19. S.B. Mulay, *Cycles and symmetries of zero-divisors*, Comm. Algebra **30** (2002), 3533–3558.

20. S.P. Redmond, *The zero-divisor graph of a non-commutative ring*, Internat. J. Commutative Rings **1** (2002), 203–211.

21. R. Wisbauer, *Modules and algebras*: *Bimodule structure and group action on algebras*, Pitman Mono. **81**, Addison-Wesley-Longman, Chicago, 1996.

22. ———, *Foundations of modules and rings theory*, Gordon and Breach, Reading, 1991.

23. T. Wu, *On directed zero-divisor graphs of finite rings*, Discrete Math. **296** (2005), 73–86.

Department of Mathematical Science, Isfahan University of Technology, P.O. Box 84156-83111, Isfahan, Iran and Institute for Studies in Theoretical Physics and Mathematics (IPM), P.O. Box 19395-5746, Tehran, Iran

Email address: mbehbood@cc.iut.ac.ir