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ARITHMETICAL RINGS SATISFY THE RADICAL FORMULA

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ABSTRACT. In this paper we prove that every arithmetical ring satisfies the radical formula.

1. Introduction. Throughout this article, rings are assumed to be commutative with unity and modules are assumed to be unitary. Let R be a ring and M an R-module. A proper submodule N of M is said to be a prime submodule of M if $ax \in N$ for $a \in R$ and $x \in M$ implies that either $aM \subseteq N$ or $x \in N$. In this case, $P = (N : M)$ is a prime ideal of R and N is said to be a P-prime submodule of M.

Let N be a proper submodule of M. The intersection of all prime submodules of M containing N is denoted by $\operatorname{rad}(N)$. If no prime submodule of M exists containing N, then $\operatorname{rad}(N)$ is defined to be M.

Also, for any subset N of M, the envelope of N, $E(N)$ is defined to be:

$$E(N) = \{x \mid x = ay, a^n y \in N, \text{ for some } a \in R, y \in M \text{ and } n \in \mathbf{N}\}.$$

In general $E(N)$ is not a submodule of M. It is clear that $\langle E(N) \rangle$, the submodule generated by $E(N)$, is contained in $\operatorname{rad}(N)$. M is said to satisfy the radical formula (M s.t.r.f.), if for every submodule N of M, $\langle E(N) \rangle = \operatorname{rad}(N)$. Furthermore, if every R-module satisfies the radical formula, then R is said to satisfy the radical formula.

A ring R is said to be an arithmetical ring if, for all ideals I, J and K of R, we have $I + (J \cap K) = (I + J) \cap (I + K)$. Obviously Prüfer domains and, in particular, Dedekind domains are arithmetical.

The question of what type of rings s.t.r.f. was considered in [**1, 4, 6–9**]. In [**1**], it was shown that every arithmetical ring with $\dim(R) \leq 1$

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satisfies the radical formula. Also, in [**3**], the authors proved that, for a Prüfer domain R, R^2 s.t.r.f. as an R-module. In this paper we show that every arithmetical ring satisfies the radical formula. Our result generalizes Theorem 2.1 and Theorem 2.3 (i), (ii) of [**2**].

2. Results. The following Lemma 2.1 is well known and can be easily proved.

Lemma 2.1. *Let R be a ring.*

i) *R s.t.r.f. if and only if, for every R-module M,* $\mathrm{rad}\,(0) \subseteq \langle E(0)\rangle$.

ii) *If R_P s.t.r.f. for every maximal ideal P of R, then the ring R satisfies the radical formula.*

Theorem 2.2. *A ring R is arithmetical if and only if, for each prime ideal P of R, every pair of ideals of the ring R_P is comparable.*

Proof. See [**5**, Theorem 1]. □

Lemma 2.3. *Let R be a local arithmetical ring and M an R-module. If, for some $s \in R$, $x \in M$ and $n \in \mathbf{N}$, $s^n x \in \langle E(0)\rangle$, then $sx \in \langle E(0)\rangle$.*

Proof. Let, if possible, $sx \notin \langle E(0)\rangle$. Since $s^n x \in \langle E(0)\rangle$, therefore $s^n x = \sum_{i=1}^m a_i y_i$ for some $a_i \in R$, $y_i \in M$ and $m \in \mathbf{N}$ such that $a_i^{k_i} y_i = 0$ for some $k_i \in \mathbf{N}$. Let $k = \max\{k_i | i = 1, 2, \dots, m\}$. Then $a_i^k y_i = 0$ for all $i = 1, 2, \dots, m$. We shall show by induction on m that $sx \in \langle E(0)\rangle$. By Theorem 2.2, every two ideals of R are comparable, then $\{Ra_i | i = 1, 2, \dots, m\}$ is a chain of ideals of R. Without loss of generality, we may suppose that Ra_1 is the minimal element of this chain. So $a_1^k y_i = 0$ for all $i = 1, 2, \dots, m$.

Assume, if possible, $s^n \in Ra_1$. Then $s^{n(k+1)} x = 0$, that is, $sx \in \langle E(0)\rangle$, a contradiction. Therefore, $s^n \notin Ra_1$. Now, by Theorem 2.2, $a_1 \in Rs^n$. So, $a_1 = s^n t_1$ for some $t_1 \in R$, which implies $s^n(x - t_1 y_1) = \sum_{i=2}^m a_i y_i$ and $(st_1)^{nk} y_1 = 0$, that is, $st_1 y_1 \in \langle E(0)\rangle$. Since $sx \notin \langle E(0)\rangle$ and $st_1 y_1 \in \langle E(0)\rangle$, then $s(x - t_1 y_1) \notin \langle E(0)\rangle$. Repeating the same argument we will get elements $t_i \in R$ such that $s^n(x - \sum_{i=1}^m t_i y_i) = 0$

and $st_iy_i \in \langle E(0)\rangle$ for all $i = 1, 2, \dots, m$, which gives $sx \in \langle E(0)\rangle$, a contradiction. Hence, $sx \in \langle E(0)\rangle$. □

Theorem 2.4. *Every arithmetical ring satisfies the radical formula.*

Proof. Let R be an arithmetical ring. By Lemma 2.1, we may assume that R is local and it is now enough to show that, for every R-module M we have $\mathrm{rad}\,(0) \subseteq \langle E(0)\rangle$. Consider $x \in \mathrm{rad}\,(0)$. We show that $x \in \langle E(0)\rangle$. Assume, if possible, $x \notin \langle E(0)\rangle$. Then, the ideal

$$I = \{a \in R \mid ax \in \langle E(0)\rangle\}$$

is a proper ideal of R. We show that I is a prime ideal of R. Suppose $ab \in I$, where $a, b \in R$. By Theorem 2.2, we may suppose that $a \in Rb$. Then $a^2 \in I$, that is, $a^2x \in \langle E(0)\rangle$. By Lemma 2.3, $ax \in \langle E(0)\rangle$, that is, $a \in I$. Hence, I is a prime ideal of R. Define

$$M(I) = \{z \in M \mid sz \in IM \text{ for some } s \in R \setminus I\}.$$

Then, by [**6**, Lemma 3.1], $\mathrm{rad}\,(0) \subseteq M(I)$. Therefore, $sx \in IM$ for some $s \in R \setminus I$. Since every pair of ideals of R is comparable, we have $sx = ay$ for some $a \in I$ and $y \in M$. Now $a \in I$, that is, $ax \in \langle E(0)\rangle$. So $a^2y = sax \in \langle E(0)\rangle$. Using Lemma 2.3, $sx = ay \in \langle E(0)\rangle$, that is, $s \in I$, which is a contradiction. Hence, $x \in \langle E(0)\rangle$. □

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