JOURNAL OF COMMUTATIVE ALGEBRA
Volume 4, Number 1, Spring 2012

# SEMI-LOCAL FORMAL FIBERS OF MINIMAL PRIME IDEALS OF EXCELLENT REDUCED LOCAL RINGS

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ABSTRACT. Let $T$ be a complete local (Noetherian) ring containing the rationals, and let $\mathcal{C}$ be a finite set of incomparable non-maximal prime ideals of $T$ partitioned into $m$ subsets $\mathcal{C}_1, \dots, \mathcal{C}_m$. We find necessary and sufficient conditions for $T$ to be the completion of an excellent reduced local ring $A$ with precisely $m$ minimal prime ideals $J_1, \dots, J_m$ such that, for each $i = 1, 2, \dots, m$, the maximal elements of the set $\{Q \in \operatorname{Spec} T \mid Q \cap A = J_i\}$ are precisely the elements of $\mathcal{C}_i$. We also find necessary and sufficient conditions for $T$ to be the completion of such a ring $A$ for the case where $A$ need not be excellent.

**1. Introduction.** Much research has been dedicated to elucidating the relationship between a local ring and its completion. A natural question is as follows: if $T$ is a complete local ring, does there exist a local subring $A$ of $T$, with some desired property, whose completion is $T$? In [**6**], Lech answered this question when the property on $A$ is that it be a local integral domain. In particular, Lech proved that a complete local ring $T$ is the completion of a local integral domain if and only if the following two conditions hold:

(1) the prime subring $I$ of $T$ is an integral domain, and $T$ has no torsion as an $I$-module;

(2) the maximal ideal of $T$ is either equal to $(0)$, or is not an associated prime ideal of $T$.

In [**4**], Heitmann proved a similar result for when the property on $A$ is that it be a unique factorization domain. Specifically, he showed the remarkable result that a complete local ring $T$ is the completion of a unique factorization domain if and only if $T$ is a field, a discrete valuation ring, or has depth at least two with no element of its prime subring a zerodivisor.

The authors thank the National Science Foundation for their support of this research via grant DMS-0850577.
Received by the editors on March 31, 2010, and in revised form on September 1, 2010.

DOI:10.1216/JCA-2012-4-1-29 Copyright ©2012 Rocky Mountain Mathematics Consortium

In this paper, we provide results when the property on $A$ is that it be an excellent reduced ring such that the formal fibers of its minimal prime ideals are semi-local and can be prescribed. We also consider the case where $A$ is not required to be excellent. Let $A$ be a local ring with maximal ideal $M$ and $T$ the $M$-adic completion of $A$. If $Q$ is a prime ideal of $A$, the formal fiber of $A$ at $Q$ is given by $\operatorname{Spec}(T \otimes_A k(Q))$, where $k(Q) = A_Q/QA_Q$. It is known that the formal fibers of a ring encode important information about the relationship between the ring and its completion. Note that there is a one-to-one correspondence between the formal fiber of $A$ at $Q$ and the inverse image of $Q$ under the surjective map $\operatorname{Spec} T \to \operatorname{Spec} A$, given by $P \mapsto P \cap A$. We will therefore (by abuse of notation) also refer to this subset of $\operatorname{Spec} T$ as the formal fiber of $A$ at $Q$. If this set has only finitely many maximal elements, we say that the formal fiber of $A$ at $Q$ is semi-local.

Past research on semi-local formal fibers has focused on formal fibers of height-one prime ideals (see, for example, [**2, 3**]), and of the zero ideal in the case when $A$ is a domain (see, for example, [**1, 8, 9**]). In [**1**], Charters and Loepp characterized all complete local rings which are completions of integral domains possessing a semi-local formal fiber at the prime ideal $(0)$. In particular, for any finite set of incomparable prime ideals $G$ of a complete local ring $T$, Charters and Loepp gave necessary and sufficient conditions for $T$ to be the completion of a local domain $A$ whose formal fiber at $(0)$ is semi-local with maximal ideals the elements of $G$. In the same paper, Charters and Loepp examined the case in which the domain $A$ is excellent, and proved a result when $T$ has characteristic zero.

In this paper, we generalize Charters' and Loepp's result in the following way. Let $T$ be a complete local ring with $n > 1$ minimal prime ideals, and let $m$ be a positive integer less than $n$. We find conditions guaranteeing that $T$ is the completion of an excellent reduced local ring $A$ with exactly $m$ minimal prime ideals, such that the formal fiber of each is semi-local with prescribed maximal elements. We also prove results in the case when the ring $A$ is not required to be excellent. To clarify what we mean by the maximal elements of the formal fibers of the minimal prime ideals of $A$ are "prescribed," suppose the ring $A$ has $m$ minimal prime ideals $J_1, J_2, \ldots, J_m$. For $i = 1, 2, \ldots, m$, let $\mathcal{C}_i$ be a finite set of prime ideals of $T$. We want to construct $A$ so that, for every $i = 1, 2, \ldots, m$, the formal fiber of $J_i$ is semi-local with maximal elements the elements of $\mathcal{C}_i$. In other words, we want to have complete control of the formal fibers of the prime ideals $J_1, J_2, \ldots, J_m$. It is not hard to see that there are necessary conditions on the sets $\mathcal{C}_i$ for such an $A$ to exist. To specify these conditions, we introduce the following definition.

**Definition 1.1.** Let $T$ be a complete local ring. Let $\mathcal{C} = \{P_1, \ldots, P_n\}$ be a finite collection of incomparable non-maximal prime ideals of $T$, and let $\mathcal{C}$ be partitioned into $m \geq 2$ subcollections $\mathcal{C}_1, \ldots, \mathcal{C}_m$. We call $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ a *feasible partition on* $\mathcal{C}$ (or simply a *feasible partition*) if, for each $Q$ in $\operatorname{Ass} T$, $\mathcal{P}$ satisfies the following conditions:

(1) $Q \subseteq P_i$ for at least one $P_i \in \mathcal{C}$;

(2) There exists exactly one $\ell$ such that whenever $Q \subseteq P_i$, $P_i \in \mathcal{C}_\ell$.

Hence, the subcollections $\mathcal{C}_i$ partition not only the elements of the collection $\mathcal{C}$, but also the associated prime ideals of $T$. It is easy to see that the set $\mathcal{C} = \cup_{j=1}^m \mathcal{C}_j$ must be a finite set of incomparable, non-maximal prime ideals of $T$ such that the sets $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ form a feasible partition on $\mathcal{C}$ for a reduced ring $A$ to exist such that, for every $i = 1, 2, \ldots, m$, the formal fiber of $J_i$ is semilocal with maximal ideals the elements of $\mathcal{C}_i$.

Note that we can find a feasible partition for any complete local ring $T$. In particular, we can define a feasible partition simply by letting $\mathcal{C}$ be the maximal elements of $\operatorname{Ass}(T)$, and partitioning $\mathcal{C}$ into disjoint subcollections satisfying the condition that if $Q, P, P' \in \operatorname{Ass}(T)$ with $Q \subseteq P$ and $Q \subseteq P'$, then $P, P' \in \mathcal{C}_i$ for some $i$. Charters and Loepp studied the case where the elements of $\mathcal{C}$ are grouped into a single class. For most complete local rings, there are many feasible partitions.

As we know that we must have a feasible partition to construct our reduced local ring $A$, we now pose the following specific question: let $T$ be a complete local ring of dimension at least one which contains the rationals, and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. Under what conditions is $T$ the completion of an excellent reduced local subring $A$ such that $A$ has exactly $m$ minimal prime ideals $\{J_1, \ldots, J_m\}$, and the formal fiber of each $J_i$ is semi-local with maximal ideals the elements of $\mathcal{C}_i$? We are able to answer this question definitively. In Theorem 4.4, we prove that such a subring exists if and only if the following conditions hold:

(1) $T$ is reduced;

(2) If $Q_i$ denotes the intersection of all minimal prime ideals contained in the elements of $\mathcal{C}_i$, then $(T/Q_i)_{\overline{P}}$ is a regular local ring for all $i = 1, 2, \ldots, m$ and every $P \in \mathcal{C}_i$;

(3) If $Q_i$ denotes the intersection of all minimal prime ideals contained in the elements of $\mathcal{C}_i$, then $T/Q_i$ is equidimensional for every $i = 1, 2, \ldots, m$.

If, moreover, $A$ is not required to be excellent, we show in Theorem 3.12 that no conditions on $T$ are needed: such a subring exists for any feasible partition $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$.

In our construction of $A$, we not only control the formal fibers of the minimal prime ideals of $A$. Indeed, we control the formal fiber of *every* prime ideal of $A$: we ensure that the formal fiber of every minimal prime ideal is semi-local, and that the formal fiber of any other prime ideal of $A$ has a single element. This property of $A$ is striking because, for most local rings, the formal fibers of the prime ideals are complex and difficult to characterize.

We base our construction on the approach of Loepp in [**10**] and of Heitmann in [**5**]. We begin with $\mathbf{Q}$, which is simply the prime subring of $T$ localized at $(0)$. We then successively adjoin elements of $T$ to our ring, to produce the desired ring $A$.

Note that, given the finite collection of incomparable non-maximal prime ideals $\mathcal{C}$ of $T$ and the feasible partition $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ on $\mathcal{C}$, we must construct $A$ so that, whenever $P \in \mathcal{C}_i$ and $P' \in \mathcal{C}_j$, we have $P \cap A = P' \cap A$ if and only if $i = j$. We also make sure to adjoin elements of $T$ so that for every prime ideal $P$ of $T$ such that $P \not\subseteq P'$ for all $P' \in \mathcal{C}$, $P \cap A$ contains a non-zerodivisor. This will guarantee that no prime ideals of $T$ outside our feasible partition are in the formal fiber of a minimal prime ideal in $A$. Furthermore, we will adjoin elements of $T$ so that our ring contains a nonzero element of every coset in $T/J$, where $J$ is an ideal of $T$ such that $J \not\subseteq P$ for all $P \in \mathcal{C}$. Thus, the map $A \to T/J$ is onto for all ideals $J$ of $T$ such that $J \not\subseteq P$ for all $P \in \mathcal{C}$. In particular, this implies that the map $A \to T/M^2$ is onto. In Lemma 3.8 we adjoin elements of $T$ to make $IT \cap A = I$ for every finitely generated ideal $I$ of $A$. This, along with the condition that the map $A \to T/M^2$ is onto, establishes that $A$ is Noetherian and that the completion of $A$ is $T$.

Throughout this paper, all rings are commutative with unity. Local rings are always Noetherian, whereas quasi-local rings may be non-Noetherian. We write $(R, M)$ is a quasi-local ring if $R$ is quasi-local with maximal ideal $M$. We then denote the $M$-adic completion of $R$ by $\widehat{R}$.

## 2. Preliminaries and definitions.

The following result, which is Proposition 1 from [**5**], gives conditions which imply that a quasi-local subring $R$ of a complete local ring $T$ is Noetherian, and that the completion of $R$ is $T$. This proposition will play a crucial role in our construction.

**Proposition 2.1.** *If $(R, M \cap R)$ is a quasi-local subring of a complete local ring $(T, M)$, the map $R \to T/M^2$ is onto, and $IT \cap R = I$ for every finitely generated ideal $I$ of $R$, then $R$ is Noetherian and the natural homomorphism $\widehat{R} \to T$ is an isomorphism.*

We recall the definition of a feasible partition.

**Definition 2.2.** Let $(T, M)$ be a complete local ring. Let $\mathcal{C} = \{P_1, \dots, P_n\}$ be a finite collection of incomparable non-maximal prime ideals of $T$, and let $\mathcal{C}$ be partitioned into $m \geq 2$ subcollections $\mathcal{C}_1, \dots, \mathcal{C}_m$. We call $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ a *feasible partition on* $\mathcal{C}$ (or simply a *feasible partition*) if, for each $Q$ in $\mathrm{Ass}\,(T)$, $\mathcal{P}$ satisfies the following conditions:

(1) $Q \subseteq P_i$ for at least one $P_i \in \mathcal{C}$;

(2) There exists exactly one $\ell$ such that whenever $Q \subseteq P_i$, $P_i \in \mathcal{C}_\ell$.

The relationship between the subcollection $\mathcal{C}_i$ and the minimal prime ideals of $T$ provides the motivation for the following definition.

**Definition 2.3.** Let $(T, M)$ be a complete local ring of dimension at least one, and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. We define the sets $\mathcal{E}_i$ by

$$\mathcal{E}_i = \{Q \in \operatorname{Min} T \mid Q \subseteq P \text{ for some } P \in \mathcal{C}_i\}.$$

Note that feasible partitions on $T$ are easy to construct. It suffices to ensure that, whenever two prime ideals $P, P'$ of $\mathcal{C}$ each contain a minimal prime ideal $Q$, they are placed in the same $\mathcal{C}_i$. The following examples help to clarify the concept of a feasible partition.

**Example 2.4.** Let

$$T = \frac{\mathbf{R}[[x, y, z]]}{\langle xyz \rangle}.$$

Suppose $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^2)$ is defined by $\mathcal{C}_1 = \{\langle x, y + z\rangle, \langle y, x\rangle\}$, $\mathcal{C}_2 = \{\langle z\rangle\}$. Then $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^2)$ is a feasible partition, with $\mathcal{E}_1 = \{\langle x\rangle, \langle y\rangle\}$ and $\mathcal{E}_2 = \{\langle z\rangle\}$.

**Example 2.5.** Let

$$T = \frac{\mathbf{R}[[x, y, z, w]]}{\langle x^\alpha y^\beta z^\gamma w^\delta \rangle},$$

with $\alpha, \beta, \gamma, \delta \in \mathbf{N}$. Let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^2)$ be defined by $\mathcal{C}_1 = \{\langle x, y\rangle, \langle x, z\rangle\}$ and $\mathcal{C}_2 = \{\langle z, w\rangle\}$. Then $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^2)$ is *not* a feasible partition, because the prime ideal $\langle z\rangle \in \operatorname{Min}(T)$ is contained in both $\mathcal{E}_1 = \{\langle x\rangle, \langle y\rangle, \langle z\rangle\}$ and $\mathcal{E}_2 = \{\langle z\rangle, \langle w\rangle\}$.

We will use feasible partitions to guide the construction of our subring $A$. In particular, we will ensure that, for a given $i$, the intersection of $A$ with any $P \in \mathcal{C}_i$ or $Q \in \mathcal{E}_i$ is the same minimal prime ideal of $A$, and that this ideal is distinct for each $i$. We use a slightly stronger condition during our construction, as given in the following definition.

**Definition 2.6.** Let $T$ be a complete local ring, $\mathcal{C}$ a finite set of incomparable non-maximal prime ideals of $T$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ a feasible partition on $\mathcal{C}$. A quasi-local subring $(R, R \cap M)$ of $T$ is called an *intersection preserving subring* (abbreviated IP-subring) if the following conditions hold:

(1) $R$ is infinite;

(2) For any $P \in \mathcal{C}$, $R \cap P = R \cap Q$ for any $Q \in \mathrm{Min}\,(T)$ satisfying $Q \subseteq P$;

(3) For $P, P' \in \mathcal{C}$, $P, P' \in \mathcal{C}_i$ if and only if $R \cap P = R \cap P'$;

(4) For each $P \in \mathcal{C}$, $r \in P \cap R$ implies $\mathrm{Ann}_T(r) \not\subseteq P$.

The ring $R$ is called *small intersection preserving* (abbreviated SIP-subring) if, additionally, $|R| < |T|$.

*Remark* 2.7. In what follows, let $(T, M)$ be a complete local ring of dimension at least one which contains the rationals. Let $\mathcal{C}$ be a finite set of incomparable non-maximal ideals of $T$. Let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition, and let $R$ be an IP-subring of $T$. Let $P \in \mathcal{C}$; then $P \cap R$ is a prime ideal of $R$, and $P \in \mathcal{C}_i$ for some $i$. Abusing notation, we denote $P \cap R$ by $\mathcal{C}_i \cap R$. This abuse of notation makes sense because, if $P, P' \in \mathcal{C}_i$, then $P \cap R = P' \cap R$.

In our construction, we create an IP-subring $A$ whose completion is $T$; furthermore, each of the prime ideals $\mathcal{C}_i \cap A$ of $A$ is, in fact, minimal. Note that, if $T$ contains the rationals, then the prime subring localized at (0) is $\mathbf{Q}$, which trivially satisfies conditions (1), (2) and (4) of Definition 2.6, and one direction of condition (3): that is, for any $P, P' \in \mathcal{C}$, if $P, P' \in \mathcal{C}_i$, then $\mathbf{Q} \cap P = \mathbf{Q} \cap P'$. We use this fact to construct an SIP-subring $R$ of $T$. We then successively adjoin elements to $R$, creating an IP-subring at each step, until the conditions of Proposition 2.1 are satisfied.

Note that any subring of $T$ satisfying condition (4) is reduced, as the following lemma (modified from Lemma 5 of [**7**]) establishes.

**Lemma 2.8.** *Let $T$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark* 2.7. *Let $R$ be a subring of $T$ such that, for each $P \in \mathcal{C}$, if $r \in P \cap R$, then $\mathrm{Ann}_T(r) \not\subseteq P$. Then $R$ is reduced.*

*Proof.* First note that, for any nonzero $t \in T$, $\mathrm{Ann}\,(t) \subseteq Q$ for some $Q \in \mathrm{Ass}\,(T)$. Then, since $Q \subseteq P$ for some $P \in \mathcal{C}$, $\mathrm{Ann}\,(t) \subseteq P$.

Now, suppose that $r \in R$ is nilpotent and nonzero. Let $\ell \in \mathbf{N}$ be the smallest positive integer such that $r^\ell = 0$. By the above, $\mathrm{Ann}_T(r^{\ell-1}) \subseteq P$ for some $P \in \mathcal{C}$.

In particular, $r \in P$ since $r \in \mathrm{Ann}_T(r^{\ell-1})$. By hypothesis there exists an $s \notin P$ such that $sr = 0$. But then $s \in \mathrm{Ann}_T(r^{\ell-1}) \subseteq P$, a contradiction. □

Hence, in particular, an IP-subring is reduced.

## 3. The construction.

We begin the construction with a few technical lemmata. First, we prove that the properties of SIP-subrings are preserved under unions of nested subrings, and under localization.

**Lemma 3.1** (Unioning lemma). *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark* 2.7. *Let $B$ be a well-ordered index set, and let $R_\beta$, $\beta \in B$, be a family of* SIP*-subrings such that if $\beta, \gamma \in B$ such that $\beta < \gamma$, then $R_\beta \subseteq R_\gamma$. Then $R = \cup_{\beta \in B} R_\beta$ is an* IP*-subring. Moreover, if there exists some $\lambda < |T|$ such that $|R_\beta| \leq \lambda$ for all $\beta$, and $|B| < |T|$, then $|R| \leq \max\{\lambda, |B|\}$, and $R$ is an* SIP*-subring of $T$.*

*Proof.* Since each $R_\beta$ is quasi-local with maximal ideal $M \cap R_\beta$, we have that $R$ is quasi-local with maximal ideal $R \cap M$. Since $R_\beta$ is infinite, $R$ is an infinite subring of $T$.

Let $I, J$ be ideals of $T$ such that $I \cap R_\beta = J \cap R_\beta$ for each $\beta \in B$. Let $a \in R \cap I$. Then $a \in R_\beta \cap I$ for some $\beta \in B$. Hence, $a \in R_\beta \cap J$ and $a \in R \cap J$. Thus, $I \cap R \subseteq J \cap R$. By a similar argument, the reverse containment holds, and $I \cap R = J \cap R$.

Let $P, P' \in \mathcal{C}_i$ for some $\mathcal{C}_i \subseteq \mathcal{C}$. Then $P \cap R_\beta = P' \cap R_\beta$ for all $\beta \in B$, because the $R_\beta$ are IP-subrings. By the above argument, $P \cap R = P' \cap R$. Similarly, $P \cap R = Q \cap R$ for every $Q \in \mathrm{Min}\,(T)$ such that $Q \subseteq P$.

Next, let $P \in \mathcal{C}_i$, and $P' \in \mathcal{C}_j$, where $i \neq j$. Let $\beta \in B$. Then $R_\beta$ is an IP-subring, so $P \cap R_\beta \neq P' \cap R_\beta$. Without loss of generality, $P \cap R_\beta \not\subseteq P' \cap R_\beta$. Hence, there exists some $b \in P \cap R_\beta$ such that $b \notin P'$. Therefore, $b \in R \cap P$ but $b \notin P'$. Thus, $P \cap R \neq P' \cap R$.

Now let $P \in \mathcal{C}$, and let $r \in P \cap R$. Then $r \in P \cap R_\beta$ for some $\beta \in B$. Since $R_\beta$ is an IP-subring, $\mathrm{Ann}_T(r) \not\subseteq P$. Hence, condition (4) of Definition 2.6 is maintained.

Finally, suppose there exists some $\lambda < |T|$ such that $|R_\beta| \leq \lambda$ for all $\beta$, and $|B| < |T|$. Then $|R| \leq \lambda |B| = \max\{\lambda, |B|\}$. Since $\lambda, |B| < |T|$, we have $|R| < |T|$ and so $R$ is an SIP-subring of $T$. □

**Lemma 3.2** (Localization lemma). *Let* $(T, M)$ *and* $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ *be as in Remark* 2.7. *Let* $R$ *be a subring of* $T$ *satisfying all conditions for an* IP*-subring except that it need not be quasi-local. Then* $R_{(R\cap M)}$ *is an* IP*-subring of* $T$ *with* $|R_{(R\cap M)}| = |R|$. *Additionally, if* $|R| < |T|$, *then* $R_{(R\cap M)}$ *is an* SIP*-subring of* $T$.

*Proof.* Since $M$ is the maximal ideal of $T$, and $M$ is a prime ideal, $R\cap M$ is a prime ideal of $R$, and each element of $R-(M\cap R)$ is a unit in $T$. Thus, $R_{(M\cap R)}$ is well-defined, and we can embed $R_{(M\cap R)}$ as a subring of $T$. Since $R$ is infinite, $R_{(M\cap R)}$ is infinite as well.

Let $P \in \mathcal{C}$ be given. Then, for any $Q \in \mathrm{Min}\,(T)$ such that $Q \subseteq P$, $(P\cap R)R_{(M\cap R)} = (Q\cap R)R_{(M\cap R)}$. Similarly, for any $P, P' \in \mathcal{C}$, $P\cap R = P'\cap R$ if and only if $(P\cap R)R_{(M\cap R)} = (P'\cap R)R_{(M\cap R)}$. Since $R$ satisfies condition (3) for an IP-subring, $(P \cap R)R_{(M\cap R)} = (P' \cap R)R_{(M\cap R)}$ if and only if $P, P' \in \mathcal{C}_i$ for some subcollection of prime ideals $\mathcal{C}_i$.

Now, let $r \in (P\cap R)R_{(M\cap R)}$. Then $r = p/s$, for some $p \in P\cap R$, and some $s \in R-(M\cap R)$. Since $R$ satisfies condition (4) for an IP-subring, $\mathrm{Ann}_T(p) \not\subseteq P$, and there exists $q \notin P$ such that $qp = 0$. Hence, $qr = 0$, and $\mathrm{Ann}_T(r) \not\subseteq P$. It follows that $R_{(R\cap M)}$ is an IP-subring of $T$.

Since $R$ is infinite, $|R_{(R\cap M)}| = |R|$. If $|R| < |T|$, it follows that $R_{(R\cap M)}$ is an SIP-subring of $T$. □

The next several lemmata will allow us to find and adjoin elements to create a nested chain of SIP-subrings whose union satisfies the conditions of Proposition 2.1. Note that if $R$ is an SIP-subring and $u \in T$, then $R[u]_{(M\cap R[u])}$ automatically satisfies condition (1) of Definition 2.6, and the backwards direction of condition (3). In order to show that $R[u]_{(M\cap R[u])}$ is an SIP-subring of $T$, we must maintain condition (2), (4), and the forward direction of (3).

We first show that, given a subring $R$, adjoining elements that are transcendental over $R/(P\cap R)$ for prime ideals $P$ maintains various properties of IP-subrings. Most notably, Lemma 3.3 implies that, if $R$ is an SIP-subring and $u + P \in T/P$ is transcendental over $R/(P\cap R)$ for all $P \in \mathcal{C} \cup \mathrm{Min}\,(T)$, then $R[u]_{(M\cap R[u])}$ is also an SIP-subring.

**Lemma 3.3** (Adjoining lemma). *Let* $R$ *be a subring of a complete local ring* $T$. *Let* $P_1, P_2$ *be prime ideals of* $T$ *such that* $P_1\cap R = P_2\cap R$.

*Suppose that, for $i = 1, 2$, $u + P_i \in T/P_i$ is transcendental over $R/(P_i \cap R)$. Then $P_1 \cap R[u] = P_2 \cap R[u]$. Furthermore, if $\mathrm{Ann}_T(p) \not\subseteq P_1$ for all $p \in R \cap P_1$, then $\mathrm{Ann}_T(p) \not\subseteq P_1$ for all $p \in R[u] \cap P_1$.*

*Proof.* Let $f \in R[u] \cap P_1$. Then, for $0 \le i \le n$, there exists an $r_i \in R$ such that

$$f = r_n u^n + \cdots + r_1 u + r_0 \in P_1.$$

Consider the coset $f + P_1$ in $T/P_1$. Then

$$\begin{aligned} f + P_1 &= (r_n + P_1)(u + P_1)^n + (r_{n-1} + P_1)(u + P_1)^{(n-1)} + \cdots \\ &\quad + (r_0 + P_1) \\ &= 0 + P_1. \end{aligned}$$

Since $u + P_1$ is transcendental over $R/(P_1 \cap R)$, we know that $r_i \in P_1$ for each $i$. Since $P_1 \cap R = P_2 \cap R$, $r_i \in P_2$ for all $i$, and so $f \in P_2$. Hence, $P_1 \cap R[u] \subseteq P_2 \cap R[u]$. Reverse inclusion follows by a similar argument, so $P_1 \cap R[u] = P_2 \cap R[u]$.

Again, let $f \in R[u] \cap P_1$. Then, as shown above, $f = r_n u^n + \cdots + r_1 u + r_0 \in P_1$, for some $r_i \in P_1 \cap R$. Hence, there exist $s_1, \ldots, s_n \in T$ such that, for each $i = 1, \ldots, n$, $s_i r_i = 0$ and $s_i \notin P_1$. Let

$$s = \prod_{i=1}^{n} s_k.$$

Then $s \notin P_1$, and $sr_i = 0$ for $i = 1, \ldots, n$. Therefore, $sf = 0$, and so $\mathrm{Ann}_T(f) \not\subseteq P_1$. □

The following is Lemma 2.4 from [**1**]. Together with the succeeding lemma, Lemma 3.4 will allow us to find transcendental elements.

**Lemma 3.4.** *Let $(T, M)$ be a complete local ring of dimension at least one, let $C$ be a finite set of incomparable non-maximal prime ideals of $T$, and $D$ a subset of $T$ such that $|D| < |T|$. Let $I$ be an ideal of $T$ such that $I \not\subseteq P$ for all $P \in C$. Then $I \not\subseteq \cup\{r + P \mid P \in C, r \in D\}$.*

**Lemma 3.5.** *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark* 2.7. *Let $R$ be a subring of $T$ such that $|R| < |T|$. Let $J$ be an ideal of $T$*

*such that $J \not\subseteq P$ for every $P \in \mathcal{C}$. Let $t, q \in T$. Then there exists an element $t' \in J$ such that, for every $P \in \mathcal{C}$ with $q \notin P$, $t+qt'+P \in T/P$ is transcendental over $R/(P \cap R)$. If, in addition, $Q \in \operatorname{Min}(T)$, $P \in \mathcal{C}$ with $Q \subseteq P$, $q \notin P$, and $R \cap P = R \cap Q$, then $t + qt' + Q \in T/Q$ is transcendental over $R/(Q \cap R)$.*

*Proof.* Let $\mathcal{G} = \{P \in \mathcal{C} \mid q \notin P\}$. Then $\mathcal{G}$ is a finite set of incomparable non-maximal prime ideals of $T$. Suppose that $t+qt'+P = t+qs'+P$ for some $P \in \mathcal{G}$. Then $(t+qt') - (t+qs') = q(t'-s') \in P$. But $q \notin P$, so $(t'-s') \in P$. These steps are reversible, so $t+qt'+P = t+qs'+P$ if and only if $t'+P = s'+P$.

For each $P \in \mathcal{G}$, let $D_{(P)}$ be a full set of coset representatives of the cosets $t'+P$ that make $t+qt'+P \in T/P$ algebraic over $R/(P \cap R)$. Let $D = \cup_{P \in \mathcal{G}} D_{(P)}$. Then $|D| = |D_{(P)}| = |R/(P \cap R)| \leq |R| < |T|$ for every $P \in \mathcal{G}$. Now use Lemma 3.4 with $I = J$ and $C = \mathcal{G}$ to show that there exists an element $t' \in J$ such that $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$ for every $P \in \mathcal{G}$. Then we have that for every $P \in \mathcal{C}$ with $q \notin P$, $t+qt'+P \in T/P$ is transcendental over $R/(P \cap R)$. Now suppose $Q \in \operatorname{Min}(T)$, $P \in \mathcal{C}$ with $Q \subseteq P$, $q \notin P$, and $R \cap P = R \cap Q$. Then, $t+qt'+P \in T/P$ is transcendental over $R/(P \cap R)$. Since $P \cap R = Q \cap R$ we have $t + qt' + Q \in T/Q$ is transcendental over $R/(Q \cap R)$ as well. □

**Corollary 3.6.** *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 2.7, and let $J$ be an ideal of $T$ such that $J \not\subseteq P$ for every $P \in \mathcal{C}$. Let $R$ be an SIP-subring of $T$ and $t + J \in T/J$. Then there exists an SIP-subring $S$ of $T$ such that $R \subseteq S \subset T$, $t + J$ is in the image of the map $S \to T/J$, and $|S| = |R|$. Moreover, if $t \in J$, then $S \cap J$ contains a non-zerodivisor of $T$.*

*Proof.* Apply Lemma 3.5 with $q = 1$. Then $q \notin P$ for every $P \in \operatorname{Spec} T$, so it is possible to choose $t' \in J$ such that $t+t'+P \in T/P$ is transcendental over $R/(P \cap R)$ for every $P \in \mathcal{C} \cup \operatorname{Min}(T)$. Consider the ring $S = R[t+t']_{(M \cap R[t+t'])}$. By Lemma 3.3, $R[t+t']$ satisfies conditions (2), (3) and (4) of being an IP-subring. Further, $|R[t+t']| = |R|$. By Lemma 3.2, $S$ is an SIP-subring of $T$, and $|S| = |R|$. Further, $(t+t') \in S$ and $(t+t') + J = t + J$, so $t + J$ is in the image of the map $S \to T/J$.

Suppose $t \in J$ and $t + t'$ is a zerodivisor. Then $t + t' \in Q$ for some $Q \in \mathrm{Ass}\,(T)$. However, $Q \subseteq P$ for some $P \in \mathcal{C}$, and so $(t + t') + P = 0 + P$. Hence, $t + t' + P \in T/P$ is algebraic over $R/(P \cap R)$, a contradiction. Thus, $t + t'$ is a non-zerodivisor contained in $S \cap J$. □

The following lemma is the heart of our construction. Given an SIP-subring $R$ of $T$, repeated application of Lemma 3.7 will enable us to construct an SIP-subring $S \supseteq R$ of $T$ which satisfies $IT \cap S = I$ for all finitely generated ideals $I$ of $S$. This condition is necessary to satisfy the hypotheses of Proposition 2.1.

**Lemma 3.7.** *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 2.7. Let $R$ be an* SIP*-subring of $T$. Then, for any finitely-generated ideal $I$ of $R$ and any $c \in IT \cap R$, there exists a subring $S$ of $T$ with the following properties*:

(1) $R \subseteq S$;

(2) $S$ is an SIP-subring of T;

(3) $|S| = |R|$;

(4) $c \in IS$.

*Proof.* We shall proceed inductively on the number of generators of $I$. First suppose $I = aR$. If $a = 0$, then $S = R$ is the desired subring. Assume $a \neq 0$, and let $c = at$ for some $t \in T$. Note that, because $a \in R$, $a$ is in some $P \in \mathcal{C}_i$ if and only if $a$ is in every $P \in \mathcal{C}_i$. If this is the case, then, abusing notation, we shall refer to $a$ as being contained in $\mathcal{C}_i$.

By condition (4) of the definition of IP-subrings, $\mathrm{Ann}_T(a) \not\subseteq P$ for all $P \in \mathcal{C}$ such that $a \in P$. By the Prime Avoidance theorem, this means that $\mathrm{Ann}_T(a) \not\subseteq \cup_{a \in P, P \in \mathcal{C}} P$. Thus, we can choose some $q \in \mathrm{Ann}_T(a)$ such that $q \notin P$ for all $P \in \mathcal{C}$ such that $a \in P$. If $a \notin P$ for every $P \in \mathcal{C}$, we let $q = 0$. By Lemma 3.5, there exists some $t' \in T$ such that, for each $P \in \mathcal{C}$ with $a \in P$, the coset $t + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$. Let $u = t + qt'$. We claim that $S = R[u]_{(R[u] \cap M)}$ is the desired subring. By Lemma 3.2 (the Localization lemma), it suffices to show that $R[u]$ satisfies conditions (1), (2), (3) and (4) of being an

SIP-subring, and that $|R[u]| = |R|$. Condition (1) of Definition 2.6 follows immediately. We now show that condition (3) holds for $R[u]$.

For any $\mathcal{C}_i$ containing $a$, if $P, P' \in \mathcal{C}_i$, then $R[u] \cap P = R[u] \cap P'$ by Lemma 3.3 (the Adjoining lemma). Next, consider any $\mathcal{C}_i$ not containing $a$. Let $P, P' \in \mathcal{C}_i$, and $f \in R[u] \cap P$. Then

$$f = r_n u^n + \cdots + r_1 u + r_0$$

for some $r_i \in R$. Multiplying both sides by $a^n$, we get

$$a^n f = r_n c^n + \cdots + a^{n-1} r_1 c + a^n r_0 \in R \cap P$$

since $au = at = c \in R$. Because $R$ is an SIP-subring, $a^n f \in R \cap P$ implies $a^n f \in P'$. However, by hypothesis $a \notin P'$ and so $f$ must be in $P'$. Consequently $f \in R[u] \cap P'$. Reverse inclusion follows by a similar argument, and so $R[u] \cap P = R[u] \cap P'$. Condition (2) of Definition 2.6 follows for $R[u]$ by a similar argument.

We will now show that condition (4) holds for $R[u]$. For each $P \in \mathcal{C}$, consider $f \in R[u] \cap P$, so that $f = r_n u^n + \cdots + r_1 u + r_0$. If $a \in P$, $u + P \in T/P$ is transcendental over $R/(P \cap R)$, so each $r_i \in P \cap R$. By assumption, for each $r_i$ there exists a $q_i \notin P$ such that $r_i q_i = 0$. Let $q = \prod q_i \notin P$, and note that $fq = 0$. Thus, $\operatorname{Ann}_T(f) \not\subseteq P$. If $a \notin P$, recall that $a^n f \in R \cap P$. By assumption, there exists a $q \notin P$ such that $qa^n f = 0$. Note that $qa^n \notin P$, so $\operatorname{Ann}_T(f) \not\subseteq P$, and condition (4) holds. Hence, $R[u]_{(R[u] \cap M)}$ is an SIP-subring. Finally, observe that $|R[u]_{(R[u] \cap M)}| = |R|$ and $c \in aR[u]_{(R[u] \cap M)}$, as desired. So the lemma holds if $I$ is generated by a single element.

Continuing inductively, suppose that the lemma holds when $I$ is generated by $k - 1$ elements where $k \geq 2$. Let $I = (a_1, \ldots, a_k)R$ and $c = a_1 t_1 + a_2 t_2 + \cdots + a_k t_k \in R$ for some $t_i \in T$. We will first show that the lemma follows in the case where

$$(*) \qquad \{\mathcal{C}_i \mid a_1 \in \mathcal{C}_i\} = \{\mathcal{C}_j \mid a_2 \in \mathcal{C}_j\}$$

We will then prove that it is always possible to define a generating set for $I$ such that $(*)$ holds, completing the proof.

Assume that $(*)$ holds. Taking $a = a_1$, define $q$ as in the principal case, and note that $a_1 q = 0$. Thus, $c$ can be rewritten as

$$c = a_1(t_1 + qt' + a_2 t'') + a_2(t_2 - a_1 t'') + a_3 t_3 + \cdots + a_k t_k$$

for any $t', t'' \in T$. Let $u = t_1 + qt' + a_2t''$. We will choose $t', t''$ such that $u + P \in T/P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$, allowing us to create an SIP-subring $R[u]_{(R[u]\cap M)}$.

Use Lemma 3.5 to find $t'$ such that, for each $P \in \mathcal{C}$ with $q \notin P$, $t_1 + qt' + P \in T/P$ is transcendental over $R/(R \cap P)$. If $q \in P$ for all $P \in \mathcal{C}$, let $t' = 0$. By our choice of $q$ and the assumption that $(*)$ holds, each $P$ in $\mathcal{C}$ contains precisely one of $q$ and $a_2$. Thus, if $P \in \mathcal{C}$ is such that $q \notin P$, then $u + P = t_1 + qt' + a_2t'' + P = t_1 + qt' + P \in T/P$ is transcendental over $R/(P \cap R)$ regardless of the choice of $t''$. Now, if $P \in \mathcal{C}$ is such that $q \in P$, then $a_2 \notin P$, and so we can use Lemma 3.5 to find $t'' \in T$ such that $t_1 + a_2t'' + P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$ satisfying $a_2 \notin P$. If $a_2 \in P$ for all $P \in \mathcal{C}$, then let $t'' = 0$. By our choice of $t'$ and $t''$, $u + P$ is transcendental over $R/(P \cap R)$ for all $P \in \mathcal{C}$. By Lemma 3.3, $R[u]$ satisfies condition (3) of Definition 2.6. Using an identical argument to the principal case, $R[u]$ satisfies condition (4). It clearly satisfies conditions (1) and (2), and $|R[u]| = |R|$. By Lemma 3.2, $R' = R[u]_{(R[u]\cap M)}$ is an SIP-subring of $T$ with $|R'| = |R|$.

Now let $J = (a_2, a_3, \dots, a_k)R'$ and

$$c^* = c - a_1u = a_2(t_2 - a_1t'') + a_3t_3 + \cdots + a_kt_k.$$

We have $c \in R \subseteq R'$ and $a_1u \in R'$, so $c^* \in JT \cap R'$. By our inductive hypothesis, there exists an SIP-subring $S$ of $T$ containing $R'$ such that $c^* \in JS$, so $c^* = a_2s_2 + \cdots + a_ks_k$ for some $s_i \in S$. It follows that $c = a_1u + a_2s_2 + \cdots + a_ks_k \in IS$, so $S$ is the desired SIP-subring.

We will now show that, given a set of generators $(a_1, a_2, \dots, a_k)$ for $I$, it is always possible to create a new set of generators for $I$ that satisfy $(*)$. Our set of generators will be of the form $(a_1 + \ell a_2, a_1 - \ell a_2, a_3, \dots, a_k)$ where $\ell \in \mathbf{N}\backslash\{0\}$. Because $T$ contains $\mathbf{Q}$, these elements generate $I$. We will choose $\ell$ such that $(*)$ holds.

First note that $\ell$ is a unit, so $\ell a_2 \in P_i$ if and only if $a_2 \in P_i$ for any prime ideal $P_i$ in $T$. It follows that $\ell a_2 \in \mathcal{C}_i$ if and only if $a_2 \in \mathcal{C}_i$. Next, note that, for each $\mathcal{C}_i$, $\mathcal{C}_i \cap R$ is an ideal of $R$. It follows that if $a_1, a_2 \in \mathcal{C}_i \cap R$, then $a_1 \pm \ell a_2 \in \mathcal{C}_i \cap R$. On the other hand, if $a_1 \in \mathcal{C}_i \cap R$ but $a_2 \notin \mathcal{C}_i \cap R$, then $a_1 \pm \ell a_2 \notin \mathcal{C}_i \cap R$. The same holds if $a_1 \notin \mathcal{C}_i \cap R$ but $a_2 \in \mathcal{C}_i \cap R$.

Finally, consider the case where $a_1, a_2 \notin \mathcal{C}_i$. Only in this case does the choice of $\ell$ determine whether $a_1 + \ell a_2 \in \mathcal{C}_i$. Suppose that $\ell, \ell' \in \mathbf{N}\backslash\{0\}$ such that $a_1 + \ell a_2, a_1 + \ell' a_2 \in \mathcal{C}_i$. Then $(a_1 + \ell a_2) - (a_1 + \ell' a_2) = a_2(\ell - \ell') \in \mathcal{C}_i$. Because $a_2 \notin \mathcal{C}_i$, it must be that $\ell - \ell' \in \mathcal{C}_i$, so $\ell - \ell' = 0$. This indicates that for each $\mathcal{C}_i$ that contains neither $a_1$ nor $a_2$, there is at most one value of $\ell$ such that $a_1 + \ell a_2 \in \mathcal{C}_i$. Similarly, there is at most one value of $\ell$ such that $a_1 - \ell a_2 \in \mathcal{C}_i$.

Choose $\ell$ such that $a_1 \pm \ell a_2 \notin \mathcal{C}_i$ for all $\mathcal{C}_i$ that contain neither of $a_1, a_2$. From the above observations, this choice ensures that $a_1 \pm \ell a_2 \in \mathcal{C}_i$ if and only if both $a_1, a_2 \in \mathcal{C}_i$. Hence, $\{\mathcal{C}_i \mid a_1 + \ell a_2 \in \mathcal{C}_i\} = \{\mathcal{C}_j \mid a_1 - \ell a_2 \in \mathcal{C}_j\}$, so we have $(*)$, and the previous argument applies. ◻

**Lemma 3.8.** *Let* $(T, M)$ *and* $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ *be as in Remark* 2.7. *Let* $J$ *be an ideal of* $T$ *such that* $J \not\subseteq P$ *for all* $P \in \mathcal{C}$, *and let* $u + J \in T/J$. *Suppose* $R$ *is an* SIP*-subring. Then there exists an* SIP*-subring* $S$ *of* $T$ *such that*

(1) $R \subseteq S \subset T$;

(2) *if* $u \in J$, *then* $S \cap J$ *contains a non-zerodivisor of* $T$;

(3) $u + J$ *is in the image of the map* $S \to T/J$;

(4) *for every finitely generated ideal* $I$ *of* $S$, *we have* $IT \cap S = I$;

(5) $|R| = |S|$.

*Proof.* First, we use Corollary 3.6 to find an SIP-subring $R'$ such that $R \subseteq R'$, $u + J$ is in the image of $R' \to T/J$, if $u \in J$, then $J \cap R'$ contains a non-zerodivisor, and $|R'| = |R|$. We will construct an $S$ such that $R' \subseteq S \subset T$ and so conditions (1)–(3) of the lemma hold for $S$.

Let $\Omega = \{(I, c) \mid I \text{ finitely generated}, c \in IT \cap R'\}$. The cardinality of the set of finitely generated ideals of $R'$ is less than or equal to $|R'|$. Hence, $|\Omega| = |R'| < |T|$. Well-order $\Omega$ so that it has no maximal element, and let 0 denote the minimal element of $\Omega$. For each $\alpha \in \Omega$, we define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$. Let $R_0 = R'$.

Let $\lambda \in \Omega$. Assume that $R_\beta$ has been defined for all $\beta < \lambda$, such that $R_\beta$ is an SIP-subring, and $|R_\beta| = |R'|$. Suppose $\gamma(\lambda) < \lambda$, and let $(I, c) = \gamma(\lambda)$. Then, using Lemma 3.7, we construct $R_\lambda$ such that $R_{\gamma(\lambda)} \subseteq R_\lambda$ and $c \in IR_\lambda$. Note that $|R_\lambda| = |R_{\gamma(\lambda)}| = |R'|$.

Next, suppose $\gamma(\lambda) = \lambda$. In this case, we define $R_\lambda = \cup_{\beta<\lambda} R_\beta$. Since $|R_\beta| = |R'|$ for all $\beta < \lambda$, and $|\Omega| = |R'|$, Lemma 3.1 implies that $R_\lambda$ is an SIP-subring, and $|R_\lambda| = |R'|$.

Define

$$R_1 = \bigcup_{\alpha\in\Omega} R_\alpha.$$

Then $|R_1| = |R'|$, and $R_1$ is an SIP-subring of $T$. If $I$ is a finitely generated ideal of $R_0$, and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha$ such that $\gamma(\alpha) < \alpha$. It follows that $c \in IR_\alpha \subseteq IR_1$. Hence, $IT \cap R_0 \subseteq IR_1$ for every finitely generated ideal $I$ of $R_0$.

We repeat this process for $R_1$, and obtain an SIP-subring $R_2$ containing $R_1$ such that $IT \cap R_1 \subseteq IR_2$ for each finitely generated ideal $I$ of $R_1$, and $|R_2| = |R'|$. Continuing in this fashion, we construct a chain of SIP-subrings $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ such that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal $I$ of $R_n$ and $|R_n| = |R'|$ for all $n \in \mathbf{N}$.

Let

$$S = \bigcup_{i=0}^{\infty} R_i.$$

Then, $|S| = |R'| = |R|$, and $S$ is an SIP-subring of $T$. Let $I = (s_1, \ldots, s_n)$ be an ideal of $S$, and let $c \in IT \cap S$. Then $c = s_1t_1 + \cdots + s_nt_n$, where $t_i \in T$, and where each $s_k \in R_{m_k}$ for some $m_k \in \mathbf{N}$. Now, there exists an $m_0 \in \mathbf{N}$ such that $c \in R_{m_0}$. Let $N = \max\{m_k \mid 0 \le k \le n\}$. Then $c \in (s_1, \ldots, s_n)T \cap R_N \subseteq (s_1, \ldots s_n)R_{N+1} \subseteq IS$. Therefore $IT \cap S = I$. It follows that $S$ is our desired SIP-subring of $T$. $\square$

Until this point, all of our lemmata have established ways of modifying IP-subrings and SIP-subrings. Now, we must show that an SIP-subring of $T$ exists.

To construct an SIP-subring, we will use the concept of a semi-SIP-subring, which satisfies a weaker version of the SIP conditions: the ideals $\mathcal{C}_i \cap R$ are not required to be distinct from each other. We will begin with a semi-SIP-subring $R$, and then adjoin elements of $T$ to make the ideals $\mathcal{C}_i \cap R$ distinct.

**Definition 3.9.** Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 2.7. We say that a quasi-local subring $(R, R \cap M)$ of $T$ is a *semi*-SIP *subring of* $T$ if the following conditions hold.

(1) R is infinite.

(2) For each $P \in \mathcal{C}$, $P \cap R = Q \cap R$ for $Q \in \mathrm{Ass}\,(T)$ with $Q \subseteq P$.

(3) For each subcollection $\mathcal{C}_i$, if $P, P' \in \mathcal{C}_i$, then $P \cap R = P' \cap R$.

(4) For each $P \in \mathcal{C}$ and $r \in P \cap R$, $\mathrm{Ann}_T(r) \not\subseteq P$.

(5) $|R| < |T|$.

**Lemma 3.10.** *Let* $(T, M)$ *and* $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ *be as in Remark* 2.7*, and fix* $\mathcal{C}_i$*. Let* $R$ *be a semi*-SIP*-subring of* $T$*, and let* $p_i \in T$ *be given such that* $p_i \in Q$ *for every* $Q \in \mathcal{E}_i$*, but* $p_i \notin P$ *for any* $P \in \mathcal{C}_j$*, where* $j \neq i$*. Suppose further that* $\mathrm{Ann}_T(p_i) \not\subseteq P$ *for any* $P \in \mathcal{C}_i$*. Then there exists a unit* $u$ *in* $T$ *such that* $R[up_i]_{(R[up_i] \cap M)}$ *is a semi*-SIP*-subring of* $T$*.*

*Proof.* Define $S = R[p_i]$, and note that $|S| = |R| < |T|$. Apply Lemma 3.5 with $t = 0$, $q = 1$ and $J = M$ to find an element $t' \in M$ such that, for each $P \in \mathcal{C}$, $t' + P \in T/P$ is transcendental over $S/(P \cap S)$. Let $u = t' + 1$. Note that $u$ is a unit, since $T$ is local, and that $u + P \in T/P$ is transcendental over $S/(P \cap S)$, because $1 \in S$.

Define $p = up_i$. Then $p \in Q$ for every $Q \in \mathcal{E}_i$, but $p \notin P$ for any $P \in \mathcal{C}_j$, where $j \neq i$. We claim that $R[p]_{(R[p] \cap M)}$ is a semi-SIP-subring. Let $f \in R[p]$. Then we can write $f$ in the form $f = r_n p^n + \cdots + r_1 p + r_0$ for some $r_i \in R$. To see that conditions (2) and (3) of Definition 3.9 hold for $P \in \mathcal{C}_i$, suppose $f \in P$ for some $P \in C_i$. Given any $Q \in \mathcal{E}_i$, we will show that $f \in Q$. Note that $p \in Q \subseteq P$, so $f \in P$ implies $r_0 \in P \cap R = Q \cap R$. Hence, $r_0 \in Q$, and so $f = r_n p^n + \cdots + r_1 p + r_0 \in Q$. Thus, $P \cap R[p] = P' \cap R[p] = Q \cap R[p]$ for every $P, P' \in \mathcal{C}_i$ and every $Q \in \mathcal{E}_i$.

Next we show that, if $f \in R[p] \cap P$ for some $P \in \mathcal{C}_i$, then $\mathrm{Ann}_T(f) \not\subseteq P$. By hypothesis, there exists some $v \notin P$ such that $vp = 0$. Since $R$ is a semi-SIP-subring, there exists a $w \notin P$ such that $wr_0 = 0$. Then $vw \notin P$, and $(wv)f = 0$. Hence, $\mathrm{Ann}_T(f) \not\subseteq P$.

Now we show that $p + P \in T/P$ is transcendental over $R/(R \cap P)$ for every $P \in \mathcal{C}_j$, where $j \neq i$. By Lemma 3.3, this implies conditions (2), (3) and (4) of Definition 3.9 hold for $R[p]$. Let $f \in R[p] \cap P$ for some

$P \in \mathcal{C}_j$, where $j \neq i$. Then

$$\begin{aligned} f &= r_n p^n + \cdots + r_1 p + r_0 \\ &= r_n (p_i u)^n + \cdots + r_1 (p_i u) + r_0 \\ &= (r_n p_i^n) u^n + \cdots + (r_1 p_i) u + r_0. \end{aligned}$$

Hence, we can express $f$ as an element of $S[u] \cap P$. Since $u + P \in T/P$ is transcendental over $S/(S \cap P)$, this implies that $r_k p_i^k \in P$, and thus $r_k \in P$ for all $k = 1, \dots, n$. We have shown that, if $f = r_n p^n + \cdots + r_1 p + r_0 \in P$, then $r_k \in P$ for $k = 1, \dots, n$. Hence, $p + P \in T/P$ is transcendental over $R/(R \cap P)$. It follows that $R[p]_{(R[p] \cap M)} = R[up_i]_{(R[up_i] \cap M)}$ is a semi-SIP-subring of $T$. □

**Lemma 3.11.** *Let* $(T, M)$ *and* $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ *be as in Remark* 2.7. *Then there exists an* SIP*-subring of* $T$.

*Proof.* Let $R_0$ be the prime subring of $T$ localized at $(0)$. Then, since $T$ contains the rationals, $R_0 = \mathbf{Q} \subseteq T$. Now, $P \cap R_0 = (0)$ for any $P \in \mathcal{C}$, and $Q \cap R_0 = (0)$ for any $Q \in \operatorname{Ass}(T)$. The other conditions of Definition 3.9 follow trivially, so $R_0$ is a semi-SIP-subring.

Consider the $m$ subcollections $\mathcal{C}_1, \dots, \mathcal{C}_m$. Using a process described below, we successively adjoin $m$ elements $p_1, \dots, p_m$ to $R_0$, such that $p_i \in P$ if and only if $P \in \mathcal{C}_i$, and such that the resulting ring $S$ remains a semi-SIP-subring. Consequently, if $P \in \mathcal{C}_i$, and $P' \in \mathcal{C}_j$ where $i \neq j$, then the construction ensures that $P \cap S$ contains some element $p_i$ which is not contained in $P'$, so that $P \cap S \neq P' \cap S$. Thus, $S$ will be an SIP-subring of $T$.

Let $R$ be a semi-SIP-subring of $T$. Let $\operatorname{Min}(T) = \{Q_1, \dots, Q_n\}$. For each $Q_i$, use the Prime Avoidance theorem to find $q_i \in Q_i - \cup\{P \in \mathcal{C} \mid Q_i \not\subseteq P\}$. Let

$$q = \prod_{i=1}^{n} q_i.$$

Then $q$ is nilpotent, so let $\ell$ be the smallest positive integer such that $q^\ell = 0$. We note that, while each of the $q_i$ is non-zero, $q$ itself may be zero. In this case, $\ell = 1$, and the argument still follows.

Fix $\mathcal{C}_k \in \mathcal{C}$. We will construct a semi-SIP-subring $R'$ containing $R$ and such that $\mathcal{C}_k \cap R' \neq \mathcal{C}_j \cap R'$ for all $j \neq k$. Consider those minimal prime ideals contained in $\mathcal{E}_k$. We define

$$p_k = \prod_{Q_i \in \mathcal{E}_k} q_i^\ell \quad \text{and } s_k = \prod_{Q_i \notin \mathcal{E}_k} q_i^\ell.$$

Note that $p_k \cdot s_k = \prod_{i=1}^n q_i^\ell = (\prod_{i=1}^n q_i)^\ell = q^\ell = 0$, and thus $\mathrm{Ann}_T(p_k) \not\subseteq P$ for each $P \in \mathcal{C}_k$. By Lemma 3.10, there exists a unit $t_k \in T$ such that $R' = R[t_k p_k]_{(R[t_k p_k] \cap M)}$ is a semi-SIP-subring. Note that, for each $j \neq k$, $p_k \notin \mathcal{C}_j$. Thus, $\mathcal{C}_k \cap R[t_k p_k]_{(R[t_k p_k] \cap M)} \neq \mathcal{C}_j \cap R[t_k p_k]_{(R[t_k p_k] \cap M)}$.

We now repeat this process for each $\mathcal{C}_i$. Our resulting subring $S$ will be a semi-SIP-subring and will, for each $\mathcal{C}_i$, contain an element $p_i$ that is not in $\mathcal{C}_j$ for all $j \neq i$. Hence $S$ is an SIP-subring of $T$. □

The following theorem characterizes the completions of reduced rings whose minimal prime ideals have semi-local formal fibers. We start with the initial subring of $T$ constructed in Lemma 3.11 and construct a local IP-subring $A$ of $T$ satisfying the conditions of Proposition 2.1.

**Theorem 3.12.** *Let $(T, M)$ be a complete local ring of dimension at least one, containing the rationals, and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. Then $T$ is the completion of a reduced local subring $A$ such that $\mathrm{Min}\, A = \{\mathcal{C}_1 \cap A, \ldots, \mathcal{C}_m \cap A\}$ and the formal fiber of $\mathcal{C}_i \cap A$ is semi-local with maximal ideals precisely the elements of $\mathcal{C}_i$. Furthermore, if $J$ is an ideal of $T$ such that $J \not\subseteq P$ for every $P \in \mathcal{C}$, then the natural map $A \to T/J$ is onto.*

*Proof.* Let $\Omega = \{u + J \mid u \in T, J \not\subseteq P \text{ for all } P \in \mathcal{C}\}$ equipped with a well-ordering $<$, such that every element has strictly fewer than $|\Omega|$ predecessors. Note that

$$|\{J \mid J \text{ is an ideal of } T \text{ with } J \not\subseteq P \text{ for every } P \in \mathcal{C}\}| \leq |T|.$$

For each $\alpha \in \Omega$, we let $|\alpha| = |\{\beta \in \Omega \mid \beta \leq \alpha\}|$, by abuse of notation. Let 0 denote the first element of $\Omega$, and let $R_0$ be the SIP-subring of $T$ constructed in Lemma 3.11. For each $\lambda \in \Omega$ after the first, we define $R_\lambda$ recursively as follows: assume $R_\beta$ is defined for all $\beta < \lambda$ such that $R_\beta$ is an SIP-subring, and $|R_\beta| \leq |\beta||R_0|$ for all $\beta < \alpha$. As before, let $\gamma(\lambda) = u + J$ denote the least upper bound of the set of predecessors of $\lambda$. If $\gamma(\lambda) < \lambda$, we use Lemma 3.8 with $R = R_{\gamma(\lambda)}$ to find an SIP-subring $R_\lambda$ such that

(1) $R_{\gamma(\lambda)} \subseteq R_\lambda \subseteq T$;

(2) if $u \in J$, then $J \cap R_\lambda$ contains a non-zerodivisor;

(3) the coset $\gamma(\lambda) = u + J$ is in the image of the map $R_\lambda \to T/J$; and

(4) for all finitely-generated ideals $I$ of $R_\lambda$, $IT \cap R_\lambda = I$.

In this case,

$$\begin{aligned} |R_\lambda| &= |R_{\gamma(\lambda)}| \\ &\leq |\gamma(\lambda)||R_0| \\ &\leq |\lambda||R_0|. \end{aligned}$$

On the other hand, if $\gamma(\lambda) = \lambda$, we let

$$R_\lambda = \bigcup_{\beta<\lambda} R_\beta.$$

Then, $|\lambda| < |\Omega| = |T|$, and $|R_\lambda| \leq |\lambda||R_0|$. By Lemma 3.1, $R_\lambda$ is an SIP-subring of $T$.

Let

$$A = \bigcup_{\alpha\in\Omega} R_\alpha.$$

Then $(A, A \cap M)$ is an IP-subring of $T$.

Note that $M^2 \not\subseteq P$ for every $P \in \mathcal{C}$ so, by our construction, the map $A \to T/M^2$ is onto. Next, let $I = (a_1, \dots, a_n)A$ be a finitely-generated ideal of $A$ and $c \in IT \cap A$. Then, for some $\delta \in \Omega$, $\{c, a_1, \dots, a_n\} \subset R_\delta$. In particular, this yields $c \in IR_\delta \subset I$. Hence $IT \cap A = I$ for all finitely-generated ideals $I$ of $A$. Since $(A, A \cap M)$ is a quasi-local subring of $T$, Proposition 2.1 implies that $A$ is Noetherian and $\widehat{A} = T$.

Now, since $T$ is faithfully flat over $A$, the ideals $\mathcal{C}_i \cap A$ are the minimal prime ideals of $A$, so that $\operatorname{Min}(A)$ has $m$ elements. By our construction, the formal fiber of $\mathcal{C}_i \cap A$ is semi-local with maximal ideals precisely the elements of $\mathcal{C}_i$. Furthermore, the natural map $A \to T/J$ is onto for any ideal $J$ such that $J \not\subseteq P$ for all $P \in \mathcal{C}$. □

It is interesting to note that, for the ring $A$ in Theorem 3.12, we know not only the formal fibers of the minimal prime ideals, but also the formal fibers of all other prime ideals of $A$. To see this, suppose $p \in \operatorname{Spec}(A)$ with $\operatorname{ht} p > 0$. Then $pT \not\subseteq P$ for every $P \in \mathcal{C}$. It follows by our construction that $A \to T/pT$ is onto. Since $pT \cap A = p$, we have $A/p \cong T/pT$. It follows that the only element in the formal fiber of $p$ is $pT$.

**4. Excellent reduced rings.** We now examine the conditions under which $A$ from Theorem 3.12 can be made excellent. Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 2.7. Our goal in this section is to find necessary and sufficient conditions on $T$ to ensure that it is the completion of an excellent reduced local subring $A$ such that $A$ has exactly $m$ minimal prime ideals $\{J_1, \dots, J_m\}$, and the formal fiber of each $J_i$ is semi-local with maximal ideals the elements of $\mathcal{C}_i$. Recall that a local ring is excellent if it is both a G-ring and universally catenary.

**Definition 4.1.** Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark 2.7. Suppose $(A, A \cap M)$ is a reduced local (Noetherian) subring of $T$ such that

(1) $\widehat{A} = T$;

(2) $\operatorname{Min}(A) = \{\mathcal{C}_1 \cap A, \dots, \mathcal{C}_m \cap A\}$, and, for $i = 1, 2, \dots, m$, the formal fiber of $\mathcal{C}_i \cap A$ is semi-local with maximal ideals precisely the elements of $\mathcal{C}_i$;

(3) For all ideals $J$ of $T$ such that $J \not\subseteq P$ for all $P \in \mathcal{C}$, the map $A \to T/J$ is onto.

Then we call $A$ a *minimal-controlled subring* (abbreviated MC-subring) of $T$.

The ring $A$ constructed in Theorem 3.12 is an MC-subring of $T$. We will show, in Theorem 4.4 that, if the complete local ring $T$ has an *excellent* subring satisfying all conditions for being an MC-subring of $T$ except for condition (3), then, for all $i$, and for all $P \in \mathcal{C}_i$, $(T/Q_i)_{\overline{P}}$ is a regular local ring where $Q_i = \cap_{Q \in \mathcal{E}_i} Q$. Lemma 4.2 helps us do this.

**Lemma 4.2.** *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be as in Remark* 2.7. *Fix $\mathcal{C}_i$, and let $Q_i = \cap_{Q \in \mathcal{E}_i} Q$. Suppose there exists an ideal $I$ of $T$ with $I \subseteq Q_i$ and $(T/I)_{\overline{P}}$ is a regular local ring for each $P \in \mathcal{C}_i$. Then $(T/Q_i)_{\overline{P}}$ is a regular local ring for all $P \in \mathcal{C}_i$.*

*Proof.* Let $P \in \mathcal{C}_i$, and let $d = \operatorname{ht} P$. Define $T' = (T/I)_{\overline{P}}$, $T'' = (T/Q_i)_{\overline{P}}$, and let $M'$ denote the maximal ideal of $T'$ and $M''$ the maximal ideal of $T''$.

We claim that $\dim T' = \dim T'' = \operatorname{ht} P = d$. To see why this holds, let $P_0 \subsetneq \cdots \subsetneq P_d$ be a maximal chain of prime ideals in $T$ such that $P_d = P$. Note that $P_0 = Q$ for some minimal prime ideal $Q \subseteq P$. Now, $Q \in \mathcal{E}_i$ and so $Q_i \subseteq Q$. Consider any ideal $J \subseteq Q$. Observe that $P_0/J \subsetneq \cdots \subsetneq P_d/J$, and so $\operatorname{ht}_{T/J} P/J \geq d$. By Theorem 15.15 in [**12**], $\operatorname{ht}_{T/J} P/J \leq \operatorname{ht}_T P = d$. Hence, $\operatorname{ht}_{TJ} P/J = d$. Finally, by [**12**, Theorem 14.18], $\dim (T/J)_{\overline{P}} = \operatorname{ht}_{T/J} P/J = d$. This argument with $J = I$ implies $\dim T' = d$, while the argument with $J = Q_i$ yields $\dim T'' = d$. This proves our claim.

Since $T'$ is a regular local ring of dimension $d$, every minimal generating set of its maximal ideal $M'$ must have exactly $d$ members. Let

$$\left\{ \frac{a_1 + I}{1 + I}, \ldots, \frac{a_d + I}{1 + I} \right\}$$

be a minimal generating set for the maximal ideal $M'$ of $T'$. Then it is not hard to show that

$$\left\{ \frac{a_1 + Q_i}{1 + Q_i}, \ldots, \frac{a_d + Q_i}{1 + Q_i} \right\}$$

generates the maximal ideal $M''$ of $T''$. It follows that $(T/Q_i)_{\overline{P}}$ is a regular local ring for all $P \in \mathcal{C}_i$. □

In Lemma 4.3, we find sufficient conditions for the complete local ring $T$ to have a subring $A$ such that $A$ is both an MC-subring of $T$ and a G-ring. We will use this lemma to construct an excellent ring $A$ in Theorem 4.4.

**Lemma 4.3.** *Let $(T, M)$ and $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}\}_{i=1}^m)$ be as in Remark* 2.7, *and suppose that $T$ is reduced. For each $i$, let $Q_i = \cap_{Q \in \mathcal{E}_i} Q$, and suppose that, for each $\mathcal{C}_i$ and each $P \in \mathcal{C}_i$, $(T/Q_i)_{\overline{P}}$ is a regular local ring. Then there exists an* MC*-subring $A$ of $T$ that is a* G*-ring. Moreover, for every $i = 1, 2, \ldots, m$, $(\mathcal{C}_i \cap A)T = Q_i$.*

*Proof.* Let $Q_i$ be given. We first claim that there exists a minimal generating set $(q_0, q_1, \ldots, q_n)$ of $Q_i$ such that, if $k \neq i$, then $q_j \notin \bigcup_{P \in \mathcal{C}_k} P$

for all $j = 0, 1, 2, \dots, n$. We will find $q_0, q_1, \dots, q_n$ inductively. First use the Prime Avoidance theorem to find $q_0$ satisfying $q_0 \in Q_i$, and

$$q_0 \notin \left\{ MQ_i \cup \bigcup_{P \in \mathcal{C}_k,\, k \neq i} P \right\}.$$

Now assume that $q_0, \dots, q_r$ have been found and $Q_i \neq (q_0, q_1, \dots, q_r)$. Then use the Prime Avoidance theorem to find $q_{r+1}$ so that $q_{r+1} \in Q_i$ and $q_{r+1} \notin MQ_i + (q_0, \dots q_r)$ and $q_{r+1} \notin \cup_{P \in \mathcal{C}_k,\, k \neq i} P$. As $T$ is Noetherian, this process must stop so that eventually we get $Q_i = (q_0, \dots, q_n)$. Note that this generating set for $Q_i$ is minimal by [**11**, Theorem 2.3].

Let $\operatorname{Min} T - \mathcal{E}_i = \{Q'_1, Q'_2, \dots, Q'_r\}$. For every $k = 1, 2, \dots, r$, use the Prime Avoidance theorem to find $v_k \in Q'_k - \cup_{P \in \mathcal{C}_i} P$. Let $v = \prod_{k=1}^{r} v_k$, and note that $v \neq 0$. For all $j = 0, 1, 2, \dots, n$, $q_j v \in \cap_{Q \in \operatorname{Min} T} Q = (0)$ since $T$ is reduced. Also, $v \notin \cup_{P \in \mathcal{C}_i} P$, so $v \notin P$ for every $P \in \mathcal{C}_i$. It follows that, for all $j = 0, 1, 2, \dots, n$ and all $P \in \mathcal{C}_i$, $\operatorname{Ann}(q_j) \not\subseteq P$.

Let $R_0$ be the SIP-subring constructed in Lemma 3.11. Then use Lemma 3.10 to find a unit $u_0$ of $T$ so that $R_0[u_0 q_0]_{(R_0[u_0 q_0] \cap M)}$ is a semi-SIP-subring of $T$. Note that $R_0[u_0 q_0]_{(R_0[u_0 q_0] \cap M)}$ is, in fact, an SIP-subring since $R_0$ is. We repeat this process for each $j = 1, \dots, n$: at each step we adjoin $q_j u_j$, where $u_j$ is a unit chosen so that the resulting ring is an SIP-subring of $T$. Since the $u_j$ are units, $(q_0 u_0, q_1 u_1, \dots, q_n u_n)$ generates $Q_i$ in $T$.

Continuing in this fashion, we adjoin a generating set for the ideal $Q_i$ corresponding to each $\mathcal{C}_i$. Let $S$ be the resulting subring. Then $S$ is an SIP-subring which has the property that, for each $\mathcal{C}_i$, $(\mathcal{C}_i \cap S)T = Q_i$. We now repeat the construction used in the proof of Theorem 3.12. For our initial subring, instead of the SIP-subring constructed in the proof of Lemma 3.11, we use the ring $S$.

Let $A$ be the resulting ring. Then $A$ is an MC-subring of $T$, and by construction, for every $i = 1, 2, \dots, m$, $(\mathcal{C}_i \cap A)T = Q_i$. We have left to show that $A$ is a G-ring. To do this, we show that for every $J \in \operatorname{Spec} A$, and for all finite field extensions $L$ of $k(J)$, where $k(J) = A_J / JA_J$, $T \otimes_A L$ is a regular ring. Since $A$ contains the rationals, it suffices to show that $T \otimes_A k(J)$ is a regular ring.

First, suppose that $\operatorname{ht} J = 0$. Then $J = \mathcal{C}_i \cap A$ for some $i = 1, 2, \dots, m$. In this case, $T \otimes_A k(J)$ localized at a maximal ideal is isomorphic to $(T/JT)_{\overline{P}}$ for some $P \in \mathcal{C}_i$. But

$$(T/JT)_{\overline{P}} = (T/(\mathcal{C}_i \cap A)T)_{\overline{P}} = (T/Q_i)_{\overline{P}}.$$

This ring is a regular local ring by hypothesis. It follows that $T \otimes_A k(J)$ is a regular ring.

Now suppose that $\operatorname{ht} J > 0$. Then $JT \not\subseteq P$ for all $P \in \mathcal{C}$, and so the map $A \to T/JT$ is onto. The kernel of this map is $JT \cap A = J$ and so $T/JT \cong A/J$. We now have $T \otimes_A k(J) \cong (T/JT)_{\overline{A-J}} \cong (A/J)_{\overline{A-J}} \cong A_J/JA_J = k(J)$, a field.

It follows that $T \otimes_A k(J)$ is a regular ring for all $J \in \operatorname{Spec} A$ and so $A$ is a G-ring. □

Theorem 4.4 is the main theorem of this section. Specifically, we demonstrate necessary and sufficient conditions for the desired subring $A$ of $T$ to exist.

**Theorem 4.4.** *Let $(T, M)$ be a complete local ring of dimension at least one, containing the rationals. Let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be a feasible partition. For each $\mathcal{C}_i \in \mathcal{P}$, let $Q_i = \cap_{Q \in \mathcal{E}_i} Q$. Then $T$ is the completion of an excellent reduced local subring $A$, such that $\operatorname{Min}(A) = \{\mathcal{C}_1 \cap A, \dots, \mathcal{C}_m \cap A\}$ and the formal fiber of $\mathcal{C}_i \cap A$ is semilocal with maximal ideals precisely the elements of $\mathcal{C}_i$, if and only if the following conditions hold:*

(1) *$T$ is reduced;*

(2) *for each $Q_i$ and each $P \in \mathcal{C}_i$, $(T/Q_i)_{\overline{P}}$ is a regular local ring;*

(3) *for each $Q_i$, $T/Q_i$ is equidimensional.*

*Proof.* Assume that such an $A$ exists. Then, since $A$ is excellent and reduced, $T$ must be reduced.

Now let $P \in \mathcal{C}_i$ for some $i$. Then $P \cap A = J$ for some minimal prime ideal $J$ of $A$. Since $A$ satisfies condition (2) of MC-subrings, for every $Q \in \mathcal{E}_i$, we have $Q \cap A = J$. It follows that $JT \subseteq \cap_{Q \in \mathcal{E}_i} Q$. Since $A$ is a G-ring, $T \otimes_A k(J)$, where $k(J) = A_J/JA_J$, is a regular ring. Now, because the formal fiber of $J$ is semilocal with maximal ideals the elements of $\mathcal{C}_i$ and $P \in \mathcal{C}_i$, $T \otimes_A k(J)$ localized at the maximal ideal

$P \otimes_A k(J)$ is isomorphic to $(T/JT)_{\overline{P}}$. Hence, $(T/JT)_{\overline{P}}$ is a regular local ring. This argument holds for all $P \in \mathcal{C}_i$, so $(T/JT)_{\overline{P}}$ is a regular local ring for all $P \in \mathcal{C}_i$. It follows from Lemma 4.2 that $(T/Q_i)_{\overline{P}}$ is a regular local ring for all $P \in \mathcal{C}_i$.

Since $A$ is excellent, it is universally catenary and hence formally catenary. Consequently, since $\mathcal{C}_i \cap A$ is a (minimal) prime ideal of $A$, $A/(\mathcal{C}_i \cap A)$ is formally equidimensional. Then its completion

$$\left(\widehat{\frac{A}{(\mathcal{C}_i \cap A)}}\right) = \frac{\widehat{A}}{(\mathcal{C}_i \cap A)\widehat{A}} = \frac{T}{(\mathcal{C}_i \cap A)T}$$

is equidimensional. We now show that, since $T/[(\mathcal{C}_i \cap A)T]$ is equidimensional, so too is $T/Q_i$. Note that

$$\operatorname{Min}\left(\frac{T}{Q_i}\right) = \left\{\frac{Q}{Q_i} \;\middle|\; Q \in \mathcal{E}_i\right\},$$

and if

$$\frac{Q}{Q_i} \in \operatorname{Min}\left(\frac{T}{Q_i}\right),$$

then $Q/[(\mathcal{C}_i \cap A)T]$ is a minimal prime ideal of $T/[(\mathcal{C}_i \cap A)T]$. Now, there exists a $Q \in \mathcal{E}_i$ such that $Q/Q_i$ is a minimal prime ideal of $T/Q_i$ satisfying $\dim T/Q_i = \dim (T/Q_i)/(Q/Q_i) = \dim T/Q$. Note that $Q/[(\mathcal{C}_i \cap A)T]$ is a minimal prime ideal of $T/[(\mathcal{C}_i \cap A)T]$. Let $Q'/Q_i$ be a minimal prime ideal of $T/Q_i$. Then $Q'/[(\mathcal{C}_i \cap A)T]$ is a minimal prime ideal of $T/[(\mathcal{C}_i \cap A)T]$. So,

$$\begin{aligned}
\dim \frac{T/Q_i}{Q'/Q_i} = \dim \frac{T}{Q'} &= \dim \frac{T/(\mathcal{C}_i \cap A)T}{Q'/(\mathcal{C}_i \cap A)T} \\
&= \dim \frac{T}{(\mathcal{C}_i \cap A)T} = \dim \frac{T/(\mathcal{C}_i \cap A)T}{Q/(\mathcal{C}_i \cap A)T} \\
&= \dim \frac{T}{Q} = \dim \frac{T}{Q_i}.
\end{aligned}$$

It follows that $T/Q_i$ is equidimensional. So conditions (1)–(3) of the theorem hold.

Conversely, suppose conditions (1)–(3) in the statement of the theorem hold. Then by Lemma 4.3, there exists a subring $A$ of $T$ that is a

G-ring and is an MC-subring of $T$. Moreover, for every $i = 1, 2, \dots, m$, $(\mathcal{C}_i \cap A)T = Q_i$.

It remains to show that $A$ is universally catenary. Let $J \in \operatorname{Spec}(A)$. If $\operatorname{ht}_A J = 0$, then $J = \mathcal{C}_i \cap A$ for some $i$, and so $\widehat{(A/J)} \cong T/((\mathcal{C}_i \cap A)T) = T/Q_i$ is equidimensional. Hence, $A/J$ is formally equidimensional.

On the other hand, if $\operatorname{ht}_A J > 0$, then $JT \not\subseteq P$ for all $P \in \mathcal{C}$. Consequently, the natural map $A \to T/JT$ is onto since $A$ satisfies condition (3) of MC-subrings. Since the kernel of the map is $JT \cap A = J$, we have $T/JT \cong A/J$, which is a domain and therefore equidimensional. Now observe that $\widehat{(A/J)} = T/JT \cong A/J$, so $A/J$ is its own completion and thus formally equidimensional. Hence $A$ is formally catenary, and therefore universally catenary. $\square$

If the three conditions of Theorem 4.4 are satisfied, then the ring $A$ constructed in Theorem 4.4 satisfies the condition that all minimal prime ideals have semi-local formal fibers. In addition, if $J$ is a prime ideal of $A$ with $J$ not a minimal prime ideal, then the formal fiber of $J$ contains exactly one element, namely $JT$. This follows since, as shown in the proof of Theorem 4.4, $A/J \cong T/JT$.

It is interesting to note that, if the complete local ring $T$ is the completion of a ring $A$ as in Theorem 4.4, then there are restrictions on the partition $\mathcal{P}$. In particular, Corollary 4.5 shows the rather restrictive condition that each $P \in \mathcal{C}$ can only contain one minimal prime ideal of $T$.

**Corollary 4.5.** *Let $(T, M)$, $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}\}_{i=1}^m)$ and $A$ be as in Theorem* 4.4. *Then each $P \in \mathcal{C}$ contains exactly one minimal prime ideal.*

*Proof.* For $P \in \mathcal{C}_i$, since $(T/Q_i)_{\overline{P}}$ is a regular local ring, it is an integral domain. It follows that $P$ contains exactly one minimal prime ideal of $T$. $\square$

To demonstrate Theorem 4.4, we present the following class of examples:

**Example 4.6.** Let $(T, M)$ be a complete local reduced ring of dimension at least one, containing the rationals. Let $\mathcal{C} = \mathrm{Min}\,(T) = \{Q_1, \ldots, Q_n\}$, and let $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ be any partition on $\mathcal{C}$. Then $\mathcal{P} = (\mathcal{C}, \{\mathcal{C}_i\}_{i=1}^m)$ is automatically a feasible partition. For any $i$ and $Q' \in \mathcal{E}_i$, we know

$$\left(\frac{T}{(\bigcap_{Q \in \mathcal{E}_i} Q)}\right)_{\overline{Q'}}$$

is a field and so a regular local ring for all $i$. If, for every $i$, $T/(\cap_{Q \in \mathcal{E}_i} Q)$ is equidimensional, then $T$ is the completion of an excellent reduced local subring $A$, with $\mathrm{Min}\,(A) = \{\mathcal{C}_i \cap A\}_{i=1}^m$ and the formal fiber of $\mathcal{C}_i \cap A$ is precisely $\mathcal{C}_i$.

**Acknowledgments.** The authors thank the referee for improvements to Lemma 3.5 and the proof of Lemma 4.3.

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