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MONOIDS OF TORSION-FREE MODULES OVER RINGS WITH FINITE REPRESENTATION TYPE

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ABSTRACT. Given a local ring R, we let $\mathcal{T}(R)$ denote the monoid of isomorphism classes of finitely generated torsion-free R-modules with operation $[M] + [N] = [M \oplus N]$. The main goal of this paper is to determine which monoids occur as $\mathcal{T}(R)$ for one-dimensional local ring-orders R with finite representation type. A byproduct of this investigation is a Krull-Remak-Schmidt theorem for finitely generated torsion-free modules over these rings.

1. Introduction and terminology. It is well known [**17**] that the Krull-Remak-Schmidt property holds for the class of all finitely generated modules over any complete local ring. That is, whenever $M_1 \oplus M_2 \oplus \cdots \oplus M_s \cong N_1 \oplus N_2 \oplus \cdots \oplus N_t$ with each M_i and N_j indecomposable, then (1) $s = t$ and (2) there exists some permutation σ of the set $\{1, 2, \dots, t\}$ such that $M_i \cong N_{\sigma(i)}$ for each i. Beginning with Evans [**7**], many authors, including Wiegand [**19**], have produced examples of non-complete local rings over which direct sum decompositions of finitely generated modules can be non-unique. In [**8**], the class of generically free modules over all local ring-orders was considered. The Krull-Remak-Schmidt property almost always fails for this larger class of modules. Thus, we restrict our rings and restrict to the nicer class of torsion-free modules. In [**2**] the first author gave a Krull-Remak-Schmidt theorem for the class of all finitely generated modules, and then for the much smaller class of finitely generated torsion-free modules over a family of equicharacteristic one-dimensional local rings $(R, \mathfrak{m})$ with *finite representation type*–those rings having, up to isomorphism, only finitely many indecomposable torsion-free modules. These results hinge on a result of Levy and Odenthal [**16**] and a list of possible ranks of indecomposable modules over the $\mathfrak{m}$-adic completion of R. Recently, the authors [**3**] gave a complete list of all possible ranks of indecomposable torsion-free modules over arbitrary one-dimensional reduced rings with finite representation type. The goal of this current

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work is to use the list of ranks from [**3**] to give a Krull-Remak-Schmidt theorem for all one-dimensional reduced rings with finite representation type. Moreover, for those rings over which the Krull-Remak-Schmidt property fails, we measure how far from unique direct-sum decompositions can be.

Throughout, $(R, \mathfrak{m})$ will denote a one-dimensional local ring with $\mathfrak{m}$-adic completion $\widehat{R}$. Since the class of all finitely generated torsion-free modules over a local ring is closed under finite direct sums and under direct summands, the monoid $\mathcal{T}(R)$ of isomorphism classes of finitely generated torsion-free R-modules with operation $[M] + [N] = [M \oplus N]$ carries all the information we need about direct sum decompositions over R.

We consider a *monoid* H to be a finitely generated commutative cancellative semigroup with identity. Furthermore, we restrict our attention to monoids that have no non-identity invertible elements and that are generated by their irreducible elements–elements that cannot be written as a sum of two non-zero elements in H. We say that $x \leq y$ in H provided there exists $z \in H$ such that $x + z = y$. A monoid H is a *Krull monoid* provided that there is a *divisor homomorphism* from H into a free monoid. That is, a map $\phi : H \hookrightarrow \mathbf{N}^t$ such that $a \leq b$ if and only if $\phi(a) \leq \phi(b)$.

A *divisor theory* is a divisor homomorphism $\phi : H \hookrightarrow \mathbf{N}^t$ such that each element in $\mathbf{N}^t$ is the greatest lower bound (with respect to $\leq$) of some finite set of elements in $\phi(H)$. It is known that each Krull monoid has a divisor theory, cf. [**10**].

The *quotient group* $\mathcal{Q}(H)$ *of a Krull monoid*, H, is the group of formal differences of elements in H, i.e., $\mathcal{Q}(H) = \{a - b : a, b \in H\}$. If H is a Krull monoid and $\phi : H \to D$ is a divisor theory for H, the *divisor class group of* H, denoted $\mathcal{C}l(H)$, is defined to be the cokernel of the induced map $\mathcal{Q}(\phi) : \mathcal{Q}(H) \to \mathcal{Q}(D)$.

For any element $h \in H$, $L(h) = \{n \mid h = a_1 + a_2 + \cdots + a_n \text{ for atoms } a_i\}$ is the *set of lengths* for the element h. A monoid H is said to be *factorial* if each element can be written uniquely (up to order of the terms) as a sum of atoms of H. A monoid H is said to be *half-factorial* if $L(h)$ is a singleton for each $h \in H$. The set $\mathcal{L}(H) = \{L(h) \mid 0 \neq h \in H\}$ is the *system of lengths of* H. The elasticity is a measure of how far from unique the factorization is in the monoid. The *elasticity of an element* $h \in H$ is $\rho(h) = \sup\{L(h)\}/\inf\{L(h)\}$. The *elasticity of the monoid* H is $\rho(H) = \sup\{\rho(h) \mid h \in H - \{0\}\}$. We note that $\rho(H) = 1$ if and only if H is half-factorial.

We now introduce the block monoid of a Krull monoid. This subject is often easier to study but yet carries a great deal of information about the original monoid. Let G be an abelian group, let $G_0 \subseteq G$ be any subset of G, and let $\mathcal{F}(G_0)$ be the free abelian monoid (written multiplicatively) over G_0. Consider the map

$$\begin{array}{cccc} c: & \mathcal{F}(G_0) & \longrightarrow & G \\ & \prod_{g\in G_0} g^{n_g} & \longmapsto & \sum_{g\in G_0} n_g g \end{array}.$$

The submonoid $\mathcal{B}(G_0) = \{s \in \mathcal{F}(G_0) : c(s) = 0\} \subseteq \mathcal{F}(G_0)$ is called the *block monoid of* G_0. Let $H \hookrightarrow \mathbf{N}^t$ be a divisor theory with H a Krull monoid. The *prime divisor classes* in $\mathcal{C}l(H)$ are the elements $q \in \mathcal{C}l(H)$ such that $q = p + \mathcal{Q}(H)$ for some atom $p \in \mathbf{N}^t$. It is shown in [**9**] that if we let $G = \mathcal{C}l(H)$ and G_0 be the set of prime divisor classes in G then the system of lengths of H is the same as the system of lengths of $\mathcal{B}(G_0)$. In particular, the elasticities of these two monoids are equal.

Since direct sum cancelation holds for R-modules [**7**], $\mathcal{T}(R)$ satisfies our definition of a monoid. Since $\widehat{R}$ is a complete local ring, the decomposition of $\widehat{R}$-modules is unique up to isomorphism, [**1**], and hence $\mathcal{T}(\widehat{R}) \cong \mathbf{N}^{\Lambda}$ where Λ is the set of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules. It is shown in [**18**] that the natural map taking M to $\widehat{R} \otimes_R M$ induces a *divisor homomorphism* $\mathcal{T}(R) \to \mathcal{T}(\widehat{R})$. Thus we may consider $\mathcal{T}(R)$ as a *full submonoid* of $\mathbf{N}^{\Lambda}$; that is, a submonoid that satisfies, for any $a, b \in \mathcal{T}(R)$, if $b = a + c$ for some $c \in \mathbf{N}^{\Lambda}$ then $c \in \mathcal{T}(R)$. Note that the atoms of $\mathcal{T}(R)$ correspond to the minimally extended $\widehat{R}$-modules–the extended $\widehat{R}$-modules for which no proper direct summand is extended.

2. $\mathcal{T}(R)$ as a Diophantine monoid. The goal of this section is to consider $\mathcal{T}(R)$ as a Diophantine monoid, that is, as the set of nonnegative integer solutions to a system of linear equations with integer coefficients. Given a list of all of the indecomposable torsion-free $\widehat{R}$-modules as well as their ranks at each of the minimal prime ideals of $\widehat{R}$, we can determine $\mathcal{T}(R)$ as a full submonoid of $\mathcal{T}(\widehat{R})$ using the following result, which follows as an immediate corollary to [**16**, Theorem 6.2]:

Proposition 2.1. *Let R and $\widehat{R}$ be local ring-orders. Let M be a finitely generated $\widehat{R}$-module. Then M is extended ($_{\widehat{R}}M \cong N \otimes_R \widehat{R}$ for some $N \in \mathcal{T}(R)$) from an R-module if and only if* $\operatorname{rank}_P(M) = \operatorname{rank}_Q(M)$ *whenever P and Q are minimal prime ideals of $\widehat{R}$ lying over the same prime ideal of R.*

In particular, if $\{M_1, M_2, \dots, M_t\}$ is a finite set of indecomposable modules over $\widehat{R}$, then the $\widehat{R}$-module $M = M_1^{n_1} \oplus M_2^{n_2} \oplus \cdots \oplus M_t^{n_t}$ is extended from an R-module if and only if $\sum_{i=1}^{t} n_i(\operatorname{rank}_P(M_i) - \operatorname{rank}_Q(M_i)) = 0$ for any pair of minimal prime ideals P and Q of $\widehat{R}$ that lie over a common prime ideal of R. Thus, given a finite set $\{M_1, M_2, \dots, M_t\}$ of indecomposable finitely generated torsion-free modules over $\widehat{R}$, we can form the free monoid $\mathbf{N}^t$ of isomorphism classes of $\widehat{R}$-modules that can be expressed as direct sums of the M_i. Then the full-submonoid of $\mathbf{N}^t$ consisting of $\widehat{R}$-modules extended from R-modules can be described as follows (cf. [**2, 8**]): Let $\{P_1, P_2, \dots, P_s\}$ denote the set of minimal prime ideals of R and, for each $i \in \{1, 2, \dots, s\}$, let $Q_{i,1}, Q_{i,2}, \dots, Q_{i,t_i}$ denote the minimal prime ideals of $\widehat{R}$ lying over P_i. Now if $q = \operatorname{spl}(R) := \#\operatorname{Spec}(\widehat{R}) - \#\operatorname{Spec}(R)$ is the *splitting number of* R, then $q = t_1 + t_2 + \cdots + t_s - s$. Set $\mathcal{A}$ to be the $q \times t$ matrix whose kth column corresponds to the indecomposable $\widehat{R}$-module M_k and is the transpose of the vector

$$(r_{1,1} - r_{1,2} \cdots r_{1,t_1-1} - r_{1,t_1} \cdots r_{s,1} - r_{s,2} \cdots r_{s,t_s-1} - r_{s,t_s})$$

where $r_{i,j} = \operatorname{rank}_{Q_{i,j}}(M_k)$. Then, the full submonoid of extended modules is isomorphic to $\ker(\mathcal{A}) \cap \mathbf{N}^t$.

In [**8**] monoids of all generically free modules are considered and hence the matrix $\mathcal{A}$ may have infinitely many columns. Since in our context R, and hence $\widehat{R}$ has only finitely many indecomposable torsion-free modules, $\mathcal{A}$ is a finite matrix.

The following lemma [**2**] allows us to calculate $\mathcal{C}l(\mathcal{T}(R))$ as well as the system of lengths of $\mathcal{T}(R)$.

Lemma 2.2. 1. *The divisor class group $\mathcal{C}l(H)$ of a finitely generated reduced Krull monoid H is trivial if and only if $H \cong \mathbf{N}^t$ for some t, i.e., H is free.*

2. *Let R and $\widehat{R}$ be as above. If* $\operatorname{spl}(R) = 0$*, then* $\mathcal{T}(R) \cong \mathcal{T}(\widehat{R})$*, and hence* $\mathcal{C}l(\mathcal{T}(R)) = 0$.

3. *If a monoid H contains a $\mathbf{Z}$-basis for a group $G \supset H$ then $G = \mathcal{Q}(H)$.*

4. *Let $H = \ker(\mathcal{A}) \cap \mathbf{N}^t \subseteq \mathbf{N}^t$ where $\mathcal{A}$ is an $s \times t$ matrix with entries in $\mathbf{Z}$. Further assume that H contains a $\mathbf{Z}$-basis for* $\ker(\mathcal{A})$ *and that the natural inclusion $i : H \to \mathbf{N}^t$ is a divisor theory. Then $\mathcal{C}l(H)$ is isomorphic to the image of $\mathcal{A} : \mathbf{Z}^t \to \mathbf{Z}^s$. Furthermore, the prime divisor classes in $\mathcal{C}l(H)$ are the elements $\{\mathcal{A}e_j\}_{j=1}^t$.*

In Section 3 we see, for each ring with finite representation type, the number of minimal prime ideals is at most three, and hence $\mathcal{A}$ has either one or two rows. Moreover, if $\operatorname{spl}(R) = 1$ and if P_1 and P_2 are two prime ideals of $\widehat{R}$ lying over a common prime ideal of R, then by setting $M_1 = \widehat{R}/P_1$ and $M_2 = \widehat{R}/P_2$ we may assume that the first two entries of $\mathcal{A}$ are 1 and -1. If $\operatorname{spl}(R) = 2$, with minimal prime ideals P_1, P_2, and P_3 of $\widehat{R}$ lying over a common prime ideal of R, then setting $M_1 = \widehat{R}/P_1$, $M_2 = \widehat{R}/P_2$, $M_3 = \widehat{R}/P_3$, $M_4 = \widehat{R}/(P_1 \cap P_2)$, $M_5 = \widehat{R}/(P_1 \cap P_3)$, and $M_6 = \widehat{R}/(P_2 \cap P_3)$ allows us to assume that the first six columns of $\mathcal{A}$ are

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}-1\\0\end{bmatrix}. \tag{1}$$

The following two lemmas use Lemma 2.2 to determine explicitly the monoid of extended modules just described. As we see in Section 3, the matrices considered in Lemmas 2.3 and 2.4 are much more general than we need. However, the process following Proposition 2.1 gives more general monoids that can occur as submonoids of $\mathcal{T}(R)$ when R has infinite representation type.

Lemma 2.3. *If $\mathcal{A} = \left[\begin{smallmatrix} 1 & -1 & a_3 & a_4 & \cdots & a_t\end{smallmatrix}\right]$ with each $a_i \in \mathbf{Z}$, then the natural inclusion* $\ker(\mathcal{A}) \cap \mathbf{N}^t \hookrightarrow \mathbf{N}^t$ *is a divisor theory if and only if there exist distinct $i, j \in \{3, 4, \ldots, t\}$ with $a_i < 0$ and $a_j > 0$. In this case,* $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^t) \cong \mathbf{Z}$*. Otherwise,* $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^t) = 0$.

Proof. First suppose that $a_i > 0$ for all $i \in \{3, 4, \ldots, t\}$. An element $h = (h_1, h_2, \ldots, h_t)$ in $\mathbf{N}^t$ is in the kernel of $\mathcal{A}$ if and only if $h_2 = h_1 + \sum_{i=3}^{t} a_i h_i$. That is, $\ker(\mathcal{A}) \cap \mathbf{N}^t \cong \mathbf{N}^{t-1}$ is free. Thus the natural inclusion $\ker(\mathcal{A}) \cap \mathbf{N}^t \hookrightarrow \mathbf{N}^t$ is not a divisor theory and $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^t) = 0$ by Lemma 2.2. The case $a_i < 0$ for all $i \in \{3, 4, \dots, t\}$ can be handled similarly.

Without loss of generality we assume that there is some number s with $3 \leq s < t$ such that $a_i < 0$ for all $i \in \{3, 4, \dots, s\}$ and $a_i > 0$ for all $i \in \{s+1, s+2, \dots, t\}$. We show that $\ker(\mathcal{A}) \cap \mathbf{N}^t$ contains a $\mathbf{Z}$-basis for $\ker(\mathcal{A})$ and that each $e_i \in \mathbf{N}^t$ is the greatest lower bound of a finite set of elements of $\ker(\mathcal{A}) \cap \mathbf{N}^t$. Suppose that $g = (g_1, g_2, \dots, g_t)$ is an element of $\ker(\mathcal{A})$. Then $g = g_1 e_1 + g_2 e_2 + \cdots + g_t e_t$ and $g_1 + \sum_{i=3}^{t} a_i g_i = g_2$. We write g as a unique $\mathbf{Z}$-linear combination of the elements $e_1 + e_2$, $-a_i e_1 + e_i$ for each $i \in \{3, 4, \dots, s\}$, and $a_i e_2 + e_i$ for each $i \in \{s+1, s+2, \dots, t\}$ as follows:

$$\begin{aligned}\sum_{i=3}^{s} g_i(-a_i e_1 + e_i) + \sum_{i=s+1}^{t} g_i(a_i e_2 + e_i) + \left(\sum_{i=3}^{s} a_i g_i + g_1\right)(e_1 + e_2) \\ = g_1 e_1 + \sum_{i=3}^{t} g_i e_i + \left(\sum_{i=3}^{t} a_i g_i + g_1\right) e_2 \\ = g_1 e_1 + \sum_{i=3}^{t} g_i e_i + g_2 e_2 = g.\end{aligned}$$

We now show that each $e_i \in \mathbf{N}^t$ is the greatest lower bound of two elements in $\ker(\mathcal{A}) \cap \mathbf{N}^t$. First note that $e_1 = \operatorname{glb}\{e_1 + e_2, -a_3 e_1 + e_3\}$ and $e_2 = \operatorname{glb}\{e_1 + e_2, a_{s+1} e_2 + e_{s+1}\}$. Now for j and k integers with $3 \leq j \leq s$ and $s+1 \leq k \leq t$ we have that $e_j = \operatorname{glb}\{-a_j e_1 + e_j, -a_j e_k + a_k e_j\}$ and $e_k = \operatorname{glb}\{a_k e_2 + e_k, -a_j e_k + a_k e_j\}$.

Since $\ker(\mathcal{A}) \cap \mathbf{N}^t$ contains a $\mathbf{Z}$-basis for $\ker(\mathcal{A})$ and each $e_i \in \mathbf{N}^t$ is the greatest lower bound of a finite set of elements of $\ker(\mathcal{A}) \cap \mathbf{N}^t$ we have a divisor theory by Lemma 2.2.

Also by Lemma 2.2, $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^t)$ is isomorphic to the image of $\mathcal{A}$ as a map from $\mathbf{Z}^t$ to $\mathbf{Z}$. Since the first two entries of $\mathcal{A}$ are 1 and -1, this map is a surjection and hence $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^t) \cong \mathbf{Z}$. $\square$

Lemma 2.4. *If* $\mathcal{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & a_7 & a_8 & \cdots & a_u \\ 0 & 1 & -1 & 1 & -1 & 0 & b_7 & b_8 & \cdots & b_u \end{bmatrix}$, *then the natural inclusion* $\ker(\mathcal{A}) \cap \mathbf{N}^u \hookrightarrow \mathbf{N}^u$ *is a divisor theory. Moreover,* $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^u) \cong \mathbf{Z}^2$.

Proof. Without loss of generality we assume the following structure on $\mathcal{A}$: For $7 \le m \le n \le p \le q \le r \le s \le t \le u$; $a_i, b_i > 0$ whenever $7 \le i \le m$; $a_i > 0, b_i < 0$ whenever $m+1 \le i \le n$; $a_i < 0, b_i > 0$ whenever $n+1 \le i \le p$; $a_i, b_i < 0$ whenever $p+1 \le i \le q$; $a_i = 0, b_i > 0$ whenever $q+1 \le i \le r$; $a_i = 0, b_i < 0$ whenever $r+1 \le i \le s$; $a_i > 0, b_i = 0$ whenever $s+1 \le i \le t$; and $a_i < 0, b_i = 0$ whenever $t+1 \le i \le u$.

We first show that $\ker(\mathcal{A}) \cap \mathbf{N}^u$ contains a $\mathbf{Z}$-basis for $\ker(\mathcal{A})$. Suppose that $g = (g_1, g_2, \dots, g_u)$ is an element of $\ker(\mathcal{A})$. Then $g = g_1e_1 + g_2e_2 + \cdots + g_ue_u$, $g_1 - g_2 + g_5 - g_6 + \sum_{i=7}^u a_ig_i = 0$ and $g_2 - g_3 + g_4 - g_5 + \sum_{i=7}^u b_ig_i = 0$. We write g uniquely as a $\mathbf{Z}$-linear combination of elements in $\ker(\mathcal{A}) \cap \mathbf{N}^u$ as follows:

$$
\begin{aligned}
g = {} & \sum_{i=7}^{m} g_i(a_ie_6 + b_ie_3 + e_i) + \sum_{i=m+1}^{n} g_i(a_ie_6 - b_ie_4 + e_i) \\
& + \sum_{i=n+1}^{p} g_i(-a_ie_1 + b_ie_3 + e_i) \\
& + \sum_{i=p+1}^{q} g_i(-a_ie_1 - b_ie_4 + e_i) \\
& + \sum_{i=q+1}^{r} g_i(b_ie_3 + e_i) + \sum_{i=r+1}^{s} g_i(-b_ie_4 + e_i) \\
& + \sum_{i=s+1}^{t} g_i(a_ie_6 + e_i) + \sum_{i=t+1}^{u} g_i(-a_ie_1 + e_i) \\
& + \left(g_1 + \sum_{i=n+1}^{p} a_ig_i + \sum_{i=p+1}^{q} a_ig_i + \sum_{i=t+1}^{u} a_ig_i\right)(e_1 + e_6) \\
& + \left(g_3 - \sum_{i=7}^{m} b_ig_i - \sum_{i=n+1}^{p} b_ig_i - \sum_{i=q+1}^{r} b_ig_i\right)(e_3 + e_4) \\
& + \left(g_4 - g_3 + \sum_{i=7}^{s} b_ig_i\right)(e_4 + e_5 + e_6) \\
& + \left(g_5 - g_4 + g_3 - \sum_{i=7}^{s} b_ig_i\right)(e_2 + e_5).
\end{aligned}
$$

We now show that each $e_i \in \mathbf{N}^u$ is the greatest lower bound of two or three elements of $\ker(\mathcal{A}) \cap \mathbf{N}^u$. Indeed, $e_1 = \mathrm{glb}\{e_1+e_6, e_1+e_2+e_3\}$, $e_2 = \mathrm{glb}\{e_2+e_5, e_1+e_2+e_3\}$, $e_3 = \mathrm{glb}\{e_3+e_4, e_1+e_2+e_3\}$, $e_4 = \mathrm{glb}\{e_3+e_4, e_4+e_5+e_6\}$, $e_5 = \mathrm{glb}\{e_2+e_5, e_4+e_5+e_6\}$, and $e_6 = \mathrm{glb}\{e_1+e_6, e_4+e_5+e_6\}$. If $7 \le i \le m$ then $e_i = \mathrm{glb}\{a_ie_6+b_ie_3+e_i, a_i(e_2+e_3)+b_ie_3+e_i, a_ie_6+b_i(e_5+e_6)+e_i\}$. If $m+1 \le i \le n$ then $e_i = \mathrm{glb}\{a_ie_6-b_ie_4+e_i, a_i(e_2+e_3)-b_ie_4+e_i, a_ie_6-b_i(e_1+e_2)+e_i\}$. If $n+1 \le i \le p$ then $e_i = \mathrm{glb}\{-a_ie_1+b_ie_3+e_i, -a_i(e_5+e_4)+b_ie_3+e_i, -a_ie_1+b_i(e_5+e_6)+e_i\}$. If $p+1 \le i \le q$ then $e_i = \mathrm{glb}\{-a_ie_1-b_ie_4+e_i, -a_i(e_5+e_4)-b_ie_4+e_i, -a_ie_1-b_i(e_1+e_2)+e_i\}$. If $q+1 \le i \le r$ then $e_i = \mathrm{glb}\{b_ie_3+e_i, b_i(e_5+e_6)+e_i\}$. If $r+1 \le i \le s$ then $e_i = \mathrm{glb}\{-b_ie_4+e_i, -b_i(e_1+e_2)+e_i\}$. If $s+1 \le i \le t$ then $e_i = \mathrm{glb}\{a_ie_6+e_i, a_i(e_2+e_3)+e_i\}$. Finally, if $t+1 \le i \le u$ then $e_i = \mathrm{glb}\{-a_ie_1+e_i, -a_i(e_5+e_4)+e_i\}$.

By Lemma 2.2, the inclusion map is a divisor theory and $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^u)$ is isomorphic to the image of $\mathcal{A}$ as a map from $\mathbf{Z}^u$ to $\mathbf{Z}^2$. Since the first two columns of $\mathcal{A}$ span $\mathbf{Z}^2$, this map is a surjection and hence $\mathcal{C}l(\ker(\mathcal{A}) \cap \mathbf{N}^u) \cong \mathbf{Z}^2$. $\square$

3. Rings with finite representation type. Throughout this section, $(R, \mathfrak{m})$ is a one-dimensional local ring with finite representation type. The characterization of these rings was completed in 1994. The classification of such rings is given by the following theorem, whose proof is summarized in [**4**].

Theorem 3.1. *Let R be a one-dimensional Cohen-Macaulay local ring with maximal ideal $\mathfrak{m}$. Then R has finite representation type if and only if*

1. *R is reduced*;

2. *The normalization $\overline{R}$ of R is generated by at most three elements as an R-module*; *and*

3. *$\mathfrak{m}\overline{R}/\mathfrak{m}$ is cyclic as an R-module.*

Over the past 40 years, an extensive series of authors, including Dade, Drozd and Roĭter, Green and Reiner, Jacobinski, Jones, and R. and S. Wiegand have studied indecomposable modules over these and other local rings, cf. [**5, 6, 11–13, 20**]. In [**2**], the first author showed that if R is in addition equicharacteristic and local with perfect residue field and characteristic not 2, 3 or 5, then the bound on the ranks is 3. Moreover, a complete list of all possible rank tuples was given for these rings. Recently, in [**3**], the authors finalized the list of ranks that occur for each of the rings classified in Theorem 3.1. The following theorem summarizes these results.

Theorem 3.2. *Let R be a Noetherian one-dimensional reduced local ring with finitely generated normalization, and assume that there is a bound on the ranks of indecomposable finitely generated torsion-free R-modules, equivalently R has finite representation type. Then R has at most three minimal prime ideals. Moreover*:

1. *If R is a domain, then every indecomposable module has rank* 1, 2 *or* 3.

2. *If R has exactly two minimal prime ideals, then every indecomposable module has rank*

$$(0,1),(1,0),(1,1),(1,2),(2,1) \quad \text{or} \quad (2,2).$$

3. *If R has exactly three minimal prime ideals then, with a suitable ordering of the minimal prime ideals, every indecomposable module has rank*

$$(0,0,1),(0,1,0),(1,0,0),(0,1,1),(1,0,1),(1,1,0),(1,1,1) \textit{ or } (2,1,1).$$

We apply the results of Section 2 to rings with finite representation type. If $(R,\mathfrak{m})$ has finite representation type, then $\operatorname{spl}(R) \in \{0,1,2\}$ and the number of minimal prime ideals of the $\mathfrak{m}$-adic completion $\widehat{R}$ is either 1, 2 or 3. Lemmas 2.3 and 2.4 and Theorem 3.2 give explicit descriptions of $\mathcal{T}(R)$ when R is a local ring-order with finite representation type.

Suppose that $\operatorname{spl}(R) = 0$. Then every torsion-free $\widehat{R}$-module is extended from an R-module and thus, $\mathcal{T}(R) \cong \mathcal{T}(\widehat{R}) \cong \mathbf{N}^t$ where t is the number of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules.

Now suppose that $\mathrm{spl}\,(R) = 1$. Then applying methods of Section 2, we see that $\mathcal{T}(R)$ is isomorphic to $\ker(\mathcal{A}) \cap \mathbf{N}^{t-s} \oplus \mathbf{N}^s$ where t is the number of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules and s is the number of indecomposable torsion-free $\widehat{R}$-modules with constant rank. We note that, in this case, each entry of matrix $\mathcal{A}$ is nonzero. Equivalently, we could allow matrix $\mathcal{A}$ to contain s zeros in which case $\mathcal{T}(R) \cong \ker(\mathcal{A}) \cap \mathbf{N}^t$. For notational ease, we stick with the first formulation of $\mathcal{T}(R)$. Considering the possible ranks given in Theorem 3.2, it must be the case that the entries of $\mathcal{A}$ are all ones and negative ones. If P and Q are two minimal prime ideals lying over a common prime ideal of R, then the number of ones in matrix $\mathcal{A}$ corresponds to the number of indecomposable torsion-free $\widehat{R}$-modules M such that $\mathrm{rank}_P(M) - \mathrm{rank}_Q(M) = 1$, and the number of negative ones in matrix $\mathcal{A}$ correspond to the number of indecomposable torsion-free $\widehat{R}$-modules M such that $\mathrm{rank}_P(M) - \mathrm{rank}_Q(M) = -1$. From Lemma 2.3 we see that $\mathcal{T}(R) \subseteq \mathcal{T}(\widehat{R})$ is a divisor theory if and only if there are at least two positive and at least two negative entries in $\mathcal{A}$. By considering the ranks listed in Theorem 3.2, we see that the only time this does not occur is when the following condition is satisfied.

($\dagger$) The local ring-order $(R, \mathfrak{m})$ is a domain, the $\mathfrak{m}$-adic completion $\widehat{R}$ has two minimal primes P and Q and up to isomorphism, either $\widehat{R}/P$ is the only indecomposable torsion-free $\widehat{R}$-module of rank (r, s) with $r - s = 1$ or the $\widehat{R}/Q$ is the only indecomposable torsion-free $\widehat{R}$-module of rank (r, s) with $r - s = -1$.

Finally, suppose that $\mathrm{spl}\,(R) = 2$. That is, $\widehat{R}$ has three minimal primes and R is a domain. In this case, after applying the methods of Section 2, we have that $\mathcal{T}(R) \cong \ker(\mathcal{A}) \cap \mathbf{N}^u$ where u is the number of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules and where $\mathcal{A}$ is as in Lemma 2.4. Moreover, considering the possible ranks given in Theorem 3.2, all nonzero columns of $\mathcal{A}$ are identical to one of the six vectors in (2) and each of these six vectors occurs as at least one column.

These results are summarized in the following proposition.

Proposition 3.3. *Let R denote a local ring-order with finite representation type. If $\mathrm{spl}\,(R) \neq 1$, then the class group of the monoid $\mathcal{T}(R)$*

depends only on the splitting number $\operatorname{spl}(R) = \#\operatorname{Spec}(\widehat{R}) - \#\operatorname{Spec}(R)$. *If* $\operatorname{spl}(R) = 1$, *then* $\mathcal{C}l(\mathcal{T}(R))$ *depends on condition* (†). *The results are summarized in Table* 1. *The integer t denotes the number of non-isomorphic indecomposable torsion-free $\widehat{R}$-modules while the integer s denotes the number of non-isomorphic indecomposable torsion-free $\widehat{R}$-modules with constant rank.*

TABLE 1. Monoids of torsion-free modules.

minimal prime ideals of $\widehat{R}$	$\operatorname{spl}(R)$	† (yes/no)	$\mathcal{T}(\widehat{R})$	$\mathcal{T}(R)$	$\mathcal{C}l(\mathcal{T}(R))$
1, 2 or 3	0		$\mathbf{N}^t$	$\mathbf{N}^t$	0
2	1	no	$\mathbf{N}^t$	$\mathbf{N}^{t-1}$	0
2	1	yes	$\mathbf{N}^t$	$(\ker(\mathcal{A}_1) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$	$\mathbf{Z}$
3	1		$\mathbf{N}^t$	$(\ker(\mathcal{A}_1) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$	$\mathbf{Z}$
3	2		$\mathbf{N}^t$	$(\ker(\mathcal{A}_2) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$	$\mathbf{Z} \oplus \mathbf{Z}$

The matrices in Table 1 *are as follows*:

• $\mathcal{A}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \end{bmatrix}_{1\times(t-s)}$ *where the number of ones corresponds to the number of indecomposable $\widehat{R}$-modules M with* $\operatorname{rank}_P(M) - \operatorname{rank}_Q(M) > 0$ *and the number of negative ones corresponds to the number of indecomposable $\widehat{R}$-modules M with* $\operatorname{rank}_P(M) - \operatorname{rank}_Q(M) < 0$ *for some predetermined order of the two minimal prime ideals P and Q of $\widehat{R}$ with $P \cap R = Q \cap R$.*

• $\mathcal{A}_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 1 & -1 & 1 & -1 & 0 & \cdots \end{bmatrix}_{2\times t-s}$ *where each of the remaining $t-s-6$ columns is identical to one of the first six columns.*

We now give a complete description of the direct sum decompositions that can occur over one-dimensional local rings with finite representation type. As we see, the monoid $\mathcal{T}(R)$ is almost always half-factorial and the elasticity of $\mathcal{T}(R)$ never exceeds 3/2.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a one-dimensional local ring with finite representation type, and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion.*

1. *If* $\operatorname{spl}(R) = 0$, *then* $\mathcal{T}(R)$ *is factorial and the Krull-Remak-Schmidt property holds for all finitely generated torsion-free R-modules.*

2. *If* $\operatorname{spl}(R) = 1$*, then* $\mathcal{T}(R)$ *is half-factorial. Moreover,* $\mathcal{T}(R)$ *is factorial if and only if* $\widehat{R}$ *has two minimal prime ideals and at least one of the following two conditions holds.*

(a) *There is, up to isomorphism, exactly one* $\widehat{R}$*-module with rank* $(1,0)$ *and there are no indecomposable* $\widehat{R}$*-modules of rank* $(2,1)$.

(b) *There is, up to isomorphism, exactly one* $\widehat{R}$*-module with rank* $(0,1)$ *and there are no indecomposable* $\widehat{R}$*-modules of rank* $(1,2)$.

3. *If* $\operatorname{spl}(R) = 2$*, then* $\rho(\mathcal{T}(R)) = 3/2$ *and thus* $\mathcal{T}(R)$ *is not half-factorial.*

Proof. If $\operatorname{spl}(R) = 0$, then by Lemma 2.2 all $\widehat{R}$-modules are extended, and hence $\mathcal{T}(R) \cong \mathcal{T}(\widehat{R})$ is free.

If $\operatorname{spl}(R) = 1$, then there are two minimal prime ideals P and Q of $\widehat{R}$ with $P \cap R = Q \cap R$. Then $\mathcal{T}(R) \cong (\ker(\mathcal{A}_1) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$ where $\mathcal{A}_1$ is as in Proposition 3.3, t is the number of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules and s is the number of isomorphism classes of indecomposable torsion-free $\widehat{R}$-modules with constant rank over the two prime ideals P and Q.

Note that the matrix $\mathcal{A}_1$ contains only one positive entry if and only if there is, up to isomorphism, exactly one indecomposable $\widehat{R}$-module M with $\operatorname{rank}_P M - \operatorname{rank}_Q M = 1$ and $\mathcal{A}_1$ contains only one negative entry if and only if there is, up to isomorphism, exactly one indecomposable $\widehat{R}$-module N with $\operatorname{rank}_P N - \operatorname{rank}_Q N = -1$. If $\widehat{R}$ has three minimal prime ideals, then neither of these conditions holds as evidenced by the existence of the indecomposable $\widehat{R}$-modules $\widehat{R}/P_i$ for $i \in \{1,2,3\}$ and $\widehat{R}/(P_i \cap P_j)$ with $i,j \in \{1,2,3\}$ distinct. Thus, we see that one of these two conditions holds precisely when one of properties 2 (a) or 2 (b) holds. In either of these cases, $\mathcal{T}(R)$ has trivial class group and is free.

Now, if $\operatorname{spl}(R) = 1$ and both of properties 2 (a) and 2 (b) fail to hold, then from Proposition 3.3 and Lemma 2.3 we see that $\mathcal{C}l(\mathcal{T}(R)) \cong \mathbf{Z}$. From Lemma 2.2, the set of prime ideal divisor classes in $\mathcal{C}l(\mathcal{T}(R))$ is $G_0 = \{0,1,-1\}$. Now $\mathcal{B}(G_0) \cong \mathbf{N}^2$ which is half-factorial, implying that $\mathcal{T}(R)$ too is half-factorial. However, $\mathcal{T}(R)$ is not factorial. In particular, if A, B, C and D are indecomposable $\widehat{R}$-modules with ranks $(1,0)$, $(2,1)$, $(0,1)$ and $(1,2)$, respectively, then $A \oplus C$, $A \oplus D$, $B \oplus C$

and $B \oplus D$ are extended $\widehat{R}$-modules, no direct summand is extended, and the isomorphism $(A \oplus C) \oplus (B \oplus D) \cong (A \oplus D) \oplus (B \oplus C)$ exhibits the failure of Krull-Remak-Schmidt over R.

Finally, suppose that $\operatorname{spl}(R) = 2$. Then $\mathcal{T}(R) \cong (\ker(\mathcal{A}_2) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$ where $\mathcal{A}_2$ is as in Proposition 3.3. From Lemma 2.4 we see that $\mathcal{C}l(\mathcal{T}(R)) \cong \mathbf{Z}^2$. Moreover, by Lemma 2.2 the set of the prime ideal divisor classes is

$$\{(1,0), (-1,0), (0,1), (0,-1), (1,-1), (-1,1)\}$$

and thus the block monoid is

$$\mathcal{B}(\mathcal{T}(R)) \cong \ker \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix} \bigcap \mathbf{N}^6.$$

Using the algorithm from [**14**, Section 2] we easily determine that $\rho(\mathcal{T}(R)) = \rho(H) = 3/2$. $\square$

The elasticity in the monoid $\mathcal{B}(\mathcal{T}(R)) \cong \ker \left[\begin{smallmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{smallmatrix}\right] \bigcap \mathbf{N}^6$ can be exhibited by the equation $\alpha_1 + \alpha_2 + \alpha_3 = \alpha_4 + \alpha_5$ where $\alpha_1 = (1,0,0,0,0,1)^t$, $\alpha_2 = (0,0,1,1,0,0)^t$, $\alpha_3 = (0,1,0,0,1,0)^t$, $\alpha_4 = (1,1,1,0,0,0)^t$ and $\alpha_5 = (0,0,0,1,1,1)^t$ are five distinct atoms. Moreover, this failure of unique factorization in the Diophantine monoid corresponds to the following isomorphism of $\widehat{R}$-modules:

$$\begin{aligned} \left(\frac{\widehat{R}}{P_1} \oplus \frac{\widehat{R}}{P_2 \cap P_3}\right) \bigoplus \left(\frac{\widehat{R}}{P_2} \oplus \frac{\widehat{R}}{P_1 \cap P_3}\right) \bigoplus \left(\frac{\widehat{R}}{P_3} \oplus \frac{\widehat{R}}{P_1 \cap P_2}\right) \\ \cong \left(\frac{\widehat{R}}{P_1} \oplus \frac{\widehat{R}}{P_2} \oplus \frac{\widehat{R}}{P_3}\right) \\ \bigoplus \left(\frac{\widehat{R}}{P_1 \cap P_2} \oplus \frac{\widehat{R}}{P_1 \cap P_3} \oplus \frac{\widehat{R}}{P_2 \cap P_3}\right). \end{aligned}$$

As is suggested by this isomorphism, the failure of $\mathcal{T}(R)$ to be half-factorial falls not on the existence of non-trivial ranks of indecomposable torsion-free $\widehat{R}$-modules, but on $\operatorname{spl}(R) > 1$. As the next result shows, we can find examples of arbitrarily large elasticities for $\mathcal{T}(R)$ simply by finding a local ring-order that is a domain and whose completion has a large number of minimal prime ideals. Such examples exist in abundance, cf. [**15**], although these rings do not have finite representation type whenever $\operatorname{spl}(R) > 2$.

Proposition 3.5. *Let $(R, \mathfrak{m})$ denote a one-dimensional local ring with reduced completion $\widehat{R}$ such that there are $n \geq 1$ minimal prime ideals lying over the same prime ideal of R. Then $\rho(\mathcal{T}(R)) \geq n/2$.*

Proof. We prove this result in the case where R is a domain and $P_1, \ldots, P_n$ are the distinct minimal prime ideals of $\widehat{R}$. The proof of the more general case is similar. If $\widehat{R}$ is also a domain, then all $\widehat{R}$-modules are extended and thus $\mathcal{T}(R)$ is factorial. Then $\rho(\mathcal{T}(R)) = 1$ and the result trivially holds. We now assume that $n \geq 2$. For each i, $1 \leq i \leq n$, there are indecomposable torsion-free $\widehat{R}$-modules $A_i = \widehat{R}/P_i$ and $B_i = \widehat{R}/(P_1 \cap \cdots P_{i-1} \cap P_{i+1} \cap \cdots \cap P_n)$. Let $A = \oplus_{i=1}^n A_i$, $B = \oplus_{i=1}^n B_i$, $C_i = A_i \oplus B_i$ for each i, and $C = \oplus_{i=1}^n C_i$. Since R is a domain, an $\widehat{R}$-module is extended if and only if it has constant rank. By considering ranks, we see that A, B and each C_i are minimally extended. Since $C = \oplus_{i=1}^n C_i = A \oplus B$, we have that $\rho(C) \geq n/2$ and thus $\rho(\mathcal{T}(R)) \geq n/2$. $\square$

We now return to ring-orders with finite representation type. Very explicit calculations of the monoids $\mathcal{T}(R)$ are given in [**2**, Table 2] in the case where R is equicharacteristic with perfect residue field having characteristic different from 2, 3 and 5. The results from Proposition 3.3 give very similar results in the more general case. However, we still have the following open question:

Question 3.6. Let R be an arbitrary local ring-order with finite representation type. How many, if any, indecomposable torsion-free R-modules are there of each possible rank?

Although probably unanswerable in general, in certain cases such as those rings considered in [**2**], we can determine exactly which ranks occur and in what numbers. Aside from the case when $\operatorname{spl}(R) = 1$ and one of properties 2 (a) or 2 (b) from Theorem 3.4 fails to hold, the answer to this question has no effect on the elasticity of $\mathcal{T}(R)$.

We conclude, in Propositions 3.7 and 3.8, with a result which gives a partial Krull-Remak-Schmidt theorem for local ring-orders with finite representation type in terms of the number of irreducible elements of $\mathcal{T}(R)$ and $\mathcal{T}(\widehat{R})$.

Proposition 3.7. *Let $(R,\mathfrak{m})$ be a local ring-order with finite representation type, and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion. Suppose that, up to isomorphism, $\widehat{R}$ has exactly b indecomposable torsion-free modules and R has exactly a indecomposable torsion-free modules. If the Krull-Remak-Schmidt theorem holds for all finitely generated torsion-free R-modules, then either $b = a$ or $b = a + 1$.*

Proof. Suppose that the Krull-Remak-Schmidt property holds for all finitely generated torsion-free R-modules. By Theorem 3.4 either $\operatorname{spl}(R) = 0$ or $\operatorname{spl}(R) = 1$ and condition (†) is satisfied.

If $\operatorname{spl}(R) = 0$, then by Proposition 2.1 all $\widehat{R}$-modules are extended from R-modules, and there is a one-to-one correspondence between the indecomposable R-modules and the indecomposable $\widehat{R}$-modules. Moreover, by faithful flatness, an R-module M is indecomposable if and only if $\widehat{M}$ is indecomposable as an $\widehat{R}$-module. Therefore, $b = a$.

Now assume that $\operatorname{spl}(R) = 1$, and that condition (†) is satisfied. Let s denote the number of indecomposable $\widehat{R}$-modules with constant rank. Suppose that $\widehat{R}/P$ is the only indecomposable $\widehat{R}$-module with rank (r,s) with $r > s$ and let $C_1, C_2, \dots, C_n$ denote the indecomposable $\widehat{R}$-modules with rank (r,s) such that $r < s$. Then $b = s + n + 1$. Let $M_1, M_2, \dots, M_a$ denote the non-isomorphic indecomposable R-modules. Then from Proposition 2.1, $\widehat{M_1}, \widehat{M_2}, \dots, \widehat{M_a}$ all have constant rank as $\widehat{R}$-modules. Suppose that $\widehat{M_i} \cong N_{i_1} \oplus N_{i_2} \oplus \cdots \oplus N_{i_u}$ for some indecomposable $\widehat{R}$-modules N_{i_j}, $1 \le j \le u$. By hypothesis, each N_{i_j} has constant rank or is isomorphic to either $\widehat{R}/P$ or C_k for some k. If N_{i_j} has constant rank for some j, then so does $L := \widehat{M_i}/N_{i_j}$. Thus $\widehat{M_i} \cong N_{i_j} \oplus L$ where both N_{i_j} and L are extended from R-modules. By faithful flatness, $L \cong 0$ since M_i was assumed to be indecomposable. If N_{i_j} does not have constant rank, then it is isomorphic to one of $\widehat{R}/P$ or C_k for some k. Note that any direct summand of $\widehat{M_i}$ which has, as a direct summand, an equal number of copies isomorphic to $\widehat{R}/P$ and of C, is extended. Therefore, for each i, either $\widehat{M_i}$ is indecomposable as an $\widehat{R}$-module, or $\widehat{M_i} \cong \widehat{R}/P \oplus C_k$ for some k. That is, $a = s + k$ and hence $b = a + 1$. The case where $\widehat{R}/Q$ is the only indecomposable $\widehat{R}$-module with rank (r, s) with $r < s$ is handled similarly. □

As the following proposition shows, the converse of Proposition 3.7 nearly always holds. Together, Propositions 3.7 and 3.8 say that, except for the three exceptions in Proposition 3.8 (2), (3) and (4), the Krull-Remak-Schmidt theorem holds if and only if $b = a$ or $b = a + 1$.

Proposition 3.8. *Let $(R, \mathfrak{m})$ be a local ring-order with finite representation type, and let $\widehat{R}$ denote its $\mathfrak{m}$-adic completion. Suppose that, up to isomorphism, $\widehat{R}$ has exactly b indecomposable torsion-free modules and R has exactly a indecomposable torsion-free modules. Further suppose that either $b = a$ or $b = a+1$. Then one of the following holds.*

1. *The Krull-Remak-Schmidt theorem holds for all finitely generated torsion-free R-modules.*

2. $\operatorname{spl}(R) = 1$ *and $\widehat{R}$ has exactly four indecomposable torsion-free modules with nonconstant rank.*

3. $\operatorname{spl}(R) = 2$ *and $\widehat{R}$ has exactly six indecomposable torsion-free modules with nonconstant rank.*

4. $\operatorname{spl}(R) = 2$ *and $\widehat{R}$ has exactly seven indecomposable torsion-free modules with nonconstant rank.*

Proof. We consider three cases. If $\operatorname{spl}(R) = 0$, then all finitely generated $\widehat{R}$-modules are extended from R-modules and hence (1) is satisfied.

Suppose now that $\operatorname{spl}(R) = 1$ with P and Q are minimal prime ideals of $\widehat{R}$ with $P \cap R = Q \cap R$. Then either $\mathcal{T}(R)$ is free or $\mathcal{T}(R) \cong (\ker(\mathcal{A}_1) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$ as given in Proposition 3.3. If we let x denote the number of ones in the matrix $\mathcal{A}_1$ and y the number of negative ones in the matrix $\mathcal{A}_1$, then the number of indecomposable R-modules is $s + xy$.

If we assume that $b = a$, then $s + x + y = s + xy$, that is, $x + y = xy$. The only nonnegative integer solutions to this equation are $x = y = 2$ or $x = y = 0$. In the former case, $\widehat{R}$ has exactly four indecomposable torsion-free modules with nonconstant rank, i.e., condition (2) is satisfied. The latter case is impossible since $\widehat{R}/P$ and $\widehat{R}/Q$ are both indecomposable $\widehat{R}$-modules with non-constant rank.

If we assume that $b = a + 1$, then we have $x + y = xy + 1$. The only positive integer solutions to this equation are $(1, y)$ for any $y \geq 1$ and $(x, 1)$ for any $x \geq 1$. In either case, we see from Theorem 3.4 that (1) is satisfied.

Finally, suppose that $\operatorname{spl}(R) = 2$. If $a = b$, we prove that $\widehat{R}$ has exactly seven indecomposable torsion-free modules with nonconstant rank. Note that $\mathcal{T}(R) \cong (\ker(\mathcal{A}_2) \cap \mathbf{N}^{t-s}) \oplus \mathbf{N}^s$ as given in Proposition 3.3. Each column of $\mathcal{A}_2$ is identical to one of the first six columns. If u, v, w, x, y and z denote the number of columns of $\mathcal{A}_2$ identical to columns one through six, respectively, then

$$u + v + w + x + y + z = uz + vy + wx + uvw + xyz$$

since the left hand side of this equation represents the number of indecomposable $\widehat{R}$-modules with non-constant rank and the right hand side represents the number of indecomposable R-modules whose completions are not indecomposable $\widehat{R}$-modules. Moreover, each of u, v, w, x, y and z are positive integers.

If each variable is at least two, then $u + z \leq uz$, $v + y \leq vy$ and $w + x \leq wx$. Since uvw and xyz are positive, we then have $u+v+w+x+y+z < uz+vy+wx+uvw+xyz$, which is impossible. Thus we assume, without loss of generality, that $u = 1$. We have

$$\begin{aligned} 1 + v + w + x + y &= vy + wx + vw + xyz \\ &= v(y + w) + x(w + yz) \\ &\geq v(y + w) + x(w + y) \\ &= (v + x)(y + w) \end{aligned}$$

since $z \geq 1$.

Setting $\alpha := y + w$ and $\beta := v + x$, we have $1 + \alpha + \beta \geq \alpha\beta$ with $\alpha, \beta \geq 2$. The only solutions here are (1) $\alpha = \beta = 2$, (2) $\alpha = 2$ and $\beta = 3$ and (3) $\alpha = 3$ and $\beta = 2$. Solution (1) gives $u = v = w = x = y = 1$ and $z = 2$. Solution (2) means all variables are one except one of v or x is two. Solution (3) means all variables are one except one of y or w is two. If $u > 1$, then some other variable must be one and a similar argument shows that all but one variable must be one, while a lone variable is two. In any of these situations, $\widehat{R}$ has exactly seven indecomposable torsion-free modules with nonconstant rank. In particular, condition (4) is satisfied.

A similar argument shows that if $a + 1 = b$, then $\widehat{R}$ has exactly six indecomposable torsion-free modules with nonconstant rank, in which case condition (3) is satisfied. $\square$

We conclude with two examples which illustrate the existence of local ring-orders satisfying (2) and (3) of Proposition 3.8, but for which the Krull-Remak-Schmidt theorem does not hold over R. The authors know of no example where condition (4) of Proposition 3.8 holds.

Example 3.9. Let k be a perfect field with characteristic different from 2, 3 and 5. Let $(R, \mathfrak{m})$ be a local integral domain with $\mathfrak{m}$-adic completion $\widehat{R} = k[[x, y]]/(x^2y - y^4)$. Such a ring exists by a theorem of Lech, cf. [**15**]. From [**2**], we see that up to isomorphism, the ring $\widehat{R}$ has exactly eight indecomposable torsion-free modules: one each of rank $(1, 0)$ and $(2, 1)$, two of rank $(0, 1)$ and four of rank $(1, 1)$. Then $\mathcal{T}(R) \cong \big(\ker \, {\scriptstyle 1 \; 1 \; -1 \; -1} \cap \mathbf{N}^4\big) \oplus \mathbf{N}^4$, and hence $\mathcal{T}(R)$ is half-factorial, but not factorial. However, up to isomorphism, both R and $\widehat{R}$ have exactly eight indecomposable torsion-free modules. In particular, both R and $\widehat{R}$ have exactly four indecomposable torsion-free modules with nonconstant rank.

Example 3.10. Let k be a perfect field with characteristic different from 2, 3 and 5. Let $(R, \mathfrak{m})$ be a local integral domain with $\mathfrak{m}$-adic completion $\widehat{R} = k[[x, y]]/(x^2y - y^3)$. Such a ring exists by a theorem of Lech, cf. [**15**]. From [**2**], we see that up to isomorphism, the ring $\widehat{R}$ has exactly nine indecomposable torsion-free modules: one each of rank $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ and three of rank $(1, 1, 1)$. Then

$$\mathcal{B}(\mathcal{T}(R)) \cong \ker \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{bmatrix} \bigcap \mathbf{N}^6 \oplus \mathbf{N}^3,$$

and hence $\mathcal{T}(R)$ is not half-factorial. However, up to isomorphism, R has exactly eight indecomposable torsion-free modules. In particular, both R and $\widehat{R}$ have exactly six indecomposable torsion-free modules with nonconstant rank.

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