JOURNAL OF COMMUTATIVE ALGEBRA
Volume 3, Number 3, Fall 2011

ON IDEAL EXTENSIONS OF IDEAL COMPLEMENTS

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ABSTRACT. In this note we give negative answers to a conjecture of Tomas Sauer. Specifically we prove that there exists an ideal $K \subset \mathbf{C}[x,y]$ that complements the space of polynomials of degree 3 such that no ideal containing K complements the space of polynomials of degree 2. We also give a characterization of zero-dimensional radical ideals in terms of extensions of ideal complements.

1. Introduction. Let $\mathbf{C}[\mathbf{x}] := \mathbf{C}[x_1, \dots, x_d]$ stand for the algebra of polynomials in d variables with complex coefficients and $\mathbf{C}_{\leq N}[\mathbf{x}]$ denote the linear subspace of $\mathbf{C}[\mathbf{x}]$ of polynomials of degree at most N. For an ideal $J \subset \mathbf{C}[\mathbf{x}]$ we use $\mathcal{V}(J)$ to denote the affine variety associated with this ideal.

The extensions of ideals complements is the object of investigations related to the multivariate Lagrange and Hermite interpolation (cf. [**3**–**5**]). Paraphrased, a result of Sauer and Xu [**5**] shows that every radical ideal that complements $\mathbf{C}_{\leq N}[\mathbf{x}]$ in $\mathbf{C}[\mathbf{x}]$ can be extended to a (zero-dimensional, radical) ideal that complements $\mathbf{C}_{\leq N-1}[\mathbf{x}]$. In other words, if $K \subset \mathbf{C}[\mathbf{x}]$ is a radical ideal such that

$$\mathbf{C}[\mathbf{x}] = \mathbf{C}_{\leq N}[\mathbf{x}] \oplus K, \tag{1.1}$$

then there exists a radical ideal $J \supset K$ such that

$$\mathbf{C}[\mathbf{x}] = \mathbf{C}_{\leq N-1}[\mathbf{x}] \oplus J. \tag{1.2}$$

Based on this result as well as some further evidence (cf. [**4**]), Sauer [**3**] made the following conjecture:

Conjecture 1.1. *If K is an arbitrary ideal in $\mathbf{C}[\mathbf{x}]$ satisfying* (1.1)*, then there exists an ideal $J \supset K$ satisfying* (1.2)*.*

2010 AMS *Mathematics subject classification.* Primary 13A15, 13E10, 41A05, 41A63.
Keywords and phrases. Ideal complement, radical ideal, zero-dimensional ideal.
Received by the editors on June 17, 2010, and in revised form on September 4, 2010.

DOI:10.1216/JCA-2011-3-3-295

In the next section of this note we will construct a counterexample to Conjecture 1.1 in the space $\mathbf{C}[x, y]$ of polynomials of two variables. (The conjecture was known to be false (cf. [**6**]) in three variables). Thus the two variable case is the first case in which Sauer's conjecture fails. In one variable, $\mathbf{C}_{\leq n}[x]$ complements the $(n+1)$-st power of maximal ideal $\langle x \rangle$ and hence complements any ideal J with $\dim \mathbf{C}[\mathbf{x}]/J = n+1$. Therefore, Sauer's conjecture holds in the univariate case.

In Section 3 we will show that a slight generalization of the above-mentioned property observed by Sauer and Xu in fact characterizes zero-dimensional radical ideals.

The best news is that all the proofs are extremely simple.

2. Counterexample to Conjecture 1.1. In this section we will construct a (curvilinear, primary) ideal $K \subset \mathbf{C}[x, y]$ such that K complements $\mathbf{C}_{\leq 3}[x, y]$ but no ideal $J \supset K$ complements $\mathbf{C}_{\leq 2}[x, y]$, thus constructing a counterexample to Conjecture 1.1 in two variables. Recall that the ideal K is curvilinear if the algebra $\mathbf{C}[\mathbf{x}]/K$ is curvilinear, i.e., is isomorphic to $\mathbf{C}[t]/\langle t^n \rangle$ for some n.

Example. Consider an algebra homomorphism $\phi : \mathbf{C}[x, y] \to \mathbf{C}[t]/\langle t^{10} \rangle$ defined by $\phi(1) = 1, \phi(x) = t$ and $\phi(y) = t^3 + t^4$. The restriction $\phi \mid_{\mathbf{C}_{\leq 3}[x,y]}$ of ϕ to the space $\mathbf{C}_{\leq 3}[x, y]$ of polynomials of degree at most three is given by the table

(2.1)

	1	t	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9
1	1	0	0	0	0	0	0	0	0	0
x	0	1	0	0	0	0	0	0	0	0
y	0	0	0	1	1	0	0	0	0	0
x^2	0	0	1	0	0	0	0	0	0	0
xy	0	0	0	0	1	1	0	0	0	0
y^2	0	0	0	0	0	0	1	2	1	0
x^3	0	0	0	1	0	0	0	0	0	0
x^2y	0	0	0	0	0	1	1	0	0	0
xy^2	0	0	0	0	0	0	0	1	2	1
y^3	0	0	0	0	0	0	0	0	0	1

Let $K := \ker \phi$. Since ϕ is a ring homomorphism $K \subset \mathbf{C}[x, y]$ is an ideal. Since the determinant of matrix (2.1) representing $\phi \mid_{\mathbf{k}_{\leq 3}[x,y]}$

is equal to $-3 \neq 0$ the homomorphism ϕ is a surjection, and thus K complements $\mathbf{C}_{\leq 3}[x,y]$. For the same reason $\mathbf{C}[x,y]/K$ is isomorphic $\mathbf{C}[t]/\langle t^{10}\rangle$ and K hence curvilinear, i.e., the subspace $\mathbf{C}_{\leq 9}[x] \subset \mathbf{C}[x,y]$ complements K. Finally, observe that $x^{10}, y^{10} \in K$, hence the associated variety

$$\mathcal{V}(K) = \{0\}$$

and K is primary.

Theorem 2.1. *No ideal $J \supset K$ complements $\mathbf{C}_{\leq 2}[x,y]$.*

Proof. Suppose that $J \supset K$ and

$$\mathbf{C}_{\leq 2}[x,y] \oplus J = \mathbf{C}[x,y]. \tag{2.2}$$

Then $\mathcal{V}(K) \supset \mathcal{V}(J) = \{0\}$ and the six-dimensional multiplication operator

$$\begin{array}{cccc} \mu_x : & \mathbf{C}[x,y]/J & \longrightarrow & \mathbf{C}[x,y]/J \\ & [f] & \longrightarrow & [xf] \end{array}$$

is nilpotent. Hence, $\mu_x^6 = 0$ and $x^6 \in J$.

By (2.1) $\phi(y^2 - x^6 - 2x^7 - x^8) = 0$. Hence, $y^2 - x^6 - 2x^7 - x^8 \in K \subset J$ and, since $x^6 \in J$, it follows that $y^2 \in J$ which contradicts (2.2). □

3. Characterization of zero-dimensional radical ideals. Recall that an ideal $J \subset \mathbf{C}[\mathbf{x}]$ is zero-dimensional if $\dim(\mathbf{C}[\mathbf{x}]/J) < \infty$. An ideal J is zero-dimensional if and only if $\#\mathcal{V}(J) < \infty$. An ideal $J \subset \mathbf{C}[\mathbf{x}]$ is radical if $f^m \in J$ implies $f \in J$. A zero-dimensional ideal is radical if and only if

$$\dim(\mathbf{C}[\mathbf{x}]/J) = \#\mathcal{V}(J).$$

Theorem 3.1 *Let K be an ideal $\mathbf{C}[\mathbf{x}]$ and G a subspace in $\mathbf{C}[\mathbf{x}]$ such that*

$$G \oplus K = \mathbf{C}[\mathbf{x}].$$

Then the following are equivalent:

(i) *K is a zero-dimensional radical ideal.*

(ii) *For every subspace $H \subset G$ there exists an ideal $J \supset K$ such that J complements H.*

Proof. (i) $\Rightarrow$ (ii): Let K be a zero-dimensional radical ideal. Then $\mathcal{V}(K) = \{\mathbf{z}_1, \dots, \mathbf{z}_N\}$ where $N = \dim G = \dim \mathbf{C}[\mathbf{x}]/K$. We will prove a slightly stronger statement than (ii). We will prove that for any $H \subset G$ there exists a radical ideal $J \supset K$ that complements H. By induction on $\dim G/H$ it is enough to prove this for a subspace $H \subset G$ of dimension $N-1$. Let H be such a subspace. For every $k = 1, \dots, N$, let $J^{(k)} \supset K$ be the radical ideal defined by

$$J^{(k)} := \{f \in \mathbf{C}[\mathbf{x}] : f(\mathbf{z}_j) = 0, j \neq k\}. \tag{3.1}$$

We claim that at least one of the ideals $J^{(k)}$ complements H. Since the colength of each $J^{(k)}$ is $N-1$, it is enough to prove that $H \cap J^{(k)} = 0$ for at least one k. If not, then for every $k = 1, \dots, N$, there exists an $h_k \in H$ such that $h_k(\mathbf{z}_j) = \delta_{k,j}$. Clearly, the functions $\{h_k, k = 1, \dots, N\}$ are linearly independent and thus span an N-dimensional space inside H, which contradicts the assumption that $\dim H = N-1$.

(ii) $\Rightarrow$ (i): G spans the algebra $\mathbf{C}[\mathbf{x}]/K$. Let $H \subset G$ be a subspace such that $H \subset \mathbf{C}[\mathbf{x}]/K$ is an ideal in $\mathbf{C}[\mathbf{x}]/K$. If $J \supset K$ complements H, then every ideal $H \subset \mathbf{C}[\mathbf{x}]/K$ is complemented by the ideal $J/K \subset \mathbf{C}[\mathbf{x}]/K$. Thus, (ii) implies that every submodule of $\mathbf{C}[\mathbf{x}]/K$ (considered as a module over itself) is a direct summand of $\mathbf{C}[\mathbf{x}]/K$. Hence the algebra $\mathbf{C}[\mathbf{x}]/K$ over $\mathbf{C}$ is semisimple. By the Wedderburn-Artin theorem [**2**] $\mathbf{C}[\mathbf{x}]/K$ is isomorphic to the finite direct sum of the copies of $\mathbf{C}$:

$$\mathbf{C}[\mathbf{x}]/J \simeq \mathbf{C} \oplus \cdots \oplus \mathbf{C}.$$

In particular, $\mathbf{C}[\mathbf{x}]/K$ is finite-dimensional and contains no nilpotent elements. Thus K is zero-dimensional and radical. $\square$

4. An open problem.

Problem 4.1. What subspaces of $\mathbf{C}[\mathbf{x}]$ have an ideal complement?

In one variable, a subspace $G \subset \mathbf{C}[x]$ has an ideal complement if and only if G is finite-dimensional or $G = \mathbf{C}[x]$. In several variables every finite-dimensional subspace of $\mathbf{C}[\mathbf{x}]$ is complemented by (a radical) ideal. No subspaces of $\mathbf{C}[\mathbf{x}]$ of finite codimension have an ideal complement since there are no finite-dimensional ideals in $\mathbf{C}[\mathbf{x}]$. Some subspaces of infinite dimension and codimension (such as $\mathbf{C}[x] \subset \mathbf{C}[x,y]$) do have an ideal complement ($y \cdot \mathbf{C}[x,y]$) and some do not; for instance no ideal in $\mathbf{C}[\mathbf{x}]$ has an ideal complement.

Perhaps the first step is to classify the subspaces of $\mathbf{C}[\mathbf{x}]$ that are complemented by a particular type of ideals, such as monomial ideals, primary ideals, etc. Here is an example of such result graciously donated to the paper by Seceleanu:

Theorem 4.1. *The following are equivalent*:

(i) *G is a complement of an Artinian monomial ideal J.*

(ii) *The support of G contains an order ideal that induces a maximal minor in the support matrix of G.*

Here by the *support of G* we mean the set of monomials appearing in the expressions of elements of G and by an *order ideal* we mean a set of monomials $\mathcal{O}$ such that if $u \in \mathcal{O}$ and $u' \mid u$ then $u' \in \mathcal{O}$. By a *support matrix* we mean a matrix with rows indexed by elements of a basis of G and columns indexed by monomials in $\operatorname{supp} G$ (the support matrix is unique up to canonical form).

Proof of Theorem 4.1. (1) $\Rightarrow$ (2) follows by letting $\mathcal{O}$ be the set of monomials that are not elements of J. Clearly this is an order ideal whose elements form a (canonical) basis for $\mathbf{C}[\mathbf{x}]/J$. Hence, the size of $\mathcal{O}$ equals the rank of G. Every basis element of G has a canonical reduction modulo J. This yields an isomorphism between G and the vector space spanned by $\mathcal{O}$ described precisely by the given maximal minor of the support matrix.

(2) $\Rightarrow$ (1) requires the process to be reversed. Let $\mathcal{O} \subset \operatorname{supp} G$ be an order ideal, and let J be a monomial ideal generated by the border of $\mathcal{O}$ defined as

$$\partial\mathcal{O} := (\cup_{i=1}^{d} x_i \mathcal{O}) \backslash \mathcal{O}.$$

Clearly

$$J \oplus \operatorname{span} \mathcal{O} = \mathbf{C}[\mathbf{x}]$$

and since both direct summands have monomial bases, it is easy to see that the monomials indexing the remaining columns of the support matrix are in J. From here and the direct sum decomposition above,

$$J \oplus G = \mathbf{C}[\mathbf{x}]$$

follows easily. □

Acknowledgments. We would like to thank Alexandra Seceleanu for many useful suggestions. In particular, Theorem 4.1 and its proof is entirely hers.

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