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BETTI NUMBERS OF SOME SEMIGROUP RINGS

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ABSTRACT. We compute the Betti numbers of all semigroup rings R corresponding to numerical semigroups of maximal embedding dimension. A description, in terms of the generators of S, precisely in which degrees the nonzero graded Betti numbers occur is given. We show that for arithmetic numerical semigroups of maximal embedding dimension, the graded Betti numbers occur symmetrically in two respects.

1. Introduction and preliminaries. By a *numerical semigroup* we mean a submonoid S of $\mathbf{N}$ such that $\mathbf{N}\setminus S$ is finite. $\mathbf{N}$ is understood to be the set of non-negative integers $\{0, 1, 2, \dots\}$. It is well known that such a semigroup is finitely generated, that is, it consists of all non-negative integer combinations of some minimal generating set $\{s_0, s_1, \dots, s_n\}$.

In this paper we only consider numerical semigroups and hence by semigroup we will always mean numerical semigroup. We denote a semigroup S minimally generated by the elements $\{s_0, s_1, \dots, s_n\}$ by $S = \langle s_0, s_1, \dots, s_n\rangle$. If nothing else is said we assume $s_0 < \dots < s_n$. Here the number $n+1$ is the *embedding dimension* of S and is denoted by $e(S)$. The number s_0 is called the *multiplicity* of S and is denoted by $m(S)$. We always have $e(S) \le m(S)$ and, if $e(S) = m(S)$, we say that S has *maximal embedding dimension*.

Given a semigroup $S = \langle s_0, \dots, s_n\rangle$ and a field k, consider the *semigroup ring* $R = k[S]$. This is the k-algebra $k[t^s\,;\, s \in S]$, t an indeterminate, defined by

$$t^s \cdot t^{s'} = t^{s+s'}, \quad s, s' \in S.$$

If A is the polynomial ring $k[x_0, \dots, x_n]$, we can define a homomorphism

$$A \xrightarrow{\phi} R$$

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by $x_i \mapsto t^{s_i}$. If we put $\deg(x_i) = s_i$ for all $i \in \{0, 1, \dots, n\}$ the map ϕ becomes homogeneous. This grading of A will be assumed throughout the paper unless otherwise is explicitly stated. By the standard grading of A we mean the grading given by $\deg(x_i) = 1$ for all $i \in \{0, 1, \dots, n\}$. Since ϕ is surjective we have a kernel $\ker \phi = I$, and consequently an isomorphism $R \cong A/I$. Clearly R is a domain, so I is a prime ideal.

2. Betti numbers. Recall that the Betti numbers of $R = k[S]$ over A are the invariants $\beta_i(R) = \dim \operatorname{Tor}_i^A(R, k)$. The Betti numbers inherit a grading from the chosen grading of A, and we denote by $\beta_{i,j}(R) = \dim \operatorname{Tor}_i^A(R, k)_j$ the ith Betti number in degree j.

Our results all rely on the following lemma.

Lemma 2.1. *Let $S = \langle s_0, \dots, s_n \rangle$ be a semigroup of maximal embedding dimension and R the corresponding semigroup ring. Put $\overline{A} = A/(x_0)$, and let $\mathfrak{m} = (x_1, \dots, x_n)$. Then*

$$\beta_{i,j}(R) = \beta_{i,j}(\overline{A}/\mathfrak{m}^2).$$

Proof. Let

$$\mathcal{G}. \quad \cdots \to \oplus_i A(-b_{2i}) \to \oplus_i A(-b_{1i}) \to A \to A/I \to 0$$

be a minimal A-free resolution of the semigroup ring A/I. Since x_0 is not a zero divisor on A or on A/I, by [**1**, Proposition 1.1.5], the tensored complex $\mathcal{G}. \otimes A/(x_0)$ is an A-free resolution of $A/I \otimes A/(x_0)$, that is in fact also minimal. Recall that $s_0 < \cdots < s_n$ and that x_0 correspond to t^{s_0} in the isomorphism $R \cong A/I$. Since S has maximal embedding dimension $\{s_0, \dots, s_n\}$ represents a full system of residue classes module s_0. This yields

$$A/I \otimes A/(x_0) \cong A/(I + (x_0)) \cong \overline{A}/\mathfrak{m}^2. \qquad \square \tag{1}$$

Remark 2.2. Because of Lemma 2.1, in the results below we will simply write $\beta_{i,j}(R)$ for the Betti numbers, even if the computations will be explicitly made for $R/(t^{s_0}) \cong \overline{A}/\mathfrak{m}^2$.

In the standard grading of A it is plain that the Hilbert series is of $\overline{A}/\mathfrak{m}^2$ is

$$H(\overline{A}/\mathfrak{m}^2; z) = 1 + nz = \frac{(1+nz)(1-z)^n}{(1-z)^n}. \tag{2}$$

It is well known (Theorem 4.1.13 in [**1**]) that the polynomial $(1+nz)(1-z)^n$ here may be written in the form

$$\sum_{i,j} (-1)^i \beta_{i,j}(\overline{A}/\mathfrak{m}^2) z^j. \tag{3}$$

Since $\overline{A}/\mathfrak{m}^2$ clearly has 2-linear resolution over A (that is, $\beta_{i,j}(\overline{A}/\mathfrak{m}^2) \neq 0$ only for $j = i+1$) when using the standard grading, we only have to identify the coefficients from the denominator of (2) with those from (3).

Proposition 2.3. *Let $S = \langle s_0, \dots, s_n \rangle$ be a semigroup of maximal embedding dimension, and let R be the corresponding semigroup ring. Then*

$$\beta_i(R) = \begin{cases} 1 & \text{if } i = 0 \\ i\dbinom{n+1}{i+1} & \text{if } i \geq 1. \end{cases}$$

Proof. From (1), (2), (3) and the equation

$$(1+nz)(1-z)^n = 1 + \sum_{k=1}^{n} \left[(-1)^k \binom{n}{k} + (-1)^{k-1} n \binom{n}{k-1} \right] z^k + (-1)^n n z^{n+1}$$

it follows that

$$\beta_i(R) = \begin{cases} 1 & \text{if } i = 0 \\ n\dbinom{n}{i} - \dbinom{n}{i+1} & \text{if } i \geq 1. \end{cases}$$

However, it is easily seen that $i\binom{n+1}{i+1} = n\binom{n}{i} - \binom{n}{i+1}$ for all $i \geq 1$. $\square$

Remark 2.4. The Betti numbers in Proposition 2.3 occur elsewhere, for example as the Betti numbers of certain graph algebras. We show what we mean by giving a second proof of the proposition.

Proof. Let $\mathcal{G}$ be a simple graph on n vertices and with edge set $\mathcal{E}(\mathcal{G})$. Recall that the *edge ideal* of $\mathcal{G}$ is the ideal

$$I(\mathcal{G}) = (x_i x_j\,;\,\{i,j\} \in \mathcal{E}(\mathcal{G})) \subseteq k[x_1, \dots, x_n].$$

For generalities about edge ideals, see [**8**, Chapter 6].

One may view (via polarization, see [**5**] for details) the ideal $\mathfrak{m}^2$ as the edge ideal $I(\mathcal{G})$ of the simple graph $\mathcal{G}$ on $2n$ vertices $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ whose edge set consists of all edges on the variables x_i, and all edges of the form $\{x_i, y_i\}$, $1 \leq i \leq n$. Also, consider the edge ideal $I(K_{n+1})$ of the complete graph K_{n+1} on $n+1$ vertices $\{z_1, \dots, z_n, w\}$. We define an onto map $S_1/I(\mathcal{G}) \xrightarrow{\varphi} S_2/I(K_{n+1})$ by

$$x_i \longmapsto z_i, \quad y_i \longmapsto w$$

for all $1 \leq i \leq n$. Here S_1 and S_2 are polynomial rings over k in $2n$ and $n+1$ variables, respectively. Clearly the ideal generated by (the images in the quotient of) the elements $y_1 - y_j$, $j > 1$, lies in the kernel which in turn must lie inside the ideal generated by (the images in the quotient of) the y_i, $1 \leq i \leq n$. Hence

$$\ker \varphi = (y_1 - y_j\,;\, j > 1).$$

It is not hard to see that the elements $y_1 - y_j$, $j > 1$, form a regular sequence on $S_1/I(\mathcal{G})$. Thus, using the same kind of arguments as in the proof of Lemma 2.1, we see that $S_1/I(\mathcal{G})$ and $S_2/I(K_{n+1})$ have the same graded Betti numbers. However, it is well known that (see [**6**, Theorem 5.1.1], or more generally [**4**, Theorem 3.1]) that these Betti numbers are

$$\beta_{i,j}(K_{n+1}) = \begin{cases} i\binom{n+1}{i+1} & \text{if } j = i+1 \\ 0 & \text{if } j \neq i+1. \end{cases} \qquad \square$$

Recall that the Betti diagram of an A-module is the diagram

	0	$\cdots$	s	$\cdots$
0	$\beta_{0,0}(M)$	$\cdots$	$\beta_{s,s}(M)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
i	$\beta_{0,i}(M)$	$\cdots$	$\beta_{s,s+i}(M)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

More about Betti diagrams may be found in for example [**2**].

As mentioned before Proposition 2.3, the standard grading of A yields a 2-linear resolution of $R \cong \overline{A}/\mathfrak{m}^2$. This is not the case if one use the grading given by $\deg(x_i) = s_i$ instead. This fact is illustrated in the following two examples.

Example 1. Consider the semigroup $S = \langle 5, 9, 13, 17, 21 \rangle$. Below we see the Betti diagram of $\overline{A}/\mathfrak{m}^2$ considering the standard grading of A.

	0	1	2	3	4
0	1	–	–	–	–
1	–	10	20	15	4

In the sequel we will collect the Betti numbers in tables of the following form instead of using the Betti diagrams. The numbers in the column to the right are $j(\beta_{i,j}(\overline{A}/\mathfrak{m}^2))$.

i	β_i	j
1	10	2(10)
2	20	3(20)
3	15	4(15)
4	4	5(4)

Example 2. We consider the same semigroup as in the previous example but instead with the grading of A defined by $\deg(x_i) = s_i$. We get the following table:

i	β_i	j							
1	10	18(1)	22(1)	26(2)	30(2)	34(2)	38(1)	42(1)	
2	20	31(1)	35(2)	39(3)	43(4)	47(4)	51(3)	55(2)	59(1)
3	15	48(1)	52(2)	56(3)	60(3)	64(3)	68(2)	72(1)	
4	4	69(1)	73(1)	77(1)	81(1)				

We now determine, in general, the degrees j for which $\beta_{i,j}(R)$ is nonzero. Since $\mathfrak{m}^2$ is a stable ideal the *Eliahou-Kervaire resolution*, see [**3**] for details, provides a minimal A-free resolution of $\overline{A}/\mathfrak{m}^2$. The minimal generators of L_i, the ith component of the Eliahou-Kervaire resolution, are the symbols $e(\sigma, u)$ where $\sigma = (q_1, \dots, q_i)$ is a sequence of integers satisfying

$$1 \leq q_1 < \cdots < q_i < \max u, \tag{4}$$

and u a minimal generator of $\mathfrak{m}^2$. Here $\max(u)$ denotes the maximal index of a variable x_i that divides u. For $i = 0$ the condition (4) is considered as void, so that the symbols of L_0 are in one-to-one correspondence with the minimal generators of $\mathfrak{m}^2$. The Eliahou-Kervaire resolution is in fact graded and in the standard grading of A the degree of a symbol $e(\sigma, u) \in L_i$ is by definition $\deg(u) + i$. Thus in our case, in the standard grading of A, the degree of $e(\sigma, u) \in L_i$ is $2 + i$.

Remark 2.5. Note that the Eliahou-Kervaire resolution resolves the ideal $\mathfrak{m}^2$. Thus, in the formulas below the homological degrees are shifted one step since we resolve $\overline{A}/\mathfrak{m}^2$.

Lemma 2.6. *Let $S = \langle s_0, \dots, s_n \rangle$ be a semigroup of maximal embedding dimension and R the corresponding semigroup ring. Then $\beta_{i+1,j}(R)$ is nonzero precisely in the degrees j that may be written*

$$j = s_k + s_l + s_{q_1} + \cdots + s_{q_i} \tag{5}$$

for some $1 \leq k \leq l \leq n$ and $1 \leq q_1 < \cdots < q_i < l$, and equals the number of different ways in which this can be done.

Proof. Recall that if $u = x_k x_l$, in the standard grading the degree of a symbol $e(\sigma, u) \in L_i$ is $\deg(x_k x_l) + i$. If we translate this via the isomorphism $R \cong A/I$ to the corresponding A-free resolution of R in the grading given by $\deg(x_i) = s_i$, we see that the degree of the symbol $e(\sigma, u)$ becomes $s_k + s_l + s_{q_1} + \cdots + s_{q_i}$.

Let $j \in \mathbf{N}$. Recall that a partition of j with i parts on a set $I \subseteq \{1, 2, \ldots, j\}$, is an expression

$$j = x_1 + \cdots + x_i$$

where $1 \leq x_1 \leq \cdots \leq x_i$ and $x_k \in I$ for all $k \in \{1, \ldots, i\}$. We denote the number of partitions of j with i parts on the set $\{1, 2, \ldots, n\}$ by $p(j, i, n)$. Motivated by Lemma 2.6, we define an *Eliahou-Kervaire partition* of an integer j with $i + 2$ parts on the set $\{1, 2, \ldots, n\}$ to be a partition $j = k + l + q_1 + \cdots + q_i$ where

(i) $1 \leq k \leq l \leq n$

(ii) $1 \leq q_1 < \cdots < q_i < l$.

Also, let $\mathrm{EKP}\,(j, i, n)$ denote the number of Eliahou-Kervaire partitions of j with $i + 2$ parts on $\{1, 2, \ldots, n\}$.

Consider a semigroup $S = \langle s_0, \ldots, s_n \rangle$ of maximal embedding dimension with semigroup ring R. Recall that assuming $s_0 < \cdots < s_n$, $\{s_0, \ldots, s_n\}$ represents a full system of congruence classes modulo s_0. Thus we may reindex so that $s_i \equiv i \pmod{s_0}$. We assume this is done for the rest of this section.

If $\beta_{i+1,j}(R) \neq 0$ (so that there is a partition $j = s_k + s_l + s_{q_1} + \cdots + s_{q_i}$) we see that there is a number m_j such that $j = (k + l + q_1 + \cdots + q_i) + m_j s_0$. This is clear since for every $i \in \{0, 1, \ldots, n\}$, s_i may be written as $s_i = i + n_i s_0$. For a given i and every degree j, denote by $M_{i,j}$ the set of all such numbers m_j.

Proposition 2.7. *Let $S = \langle s_0, \ldots, s_n \rangle$ be a semigroup of maximal embedding dimension and R the corresponding semigroup ring. Then*

$$\beta_{i+1,j}(R) = \sum_{m_j \in M_{i,j}} \mathrm{EKP}\,(j - m_j s_0, i, n). \tag{6}$$

Proof. Assume $\beta_{i+1,j}(R)$ is nonzero and that $j = s_k + s_l + s_{q_1} + \cdots + s_{q_i} = (k + l + q_1 + \cdots + q_i) + m_j s_0$. Clearly every Eliahou-Kervaire partition of $j - m_j s_0$ with $i+2$ parts on $\{1, 2, \ldots, n\}$ will contribute by 1 to the number $\beta_{i+1,j}(R)$. Hence, taking the sum over all elements in $M_{i,j}$ yields $\beta_{i+1,j}(R)$. □

Example 3. Let $S = \langle 3, 5, 7\rangle$. We may use Lemma 2.6 directly to find the degrees where the relations generating the ideal I lie. The Betti numbers that keep track of this information lie in homological degree 1, so we have $i = 0$. Thus the degrees j are

$$\begin{cases} j = 5 + 5 = (2+2) + 2 \cdot 3 \\ j = 5 + 7 = (2+1) + 3 \cdot 3 \\ j = 7 + 7 = (1+1) + 4 \cdot 3. \end{cases}$$

From the expressions to the right we see that the corresponding Eliahou-Kervaire partitions are $4 = 2 + 2$, $3 = 2 + 1$ and $2 = 1 + 1$.

Assume we have an Eliahou-Kervaire partition $k + l + q_1 + \cdots + q_i$ of j. By subtracting h from q_{h+1} for every $h \in \{1, 2, \ldots, i-1\}$ we obtain a partition $q'_1 + \cdots + q'_i$ of $j - k - l - \binom{i}{2}$ with i parts on $\{1, 2, \ldots, n - i\}$. The number k still satisfies $1 \le k \le l$. This yields:

Lemma 2.8. *For the number* EKP (j, i, n) *we have the equality*

$$\mathrm{EKP}\,(j, i, n) = \sum_{\substack{i+1 \le l \le n \\ 1 \le k \le l}} p\Big(j - k - l - \binom{i}{2}, i, n - i\Big).$$

By inserting this into (6) we get

Theorem 2.9. *Let $S = \langle s_0, \ldots, s_n\rangle$ be a semigroup of maximal embedding dimension and R the corresponding semigroup ring. Then*

$$\beta_{i+1,j}(R) = \sum_{\substack{i+1 \le l \le n \\ 1 \le k \le l \\ m_j \in M_{i,j}}} p\Big(j - m_j s_0 - k - l - \binom{i}{2}, i, n - i\Big).$$

To make this theorem more explicit we give, in Proposition 2.12 below, a generating series for the numbers EKP (j, i, n) for fixed i and n. We will use the following two lemmas.

The following lemma describes, for any integer l, the bivariate generating series

$$G(x,y) = \sum_{\substack{n \geq 0 \\ k \geq 0}} p(n,k,l) y^k x^n.$$

Lemma 2.10. *Given a positive integer l, the bivariate generating series $G(x,y)$ of the numbers $p(n,k,l)$ is given by*

$$G(x,y) = \prod_{j=1}^{l} \frac{1}{1 - yx^j}. \tag{7}$$

If we consider the above generating series for fixed k, we have the *vertical generating series*

$$G_k(x) = \sum_{n \geq k} p(n,k,l) x^n.$$

We may use the vertical generating series to rewrite $G(x,y)$ as

$$G(x,y) = \sum_{k \geq 0} G_k(x) y^k. \tag{8}$$

Lemma 2.11. *With k fixed and the notations as above, we have*

$$G_k(x) = x^k \prod_{j=1}^{k} \frac{1 - x^{j+l-1}}{1 - x^k}.$$

Proof. It is easy to verify that $G(x,xy)(1 - yx^{l+1}) = G(x,y)(1 - yx)$. From this and equation (8) we get that

$$G_k(x) = \frac{x(1 - x^{k+l-1})}{1 - x^k} \cdot G_{k-1}(x),$$

which proves the assertion by induction. □

In light of Proposition 2.7 and Lemma 2.8 our interest lies in the coefficients of $G_i(x) = x^i \prod_{j=1}^{i}(1-x^{j+n-i-1})/(1-x^i)$ (we use $l = n-i$) that stand before powers of x with exponents of the form $j-k-l-\binom{i}{2}$, where $1 \le k \le l \le n$. Considering the double sum in Lemma 2.8 we see that the maximal value the expression $j-k-l-\binom{i}{2}$ takes there, is $j-i-\binom{i}{2}-2$. For each pair k, l of summation indices, we let $\alpha_{k,l}$ denote the number $k+l-i-2 = (j-i-\binom{i}{2}-2)-(j-k-l-\binom{i}{2})$. Observe that $\alpha_{k,l}$ does not depend on j. Using this we obtain the generating series for the numbers $\mathrm{EKP}\,(r,i,n)$:

Proposition 2.12. *The generating series for the numbers* $\mathrm{EKP}\,(j,i,n)$, *i and n fixed, is given by*

$$\sum_{j\ge 0} \mathrm{EKP}\,(j,i,n)x^r = x^{2i+\binom{i}{2}+2} \sum_{\substack{i+1\le l\le n\\ 1\le k\le l}} \left[x^{\alpha_{k,l}} \prod_{r=1}^{i} \frac{1-x^{r+n-i-1}}{1-x^i} \right].$$

Proof. We have added generating series of the summands that occur in Lemma 2.8 giving them an additional "weight," $x^{\alpha_{k,l}+2i+\binom{i}{2}+2}$, so that the coefficient before x^j in the sum counts precisely $\mathrm{EKP}\,(j,i,n)$. □

Example 4. Consider the semigroup $S = \langle 3,5,7\rangle$ from Example 3. We showed there that $\beta_{1,10}(R) = \beta_{1,12}(R) = \beta_{1,14}(R) = 1$. If we do the same kind of computations but with $i = 1$ instead we get

$$\begin{cases} = 5+5+5 = (2+2+2)+3\cdot 3\\ j = 5+5+7 = (2+2+1)+4\cdot 3\\ j = 5+7+7 = (2+1+1)+5\cdot 3\\ J = 7+7+7 = (1+1+1)+6\cdot 3. \end{cases}$$

Considering the right hand side expressions we see that only the middle two degrees j are in fact Betti degrees. We have $\beta_{2,17}(R) = \beta_{2,19}(R) =$

1. Computing the generating series from the above proposition (with $i = 1$ and $n = 2$) gives

$$\sum_{j\geq 0} \mathrm{EKP}\,(j,1,2)x^j = x^4 + x^5.$$

The exponents 4 and 5 here correspond to the Eliahou-Kerviare partitions $4 = 2+1+1$ and $5 = 2+2+1$.

We may also confirm the fact that $\mathrm{pd}_A(R) = 2$: Consider the following computations

$$\begin{cases} j = 5+5+5+5 = (2+2+2+2) + 4\cdot 3 \\ j = 5+5+5+7 = (2+2+2+1) + 5\cdot 3 \\ j = 5+5+7+7 = (2+2+1+1) + 6\cdot 3 \\ j = 5+7+7+7 = (2+1+1+1) + 7\cdot 3 \\ j = 7+7+7+7 = (1+1+1+1) + 8\cdot 3. \end{cases}$$

Non of the expressions on the right hand side give Eliahou-Kervaire partitions, hence $\beta_3(R) = 0$.

3. Arithmetic semigroups. We no longer assume that the minimal generators of S satisfy $s_i \equiv i \pmod{s_0}$. Consider the sets $M_{i,j}$ from Proposition 2.7. If $|M_{i,j}| = 1$ for every i and j, the description of the Betti numbers can be made more explicit. We call a semigroup $S = \langle s_0, \dots, s_n\rangle$ *arithmetic* if $s_i = s_0 + id$ for all $i \in \{0, \dots, n\}$, where d is some integer $1 \leq d < s_0$ with $\gcd(d, s_0) = 1$.

For $n = 3$, the following results are included in more general results by Sengupta, [**7**]. In [**7**] minimal resolutions for all monomial curves in $\mathbf{A}^4$ defined by an arithmetic sequence are given. Hence the information about the Betti numbers below can be obtained from these resolutions in the case where $n = 3$.

Proposition 3.1. it Let $S = \langle s_0, \dots, s_n\rangle$ be an arithmetic semigroup of maximal embedding dimension and R the corresponding semigroup ring. Assume $s_i = s_0 + id$ for every $i \in \{0, 1, \dots, n\}$. Then the following holds.

(i) *The nonzero Betti numbers $\beta_{i+1,j}(R)$ lie in degrees j that are of the form $(i+2)s_0 + m_j d$. The integer m_j is uniquely determined* *by j and m_j has an Eliahou-Kervaire partition with $i+2$ parts on $\{1,2,\dots,n\}$.*

(ii) *$\beta_{i+1,j}(R)$ equals the number of Eliahou-Kervaire partitions of m_j with $i+2$ parts on $\{1,2,\dots,n\}$.*

(iii) *The minimal and maximal degrees, $j_{\min}$ and $j_{\max}$, respectively, for which $\beta_{i+1,j}(R)$ is nonzero are*

$$j_{\min} = (i+2)s_0 + \left(1 + \binom{i+2}{2}\right)d$$
$$j_{\max} = (i+2)s_0 + \left((i+2)n - \binom{i+2}{2}\right)d.$$

(iv) *$\beta_{i+1,j}(R)$ is nonzero in every degree $j = (i+2)s_0 + m_j d$ for which*

$$1 + \binom{i+2}{2} \leq m_j \leq (i+2)n - \binom{i+2}{2}.$$

Remark 3.2. Part (iv) of the proposition says that there is a certain kind of symmetry in the Betti numbers. Namely, if $s_i = s_0 + id$ for every $i \in \{0,1,\dots,n\}$, then $\beta_{i+1,j}(R)$ is nonzero in every dth degree j between two specific degrees $j_{\min}$ and $j_{\max}$.

Proof. (i), (iii) and (iv) follow directly by considering (5), so let us prove the second assertion. Let m_j be the unique number for which $j = (i+2)s_0 + m_j d$. By mapping the partition $j = s_k + s_l + s_{q_1} + \cdots + s_{q_i}$ to the Eliahou-Kervaire partition $k + l + q_1 + \cdots + q_i$ of m_j, we obtain not only an injection, but in fact a bijection between the set $B_{i+1,j}$ consisting of partitions $j = s_k + s_l + s_{q_1} + \cdots + s_{q_i}$ of j and the set of Eliahou-Kervaire partitions of integers with $i+2$ parts on $\{1,2,\dots,n\}$. □

Example 3.3. A simple example of an arithmetic semigroup is $S = \langle 5, 9, 13, 17, 21\rangle$. In this case the number d from the definition of arithmetic semigroup equals 4. Another example is $S = \langle 5,6,7,8,9\rangle$ where d instead equals 1.

Proposition 3.4. *Let $S = \langle s_0, \dots, s_n\rangle$ be an arithmetic semigroup of maximal embedding dimension and R the corresponding semigroup* *ring. If* $j_1, \dots, j_r$ *are the degrees in which* $\beta_{i+1,j}(R)$ *is nonzero, then*

$$\beta_{i+1,j_k}(R) = \beta_{i+1,j_{r-k+1}}(R), \tag{9}$$

for every $0 \le k \le r$.

Proof. Define a map ϕ on the set of generators of S by $s_i \mapsto s_{n-i+1}$, $0 \le i \le n$. If $e(\sigma, u)$ is a generator of L_i corresponding to the sequence $\sigma = (q_1, \dots, q_i)$ and the minimal generator $u = x_k x_l$ of $\mathfrak{m}^2$, $1 \le k \le l \le n$, we consider the set

$$M_{\sigma,u} = \{s_k, s_l, s_{q_1}, \dots, s_{q_i}\}.$$

Here all but possibly two elements are distinct, and ϕ maps $M_{\sigma,u}$ to a set $\phi(M_{\sigma,u}) = \{\phi(s_k), \phi(s_l), \phi(s_{q_1}), \dots, \phi(s_{q_i})\}$, that in turn correspond to some other generator of L_i. In light of Proposition 3.1 and the fact that ϕ preserves, but reverses, all relations $s_t \le s_u$, $s_t, s_u \in M_{\sigma,u}$, the proposition follows. □

Thus, if S is arithmetic of maximal embedding dimension equation (9) tells us that the contributions to $\beta_i(R)$ from its graded components $\beta_{i+1,j}(R)$ are symmetric relative to the nonzero degrees j.

Note that for arithmetic semigroups of maximal embedding dimension, there are two kinds of symmetries in the Betti numbers. One described just above, and one in Remark 3.2. It is natural to ask for which other semigroups both these symmetries hold.

Example 3.5. Let $S = \langle 5, 7, 9 \rangle$. Below are the Betti numbers of A/I using the grading of A given by $\deg(x_i) = s_i$. Clearly the Betti numbers are symmetric in the sense of Proposition 3.4, but not in the sense of (iv) in Proposition 3.1.

i	β_i	j		
1	3	14(1)	25(1)	27(1)
2	2	28(1)	30(1)	

Example 3.6. The symmetry in the Betti numbers in Proposition 3.4 does not hold in general. Consider for example the semigroup $S = \langle 5, 11, 17, 18, 19 \rangle$. Below are the Betti numbers of A/I.

i	β_i	j								
1	10	22(1)	28(1)	29(1)	30(1)	34(1)	35(1)	36(1)	37(1)	38(1)
2	20	39(1)	40(1)	41(1)	45(1)	46(2)	47(3)	48(2)	49(1)	52(1)
		53(2)	54(2)	55(2)	56(1)					
3	15	57(1)	58(1)	59(1)	63(1)	64(2)	65(3)	66(2)	67(1)	71(1)
		72(1)	73(1)							
4	4	76(1)	82(1)	83(1)	84(1)					

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REFERENCES

1. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised edition, Cambridge University Press, Cambridge, 1998.

2. D. Eisenbud, *The geometry of syzygies*, Grad. Texts Math. **229**, Springer, New York, 2005.

3. S. Eliahou and M. Kervaire, *Minimal resolutions of some monomial ideals*, J. Algebra **129** (1990), 1–25.

4. E. Emtander, *Betti numbers of hypergraphs*, Comm. Algebra **37** (2009), 1545–1571.

5. R. Fröberg, *A study of graded extremal rings and of monomial rings*, Math. Scand. **51** (1982), 22–34.

6. S. Jacques, *Betti numbers of graph ideals*, Ph.D. thesis, University of Sheffield, 2004.

7. I. Sengupta, *A minimal free resolution for certain monomial curves in* $\mathbf{A}^4$, Comm. Algebra **31** (2003), 2791–2809.

8. R.H. Villarreal, *Monomial algebras*, Marcel Dekker, Inc., New York, 2001.

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