JOURNAL OF COMMUTATIVE ALGEBRA
Volume 2, Number 2, Summer 2010

ZASSENHAUS RINGS AS IDEALIZATIONS OF MODULES

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ABSTRACT. A ring R is called a Zassenhaus ring if any homomorphism φ of the additive group of R that leaves all left ideals of R invariant, is a left multiplication by some element a of R, i.e., $\varphi(x) = ax$ for all $x \in R$. Let M be an R-R-bimodule. Then the direct sum $R \oplus M$ turns naturally into a ring $R(+)M$ by defining $MM = \{0\}$. This ring is called the idealization of the module M, which is an ideal of $R(+)M$. We will investigate conditions under which $R(+)M$ is a Zassenhaus ring.

1. Introduction. Let R be a ring and ${}_RM_R$ an R-R-bimodule.

Then $R(+)M = \left\{ \left[\begin{smallmatrix} r \\ m \end{smallmatrix} \right] : r \in R, m \in M \right\}$ is a ring with vector addition and multiplication $\left[\begin{smallmatrix} r \\ m \end{smallmatrix} \right] \left[\begin{smallmatrix} r' \\ m' \end{smallmatrix} \right] = \left[\begin{smallmatrix} rr' \\ rm'+mr' \end{smallmatrix} \right]$, i.e., $R(+)M$ is naturally isomorphic to the ring of matrices $\left\{ \left[\begin{smallmatrix} r & 0 \\ m & r \end{smallmatrix} \right] : r \in R, m \in M \right\}$. This ring was first introduced in [**11**] and is called the idealization of M or a trivial extension of the ring R. The very first paper [**1**] in this journal is an excellent survey article on idealizations, where the ring R is commutative. In this case, any R-module M is automatically an R-R-bimodule. As was pointed out in [**1**], idealizations provide many nice examples of interesting rings and there is usually some intriguing connection between algebraic properties of R, M and $R(+)M$. We will concern ourselves in this paper with the Zassenhaus property of a ring. Several variations of this theme have been studied in [**2**–**6**].

A ring R is called a Zassenhaus ring if any additive endomorphism $\varphi : R \to R$ such that $\varphi(X) \subseteq X$ for any left ideal X of R is the (left) multiplication by some element of R. On the other hand, if M_R is a right R-module, we define $H(R, M) = \{\varphi \in \mathrm{Hom}_{\mathbf{Z}}(R, M) : \varphi(r) \in Mr$

2010 AMS *Mathematics subject classification.* Primary 13A02, 13A15, Secondary 20k20.
Keywords and phrases. Idealization of modules, trivial extensions, Zassenhaus rings and modules.
Received by the editors on July 23, 2009, and in revised form on September 23, 2009.

DOI:10.1216/JCA-2010-2-2-139

for all $r \in R\}$ and call the module M a Zassenhaus module, if each $\varphi \in H(R, M)$ is actually the (left) multiplication by some $\mu \in M$, i.e., $\varphi(r) = \mu r$ for all $r \in R$. (We refer to [**8**] for some motivation for this nomenclature.)

Here is a partial list of our results:

• If $R(+)M$ is a Zassenhaus ring, then M_R is a Zassenhaus module. If M_R is also faithful, then R is a Zassenhaus ring.

• $R(+)M$ need not be a Zassenhaus ring, even if R is a Zassenhaus ring and M is a Zassenhaus module.

• There exist Zassenhaus modules M_R such that M_R is not faithful.

• Let R be a left Ore domain and ${}_RM_R$ a bimodule such that ${}_RM$ has rank at least 2 and M_R is an R-reduced module. Then $R(+)M$ is a Zassenhaus ring if and only if R is a Zassenhaus ring and M_R is a Zassenhaus module.

• Let R be an integral domain and M an R-reduced R-module. Then $R(+)M$ is a Zassenhaus ring if and only if R is a Zassenhaus ring and M is a Zassenhaus module. (Corollary 1 shows that "R-reduced" is needed.)

• Assume that the additive group of R is $\mathbf{Z}$-reduced and torsion-free and M contains a strongly pure element. Then $R(+)M$ is a Zassenhaus ring if and only if M_R is a Zassenhaus module.

• There are subrings of algebraic number fields that are not Zassenhaus rings and neither are their epimorphic images.

• There are subrings of algebraic number fields that are Zassenhaus rings but not E-rings.

• If $R(+)M$ is a Zassenhaus ring, then R need not be a Zassenhaus ring.

2. Definitions and some general results.

Definition 1. Let R be a ring, $1 \in R$, and ${}_RM_R = M$ an R-R-bimodule. We define

$$\begin{aligned}\widehat{R} &= \{\varphi \in \operatorname{End}_{\mathbf{Z}}(R) : \varphi(X) \subseteq X \text{ for all left ideals } X \text{ of } R\} \\ &= \{\varphi \in \operatorname{End}_{\mathbf{Z}}(R) : \varphi(r) \in Rr \text{ for all } r \in R\}\,.\end{aligned}$$

Note that $R\cdot = \{x \mapsto rx : r \in R\} \subseteq \widehat{R}$. We call R a Zassenhaus ring if $\widehat{R} = R\cdot$.

For future reference, we define $\widetilde{R} = \{\varphi \in \widehat{R} : \varphi(r) \in Rr^2$ for all $r \in R\}$.

It is easy to see that $\widetilde{R}$ is a left ideal of $\widehat{R}$ and, if R is commutative, then $\widetilde{R}$ is an ideal of $\widehat{R}$.

Moreover, if R is an integral domain, not a field, then $\widetilde{R} \cap R\cdot = \{0\}$.

In addition, we define

$$\widehat{M} = \{\varphi \in \mathrm{End}_{\mathbf{Z}}(M) : \varphi(m) \in Rm \text{ for all } m \in M\}.$$

Finally, let

$$H(R, M) = \{\varphi \in \mathrm{Hom}_{\mathbf{Z}}(R, M) : \varphi(r) \in Mr \text{ for all } r \in R\}.$$

We call M a Zassenhaus module if $H(R, M) = M\cdot = \{x \mapsto mx : m \in M\}$.

Definition 2. A ring R with identity is called a left Ore domain, if R has no zero-divisors, i.e., whenever $rs = 0$ for some $r, s \in R$, then $r = 0$ or $s = 0$, and for any two non-zero elements $u, v \in R$ we have $Ru \cap Rv \neq \{0\}$.

Let ${}_RM$ be a left module and $m \in M$. We call the element m torsion-free, if $r \in R$ and $rm = 0$ implies $r = 0$.

We say that ${}_RM$ has rank at least 2, if ${}_RM$ contains two linearly independent, torsion-free elements. Note that this condition implies that ${}_RM$ be faithful.

Proposition 1. *If M_R is a faithful Zassenhaus R-module, then R is a Zassenhaus ring.*

Proof. Let $\alpha \in \widehat{R}$. Let $0 \neq m_0 \in M$ and define $\beta : R \to M$ by $\beta(r) = m_0\alpha(r)$. Obviously, $\beta \in H(R, M)$. Thus there is some $m \in M$ such that $\beta(r) = mr$ for all $r \in R$. Note that $\alpha(r) = \rho_r r$ for some $\rho_r \in R$. We infer that $(m_0\rho_r - m)r = 0$ and, for $r = 1$, we have

$m = m_0\rho_1$. It follows that $m_0(\rho_r - \rho_1)r = 0$ for all $m_0 \in M$. Since M is faithful, we have $\alpha(r) - \rho_1 r = 0$ for all $r \in R$. Thus $\alpha = \rho_1 \cdot \in R\cdot$ and R is a Zassenhaus ring. □

Remark 1. Let R be a ring and M_R an R-module such that $MJ = \{0\}$ for some ideal J of R. Then M_R is a Zassenhaus module if and only if $M_{R/J}$ is a Zassenhaus module.

Proof. Assume that M_R is a Zassenhaus module. Let $\beta \in H(R/J, M)$. Then there exist $m_{r+J} \in M$ such that $\beta(r+J) = m_{r+J}(r+J) = m_{r+J}r$ for all $r \in R$. Now define $\alpha : R \to M$ by $\alpha(r) = m_{r+J}r$. It is easy to verify that α is well-defined and $\alpha \in H(R, M)$. This shows that $\alpha(r) = mr$ for a fixed $m \in M$ and all $r \in R$. Thus $\beta(r+J) = \alpha(r) = mr = m(r+J)$, and it follows that the R/J-module $M_{R/J}$ is Zassenhaus.

Now assume that $M_{R/J}$ is a Zassenhaus module, and let $\varphi \in H(R, M)$. Then there exist $\mu_r \in M$ such that $\varphi(r) = \mu_r r$. Note that $\varphi(J) = \{0\}$. Now define $\overline{\varphi} : R/J \to M$ by $\overline{\varphi}(r+J) = \varphi(r)$. Then φ is well defined and $\varphi \in H(R/J, M) = M\cdot$, and there exists some $\mu \in M$ such that $\varphi(r) = \overline{\varphi}(r+J) = \mu(r+J) = \mu r$ for all $r \in R$. This shows that M_R is a Zassenhaus module. □

Let M_R be a Zassenhaus module and $J = \mathrm{ann}_R(M)$. Then $M_{R/J}$ is a faithful Zassenhaus module. By Proposition 1, R/J is a Zassenhaus ring. Now let $\alpha \in \widehat{R}$, $\alpha(r) = \rho_r r$ for all $r \in R$. Define $\beta : R/J \to R/J$ by $\beta(r+J) = \rho_r(r+J) = \alpha(r) + J$. Note that $\alpha(J) \subseteq J$, which implies that β is well defined, and thus $\beta \in \widehat{R/J}$. It follows that there exists some $\rho \in R$ such that $\alpha(r) + J = \beta(r+J) = (\rho + J)(r+J) = \rho r + J$ and thus $(\alpha - \rho\cdot) \in \widehat{R}$. This shows that $(\alpha - \rho\cdot)(R) \subseteq J$. We conclude that R is a Zassenhaus ring provided that $\{\varphi \in \widehat{R} : \varphi(R) \subseteq J\} = \{0\}$.

Definition 3. If R is a ring, then R^+ denotes the additive group of R. Then R^+ is $\mathbf{Z}$-reduced, if $\cap_{n\in\mathbf{N}} nR = \{0\}$.

Proposition 2. *Let R be a ring such that R^+ is $\mathbf{Z}$-reduced and torsion-free. Then $\widetilde{R} = \{0\}$.*

Proof. Let $\varphi \in \widetilde{R}$. Then there exists an $\rho_r \in R$ such that $\varphi(r) = \rho_r r^2$ for all $r \in R$. Let n be a positive integer. Then $n\rho_r r^2 = n\varphi(r) = \varphi(nr) = \rho_{nr} n^2 r^2$. Thus $n(\rho_r r^2 - \rho_{nr} n r^2) = 0$ for all $r \in R$ and all positive integers n. This implies $\varphi(r) = \rho_r r^2 \in \cap_{1 \leq n} nR = \{0\}$, since R^+ is $\mathbf{Z}$-reduced. □

Proposition 3. *Let R be a Zassenhaus ring, I an index set and M_R a submodule of the Cartesian product $\Pi = (\prod_I R)_R$. Then M_R is a Zassenhaus module.*

Proof. Let $\beta \in H(R, M)$, and β_i is the map β followed by the projection in the ith coordinate of the cartesian product. Then there exists a $\mu_r = (\rho_i^{(r)})_{i \in I} \in \Pi$ such that $\beta(r) = \mu_r r = (\rho_i^{(r)})_{i \in I} r = (\rho_i^{(r)} r)_{i \in I}$ for all $r \in R$. This implies that $\beta_i(r) = \rho_i^{(r)} r$ for all $r \in R$ and $\beta \in \widehat{R}$. Thus $\beta_i(r) = \rho_i r$ for some $\rho_i \in R$ and all $r \in R$. This shows that $\beta(r) = (\rho_i)_{i \in I} r$, and since $\beta(1) = (\rho_i)_{i \in I} \in M$ we infer that $\beta \in M \cdot$ and M is a Zassenhaus module. □

Remark 2. The above Proposition and the main result in [**7**] immediately show the following:

Let κ be a cardinal less than the first measurable cardinal and R a Zassenhaus ring with identity such that the additive group of R is slender and $|R| < \kappa$. Then there exist Zassenhaus R-modules G of arbitrarily large cardinalities. Moreover, the additive group of G is slender and $\mathrm{End}_{\mathbf{Z}}(G) = R$. This shows that Zassenhaus modules M_R exist in abundance if R is a Zassenhaus ring.

Definition 4. Let R be a ring and M an R-R-bimodule.

The element $m \in M_R$ is *pure* in M provided that $m \in Mu$, $u \in R$, implies that u is a unit of R. The element $m \in M$ is called *strongly pure* in M if, whenever s, r are non-zero elements of R such that $sm \in Mr$, then $s \in Rr$. It is easy to see that any strongly pure element is pure. Moreover, if R_R is R-reduced and $m \in {}_R M_R$ is strongly pure, then $m \in_R M$ is a torsion-free element.

It can happen that pure implies strongly pure:

Let R be a commutative ring with identity and view R as a module over itself. If $s \in R$ is a pure element of this module, then s is a unit and thus s is also strongly pure.

There are more examples of modules where pure implies strongly pure:

Let R be a commutative valuation domain, M a torsion-free R-module and $m \in M$ a pure element of M. Assume $0 \neq s, r \in R$ such that $sm = m'r$ for some $m' \in M$. If $s \notin Rr$, then $r \in Rs$ and thus $r = as$ for some $a \in R$. Then $sm = m'as$ and $m = m'a$ for the pure element m implies that a is a unit of R and we get the contradiction $s \in Rr$. This shows that the pure element $m \in M$ is strongly pure.

Let $S = R(+)M = \left\{\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] : r \in R, m \in M\right\}$. We want to compute $\widehat{S}$. To this end, note that $\left[\begin{smallmatrix} 0 \\ M \end{smallmatrix}\right]$ is an ideal of S.

Let $\psi \in \widehat{S}$. Then there exist $\alpha \in \mathrm{End}_{\mathbf{Z}}(R)$, $\beta \in \mathrm{Hom}_{\mathbf{Z}}(R, M)$ and $\gamma \in \mathrm{End}_{\mathbf{Z}}(M)$ such that ψ may be presented as $\psi = \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right]$. Note that $\psi\left(\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right]\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} \alpha(r) \\ \beta(r)+\gamma(m) \end{smallmatrix}\right]$. It is easy to see that $\psi \in S\cdot$ if and only if $\psi = \left[\begin{smallmatrix} \rho\cdot & 0 \\ \mu\cdot & \rho\cdot \end{smallmatrix}\right]$ for some $\rho \in R$ and $\mu \in M$.

First we need:

Lemma 1. *Let R be a left Ore domain and ${}_RM$ a left R-module of rank at least 2. Then $\widehat{M} = R\cdot$, i.e., for any $\varphi \in \widehat{M}$, there is some $\rho \in R$ such that $\varphi(m) = \rho m$ for all $m \in M$.*

Proof. Fix a torsion-free element $m \in M$. Then $\varphi(m) = \rho m$ for some $\rho \in R$. Let $m_1 \in M$ such that $\{m, m_1\}$ is linearly independent over R. Then $\varphi(m_1) = \rho_1 m_1$ for some $\rho_1 \in R$ and there is some $\sigma \in R$ such that $\varphi(m + m_1) = \sigma(m + m_1)$. Since m, m_1 are R-linearly independent we infer that $\rho = \sigma = \rho_1$. Now let $\mu \in M$ be another torsion-free element such that $\{m, \mu\}$ is linearly dependent. Then there exist $r, \rho \in R$ such that $rm + \rho\mu = 0$ and $r \neq 0 \neq \rho$. We want to show that $\{\mu, m_1\}$ is R-linearly independent. To this end, let $r_0, r_1 \in R$ be such that $r_0\mu + r_1 m_1 = 0$. We may assume that

$r_0 \neq 0$. Since R is left Ore, we have $Rr_0 \cap R\rho \neq \{0\}$, and there exist $s_0, \sigma \in S$ such that $s_0 r_0 = \sigma\rho \neq 0$. Note that $s_0 r_0 \mu + s_0 r_1 m_1 = 0$, and it follows that $\sigma\rho\mu + s_0 r_1 m_1 = 0$ and an obvious substitution yields $\sigma(-rm) + s_0 r_1 m_1 = 0$ and we conclude $\sigma r = 0 = s_0 r_1$. Since R is a domain and $\sigma \neq 0 \neq s_0$, we have that $r = 0 = r_1$, which shows that $\{\mu, m_1\}$ is R-linearly independent and $\varphi(m_1) = \rho m_1$. Now the first argument shows that $\varphi(\mu) = \rho\mu$ as well and we have that $\varphi(v) = \rho v$ for all torsion-free elements $v \in M$. Let $\mu \in M$ be a non-torsion-free element, i.e., there is some $0 \neq t \in R$ such that $t\mu = 0$. By way of contradiction, we assume that there is some $0 \neq s \in R$ such that $s(m + \mu) = 0$. Since R is Ore, there are non-zero elements $x, y \in R$ with $xs = yt \neq 0$. Now $0 = x0 = xsm + xs\mu = xsm + yt\mu = xsm$, which contradicts the choice of m being torsion-free. Thus $m + \mu$ is torsion-free and we get $\varphi(\mu) = \varphi((m+\mu)-m) = \rho(m+\mu)-\rho(m) = \rho\mu$. This shows that $\varphi = \rho\cdot$. □

We are now ready for the following:

Lemma 2. *Let R be a left Ore domain, ${}_RM_R$ an R-R-bimodule such that ${}_RM$ is an R-module of rank at least 2. Let $\alpha \in \mathrm{End}_{\mathbf{Z}}(R)$, $\beta \in \mathrm{Hom}_{\mathbf{Z}}(R, M)$ and $\gamma \in \mathrm{End}_{\mathbf{Z}}(M)$. Then $\psi = \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right] \in \widehat{R(+)M}$ if and only if*

(a) *$\alpha \in \widehat{R}$ and there are $\rho_r \in R$ such that $\alpha(r) = \rho_r r$ for all $0 \neq r \in R$ and*

(b) *There is some $\rho_0 \in R$ such that $\gamma(m) = \rho_0 m$ for all $m \in M$, i.e., $\gamma \in R\cdot$ and*

(c) *There are $\mu_r \in M$ such that $\beta(r) = \mu_r r$, i.e., $\beta \in H(R, M)$ and*

(d) *$(\rho_0 - \rho_r)m \in Mr$ for all $0 \neq r \in R$ and all $m \in M$, i.e., $(\rho_0 - \rho_r)M \subseteq Mr$ for all $0 \neq r \in R$.*

Proof. For $r, \rho \in R$ and $m, \mu \in M$ we have $\left[\begin{smallmatrix} \rho \\ \mu \end{smallmatrix}\right]\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho\cdot & 0 \\ \mu\cdot & \rho\cdot \end{smallmatrix}\right]\left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho r \\ \mu r + \rho m \end{smallmatrix}\right]$. Now let $\psi = \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right] \in \widehat{R(+)M}$. Then $\psi\left(\begin{smallmatrix} r \\ m \end{smallmatrix}\right) = \left[\begin{smallmatrix} \alpha(r) \\ \beta(r)+\gamma(m) \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho_{r,m} r \\ \mu_{r,m} r + \rho_{r,m} m \end{smallmatrix}\right]$ for some $\rho_{r,m} \in R$ and $\mu_{r,m} \in M$, and it follows that

$$\beta(r) + \gamma(m) = \mu_{r,m} r + \rho_{r,m} m \text{ for all } r \in R \text{ and } m \in M.$$

Moreover, $\alpha(r) = \rho_r r$ and $\rho_r = \rho_{r,m}$ is independent of m for all $0 \neq r \in R$ and all $m \in M$ since R is a domain, which shows (a).

For $r = 0$ we get $\gamma(m) = \rho_{0,m} m$ which shows that $\gamma \in \widehat{M}$ and thus, by Lemma 1, we have $\gamma(m) = \rho_0 m$ for all $m \in M$ and $\rho_0 m = \rho_{0,m} m$ for all $m \in M$. This proves (b).

For $m = 0$, we get $\beta(r) = \mu_{r,0} r$, which shows (c).

We now have $\mu_{r,0} r + \rho_0 m = \mu_{r,m} r + \rho_r m$ for all $0 \neq r \in R$ and all $m \in M$, i.e.,

$$(\mu_{r,m} - \mu_{r,0}) r = (\rho_0 - \rho_r) m,$$

which shows (d).

To show the converse, assume that (a)–(d) hold. Then there exist $\sigma_{r,m}$ such that $(\rho_0 - \rho_r)m = \sigma_{r,m} r$. Define $\mu_{r,m} = \mu_r + \sigma_{r,m}$. The above computations now show that $\psi \begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} \alpha(r) \\ \beta(r)+\gamma(m) \end{bmatrix} = \begin{bmatrix} \rho_r r \\ \mu_r r + \rho_0 m \end{bmatrix} = \begin{bmatrix} \rho_r & 0 \\ \mu_{r,m} & \rho_r \end{bmatrix} \begin{bmatrix} r \\ m \end{bmatrix}$, since

$$\begin{aligned} \mu_{r,m} r + \rho_r m &= (\mu_r + \sigma_{r,m}) r + \rho_r m = \mu_r r + \sigma_{r,m} r + \rho_r m \\ &= \mu_r r + (\rho_0 - \rho_r) m + \rho_r m = \mu_r r + \rho_0 m. \end{aligned}$$

This shows that $\psi \in \widehat{R(+)M}$. □

Corollary 1. *Let $R = \mathbf{Z}$ and $M = (\mathbf{Q} \oplus \mathbf{Q})_{\mathbf{Z}}$. Then R is a Zassenhaus ring and M_R is a Zassenhaus module, but $R(+)M$ is not a Zassenhaus ring.*

Proof. Let $\psi = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_M \end{bmatrix}$. Then $\psi \in \widehat{R(+)M}$ by Lemma 2 since M is divisible. It is easy to see that $\psi \notin (R(+)M)\cdot$. □

The following will come in handy.

Proposition 4. *Let R be a ring, M an R-R-bimodule and $\beta \in H(R, M)$. Let $\psi = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$. The following hold:*

(a) *$\psi \in \widehat{R(+)M}$ and*

(b) $\psi \in (R(+)M)\cdot$ *if and only if* $\beta \in M\cdot$.

(c) *If* $R(+)M$ *is a Zassenhaus ring, then* M_R *is a Zassenhaus module. If* M_R *is also faithful, then* R *is a Zassenhaus ring.*

Proof. Since $\beta \in H(R,M)$ there exist $\mu_r \in M$ such that $\beta(r) = \mu_r r$ for all $r \in R$. Thus $\psi \begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_r r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu_r & 0 \end{bmatrix} \begin{bmatrix} r \\ m \end{bmatrix}$ and (a) follows.

We now show (b). If $\beta = \mu\cdot$, then we can use the last equation to infer that $\psi \in (R(+)M)\cdot$. Assume that $\psi = \begin{bmatrix} \rho & 0 \\ \mu & \rho \end{bmatrix} \cdot \in (R(+)M)\cdot$. Then $\psi \begin{bmatrix} r \\ m \end{bmatrix} = \begin{bmatrix} \rho r \\ \mu r + \rho m \end{bmatrix} = \begin{bmatrix} 0 \\ \beta(r) \end{bmatrix}$. For $r = 1$, we get $\rho = 0$ and thus $\beta(r) = \mu r$ for all $r \in R$, i.e., $\beta \in M\cdot$.

Part (c) is an immediate consequence of parts (a), (b) and Proposition 1. □

Definition 3. Let R be a ring and M_R an R-module. Then M_R is called R-reduced, if $\cap_{0 \neq r \in R} Mr = \{0\}$.

We have:

Proposition 5. *Let* R *be a Zassenhaus Ore domain,* M_R R*-reduced and* ${}_RM$ *of rank at least* 2. *Then*

$$\widehat{R(+)M} = \left\{ \begin{bmatrix} \rho\cdot & 0 \\ \beta & \rho\cdot \end{bmatrix} : \rho \in R, \beta \in H(R,M) \right\} = R(+)(H(R,M)).$$

Proof. Let $\psi \in \widehat{R(+)M}$. Condition (d) of Lemma 2 now becomes $(\mu_{r,m} - \mu_{r,0})r = (\rho_0 - \rho_1)m$ for all $m \in M$ and $0 \neq r \in R$ since R is a Zassenhaus ring. Since M_R is R-reduced and ${}_RM$ is faithful, we infer that $\rho_0 = \rho_1 =: \rho$ and ψ has the desired form. □

Thus we have:

Proposition 6. *Let* R *be a Zassenhaus left Ore domain,* M_R *reduced and* ${}_RM$ *of rank at least* 2. *Then* $R(+)M$ *is a Zassenhaus ring if and only if* M *is a Zassenhaus module.*

Theorem 1. *Let* R *be a left Ore domain and* ${}_RM_R$ *an* R*-*R*-bimodule such that* ${}_RM$ *has rank at least* 2.

(a) *Assume that M_R is R-reduced and faithful.*

Then $R(+)M$ is a Zassenhaus ring if and only if R is a Zassenhaus ring and M_R is a Zassenhaus module.

(b) *Let R^+ be $\mathbf{Z}$-reduced and torsion-free. Assume that there is some strongly pure element $m_0 \in M$.*

Then $\widehat{R(+)M} = \left\{ \begin{bmatrix} \rho\cdot & 0 \\ \beta & \rho\cdot \end{bmatrix} : \rho \in R, \beta \in H(R,M) \right\}$.

Thus, $R(+)M$ is a Zassenhaus ring if and only if M_R is a Zassenhaus module.

Proof. First we prove (a). If $R(+)M$ is a Zassenhaus ring, then M_R and R are Zassenhaus by Proposition 4 (c). To show the converse, assume $\psi = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} \in \widehat{R(+)M}$. By Lemma 2, there is some $\rho, \rho_0 \in R, \mu \in M$ such that $\alpha(r) = \rho r, \beta(r) = \mu r$ for all $r \in R$ and $\gamma(m) = \rho_0 m$ for all $m \in M$. Moreover, $(\rho_0 - \rho)m \in Mr$ for all $r \in R, m \in M$. Since ${}_RM$ is faithful and M_R is R-reduced, we infer $\rho_0 = \rho$ and thus $\psi = \begin{bmatrix} \rho & 0 \\ \mu & \rho \end{bmatrix} \cdot$, which shows that $R(+)M$ is a Zassenhaus ring.

We now prove (b). Let $\psi = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix} \in \widehat{R(+)M}$ with $\alpha(r) = \rho_r r$ for all $0 \neq r \in R$ and $\beta(r) = \mu_r r$ for some $\mu_r \in M$. Moreover, $\gamma(m) = \rho_0 m$ as in Lemma 2. By Lemma 2 (d), we have that $(\rho_0 - \rho_r)m_0 \in Mr$, and it follows that $\rho_0 - \rho_r \in Rr$ for all $r \in R$ since m_0 is strongly pure. We infer that $(\rho_0 r - \rho_r r) = (\rho_0 \cdot - \alpha)(r) \in Rr^2$. Thus $(\rho_0 \cdot) - \alpha \in \widetilde{R} = \{0\}$ by Proposition 2. This shows that $\psi = \begin{bmatrix} \rho_0\cdot & 0 \\ \beta & \rho_0\cdot \end{bmatrix}$ for some $\beta \in H(R,M)$ has the desired form. By Lemma 2, any ψ of this form is in $\widehat{R(+)M}$. We infer that $\widehat{R(+)M} = \left\{ \begin{bmatrix} \rho\cdot & 0 \\ \beta & \rho\cdot \end{bmatrix} : \rho \in R, \beta \in H(R,M) \right\}$. Moreover, $R(+)M$ is a Zassenhaus ring if and only if M_R is a Zassenhaus module. □

Corollary 1 shows that the hypothesis "R-reduced" is needed in the following:

Corollary 2. *Let R be an integral domain and M an R-reduced R-module such that M has rank at least 2. Then $R(+)M$ is a Zassenhaus ring if and only if R is a Zassenhaus ring and M is a Zassenhaus module.*

Note that if $\mathbf{Q} \subsetneqq R$ is a field and M is an R-vector space, then $H(R, M) = \mathrm{Hom}_{\mathbf{Z}}(M)$. This is an example where neither R nor M is Zassenhaus. We will now show that even Zassenhaus rings may have non-Zassenhaus modules.

Example 1. Let R be a Dedekind domain, not a field, but a $\mathbf{Q}$-algebra. Then there exists an R-module M such that M is not Zassenhaus.

Proof. Let Π be the set of prime ideals of R. For $P \in \Pi$ let R_P denote the localization of R at P and $\pi_P \in R$ such that $\pi_P R_P$ is the maximal ideal of the discrete valuation domain R_P. Note that there are $\mathbf{Q}$-subspaces $C_{P,i}$ of R_P such that $\pi_P^n R_p = \oplus_{i \geq n} C_{P,i}$ for all $n = 0, 1, 2, \dots$. Pick any $\alpha_P \in \mathrm{End}_{\mathbf{Q}}(R_P)$ such that $\alpha_P(C_{P,i}) \subseteq C_{P,2i}$ for all $i \geq 0$. Note that for any $r \in R_P$, there is some n and a unit $u \in R_P$ such that $r = \pi_P^n u$. This implies that $\alpha_P(r) = \alpha_P(\pi_P^n u) = \pi_P^{2n} y$ for some $y \in R_P$. Thus $\alpha_P(r) = \pi_P^n y u^{-1}(\pi_P u) = m_{P,r} r$ for $m_{P,r} = \pi_P^n y u^{-1}$. This shows that $\alpha_P \in \widehat{R_P}$ but $\alpha_P \notin R_P \cdot$. Now let $M = \prod_{P \in \Pi} R_P$, and define $\alpha \in \mathrm{End}_{\mathbf{Q}}(M)$ by $\alpha = (\alpha_P)_{P \in \Pi}$. Let $\widehat{\alpha}$ denote the natural embedding from R into M followed by α, i.e., $\widehat{\alpha}(r) = (m_{P,r} r) = (m_{P,r}) r = m_r r$ for $m_r = (m_{P,r})_{P \in \Pi}$. Note that $m_{P,r} \in R_P r$, and thus there is no $m_P \in R_P$ such that $m_{P,r} = m_P$ for all $r \in R$. This shows that $\widehat{\alpha} \in H(R, M)$, but $\widehat{\alpha} \notin M \cdot$. If $R = \mathbf{Q}[x]$ is the rational polynomial ring, then, by [**3**, Corollary 4], R is a Zassenhaus ring and M is a torsion-free, R-reduced R-module but not Zassenhaus. □

Proposition 7. *Let the* $\mathbf{Q}$*-algebra* R *be a discrete valuation domain and* M *an* R*-module. Then* $R(+)M$ *is not a Zassenhaus ring.*

Proof. The case where R is a field follows from [**6**, Proposition 5]. If R is not a field, we have seen in the proof of Example 1, that there is some $\alpha \in \widehat{R}$ such that $\alpha(r) \in Rr^2$ for all $r \in R$. It follows from Lemma 2 that $\psi = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \in \widehat{R(+)M}$ but $\psi \notin (R(+)M)\cdot$. □

3. Subrings of algebraic number fields.

Notation 1. *Let $F = \mathbf{Q}(\omega)$ be an n-dimensional Galois extension of $\mathbf{Q}$ with primitive element ω and Galois group $G = \{g_1, g_2, \dots, g_n\}$ and $\mathrm{id}_F = g_1$.*

Let $\mathfrak{O}_F$ denote the ring of algebraic integers of F, and let $\{a_1, a_2, \dots, a_n\}$ be an integral basis of $\mathfrak{O}_F$. Let $\Delta = [g_i(a_j)]_{1 \le i,j \le n}$.

Note that Δ is an $n \times n$-matrix with entries in $\mathfrak{O}_F$.

Let p be a prime integer such that p does not divide $m_\Delta = \det(\Delta)$.

Let R be a full, integrally closed subring of F and N a finite rank torsion-free R-module.

For any (prime) ideal P of $\mathfrak{O}_F$, let $\mathrm{Fix}\,(P) = \{g \in G : g(P) = P\}$.

Note that for any $\varphi \in \mathrm{End}_{\mathbf{Q}}(F)$ there are unique $r_i \in F$ such that $\varphi = \sum_{1 \le i \le n} r_i g_i$, i.e., $\mathrm{End}_{\mathbf{Q}}(F) = F[G]$, the group ring of G over F.

We need the following

Claim 1 [**10**, Lemma 2.5] (see also [**3**, Proposition 3]). *With the notations as above, let $\varphi = \sum_{1 \le i \le n} r_i g_i \in \mathrm{End}_{\mathbf{Z}}(\mathfrak{O}_F)$, and let P be a prime (maximal) ideal of $\mathfrak{O}_F$ lying above the prime integer p (i.e., $p \in P$) such that $\varphi(P^k) \subseteq P^k$ for all positive integers k. Then $r_i = 0$ for all i such that $g_i \notin \mathrm{Fix}\,(P)$.*

Claim 2. *With the notations as above, let $S = (\mathfrak{O}_F)_P \supset \mathfrak{O}_F$ be the localization of $\mathfrak{O}_F$ at the prime ideal P. Then $\widehat{S} = S[\mathrm{Fix}\,(P)]$, the group ring of $\mathrm{Fix}\,(P)$ over S.*

Proof. Since $\{P^k S : k \ge 1\}$ is the list of all non-trivial ideals of the discrete valuation domain S, we have that $S[\mathrm{Fix}\,(P)] \subseteq \widehat{S}$. For the other inclusion, let $\varphi \in \widehat{S}$. There exists a unit $u \in S$ such that $u\varphi(\mathfrak{O}_F) \subseteq \mathfrak{O}_F$. Note that $u\varphi(P^k) = u\varphi(P^k S \cap \mathfrak{O}_F) \subseteq u\varphi(P^k S) \cap u\varphi(\mathfrak{O}_F) \subseteq P^k S \cap \mathfrak{O}_F = P^k$. This shows that $u\varphi \in \mathrm{End}_{\mathbf{Q}}(\mathfrak{O}_F)$ is such that $u\varphi(P^k) \subseteq P^k$ for all $k \ge 1$ and we have $u\varphi \in \mathfrak{O}_F[Fix(P)]$ by Claim 1. Thus $\varphi \in u^{-1}\mathfrak{O}_F[\mathrm{Fix}\,(P)] \cap \widehat{S} \subseteq S[\mathrm{Fix}\,(P)]$. □

Example 2. We will construct a finite rank discrete valuation domain S that is not Zassenhaus and not a $\mathbf{Q}$-algebra. Moreover, S does not admit any Zassenhaus modules M_S.

Proof. Let $F = \mathbf{Q}(\sqrt{3}, i) = \mathbf{Q}(i+\sqrt{3}) = \mathbf{Q}(\sqrt{-1}, \sqrt{-3})$. Let $K = \mathbf{Q}(\sqrt{-1})$ and $L = \mathbf{Q}(\sqrt{-3})$. Then $F = KL$. Note that and $-1 \equiv 3 \bmod 4$ and thus K has discriminant -4. Moreover, $-3 \equiv 1 \bmod 4$ which implies that the discriminant of L is -3. This shows that K, L have relatively prime discriminants whose product squared is the discriminant of F and $\mathfrak{O}_F = \mathfrak{O}_K\mathfrak{O}_L$, by [**9**, page 68, Proposition 17]. Moreover, 5 does not divide the discriminant of F, which means that the prime 5 is unramified in $\mathfrak{O}_F$. The primitive element $\omega = i+\sqrt{3}$ has minimal polynomial $m(x) = x^4 - 4x^2 + 16$ and $m(x) \equiv (x^4 + x^2 + 1) \bmod 5$. Note that $x^4 + x^2 + 1 = u(x)v(x)$ where $u(x) = x^2 + x + 1$ and $v(x) = x^2 - x + 1$ are irreducible mod5. Let $D = \mathfrak{O}_F$, $P = u(\omega)D + 5D$, and $Q = v(\omega)D + 5D$. Then P, Q are prime ideals of D such that $5D = PQ$ is the prime factorization of $5D$.

Note that $u(\omega) = 3 + \sqrt{3} + i + 2i\sqrt{3}$ and $v(\omega) = 3 - \sqrt{3} - i - 2i\sqrt{3}$. Let $G = \{\mathrm{id}_F, \alpha, \beta, \gamma\}$ be the Galois group of F where $\alpha(\sqrt{3}) = -\sqrt{3}, \alpha(i) = i$ and $\beta(\sqrt{3}) = \sqrt{3}, \beta(i) = -i$. Of course, $\gamma = \alpha\beta$. Obviously, $\gamma(u(\omega)) = v(\omega)$, which implies $\gamma(P) = Q$. It is easy to verify that $13\alpha(u(\omega)) = 13\alpha(3 + \sqrt{3} + i + 2i\sqrt{3}) = 13(3 - \sqrt{3} + i - 2i\sqrt{3}) = (3+\sqrt{3}+i+2i\sqrt{3})(-5+2\sqrt{3}+12i-10i\sqrt{3}) \in P$ and we infer $\alpha(P) = P$ and $\mathrm{Fix}\,(P) = \{\mathrm{id}_F, \alpha\}$.

Now let $S = D_P$ be the localization of D at the prime ideal P. Then S is a discrete valuation domain and all non-trivial ideals J of S have the form $J = P^kS$ for some $k \geq 1$. Moreover, $\widehat{S} = S[Fix(P)] \neq S\cdot$. Note that none of the rings $S_n = S/(P^nS)$ is a Zassenhaus ring. By Proposition 1 and Remark 1, S has no Zassenhaus modules. □

Recall that a ring R is an E-ring if $R\cdot = \mathrm{Hom}_{\mathbf{Z}}(R, R)$. Of course, every E-ring is a Zassenhaus ring. The results in this section and in [**10**] allow us to find many examples of Zassenhaus rings that are not E-rings. We still use Notation 1. It is well known that $S = \mathfrak{O}_F$ is not an E-ring but a Zassenhaus ring. Let Π be a (finite) set of prime ideals of S such that $\sigma \in G$ and $\sigma(P) = P$ for all $P \in \Pi$ implies that $\sigma = \mathrm{id}_F$. Then the localization $R = S_\Pi$ is a Zassenhaus ring. It can easily be arranged that $\rho(\Pi) = \Pi$ for some $\mathrm{id}_F \neq \rho \in G$. In this case, R is not an E-ring.

The module N over an (E-ring) R is called an E-module, if $\mathrm{Hom}_{\mathbf{Z}}(R, N) = N\cdot$. Trivially, any E-module is a Zassenhaus module. E-modules of finite rank were studied in [**10**]. It is easy to check that the results in [**10**, Section 2] all hold if one replaces "E-module" by "Zassenhaus module" and "$\mathrm{Hom}_{\mathbf{Z}}(R, N)$" by "$H(R, N)$". The same can be said about the results in [**10**, Section 3]. We illustrate this with the following:

Example 3. Let F be a quadratic number field and p a prime integer such that $p\mathfrak{O}_F = PQ$ for two distinguished prime ideals of $\mathfrak{O}_F$. Let $G = \{\mathrm{id}_F, \sigma\}$ be the Galois group of F. Then $\sigma(P) = Q$ and it follows that $S = (\mathfrak{O}_F)_{\{P,Q\}}$ is a Zassenhaus ring but *not* an E-ring. The ring S is a subring of the ring $R = (\mathfrak{O}_F)_P$ and thus R is an S-module. We will show that R_S is a Zassenhaus module. It is enough to show that $\sigma \notin H(S, R)$. By way of contradiction, assume otherwise and pick $0 \neq x \in P - \sigma^{-1}(P \cap Q)$. Then $\sigma(x) = [\sigma(x)x^{-1}]x \in Rx$, which implies that $\sigma(x)x^{-1} \in R$ and $\sigma(x) \in Q - P$ is a unit in R. Thus $x^{-1} \in R$ and we get the contradiction $1 = x^{-1}x \in P$. Of course, this example can be vastly generalized.

4. The case of $S = \mathbf{Z}[x]$. In this section, S will always denote the integer polynomial ring $S = \mathbf{Z}[x]$. We define $J = \{(f(x)/g(x)) : f(x), g(x) \in S,\ g(x) \text{ primitive }\}$. Recall that S is a subring of the integral domain J, and all ideals I of J have the form $I = nJ$ for some integer n.

Here is another Zassenhaus ring which admits a non-Zassenhaus module:

Example 4. There exists a commutative ring R such that R is not a Zassenhaus ring, but some epimorphic image of R is a Zassenhaus ring.

Proof. Note that J is a ring and every element of J is of the form of an integer times a unit of J. Define $\varphi \in \mathrm{Hom}_{\mathbf{Z}}(S, J)$ by $\varphi(f(x)) = f(x^2)$. Let $y = ng(x) \in S$ with $g(x)$ a primitive polynomial. Then $g(x)$ is a unit in J and we have $\varphi(y) = ng(x^2)g(x)^{-1}g(x) = (g(x^2)/g(x))y$ where

$(g(x^2)/g(x)) \in J$ and it follows that $\varphi \in H(S,J) - (J\cdot)$. Now consider $R = S(+)J$. By Proposition 4 (c), the ring R is not a Zassenhaus ring, but $S \cong R/J$ is a Zassenhaus ring. □

Let $S \subset J$ be as above, and let ${}_SM_J$ be an S-J-bimodule. We will show that $H(S, M_S) = \mathrm{Hom}_{\mathbf{Z}}(S, M)$:

Assume that $\varphi \in \mathrm{Hom}_{\mathbf{Z}}(S, M)$. Let $y = ng \in S$ be such that $n \in \mathbf{N}$ and $g \in S$ is primitive. Then $\varphi(y) = \varphi(g)g^{-1}ng = (\varphi(g)g^{-1})y$ and $\varphi(g)g^{-1} \in M$ since $g^{-1} \in J$. This shows that $\varphi \in H(S, M)$.

If R is a ring with identity, then R is naturally a subring of $\widehat{R}$. This allows us to use transfinite induction to define an ascending chain of rings $\{R^{(\alpha)} : \alpha \text{ an ordinal}\}$ as follows: Let $R^{(0)} = R$ and $R^{(\alpha+1)} = \widehat{R^{(\alpha)}}$. For limit ordinals λ, we define $R^{(\lambda)} = \cup_{\alpha<\lambda} R^{(\alpha)}$. There is an example in [**3**] for which this transfinite chain never terminates, i.e., $R^{(\alpha)} \subsetneqq R^{(\alpha+1)}$ for all ordinals α. We will present another such example, where all the rings in the transfinite chain are idealizations of $S = \mathbf{Z}[x]$-modules.

Recall that by Proposition 5, we have

$$\widehat{S(+)M} = \left\{ \begin{bmatrix} \rho\cdot & 0 \\ \beta & \rho\cdot \end{bmatrix} : \rho \in S, \beta \in H(S, M_S) \right\} = S(+)(\mathrm{Hom}_{\mathbf{Z}}(S, M)).$$

For $s \in S, \varphi \in \mathrm{Hom}_{\mathbf{Z}}(S, M)$, $j \in J$, define $(s\varphi j)(x) = s\varphi(x)j$ for all $x \in S$. Then $\varphi \in \mathrm{Hom}_{\mathbf{Z}}(S, M)$ and $\mathrm{Hom}_{\mathbf{Z}}(S, M)$ becomes an S-J-bimodule. We may define $R^{(0)} = S(+)J$ and $R^{(1)} = \widehat{R^{(0)}} = S(+)(\mathrm{Hom}_{\mathbf{Z}}(S, J))$. Note that J naturally embeds into $\mathrm{Hom}_{\mathbf{Z}}(S, J)$ via $j(s) = sj$ for all $s \in S$. This induces a natural embedding of $R^{(0)}$ into $R^{(1)}$. More generally, given ${}_SM_J$ there is a natural embedding of M into $\mathrm{Hom}_{\mathbf{Z}}(S, M)$ by $m(s) = ms$ for all $m \in M, s \in S$. This allows us to define $R^{(\alpha+1)} = \widehat{S(+)M^{(\alpha)}} = S(+)\mathrm{Hom}_{\mathbf{Z}}(S, M^{(\alpha)}) = S(+)M^{(\alpha+1)}$ with $M^{(0)} = J$. Note that $M^{(\alpha)} \subsetneqq M^{(\alpha+1)}$ via the natural embedding. Note that the chain $\{R^{(\alpha)} : \alpha \text{ an ordinal}\}$ never terminates.

On the other hand we have the somewhat surprising:

Lemma 3. *Let A be a torsion-free, $\mathbf{Z}$-reduced abelian group. Then $M_S = A \otimes_{\mathbf{Z}} S$ is a Zassenhaus module.*

Proof. Let $s = \sum_{0\le i\le N} k_i x^i \in S$ be such that $k_0 \neq 0$. Let $\varphi \in H(S,M)$. Then there are $a_{n,\alpha} \in A$ such that

$$\varphi(x^n) = \sum_{0\le\alpha\le d_n} a_{n,\alpha} \otimes x^\alpha \in M = \oplus_{\alpha\ge 0}(A\otimes x^\alpha).$$

Since $\varphi \in H(S,M)$, there is a $c_s \in M$ such that $\varphi(s) = c_s s$ for all $s \in S$. Let $c_s = \sum_{0\le\beta\le N_s} \ell_{s,\beta}\otimes x^\beta$.

We compute

$$\begin{aligned}\varphi(s) &= \sum_i k_i\varphi(x^i) = \sum_i k_i\Big(\sum_\alpha a_{i,\alpha}\otimes x^\alpha\Big)\\ &= \sum_\alpha \bigg(\Big(\sum_i k_i a_{i,\alpha}\Big)\otimes x^\alpha\bigg).\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi(s) = c_s s &= \bigg(\sum_{0\le\beta\le N_s} \ell_{s,\beta}\otimes x^\beta\bigg)\bigg(\sum_{0\le i\le N} k_i x^i\bigg)\\ &= \sum_{i,\alpha} \ell_{s,\beta} k_i \otimes x^{i+\beta}\\ &= \sum_\alpha \bigg(\Big(\sum_i \ell_{s,\alpha-i} k_i\Big)\otimes x^\alpha\bigg).\end{aligned}$$

Thus, for all $\alpha \ge 0$, we have

(*) $\sum_{0\le i\le\alpha} \ell_{s,\alpha-i}k_i = \sum_{i\ge 0} k_i a_{i,\alpha} = \sum_{0\le i\le N} k_i a_{i,\alpha}$.

Note that $\varphi(k_0) = \sum_\alpha \ell_{1,\alpha}k_0\otimes x^\alpha = \sum_\alpha k_0 a_{0,\alpha}\otimes x^\alpha$ and it follows that $a_{0,\alpha} = \ell_{1,\alpha}$ for all α.

Now let $t(\alpha,s) = -\sum_{1\le i\le\alpha} k_i a_{0,\alpha-i} + \sum_{0\le i\le N} k_i a_{i,\alpha} \in A$. Since A is **Z**-reduced, there is some natural number $\|t(s,\alpha)\|$ such that $t(\alpha,s)\notin \|t(\alpha,s)\|A$ provided that $t(\alpha,s)\neq 0$.

Let $w_s = \operatorname{lcm}\{\|t(\alpha,s)\| : t(\alpha,s)\neq 0, 1\le\alpha\le N_s\}$.

(**) Assume that w_s divides the integer $k_0 = s(0)$.

We will show that

$(***)$ $\ell_{s,\alpha} = a_{0,\alpha}$ for all $\alpha \geq 0$.

We proceed by induction over α. For $\alpha = 0$ we have the equation $\ell_{s,0}k_0 = k_0 a_{0,0} + \sum_{i\geq 1} k_i a_{i,0}$, and it follows that $k_0^{-1}(\sum_{i\geq 1} k_i a_{i,0}) \in A$ no matter how the k_i's are chosen. Since A is **Z**-reduced, we infer that $a_{i,0} = 0$ for all $i \geq 1$, and we have that $\ell_{s,0} = a_{0,0}$ for all $s \in S$. This shows that $(***)$ holds for $\alpha = 0$.

Now assume that $(***)$ holds for all $0 \leq \beta < \alpha$. Now $(*)$ becomes $\ell_{s,\alpha}k_0 = k_0 a_{0,\alpha} - \sum_{1\leq i\leq \alpha} k_i a_{0,\alpha-i} + \sum_{0\leq i\leq N} k_i a_{i,\alpha}$, and thus $k_0(\ell_{s,\alpha} - a_{0,\alpha}) = t(\alpha, s)$. If $\ell_{s,\alpha} - a_{0,\alpha} \neq 0$, we get the contradiction $k_0^{-1}t(\alpha, s) \in A$ by $(**)$. This shows that $\ell_{s,\alpha} = a_{0,\alpha}$ for all $s \in S$ that satisfy $(**)$, i.e., $s(0)$ is "big enough."

For such an element $s \in S$ we have that $c_s = \sum_\alpha \ell_{s,\alpha} \otimes x^\alpha = \sum_\alpha a_{0,\alpha} \otimes x^\alpha = \varphi(1) = \varphi(x^0)$. Now let $v \in S$. Then there exists some $k \in \mathbf{Z}$ such that $k + v$ satisfies $(**)$. As we just have seen, this implies $\varphi(1)k + \varphi(v) = \varphi(k + v) = \varphi(1)(k + v) = \varphi(1)k + \varphi(1)v$ and the desired equation $\varphi(v) = \varphi(1)v$ follows for all $v \in S$. Thus M_S is a Zassenhaus module. □

We also need:

Lemma 4. *Let $S = \mathbf{Z}[x] \subseteq R \subseteq V$ be rings with torsion-free additive groups and ${}_RM_S = V \otimes_{\mathbf{Z}} S$. Let $0 \neq t \in R$ and $s \in S$ be such that $t \otimes 1 = ms$ for some $m \in M$. Then $s \in \mathbf{Z}$, $m = u \otimes 1$ for some $u \in V$ and $t = us$.*

Proof. Let $s = \sum_{0\leq i\leq N} k_i x^i$. There exist finitely many $v_j \in V$ such that $m = \sum_j v_j \otimes x^j$. Then $t \otimes 1 = ms = (\sum_j v_j \otimes x^j)(\sum_{0\leq i\leq N} k_i x^i) = \sum_\alpha(\sum_{0\leq i\leq \alpha} v_{\alpha-i}k_i) \otimes x^\alpha$. This implies that $t = v_0 k_0$ and $k_0 \neq 0$ since $t \neq 0$. We have

$(*)$ $0 = \sum_{0\leq i\leq \alpha} v_{\alpha-i}k_i$ for all $\alpha \geq 1$. An easy induction shows that $v_j = v_0 q_j$ for some $q_j \in \mathbf{Q}$ with $q_0 = 1$. Now $t = v_0(q_0 k_0)$ and $v_0(\sum_{0\leq i\leq \alpha} q_{\alpha-i}k_i) = 0$. Let $g(x) = \sum_j q_j x^j \in \mathbf{Q}[x]$. The equations $(*)$ imply that $g(x)s = q_0 k_0 = k_0$. We infer that $g(x) = 1$ and $s = k_0$ are constant polynomials. It follows that $m = v_0 \otimes 1$, $s = k_0$ and $t = v_0 k_0$ as claimed. □

We also need

Lemma 5. *Let* $M = (R \otimes_{\mathbf{Z}} S)e_1 \oplus (R \otimes_{\mathbf{Z}} S)e_2$ *and* $\widehat{M} = \{\varphi \in \mathrm{Hom}_{\mathbf{Z}}(M, M) : \varphi(m) \in Rm$ *for all* $m \in M\}$. *If* $\varphi \in \widehat{M}$, *then there exists some* $\rho \in R$ *such that* $\varphi(m) = \rho m$ *for all* $m \in M$.

Proof. Let $\varphi \in \widehat{M}$. Then there exist $\rho_{s,i} \in R$ such that $\varphi((1 \otimes s)e_i) = (\rho_{s,i} \otimes s)e_i$ for $i = 1, 2$ and $\varphi((1 \otimes s)e_1 + (1 \otimes s)e_2) = \rho_s((1 \otimes s)e_1 + (1 \otimes s)e_2)$, and it follows that $\rho_{s,1} = \rho_s = \rho_{s,2}$ for all $s \in S$. Now $\varphi((1 \otimes s)e_1 + (1 \otimes t)e_2) = \rho_{s,t}((1 \otimes s)e_1 + (1 \otimes t)e_2) = \rho_s(1 \otimes s)e_1 + \rho_t(1 \otimes t)e_2$, and it follows that $\rho_s = \rho_t$ for all $s, t \in R$. Thus there is an element $\rho \in R$ such that $\varphi(1 \otimes s) = \rho(1 \otimes s) = \rho \otimes s$ for all $s \in S$. Let $r \in R$, and compute $\varphi((1 \otimes s)e_1 + (r \otimes s)e_2) = \tau_{r,s}((1 \otimes s)e_1 + (r \otimes s)e_2 = (\rho \otimes s)e_1 + t_{r,s}(r \otimes s)e_2$ where $\varphi((r \otimes s)e_2 = t_{r,s}(r \otimes s)e_2$. It follows that $\rho = \tau_{r,s}$ and $\rho r = t_{r,s}r$. Therefore, $\varphi((r \otimes s)e_2 = (t_{r,s}r \otimes s)e_2 = (\rho r \otimes s)e_2 = \rho((r \otimes s)e_2)$. In a similar fashion, one can show that $\varphi((r \otimes s)e_1 = \rho((r \otimes s)e_1)$ for all $r \in R, s \in S$ and $R \otimes_{\mathbf{Z}} S$ is additively generated by elements of this form. This shows that $\varphi(m) = \rho m$ for all $m \in M$. $\square$

Now we are ready to prove:

Theorem 2. *There exists a commutative ring* R *and* R*-module* M *of rank at least* 2, *such that* R *is not a Zassenhaus ring, but* $R(+)M$ *is a Zassenhaus ring.*

Proof. Let $S = \mathbf{Z}[x]$, and let J be as defined at the beginning of this section. By Example 4, the ring $R = S(+)J$ is not a Zassenhaus ring.

Let ${}_RM_S = (R \otimes_{\mathbf{Z}} S)e_1 \oplus (R \otimes_{\mathbf{Z}} S)e_2$, which is naturally a R-S-bimodule, which turns into an R-R-bimodule ${}_RM_R$ by setting $MJ = \{0\}$, i.e., M_R is not faithful but ${}_RM$ has rank at least 2.

Define $T = R(+)M$.

Recalling the notations of Lemma 2, let $\psi = \left[\begin{smallmatrix} \alpha & 0 \\ \beta & \gamma \end{smallmatrix}\right] \in \widehat{T}$ and $m_0 = (1 \otimes 1)e_1 \in M$. Then $\psi \left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho_{r,m} & 0 \\ \mu_{r,m} & \rho_{r,m} \end{smallmatrix}\right] \left[\begin{smallmatrix} r \\ m \end{smallmatrix}\right] = \left[\begin{smallmatrix} \rho_{r,m} r \\ \mu_{r,m} r + \rho_{r,m} m \end{smallmatrix}\right]$, and it follows that $\alpha(r) = \rho_{r,m}r$ and $\beta(r) + \gamma(m) = \mu_{r,m}r + \rho_{r,m}m$ for all $r \in R, m \in M$.

For $r = 0$, we get $\gamma(m) = \mu_{0,m}m$, which means that $\gamma \in \widehat{M}$ and by Lemma 5, there is some $\rho_0 \in R$ such that $\gamma(m) = \rho_0 m = \mu_{0,m}m$ for all $m \in M$.

For $m = 0$, we get $\beta(r) = \mu_{r,0}r$ for all $r \in R$ and thus $\beta \in H(R, M)$. By Lemma 3 and Remark 1, M_R is a Zassenhaus module and thus there is some $\mu_0 \in M$ such that $\beta(r) = \mu_0 r = \mu_{r,0}r$ for all $r \in R$.

Now we have $\mu_0 r + \rho_0 m = \mu_{r,m}r + \rho_{r,m}m$.

It follows $(\rho_0 - \rho_{r,m})m = (\mu_{r,m} - \mu_0)r$ for all $r \in R, m \in M$. We choose $m = m_0$ and obtain $(\rho_0 - \rho_{r,m_0})(1 \otimes 1) = br$ for some $b \in R \otimes_{\mathbf{Z}} S$. Now apply Lemma 4 and infer that $\rho_0 = \rho_{r,m_0}$ for all $r \in R - (\mathbf{Z} \oplus J)$. This shows that $\rho_0 = \rho_{r,m_0}$ and $\alpha(r) = \rho_0 r$ for all $r = s + j \in R$ such that $s \in S$ is not constant. Let $z \in \mathbf{Z}$, $j \in J$ and $\sigma \in S$ any polynomial of positive degree.

Then $\alpha(z + j) = \alpha((z - \sigma + j) + \sigma) = \alpha(z - \sigma + j) + \alpha(\sigma) = \rho_0(z - \sigma + j) + \rho_0\sigma = \rho_0(z + j)$. This shows that $\alpha = \rho_0 \cdot \in R\cdot$. It follows that $\psi = \begin{bmatrix} \rho_0 \cdot & 0 \\ \mu \cdot & \rho_0 \cdot \end{bmatrix} \in T\cdot$, and we have that T is a Zassenhaus ring. $\square$

REFERENCES

1. D.D. Anderson and M. Winders, *Idealization of a module*, J. Comm. Algebra. **1** (2009), 3–56.

2. J. Buckner and M. Dugas, *Quasi-localizations of* **Z**, Israel J. Math. **160** (2007), 349–370.

3. ———, *Left rigid rings*, J. Algebra **309** (2007), 192–206.

4. ———, *Rings with Zassenhaus families of ideals*, Comm. Algebra **36** (2008), 2133–2142.

5. ———, *Group algebras with Zassenhaus families of right ideals*, Houston J. Math., to appear.

6. ———, *Zassenhaus algebras*, Rocky Mountain J. Math., to appear.

7. M. Dugas, *Large E-modules exist*, J. Algebra **142** (1991), 405–413.

8. M. Dugas and R. Göbel, *An extension of Zassenhaus' theorem on endomorphism rings*, Fund. Math. **194** (2007), 239–251.

9. S. Lang, *Algebraic number theory*, Second edition, Grad. Texts Math., Springer-Verlag, New York, 1994.

10. A. Mader and C. Vinsonhaler, *Torsion-free E-modules*, J. Algebra **115** (1988), 401–411.

11. M. Nagata, *Local rings*, Interscience Tracts Pure Appl. Math. **13**, John Wiley and Sons, New York, 1962.

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