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MINIMAL RELATIONS AND CATENARY DEGREES IN KRULL MONOIDS

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ABSTRACT. Let H be a Krull monoid with class group G. Then, H is factorial if and only if G is trivial. Sets of lengths and sets of catenary degrees are well studied invariants describing the arithmetic of H in the non-factorial case. In this note, we focus on the set $Ca(H)$ of catenary degrees of H and on the set $\mathcal{R}(H)$ of distances in minimal relations. We show that every finite nonempty subset of $\mathbb{N}_{\geq 2}$ can be realized as the set of catenary degrees of a Krull monoid with finite class group. This answers **[20,** Problem 4.1]. Suppose, in addition, that every class of G contains a prime divisor. Then, $Ca(H) \subset \mathcal{R}(H)$ and $\mathcal{R}(H)$ contain a long interval. Under a reasonable condition on the Davenport constant of G, $\mathcal{R}(H)$ coincides with this interval, and the maximum equals the catenary degree of H.

1. Introduction. In this note, we study the arithmetic of atomic monoids with a focus on Krull monoids. This setting includes Krull domains and hence all integrally closed Noetherian domains. By an atomic monoid, we mean a commutative cancelative semigroup with unit element with the property that every non-unit can be written as a finite product of atoms (irreducible elements). Let H be an atomic monoid. Then, H is factorial if and only if every equation $u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell$, with $k, \ell \in \mathbb{N}$ and atoms $u_1, \ldots, u_k, v_1, \ldots, v_\ell$, implies that $k = \ell$ and that, after renumbering if necessary, u_i and v_i just differ by a unit for all $i \in [1, k]$. It is well known that H is factorial if and only if it is a Krull monoid with trivial class group.

Suppose that H is atomic, but not factorial. Then, there is an element $a \in H$ having two distinct factorizations z and z', which can

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be written in the form

$$z = u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_\ell$$

and

$$z' = u_1 \cdot \ldots \cdot u_k w_1 \cdot \ldots \cdot w_m,$$

where $k \in \mathbb{N}_0$, $\ell, m \in \mathbb{N}$, all u_r, v_s, w_t are atoms, and no v_s is associated to any w_t with $r \in [1,k]$, $s \in [1,\ell]$, and $t \in [1,m]$. Then, we call $\mathsf{d}(z,z') = \max\{\ell, m\}$ the distance between the factorizations z and z'. The catenary degree $\mathsf{c}(a)$ of a is the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that, for any two factorizations y, y' of a, there are factorizations $y = y_0, y_1, \ldots, y_n = y'$ of a such that the distance $\mathsf{d}(y_{i-1}, y_i)$, where $i \in [1,n]$, of each two subsequent factorizations is bounded by N. The catenary degree $\mathsf{c}(H)$ of H is defined as the supremum of the catenary degrees $\mathsf{c}(a)$ over all elements $a \in H$.

In this note, we study the set of catenary degrees and the set $\mathcal{R}(H)$ of distances in minimal relations of H. More precisely, $\mathcal{R}(H)$ is defined as the set of all $d \in \mathbb{N}$ having the following property:

> There is an element $a \in H$ having two distinct factorizations z and z' with distance $\mathsf{d}(z,z') = d$, but there are no factorizations $z = z_0, z_1, \ldots, z_n = z'$ of a such that $\mathsf{d}(z_{i-1}, z_i) < d$ for all $i \in [1,n]$.

The key idea of the present note is to study $\mathcal{R}(H)$ with the aid of a crucial subset $\daleth^*(H)$, defined as

$$\begin{aligned}\daleth^*(H) = \{\min(\mathsf{L}(uv) \setminus \{2\}) \mid \\ u, v \text{ are atoms and the set of lengths } \mathsf{L}(uv) \\ \text{has at least two elements}\}.\end{aligned}$$

Thus, we consider four closely related sets of invariants, namely, $\mathcal{R}(H)$, $Ca(H)$ and $\daleth^*(H)$, together with the well-studied set of distances $\Delta(H)$ (also called the delta set of H). These sets satisfy some straightforward inclusions (see equation (2.2) and Lemma 2.1), but they are different in general (Example 2.3). First, we show that every finite nonempty subset $C \subset \mathbb{N}_{\geq 2}$ can be realized as the set of catenary degrees of a Krull monoid with finite class group (Proposition 3.2). This answers Problem 4.1 of **[20]**.

In contrast to the wildness provided by this realization result, each of the sets $\mathcal{R}(H)$, $Ca(H)$, $\daleth^*(H)$, and $\Delta(H)$ is very structured if H is a Krull monoid such that each class contains a prime divisor. This assumption holds true, among others, for holomorphy rings in global fields and for all semigroup rings which are Krull.

Indeed, under this assumption on the prime divisors, the set $\daleth^*(H)$ is an interval (Proposition 3.3). Under a reasonable condition on the Davenport constant of the class group, the sets $\daleth^*(H) \cup \{2\}$ and $\mathcal{R}(H)$ coincide, they are intervals, and their maxima equal the catenary degree. In order to formulate our main result, we recall that, for an abelian group G, $\mathsf{D}(G)$ denotes the Davenport constant of G (the assumption $\mathsf{D}(G) = \mathsf{D}^*(G)$ will be analyzed in Remark 3).

Theorem 1.1. *Let H be a Krull monoid with class group G such that every class contains a prime divisor. Then, $\daleth^*(H)$ is an interval with $\daleth^*(H) \subset \mathcal{R}(H)$. Moreover, if $\mathsf{D}(G) = \mathsf{D}^*(G) \in \mathbb{N}_{\ge 4}$, then*

$$\mathcal{R}(H) = \daleth^*(H) \cup \{2\} = \big(2 + \Delta(H)\big) \cup \{2\} = [2, \mathsf{c}(H)]. \tag{1.1}$$

Specifically, we have:

(i) *if $|G| = 1$, then $\mathcal{R}(H) = \emptyset$, and if $|G| = 2$, then $\mathcal{R}(H) = \{2\}$. If G is infinite, then $\mathcal{R}(H) = \mathbb{N}_{\ge 2}$;*

(ii) *if $\mathsf{D}(G) = 3$ and every nonzero class contains precisely one prime divisor, then $\mathcal{R}(H) = \daleth^*(H) = \{3\}$;*

(iii) *if either* ($\mathsf{D}(G) = 3$ *and there is a nonzero class containing at least two distinct prime divisors*) *or, if $\mathsf{D}(G) \in \mathbb{N}_{\ge 4}$, then* $\min \mathcal{R}(H) = 2$ *and* $\min \daleth^*(H) = 3$.

2. Background on factorizations and Krull monoids. We denote by $\mathbb{N}$ the set of positive integers, and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{c \in \mathbb{Z} \mid a \le c \le b\}$ the discrete interval. For subsets $A, B \subset \mathbb{Z}$, $A + B = \{a + b \mid a \in A,\ b \in B\}$ denotes the sumset, $-A = \{-a \mid a \in A\}$, and $y + A = \{y\} + A$ for every $y \in \mathbb{Z}$. If $A = \{a_1, \ldots, a_k\}$ with $k \in \mathbb{N}_0$ and $a_1 < \cdots < a_k$, then $\Delta(A) = \{a_{\nu+1} - a_\nu \mid \nu \in [1, k-1]\}$ is the set of distances of A. Thus, $|A| \le 1$ if and only if $\Delta(A) = \emptyset$. For $A \subset \mathbb{N}$, we denote by $\rho(A) = \sup A / \min A \in \mathbb{Q}_{\ge 1} \cup \{\infty\}$ the elasticity of A, and we set $\rho(\{0\}) = 1$. Let G be an additive abelian group and $r \in \mathbb{N}$. An r-tuple

$(e_1, \ldots, e_r)$ of elements from G is said to be independent if each e_i is nonzero and $\langle e_1, \ldots, e_r \rangle = \langle e_1 \rangle \oplus \cdots \oplus \langle e_r \rangle$.

By a monoid, we mean a commutative cancelative semigroup with unit element, and we will use multiplicative notation. Let H be monoid. We denote by $H^\times$ the group of units, by $H_{\rm red} = H/H^\times$ the associated reduced monoid, by $\mathcal{A}(H)$ the set of atoms of H, and by $\mathsf{q}(H)$ the quotient group of H. For a set P, we denote by $\mathcal{F}(P)$ the free abelian monoid with basis P. An element $a \in \mathcal{F}(P)$ will be written in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \quad \text{with} \quad \mathsf{v}_p(a) \in \mathbb{N}_0$$

and

$$\mathsf{v}_p(a) = 0 \quad \text{for almost all } p \in P,$$

and we call

$$|a|_F = |a| = \sum_{p \in P} \mathsf{v}_p(a) \in \mathbb{N}_0$$

the length of a.

Factorizations and sets of lengths. The monoid $\mathsf{Z}(H) = \mathcal{F}(\mathcal{A}(H_{\rm red}))$ is the factorization monoid of H, and the factorization homomorphism $\pi \colon \mathsf{Z}(H) \to H_{\rm red}$ maps a factorization onto the element it factors. For an element $a \in H$, we call

- $\mathsf{Z}_H(a) = \mathsf{Z}(a) = \pi^{-1}(aH^\times) \subset \mathsf{Z}(H)$ the set of factorizations of a, and
- $\mathsf{L}_H(a) = \mathsf{L}(a) = \{|z| \mid z \in \mathsf{Z}(a)\} \subset \mathbb{N}_0$ the set of lengths of a.

Note that $\mathsf{L}(a) = \{0\}$ if and only if $a \in H^\times$, and that $1 \in \mathsf{L}(a)$ if and only if $a \in \mathcal{A}(H)$, and then, $\mathsf{L}(a) = \{1\}$. The monoid is said to be

- *atomic* if $\mathsf{Z}(a) \ne \emptyset$ for all $a \in H$ (equivalently, every nonunit is a finite product of atoms);
- *factorial* if $|\mathsf{Z}(a)| = 1$ for all $a \in H$ (equivalently, $H_{\rm red}$ is free abelian);
- *half-factorial* if $|\mathsf{L}(a)| = 1$ for all $a \in H$;
- a BF-*monoid* if $\mathsf{L}(a)$ is finite and nonempty for all $a \in H$.

From now on, we suppose that H is a BF-monoid (which holds true if H is v-Noetherian). We denote by $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ the *system* *of sets of lengths* of H, by

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \subset \mathbb{N}$$

the *set of distances* of H (also called the *delta set* of H), and, if H is not half-factorial, then it follows from [**15,** Proposition 1.4.4] that

$$\min \Delta(H) = \gcd \Delta(H). \tag{2.1}$$

We study sets of distances and sets of catenary degrees via the following set $\daleth^*(H)$, which is defined as

$$\begin{aligned}\daleth^*(H) &= \{\min(\mathsf{L}(uv) \setminus \{2\}) \mid u, v \in \mathcal{A}(H), |\mathsf{L}(uv)| > 1\} \\ &= \{\min(L \setminus \{2\}) \mid 2 \in L \in \mathcal{L}(H), |L| > 1\} \subset \mathbb{N}_{\ge 3}.\end{aligned}$$

By definition, we have

$$\daleth^*(H) \subset 2 + \Delta(H) \subset \mathbb{N}_{\ge 3}, \tag{2.2}$$

H is half-factorial if and only if $\Delta(H) = \emptyset$, and if this holds, then $\daleth^*(H) = \emptyset$.

Catenary degrees and minimal relations. Any two factorizations $z, z' \in \mathsf{Z}(H)$ can be written as

$$z = u_1 \cdot \ldots \cdot u_k v_1 \cdot \ldots \cdot v_\ell \quad \text{and} \quad z' = u_1 \cdot \ldots \cdot u_k w_1 \cdot \ldots \cdot w_m,$$

where $k, \ell, m \in \mathbb{N}_0$, $u_1, \ldots, u_k, v_1, \ldots, v_\ell$, $w_1, \ldots, w_m \in \mathcal{A}(H_{\mathrm{red}})$, and $\{v_1, \ldots, v_\ell\} \cap \{w_1, \ldots, w_m\} = \emptyset$, and then, $\mathsf{d}(z, z') = \max\{\ell, m\}$ is the *distance* between z and z'. Thus, $z = z'$ if and only if $\mathsf{d}(z, z') = 0$. If $z, z' \in \mathsf{Z}(a)$ for some $a \in H$ and $N \in \mathbb{N}_0$, then a finite sequence $z_0, z_1, \ldots, z_k$ in $\mathsf{Z}(a)$ is called an N-chain of factorizations from z to z' if $z = z_0$, $z' = z_k$, and $\mathsf{d}(z_{i-1}, z_i) \le N$ for each $i \in [1, k]$. The *catenary degree* $\mathsf{c}(a) \in \mathbb{N}_0 \cup \{\infty\}$ of a is defined as the smallest $N \in \mathbb{N}_0 \cup \{\infty\}$ such that any two factorizations of a can be concatenated by an N-chain, and

$$\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H\} \in \mathbb{N}_0 \cup \{\infty\}$$

denotes the *catenary degree* of H. Clearly, $\mathsf{c}(a) = 0$ if and only if $|\mathsf{Z}(a)| = 1$, and hence, $\mathsf{c}(H) = 0$ if and only if H is factorial. If $z, z' \in \mathsf{Z}(a)$ are distinct, then $2 + ||z| - |z'|| \le \mathsf{d}(z, z')$, and $|\mathsf{Z}(a)| > 1$ implies that $2 + \sup \Delta(\mathsf{L}(a)) \le \mathsf{c}(a)$ ([**15,** Lemma 1.6.2]). Thus, if H

is not factorial, then ([**15,** Theorem 1.6.3]

$$2 + \sup \Delta(H) \le \mathsf{c}(H). \tag{2.3}$$

Since H is a BF-monoid and $\mathsf{c}(a) \le \sup \mathsf{L}(a) < \infty$, the catenary degrees of all elements are finite, and we denote by

$$\mathit{Ca}(H) = \{\mathsf{c}(a) \mid a \in H \text{ with } |\mathsf{Z}(a)| > 1\} \subset \mathbb{N}_{\ge 2}$$

the *set of catenary degrees* of all elements having at least two distinct factorizations. If $a \in \mathcal{A}(H)$, then $|\mathsf{Z}(a)| = 1$, and, if $|\mathsf{Z}(a)| = 1$, then $\mathsf{c}(a) = 0$. In order to simplify the statements of our results, we define $\mathit{Ca}(H)$ as the set of positive catenary degrees. Furthermore, let $\mathcal{R}(H)$ be the set of all $d \in \mathbb{N}_{\ge 2}$ with the following property:

> There are an element $a \in H$ and two distinct factorizations $z, z' \in \mathsf{Z}(a)$ with $\mathsf{d}(z, z') = d$ such that there is no $(d-1)$-chain of factorizations concatenating z and z'.

Our first lemma gathers some elementary properties of all these concepts.

Lemma 2.1. *Let H be a* BF*-monoid.*

(i) *H is factorial if and only if $\mathsf{c}(H) = 0$ if and only if $\mathit{Ca}(H) = \emptyset$ if and only if $\mathcal{R}(H) = \emptyset$;*
(ii) *$\mathit{Ca}(H) \subset \mathcal{R}(H)$. If H is not factorial, then $2 \le \min \mathit{Ca}(H)$ and $\mathsf{c}(H) = \sup \mathit{Ca}(H) = \sup \mathcal{R}(H)$;*
(iii) *$\urcorner^*(H) \subset \mathcal{R}(H) \subset \mathbb{N}_{\ge 2}$.*

Proof.

(i) Since H is factorial if and only if $|\mathsf{Z}(a)| = 1$ for all $a \in H$, the assertion follows.

(ii) Let $a \in H$ have at least two distinct factorizations. By definition of $\mathsf{c}(a)$, there are factorizations $z, z' \in \mathsf{Z}(a)$ with $\mathsf{d}(z, z') = \mathsf{c}(a)$ which cannot be concatenated by a $(\mathsf{c}(a) - 1)$-chain of factorizations. Thus, $\mathsf{c}(a) \in \mathcal{R}(H)$. Suppose that H is not factorial. Then, there is an element $a \in H$ with $|\mathsf{Z}(a)| > 1$, and, for every such element, we have $\mathsf{c}(a) \ge 2$, whence $\min \mathit{Ca}(H) \ge 2$. Since $\mathsf{c}(H) = \sup\{\mathsf{c}(a) \mid a \in H \text{ with } |\mathsf{Z}(a)| > 1\}$, it follows that $\mathsf{c}(H) = \sup \mathcal{R}(H)$.

(iii) Suppose that $\daleth^*(H) \neq \emptyset$, and let $a \in H$ with $2 \in \mathsf{L}(a)$ and $|\mathsf{L}(a)| > 1$. Then, there are $u_1, u_2, v_1, \ldots, v_\ell \in \mathcal{A}(H)$ with $u_1u_2 = v_1 \cdot \ldots \cdot v_\ell$, where $\ell = \min(\mathsf{L}(a) \setminus \{2\})$. Since, for any distinct two factorizations $x, x' \in \mathsf{Z}(a)$, we have $\mathsf{d}(x, x') \geq 2 + ||x| - |x'||$ and $\ell = \min(\mathsf{L}(a) \setminus \{2\})$, the factorizations $z = u_1u_2$ and $z' = v_1 \cdot \ldots \cdot v_\ell$ cannot be concatenated by an $(\ell - 1)$-chain of factorizations. Thus, $\ell = \mathsf{d}(z, z') \in \mathcal{R}(H)$. $\square$

Krull monoids and zero-sum sequences. A monoid homomorphism $\varphi\colon H \to D$ is called

- a *divisor homomorphism* if $\varphi(a) \mid \varphi(b)$ implies that $a \mid b$ for all $a, b \in H$;
- a *divisor theory* (for H) if φ is a divisor homomorphism, D is free abelian, and, for every $\alpha \in D$, there are $a_1, \ldots, a_m \in H$ such that $\alpha = \gcd(\varphi(a_1), \ldots, \varphi(a_m))$.

A monoid H is a *Krull monoid* if it satisfies one of the following equivalent conditions ([**15,** Theorem 2.4.8]):

(a) H is completely integrally closed and satisfies the ACC on divisorial ideals;
(b) H has a divisor theory;
(c) there is a divisor homomorphism from H to a factorial monoid.

Suppose that H is a Krull monoid. Then, there is a free abelian monoid $F = \mathcal{F}(P)$ such that the embedding $H_{\text{red}} \hookrightarrow F$ is a divisor theory. The group $\mathcal{C}(H) = \mathsf{q}(F)/\mathsf{q}(H_{\text{red}})$ is the (divisor) class group of H and $G_P = \{[p] = p\mathsf{q}(H_{\text{red}}) \mid p \in P\} \subset \mathcal{C}(H)$ is the set of classes containing prime divisors. We refer to [**15, 19**] for detailed presentations of the theory of Krull monoids. Here, we merely recall that a domain R is a Krull domain if and only if its monoid of nonzero elements is a Krull monoid, and property (a) shows that every integrally closed Noetherian domain is Krull. Holomorphy rings in global fields and regular congruence monoids in these domains are Krull monoids with finite class group and every class contains a prime divisor ([**15**]). Furthermore, semigroup rings which are Krull have the property that every class contains a prime divisor ([**7**]).

We continue with a Krull monoid having a combinatorial flavor, whose significance will become obvious in Lemma 2.2. Let G be an additive abelian group and $G_0 \subset G$ a subset. In additive combinatorics ([**18**]), a *sequence* (over G_0) means a finite sequence of terms from G_0 where repetition is allowed and the order of the elements is disregarded, and (as is standard) we consider sequences as elements of the free abelian monoid with basis G_0. Let

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)$$

be a sequence over G_0. Then, $\operatorname{supp}(S) = \{g_1, \ldots, g_\ell\} \subset G_0$ is the support of S, $\sigma(S) = g_1 + \ldots + g_\ell \in G$ is the sum of S, $|S| = \ell \in \mathbb{N}_0$ is the length of S, and $-S = (-g_1) \cdot \ldots \cdot (-g_\ell)$. We denote by

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$$

the monoid of zero-sum sequences over G_0. Since the embedding $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, $\mathcal{B}(G_0)$ is a Krull monoid by property (c). We follow the convention of writing $*(G_0)$ instead of $*(\mathcal{B}(G_0))$ for all arithmetical concepts $*(H)$ defined for a monoid H. In particular, we have $\mathcal{R}(G_0) = \mathcal{R}(\mathcal{B}(G_0))$, $\mathcal{C}a(G_0) = \mathcal{C}a(\mathcal{B}(G_0))$, etc. If G_0 is finite, then the set of atoms $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is finite, and

$$\mathsf{D}(G_0) = \sup\{|A| \mid A \in \mathcal{A}(G_0)\} \in \mathbb{N}$$

is the *Davenport constant* of G_0. Let G be a finite abelian group, say $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$, where $r = \mathsf{r}(G) \in \mathbb{N}_0$ is the rank of G, $n_1, \ldots, n_r \in \mathbb{N}$ with $1 < n_1 \mid \cdots \mid n_r$. If $\mathsf{D}^*(G) = 1 + \sum_{i=1}^r (n_i - 1)$, then

$$\mathsf{D}^*(G) \le \mathsf{D}(G) \le |G|. \tag{2.4}$$

We have equality on the left hand side for p-groups, for groups with rank $\mathsf{r}(G) \le 2$, and others (see [**13,** Corollary 4.2.13] for groups close to p-groups, and [**5, 21**] for groups of rank three).

The next lemma reveals the universal role of monoids of zero-sum sequences in the study of general Krull monoids.

Lemma 2.2. *Let H be a reduced Krull monoid, $F = \mathcal{F}(P)$ a free abelian monoid such that the embedding $H \hookrightarrow F$ is a cofinal divisor homomorphism, and $G = \mathsf{q}(F)/\mathsf{q}(H)$. Let $G_P = \{[p] \mid p \in P\} \subset G$ denote the set of classes containing prime divisors, $\widetilde{\boldsymbol{\beta}} \colon F \to \mathcal{F}(G_P)$ be* *the unique homomorphism satisfying* $\widetilde{\beta}(p) = [p]$ *for all* $p \in P$*, and let* $\boldsymbol{\beta} = \widetilde{\boldsymbol{\beta}} \mid H \colon H \to \mathcal{B}(G_P)$*.*

(i) $\mathcal{L}(H) = \mathcal{L}(G_P)$*. In particular,* $\daleth^*(H) = \daleth^*(G_P)$*;*
(ii) $\mathsf{c}(G_P) \le \mathsf{c}(H) \le \max\{\mathsf{c}(G_P), 2\} \le \mathsf{D}(G_P)$*;*
(iii) $\mathcal{R}(G_P) \subset \mathcal{R}(H)$*.*

Proof. The assertions on $\mathcal{L}(H)$ and $\mathsf{c}(H)$ follow from [**15,** Theorem 3.4.10]. Since $\mathcal{L}(H) = \mathcal{L}(G_P)$, we infer that

$$\begin{aligned}\daleth^*(H) &= \{\min(L \setminus \{2\}) \mid 2 \in L \in \mathcal{L}(H)\} \\ &= \{\min(L \setminus \{2\}) \mid 2 \in L \in \mathcal{L}(G_P)\} = \daleth^*(G_P).\end{aligned}$$

In order to verify the assertion on $\mathcal{R}(H)$, let $d \in \mathcal{R}(G_P)$ be given. Then, there is an $A \in \mathcal{B}(G_P)$ and factorizations $Z = U_1 \cdot \ldots \cdot U_k$, $Z' = V_1 \cdot \ldots \cdot V_\ell \in \mathsf{Z}(A)$, where $U_1, \ldots, U_k, V_1, \ldots, V_\ell \in \mathcal{A}(G_P)$, such that $\mathsf{d}(Z, Z') = d$ and such that there is no $(d-1)$-chain of factorizations of A concatenating Z and Z'. Suppose that $A = g_1^{k_1} \cdot \ldots \cdot g_m^{k_m}$, where $g_1, \ldots, g_m \in G_P$ are pairwise distinct and $k_1, \ldots, k_m \in \mathbb{N}$. For each $i \in [1, m]$ we choose a prime divisor $p_i \in g_i \cap P$, and set $a = p_1^{k_1} \cdot \ldots \cdot p_m^{k_m}$. Then, there are $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in \mathcal{A}(H)$ such that $\boldsymbol{\beta}(a) = A$, $\boldsymbol{\beta}(u_i) = U_i$ and $\boldsymbol{\beta}(v_j) = V_j$ for each $i \in [1, k]$ and each $j \in [1, \ell]$ (recall that, for every element $c \in H$, we have that c is an atom of H if and only if $\boldsymbol{\beta}(c)$ is an atom of $\mathcal{B}(G_P)$ by [**15,** Proposition 3.2.3]). Then, $z = u_1 \cdot \ldots \cdot u_k, z' = v_1 \cdot \ldots \cdot v_\ell \in \mathsf{Z}(a)$ with $\mathsf{d}(z, z') = d$. Since there is no $(d-1)$-chain of factorizations of A concatenating Z and Z', there is no $(d-1)$-chain of factorizations of a concatenating z and z'. This implies that $d \in \mathcal{R}(H)$. $\square$

As previously mentioned in the introduction, the four sets $\daleth^*(H)$, $\Delta(H)$, $Ca(H)$ and $\mathcal{R}(H)$, are often pairwise significantly different. In general, only the (trivial) relations, as gathered in Lemma 2.1, hold, and all sets are far from being intervals. Indeed, in Proposition 3.2, we will see that the sets $\daleth^*(H)$, $\mathcal{R}(H)$ and $Ca(H)$ may simultaneously coincide with any given finite subset of $\mathbb{N}_{\ge 2}$. There is an abundance of examples in the literature demonstrating the diverging behavior of these sets. Thus, we only provide three basic examples, and, since numerical monoids have been studied in detail in [**6, 9, 11, 20**], we restrict the examples in this note to the setting of Krull monoids.

Example 2.3.

1. By Lemma 2.1 (iii), we have $\daleth^*(H) \subset \mathcal{R}(H)$. Here, we provide an example of a Krull monoid H for which $\daleth^*(H) = \emptyset$, but $\mathcal{R}(H) \neq \emptyset$. Let $r, n \in \mathbb{N}$ with $r \geq 2$ and $n \geq 3$, $(e_1, \ldots, e_r) \in G^r$ be independent with $\operatorname{ord}(e_1) = \cdots = \operatorname{ord}(e_r) = n$, $e_0 = -(e_1 + \cdots + e_r)$ and $G_0 = \{e_0, \ldots, e_r\}$. We set $H = \mathcal{B}(G_0)$ and observe that $\mathcal{A}(G_0) = \{W, U_0, \ldots, U_r\}$, where $W = e_0 \cdot \ldots \cdot e_r$ and $U_i = e_i^n$ for all $i \in [0, r]$. Clearly, we have $W^n = U_0 \cdot \ldots \cdot U_r$, and it is easy to check that $\daleth^*(G_0) = \emptyset$, but $\mathit{Ca}(G_0) \neq \emptyset$. A detailed discussion of the arithmetic of this monoid may be found in [**15,** Proposition 4.1.2]. In particular, $\Delta(G_0) = \emptyset$ if and only if $n = r + 1$.

2. By (2.2), we have $\daleth^*(H) \subset 2 + \Delta(H)$ and, by Lemma 2.1, we have $\daleth^*(H) \cup \mathit{Ca}(H) \subset \mathcal{R}(H)$. An example of a Krull monoid H with $\Delta(H) = \emptyset$ (and, hence, $\daleth^*(H) = \emptyset$), but for which $\mathit{Ca}(H)$, and hence, $\mathcal{R}(H)$ are both infinite, is discussed in [**15,** Example 4.8.11].

3. We provide an example of a reduced Krull monoid H with $\mathit{Ca}(H) \subsetneq \mathcal{R}(H)$. We proceed in two steps. To begin, let G be an additive abelian group, $e_1, e_2 \in G$ two independent elements with $\operatorname{ord}(e_1) = \operatorname{ord}(e_2) = 4$ and $G_0 = \{e_1, e_2, -e_1 - e_2\}$. Then, $H_0 = \mathcal{B}(G_0)$ is a reduced Krull monoid, and clearly, we have $\mathcal{A}(H_0) = \{u_1, u_2, u_3, u_4\}$, where

$$u_1 = e_1^4, u_2 = e_2^4, u_3 = (-e_1 - e_2)^4,$$

and

$$u_4 = e_1 e_2 (-e_1 - e_2).$$

Then, $u_4^4 = u_1 u_2 u_3$ and $\mathit{Ca}(H_0) = \mathcal{R}(H_0) = \{4\}$.

Now, we use the construction presented in [**15,** Lemma 4.8.1] (we use the notation of that lemma, with $d = 3$ and $n = 4$). Let $\Gamma = \langle v \rangle$ be an infinite cyclic group, $w = u_1 u_2 u_3 v^{-1} \in \mathsf{q}(H_0) \times \langle v \rangle$, and

$$H = [u_1, u_2, u_3, u_4, v, w] \subset \mathsf{q}(H_0) \times \langle v \rangle.$$

Then, by [**15,** Lemma 4.8.1], H is a reduced Krull monoid, $\mathcal{A}(H) = \{u_1, u_2, u_3, u_4, v, w\}$, and, for all $k_1, \ldots, k_4, k, \ell \in \mathbb{N}_0$, we have $u_1^{k_1} \cdot \ldots \cdot u_4^{k_4} v^k w^\ell \in H_0$ if and only if $k = \ell$. Obviously, we have $vw = u_1 u_2 u_3 = u_4^4$, and it follows that $\mathit{Ca}(H) = \{4\}$ and $\mathcal{R}(H) = \{3, 4\}$.

3. Main results. In this section, we establish a realization theorem for finite nonempty subsets of $\mathbb{N}_{\geq 2}$ as sets of catenary degrees (answering [**20,** Problem 4.1] in the affirmative), and we give the proof of our main result, stated in the introduction.

Lemma 3.1. *Let* $n \in \mathbb{N}$, $(H_i)_{i=1}^n$ *be a family of* BF*-monoids, and* $H = H_1 \times \ldots \times H_n$.

(i) $\mathcal{R}(H) = \bigcup_{i=1}^n \mathcal{R}(H_i)$;
(ii) $\mathit{Ca}(H) = \bigcup_{i=1}^n \mathit{Ca}(H_i)$;
(iii) $\daleth^*(H) = \bigcup_{i=1}^n \daleth^*(H_i)$.

Proof. Without restriction, we may suppose that $H_1, \ldots, H_n$ are reduced. Then, $H_1, \ldots, H_n$ are divisor-closed submonoids of H, whence

$$\bigcup_{i=1}^n \mathcal{R}(H_i) \subset \mathcal{R}(H),$$

$$\bigcup_{i=1}^n \mathit{Ca}(H_i) \subset \mathit{Ca}(H),$$

and

$$\bigcup_{i=1}^n \daleth^*(H_i) \subset \daleth^*(H).$$

It remains to verify the reverse inclusions. We may suppose that $n = 2$. Then, the general case follows by an inductive argument.

(i) Let $d \in \mathcal{R}(H)$. Then, there are $a = a_1 a_2 \in H_1 \times H_2$ and factorizations $z = z_1 z_2, z' = z_1' z_2' \in \mathsf{Z}(a)$ with $\mathsf{d}(z_1 z_2, z_1' z_2') = d$ such that there is no $(d-1)$-chain of factorizations concatenating z and z', where $a_i \in H_i$ and $z_i, z_i' \in \mathsf{Z}(a_i)$ for $i \in [1,2]$. We set $d_i = \mathsf{d}(z_i, z_i')$ and observe that $d_i \leq d$ for each $i \in [1,2]$. Since $z_1 z_2, z_1 z_2', z_1' z_2'$ is a $\max\{d_1, d_2\}$-chain from z to z', it follows that $d \leq \max\{d_1, d_2\}$, whence $d = \max\{d_1, d_2\}$. If there were $(d-1)$-chains between z_1 and z_1' and between z_2 and z_2', there would be a $(d-1)$-chain between $z_1 z_2$ and $z_1' z_2'$. Without restriction, we may suppose that there is no $(d-1)$-chain between z_1 and z_1'. This implies that $d = d_1 \in \mathcal{R}(H_1)$.

(ii) Let $a = a_1 a_2 \in H$ with $\mathsf{c}(a) \ge 2$, where $a_1 \in H_1$ and $a_2 \in H_2$. Since $\mathsf{c}(a) = \max\{\mathsf{c}(a_1), \mathsf{c}(a_2)\}$, say, $\mathsf{c}(a) = \mathsf{c}(a_1)$, it follows that $\mathsf{c}(a) = \mathsf{c}(a_1) \in Ca(H_1)$.

(iii) Let $d \in \daleth^*(H) \subset \mathbb{N}_{\ge 3}$. Then, there are atoms $u, v \in \mathcal{A}(H)$ such that $\min(\mathsf{L}(uv) \setminus \{2\}) = d$. Note that $\mathcal{A}(H) = \mathcal{A}(H_1) \cup \mathcal{A}(H_2)$. If u is an atom of H_1, and v is an atom of H_2, or conversely, then $\mathsf{L}(uv) = \mathsf{L}(u) + \mathsf{L}(v) = \{2\}$. Thus, there is an $i \in [1,2]$ such that u, v are atoms of H_i and $d = \min(\mathsf{L}_H(uv) \setminus \{2\}) = \min(\mathsf{L}_{H_i}(uv) \setminus \{2\}) \in \daleth^*(H_i)$. □

Proposition 3.2. *For every finite nonempty subset $C \subset \mathbb{N}_{\ge 2}$, there is a finitely generated Krull monoid H with finite class group such that $\mathcal{R}(H) = Ca(H) = C$, and $\daleth^*(H) = C \setminus \{2\}$.*

Proof. Let $s \in \mathbb{N}$ and $C = \{d_1, \ldots, d_s\} \subset \mathbb{N}_{\ge 2}$ be given. We start with the following assertion.

A. *For every $i \in [1, s]$, there is a finitely generated reduced Krull monoid H_i with finite class group such that $\mathcal{R}(H_i) = Ca(H_i) = \{d_i\}$, and, if $d_i > 2$, then $\daleth^*(H_i) = \{d_i\}$.*

Proof of **A**. Let $i \in [1, s]$. First, we handle the case where $d_i \ge 3$. Let G_i be a cyclic group of order d_i and $g_i \in G_i$ an element with $\operatorname{ord}(g_i) = d_i$. We set $H_i = \mathcal{B}(\{g_i, -g_i\})$, $U_i = g_i^{d_i}$, $V_i = (-g_i)g_i$ and $A_i = U_i(-U_i)$. Then, $\mathsf{Z}(A_i) = \{U_i(-U_i), V_i^{d_i}\}$, which implies that $\mathsf{c}(A_i) = d_i$. Since $\mathcal{A}(H_i) = \{U_i, -U_i, V_i\}$, we infer that $\daleth^*(H_i) = \mathcal{R}(H_i) = Ca(H_i) = \{d_i\}$. Clearly, H_i is reduced and finitely generated. Since $\{g_i, -g_i\}$ is a generating set of G_i, H_i is a Krull monoid with class group isomorphic to G_i by [**15,** Proposition 2.5.6].

Now, suppose that $d_i = 2$. Let H_i be any finitely generated Krull monoid whose class group G_i has exactly two elements. Then, H_i is not factorial and $\mathsf{c}(H_i) = 2$. Therefore, Lemma 2.1 implies that $Ca(H_i) = \mathcal{R}(H_i) = \{2\}$. □

Now, $H = H_1 \times \ldots \times H_s$ is a finitely generated Krull monoid with finite class group, and Lemma 3.1 implies that $\mathcal{R}(H) = Ca(H) = \{d_1, \ldots, d_s\} = C$ and that $\daleth^*(H) = C \setminus \{2\}$. □

Let $H = H_1 \times \ldots \times H_n$ be a finite direct product of BF-monoids. Then, we have $\Delta(H_1) \cup \ldots \cup \Delta(H_n) \subset \Delta(H)$, but, in general, this inclusion is strict. For every finite nonempty set $L \subset \mathbb{N}_{\ge 2}$, there is a finitely generated Krull monoid H and an element $a \in H$ such that $L = \mathsf{L}(a)$ (**[15,** Proposition 4.8.3]). This implies that, for every finite set $C \subset \mathbb{N}$, there is a finitely generated Krull monoid H and an element $a \in H$ such that $\Delta(\mathsf{L}(a)) = C$. However, it is an open problem whether every finite set $C \subset \mathbb{N}$ with $\min C = \gcd C$ (recall equation (2.1)) can be realized as a set of distances of some monoid. For recent progress in this direction, we refer to **[11]**. On the other hand, if H is a Krull monoid and every class contains a prime divisor, then $\Delta(H)$ is an interval (**[16]**), and the next proposition reveals that the same holds for $\daleth^*(H)$.

Proposition 3.3. *Let G be a finite abelian group. Then, $\daleth^*(G) = \emptyset$ if and only if $|G| \le 2$. If $|G| \ge 3$, then $\daleth^*(G)$ is a finite interval with $\min \daleth^*(G) = 3$.*

Proof. Since $\mathcal{B}(G)$ is half-factorial if and only if $|G| \le 2$ **[15,** Corollary 3.4.12], it follows that $\daleth^*(G) = \emptyset$ if and only if $|G| \le 2$. Suppose that $|G| \ge 3$. Since $\mathsf{c}(G) \le \mathsf{D}(G) \le |G|$, by Lemma 2.2 and $\sup \daleth^*(G) \le \sup \mathcal{R}(G) = \mathsf{c}(G)$, by Lemma 2.1, it follows that $\daleth^*(G)$ is finite. We define a function $f\colon \daleth^*(G) \to \mathbb{N}$ as follows. If $\ell \in \daleth^*(G)$, then there are $U_1, U_2, V_1, \ldots, V_\ell \in \mathcal{A}(G)$ such that

$$U_1 U_2 = V_1 \cdot \ldots \cdot V_\ell\,,$$

where U_1U_2 has no factorization with length in $[3, \ell-1]$. Let $f(\ell)$ be defined as the minimum over all $|U_1U_2|$, where U_1 and U_2 stem from such a configuration. We continue with the following assertion.

- **B.** *For every $\ell \in [3, \max \daleth^*(G)]$, the interval $[\ell, \max \daleth^*(G)]$ is contained in $\daleth^*(G)$, and the function $f \mid [\ell, \max \daleth^*(G)]\colon [\ell, \max \daleth^*(G)] \to \mathbb{N}$ is strictly increasing.*

Clearly, **B** implies that $\daleth^*(G) = [3, \max \daleth^*(G)]$ is an interval with $\min \daleth^*(G) = 3$.

Proof of **B**. We proceed by induction on ℓ. Obviously, the assertion holds for $\ell = \max \daleth^*(G)$. Now suppose that the assertion holds for some $\ell \in [4, \max \daleth^*(G)]$. In order to show that it holds for $\ell - 1$, let

$U_1, U_2, V_1, \ldots, V_\ell \in \mathcal{A}(G)$ be such that

$$U_1 U_2 = V_1 \cdot \ldots \cdot V_\ell,$$

where $U_1 U_2$ has no factorization with length in $[3, \ell - 1]$, and $f(\ell) = |U_1 U_2|$. For every $i \in [1, \ell]$, we set $V_i = U_{1,i} U_{2,i}$ with $U_{1,i}, U_{2,i} \in \mathcal{F}(G)$ such that

$$U_1 = U_{1,1} \cdot \ldots \cdot U_{1,\ell} \quad \text{and} \quad U_2 = U_{2,1} \cdot \ldots \cdot U_{2,\ell}.$$

Note that $f(\ell) = |U_1 U_2| = |V_1 \cdot \ldots \cdot V_\ell| \ge 2\ell \ge 8$. Thus, after renumbering, if necessary, we may suppose that $|U_1| \ge 4$, $|U_{1,1}| \ge 1$, and $|U_{1,2}| \ge 1$, say, $g_1 \mid U_{1,1}$ and $g_2 \mid U_{1,2}$ with $g_1, g_2 \in G$. Clearly,

$$U_1' = (g_1 + g_2) g_1^{-1} g_2^{-1} U_1$$

and

$$V_1' = (g_1 + g_2) g_1^{-1} g_2^{-1} V_1 V_2$$

are zero-sum sequences, $U_1' \in \mathcal{A}(G)$, and

$$U_1' U_2 = V_1' V_3 \cdot \ldots \cdot V_\ell.$$

First, we assert that $U_1' U_2$ has no factorization with length in $[3, \ell - 2]$. Assume, to the contrary, that

$$U_1' U_2 = W_1 \cdot \ldots \cdot W_t,$$

where $t \in [3, \ell - 2]$, $W_1, \ldots, W_t \in \mathcal{A}(G)$ and $(g_1 + g_2) \mid W_1$. Since $(g_1 + g_2)^{-1} g_1 g_2 W_1$ is either an atom or a product of two atoms, $U_1 U_2$ would have a factorization with length in $[3, \ell - 1]$, a contradiction.

Now, we assume, to the contrary, that $U_1' U_2$ has no factorization with length $\ell - 1$. Then, there exists a $t \ge \ell$ such that $U_1' U_2$ has a factorization with length t, but no factorization with length in $[3, t-1]$. Therefore, $t \in \daleth^*(G)$, and

$$f(t) \le |U_1' U_2| = |U_1 U_2| - 1 = f(\ell) - 1,$$

a contradiction to the induction hypothesis that f is strictly increasing on $[\ell, \max \daleth^*(G)]$. Thus, $U_1' U_2$ has a factorization with length $\ell - 1$ which implies that $\ell - 1 \in \daleth^*(G)$ and $f(\ell - 1) \le |U_1' U_2| < |U_1 U_2| = f(\ell)$. $\square$

Proposition 3.4. *Let H be a Krull monoid with finite nontrivial class group such that every class contains a prime divisor. If $\mathsf{D}(G) = 3$ and each nonzero class contains precisely one prime element, then $\mathcal{R}(H) = Ca(H) = \{3\}$. In all other cases, we have $\min Ca(H) = \min \mathcal{R}(H) = 2$.*

Proof. Without restriction, we may suppose that H is reduced, and we consider a divisor theory $H \hookrightarrow F = \mathcal{F}(P)$. Inequality (2.4) and the subsequent remark show that $\mathsf{D}(G) = 3$ if and only if G is cyclic of order three or an elementary 2-group of rank two. Since G is nontrivial, H is not factorial, and Lemma 2.1 implies that $Ca(H) \subset \mathcal{R}(H)$ and that $2 \le \min Ca(H)$. We distinguish several cases.

Case 1. G is an elementary 2-group. Suppose that $\mathsf{r}(G) = 1$ (equivalently, $|G| = 2$). Since H is not factorial, the nonzero class contains two distinct prime divisors. Thus it is sufficient to consider the following three cases.

Case 1.1. There is a nonzero class containing two distinct prime divisors. Let $g \in G \setminus \{0\}$ contain two distinct prime divisors $p, q \in P \cap g$. Then, $u_1 = p^2$, $u_2 = q^2$, and $v = pq$ are pairwise distinct atoms, and we set $a = u_1 u_2$. Then, $\mathsf{Z}(a) = \{u_1 u_2, v^2\}$, and hence, $2 = \mathsf{c}(a) \in Ca(H)$.

Case 1.2. $\mathsf{r}(G) \ge 3$. Let $(e_1, e_2, e_3) \in G^3$ be independent, and set $e_0 = e_1 + e_2 + e_3$. We choose prime divisors $p_i \in e_i \cap P$ for $i \in [0, 3]$, $q_1 \in (e_1 + e_2) \cap P$, $q_2 \in (e_1 + e_3) \cap P$, and $q_3 \in (e_2 + e_3) \cap P$. Then, $u_1 = p_0 \cdot \ldots \cdot p_3$, $u_2 = q_1 q_2 q_3$, $v_1 = p_1 p_2 q_1$, and $v_2 = p_0 p_3 q_2 q_3$ are pairwise distinct atoms, and we set $a = u_1 u_2 = v_1 v_2$. Then, $|\mathsf{Z}(a)| > 1$ and $2 \le \mathsf{c}(a) \le \max \mathsf{L}(a) = 2$, whence $2 = \mathsf{c}(a) \in Ca(H)$.

Case 1.3. $\mathsf{r}(G) = 2$, and every nonzero class contains precisely one prime divisor. Suppose that $G = \{0, e_0, e_1, e_2\}$, and let $p_i \in e_i \cap P$ for $i \in [0, 2]$. Then, $\{p_0^2, p_1^2, p_2^2, p_0 p_1 p_2\}$ is the set of atoms which are not prime, and hence, $Ca(H) = \mathcal{R}(H) = \{3\}$.

Case 2. G is an elementary 3-group. We distinguish three cases.

Case 2.1. There is a nonzero class containing two distinct prime divisors. Let $g \in G \setminus \{0\}$ contain two distinct prime divisors $p, q \in g \cap P$. Then, $u_1 = p^3$, $u_2 = q^3$, $v_1 = p^2 q$ and $v_2 = p q^2$ are pairwise distinct atoms, and we set $a = u_1 u_2$. Then, $\mathsf{Z}(a) = \{u_1 u_2, v_1 v_2\}$, and hence, $2 = \mathsf{c}(a) \in Ca(H)$.

Case 2.2. $\mathsf{r}(G) \ge 2$. Let $(e_1, e_2) \in G^2$ be independent, and set $e_0 = e_1 + e_2$. We choose prime divisors $p_2 \in e_2 \cap P$, $p_i' \in (2e_i) \cap P$ for $i \in [1,2]$, and $q \in e_0 \cap P$. Then, $u_1 = qp_1'p_2'$, $u_2 = qp_1'p_2p_2$, $v_1 = p_2p_2'$ and $v_2 = q^2p_1'p_1'p_2$ are pairwise distinct atoms, and we set $a = u_1u_2$. Then, $\mathsf{Z}(a) = \{u_1u_2, v_1v_2\}$ and hence $2 = \mathsf{c}(a) \in Ca(H)$.

Case 2.3. $\mathsf{r}(G) = 1$, and every nonzero class contains precisely one prime divisor. Suppose that $G = \{0, g, -g\}$, $g \cap P = \{p\}$ and $(-g) \cap P = \{q\}$. Then, $\{p^3, q^3, pq\}$ is the set of atoms of H which are not prime, and hence, $Ca(H) = \mathcal{R}(H) = \{3\}$.

Case 3. There is an element $g \in G$ with $\operatorname{ord}(g) = 2m + 1$ for some $m \ge 2$. We choose prime divisors $p \in g \cap P$ and $q \in (2g) \cap P$. Then, $u_1 = p^{2m+1}, u_2 = pq^m$, $v_1 = p^{2m-1}q$ and $v_2 = q^{m-1}p^3$ are pairwise distinct atoms, and we set $a = u_1u_2 = v_1v_2$. Then, $|\mathsf{Z}(a)| > 1$ and $2 \le \mathsf{c}(a) \le \max \mathsf{L}(a) = 2$, whence $2 = \mathsf{c}(a) \in Ca(H)$.

Case 4. There is an element $g \in G$ with $\operatorname{ord}(g) = 2m$ for some $m \ge 2$. We choose prime divisors $p \in g \cap P$ and $q \in (2g) \cap P$. Then, $u_1 = p^{2m}, u_2 = q^m$, $v_1 = p^{2m-2}q$ and $v_2 = q^{m-1}p^2$ are pairwise distinct atoms, and we set $a = u_1u_2 = v_1v_2$. Then, $|\mathsf{Z}(a)| > 1$ and $2 \le \mathsf{c}(a) \le \max \mathsf{L}(a) = 2$, whence $2 = \mathsf{c}(a) \in Ca(H)$. $\square$

Proof of Theorem 1.1. Let H be a Krull monoid with class group G such that every class contains a prime divisor. Then, Lemma 2.2 implies that $\daleth^*(H) = \daleth^*(G)$, and, by Lemma 2.1, we have $\daleth^*(H) \subset \mathcal{R}(H) \subset \mathbb{N}_{\ge 2}$. If G is finite, then $\daleth^*(G)$ is a finite interval by Proposition 3.3. If G is infinite, then $\daleth^*(H) = \mathbb{N}_{\ge 2}$ by [**15,** Theorem 7.4.1], and hence, $\mathcal{R}(H) = \mathbb{N}_{\ge 2}$. The details given in Theorem 1.1 (i)–(iii) follow from Lemma 2.1 and from Propositions 3.3 and 3.4. Note that, if $|G| = 2$, then $\mathsf{c}(H) = 2$ by [**15,** Corollary 3.4.12].

It remains to prove the equalities given in (1.1). Thus, suppose that G is finite and $\mathsf{D}(G) = \mathsf{D}^*(G) \ge 4$. Since $|G| \ge 3$, $\mathcal{B}(G)$ is a Krull monoid with class group isomorphic to G, and every class contains a prime divisor ([**15,** Proposition 2.5.6]). In particular, $\mathcal{B}(G)$ is not factorial, whence $\mathsf{c}(G) \ge 2$ and $\mathsf{c}(H) = \mathsf{c}(G)$ by Lemma 2.2. Furthermore, again by Lemma 2.2, and by Proposition 3.4 (applied to $\mathcal{B}(G)$), we have

$$2 \in \mathcal{R}(G) \subset \mathcal{R}(H).$$

Combining all of the above together, we obtain that

$$\daleth^*(H) \cup \{2\} = \daleth^*(G) \cup \{2\} \subset \mathcal{R}(G) \subset \mathcal{R}(H) \subset [2, \mathsf{c}(H)] = [2, \mathsf{c}(G)],$$

and, using equations (2.2) and (2.3), that

$$\daleth^*(H) \subset 2 + \Delta(H) \subset [2, \mathsf{c}(H)].$$

Finally, since $\mathsf{D}(G) = \mathsf{D}^*(G) \geq 4$, [**14,** Corollary 4.1] implies that $\max \daleth^*(H) = \mathsf{c}(H)$, whence equality holds in the above inclusions, and the equalities given in (1.1) follow. $\square$

Remarks 3.5.

1. Let G be a finite nontrivial abelian group, say, $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$, where $r, n_1, \ldots, n_r \in \mathbb{N}$ with $1 < n_1 \mid \cdots \mid n_r$. The equation $\max \daleth^*(H) = \mathsf{c}(H)$ has not only been proved in the case where $\mathsf{D}^*(G) = \mathsf{D}(G)$, but under the weaker assumption that

$$(*) \qquad \left\lfloor \frac{1}{2}\mathsf{D}(G) + 1 \right\rfloor \leq \max\left\{ n_r,\, 1 + \sum_{i=1}^{r} \left\lfloor \frac{n_i}{2} \right\rfloor \right\}.$$

There is no known group where $\max \daleth^*(H) = \mathsf{c}(H)$ does not hold. The only groups known thus far, which satisfy $\mathsf{D}(G) > \mathsf{D}^*(G)$, and for which the precise value of $\mathsf{D}(G)$ is known, are of the form $G = C_2^4 \oplus C_{2k}$ with $k \geq 71$ odd [**10,** Theorem 5.8]. They satisfy $\mathsf{D}(G) = \mathsf{D}^*(G) + 1$, whence, also, the weaker condition in $(*)$ holds.

2. Let H be as in Theorem 1.1, and suppose that $\mathsf{D}(G) = \mathsf{D}^*(G) \in \mathbb{N}_{\geq 4}$. The catenary degree $\mathsf{c}(H)$ is explicitly known only in very special cases ([**17**]) and, in all of these cases, we have $Ca(H) = \mathcal{R}(H) = [2, \mathsf{c}(H)]$. We poset the conjecture that this equation holds for all groups G with $\mathsf{D}(G) = \mathsf{D}^*(G) \in \mathbb{N}_{\geq 4}$.

3. The set of elasticities $\{\rho(L) \mid L \in \mathcal{L}(H)\}$ has been studied by Chapman, et al., in a series of papers, see [**2, 3, 4, 8**]. They showed that, if H is a Krull monoid with finite nontrivial class group G, then, for every rational number q with $1 \leq q \leq \mathsf{D}(G)/2$, there is an $L \in \mathcal{L}(H)$ with $q = \rho(L)$.

4. All results for Krull monoids dealing with lengths of factorizations carry over to transfer Krull monoids, as studied in [**12**]. In particular,

they hold for certain maximal orders in central simple algebras over global fields [**1, 22**].

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