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COHEN-MACAULAY PROPERTIES UNDER THE AMALGAMATED CONSTRUCTION

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ABSTRACT. Let A and B be commutative rings with unity, $f : A \to B$ a ring homomorphism and J an ideal of B. Then, the subring $A \bowtie^f J := \{(a, f(a)+j) \mid a \in A \text{ and } j \in J\}$ of $A \times B$ is called the amalgamation of A with B along J with respect to f. In this paper, we study the property of Cohen-Macaulay in the sense of ideals, which was introduced by Asgharzadeh and Tousi [**2**], a general notion of the usual Cohen-Macaulay property (in the Noetherian case), on the ring $A \bowtie^f J$. Among other things, we obtain a generalization of the well-known result of when Nagata's idealization is Cohen-Macaulay.

1. Introduction. The theory of Cohen-Macaulay rings is a major area of study in commutative algebra and algebraic geometry. From the appearance of the notion of Cohen-Macaulayness, this notion admits a rich theory in commutative *Noetherian* rings. There have been attempts to extend this notion to commutative *non-Noetherian* rings, since Glaz raised the question whether there exists a generalization of the notion of Cohen-Macaulayness with certain desirable properties to non-Noetherian rings [**13, 14**]. In order to provide an answer to Glaz's question [**14,** page 220], recently, several notions of Cohen-Macaulayness for non-Noetherian rings and modules were introduced in [**2, 15, 16**]. Among those is Cohen-Macaulay in the sense of $\mathcal{A}$, introduced by Asgharzadeh and Tousi [**2**], where $\mathcal{A}$ is a non-empty subclass of ideals of a commutative ring (the definition will be given later in Section 2).

In [**7, 8**], D'Anna, Finocchiaro and Fontana introduced the following new ring construction. Let A and B be commutative rings with unity, let J be an ideal of B, and let $f : A \to B$ be a ring homomorphism. The

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amalgamation of A with B along J with respect to f is the following subring

$$A \bowtie^f J := \{(a, f(a)+j) \mid a \in A \text{ and } j \in J\}$$

of $A \times B$. This construction generalizes the amalgamated duplication of a ring along an ideal (introduced and studied in [**6, 10**]). Moreover, several classical constructions, such as Nagata's idealization (cf., [**17,** Chapter 6, Section 25], [**19,** page 2]), the $A + XB[X]$ and the $A + XB[[X]]$ constructions can be studied as particular cases of this new construction (see [**7,** Examples 2.5, 2.6]).

Below, we briefly review some known results regarding the behavior of Cohen-Macaulayness under the amalgamated construction and its particular cases.

Let M be an A-module. In 1955, Nagata introduced a ring extension of A called the *trivial extension* of A by M (or the *idealization* of M in A), denoted here by $A \ltimes M$. Now, assume that A is Noetherian local and that M is finitely generated. It is well known that the trivial extension $A \ltimes M$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and M is maximal Cohen-Macaulay, see [**1,** Corollary 4.14].

Let A be a Noetherian local ring and I an ideal of A. Consider the amalgamated duplication

$$A \bowtie I := \{(a, a+i) \mid a \in A \text{ and } i \in I\}$$

as in [**6, 10**]. The properties of being Cohen-Macaulay, generalized Cohen-Macaulayness, Gorenstein, quasi-Gorenstein, (S_n), (R_n) and normality under the construction of amalgamated duplication were further studied in many research papers, such as [**3, 6, 9, 21**].

In [**9**], under the condition that A is Cohen-Macaulay (Noetherian local) and J is a finitely generated A-module, it is observed that $A \bowtie^f J$ is a Cohen-Macaulay ring if and only if it is a Cohen-Macaulay A-module if and only if J is a maximal Cohen-Macaulay module. Then, in [**22**], assuming $(A, \mathfrak{m})$ is Noetherian local, J is contained in the Jacobson radical of B such that $\operatorname{depth}_A J < \infty$ and that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$, for each $\mathfrak{q} \in \operatorname{Spec}(B) \backslash V(J)$, it is shown that $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a big Cohen-Macaulay A-module (i.e., $\operatorname{depth}_A J = \dim A$).

The next natural step is to seek when the amalgamated algebra $A \bowtie^f J$ is Cohen-Macaulay without the Noetherian assumption.

In this paper, we investigate the property of Cohen-Macaulayness in the sense of ideals (respectively, maximal ideals, finitely generated ideals) on the amalgamation. More precisely, in Section 2, we recall some essential definitions and results on which we base our approach. In Section 3, we fix our notation and give some elementary results on the behavior of the Koszul grade with respect to amalgamation. In Section 4, we classify some necessary and sufficient conditions for the amalgamated algebra $A \bowtie^f J$ to be Cohen-Macaulay in the sense of ideals (respectively, maximal ideals, finitely generated ideals) (Theorems 4.1, 4.6 and 4.11). Among the applications of our results are the classification of when the trivial extension $A \ltimes M$ and the amalgamated duplication $A \bowtie I$ are Cohen-Macaulay in the sense of ideals (Corollaries 4.8 and 4.16).

2. Preliminaries. To facilitate the reading of the paper, we recall in this section some preliminary definitions and properties to be used later.

Let $\mathfrak{b}$ be a finitely generated ideal of a commutative ring A and M an A-module. Assume that $\mathfrak{b}$ is generated by the sequence $\mathbf{x} = x_1, \ldots, x_\ell$. We denote the Koszul complex related to $\mathbf{x}$ by $\mathbb{K}_\bullet(\mathbf{x})$. The *Koszul grade of* $\mathfrak{b}$ *on* M is defined by

$$\mathrm{K.grade}_A(\mathfrak{b}, M) := \inf\{i \in \mathbb{N} \cup \{0\} \mid H^i(\mathrm{Hom}_A(\mathbb{K}_\bullet(\mathbf{x}), M)) \neq 0\}.$$

It follows from [**5,** Proposition 1.6.10(d), Corollary 1.6.22] that this does not depend upon the choice of generating sets of $\mathfrak{b}$.

Let $\mathfrak{a}$ be an arbitrary ideal of A. The Koszul grade of $\mathfrak{a}$ on M can then be defined by setting

$$\begin{aligned}&\mathrm{K.grade}_A(\mathfrak{a}, M)\\ &\quad := \sup\{\mathrm{K.grade}_A(\mathfrak{b}, M) \mid \mathfrak{b} \text{ is a finitely generated subideal of } \mathfrak{a}\}.\end{aligned}$$

In view of [**5,** Proposition 9.1.2(f)], this definition coincides with the original one for finitely generated ideals. In particular, when $(A, \mathfrak{m})$ is locally Noetherian, $\mathrm{depth}_A M$ was defined by $\mathrm{K.grade}_A(\mathfrak{m}, M)$ in [**5,** subsection 9.1].

The *Čech grade of* $\mathfrak{b}$ *on* M is defined by

$$\check{\mathrm{C}}.\mathrm{grade}_A(\mathfrak{b}, M) := \inf\{i \in \mathbb{N} \cup \{0\} \mid H^i_{\mathbf{x}}(M) \neq 0\}.$$

Here, $H^i_{\mathbf{x}}(M)$ denotes the ith cohomology of the *Čech* complex of M related to $\mathbf{x}$. It follows from [**16,** Proposition 2.1(e)] that $H^i_{\mathbf{x}}(M)$ is independent of the choice of sequence of generators for $\mathfrak{b}$. One can then define

$$\begin{aligned}&\check{\mathrm{C}}.\operatorname{grade}_A(\mathfrak{a}, M)\\ &\quad := \sup\{\check{\mathrm{C}}.\operatorname{grade}_A(\mathfrak{b}, M) \mid \mathfrak{b} \text{ is a finitely generated subideal of } \mathfrak{a}\}.\end{aligned}$$

By virtue of [**16,** Proposition 2.7], we have $\check{\mathrm{C}}.\operatorname{grade}_A(\mathfrak{a}, M) = \mathrm{K}.\operatorname{grade}_A(\mathfrak{a}, M)$.

Let $\mathfrak{p}$ be a prime ideal of R. By $\operatorname{ht}_M \mathfrak{p}$, we mean the Krull dimension of the $R_\mathfrak{p}$-module $M_\mathfrak{p}$. Also,

$$\operatorname{ht}_M \mathfrak{a} := \inf\{\operatorname{ht}_M \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_A(M) \cap V(\mathfrak{a})\}.$$

Let $\mathcal{A}$ be a non-empty subclass of the class of all ideals of the ring A and M an A-module. We say that M is *Cohen-Macaulay in the sense of* $\mathcal{A}$ if $\operatorname{ht}_M(\mathfrak{a}) = \mathrm{K}.\operatorname{grade}_A(\mathfrak{a}, M)$ for all ideals $\mathfrak{a}$ in $\mathcal{A}$, see [**2,** Definition 3.1]. The classes we are interested in include the class of all maximal ideals, the class of all ideals and the class of all finitely generated ideals. Assume that A is Noetherian. It is well known that A is Cohen-Macaulay (in the sense of the original definition in the Noetherian setting) if and only if it is Cohen-Macaulay in the sense of ideals (respectively, maximal ideals, finitely generated ideals), see [**5,** Corollary 2.1.4].

3. The Koszul grade on amalgamation. We fix some notation which we shall use frequently throughout the paper: A and B are two commutative rings with unity, $f : A \to B$ is a ring homomorphism and J denotes an ideal of B so that J is an A-module via the homomorphism f. In the sequel, we consider contraction and extension with respect to the natural embedding

$$\iota_A : A \to A \bowtie^f J,$$

defined by $\iota_A(x) = (x, f(x))$, for every $x \in A$. In particular, for every ideal $\mathfrak{a}$ of A, $\mathfrak{a}^e$ means $\mathfrak{a}(A \bowtie^f J)$.

This section is devoted to proving some lemmas on the behavior of the Koszul grade on amalgamation. These lemmata provide the key for some crucial arguments later in this paper. In the proof of the next lemma, we use $H_i(\mathbf{x}, M)$ to denote the ith Koszul homology of an A-module M with respect to a finite sequence $\mathbf{x} \subset A$.

Lemma 3.1. *Let the notation and hypotheses be as at the beginning of this section. Then*:

(i) *for any finitely generated ideal* $\mathfrak{b}$ *of* A*, we have the equality*

$$\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{b}^e, A \bowtie^f J) = \min\{\mathrm{K.grade}_A(\mathfrak{b}, A), \mathrm{K.grade}_A(\mathfrak{b}, J)\};$$

(ii) *for any ideal* $\mathfrak{a}$ *of* A*, we have the inequality*

$$\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A \bowtie^f J) \leq \min\{\mathrm{K.grade}_A(\mathfrak{a}, A), \mathrm{K.grade}_A(\mathfrak{a}, J)\}.$$

Proof. Assume that $\mathfrak{b}$ is a finitely generated ideal of A and that $\mathfrak{b}$ is generated by a finite sequence $\mathbf{x}$ of length ℓ. Then, using [**2,** Proposition 2.2(iv)], together with [**16,** Proposition 2.7], we have

$$\begin{aligned}
&\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{b}^e, A \bowtie^f J)\\
&\quad = \mathrm{K.grade}_A(\mathfrak{b}, A \bowtie^f J)\\
&\quad = \sup\{k \geq 0 \mid H_{\ell-i}(\mathbf{x}, A \bowtie^f J) = 0 \text{ for all } i < k\}\\
&\quad = \sup\{k \geq 0 \mid H_{\ell-i}(\mathbf{x}, A) \oplus H_{\ell-i}(\mathbf{x}, J) = 0 \text{ for all } i < k\}\\
&\quad = \min\{\mathrm{K.grade}_A(\mathfrak{b}, A), \mathrm{K.grade}_A(\mathfrak{b}, J)\}.
\end{aligned}$$

For the third equality, note that the amalgamation $A \bowtie^f J$, as an A-module, is isomorphic to the direct sum of $A \oplus J$ using [**7,** Lemma 2.3(4)]. This proves (i).

In order to obtain (ii), assume that $\mathfrak{a}$ is an ideal of A. Let Σ be the class of all finitely generated subideals of $\mathfrak{a}$. It follows from the definition that

$$\begin{aligned}
&\mathrm{K.grade}_A(\mathfrak{a}, A \bowtie^f J)\\
&\quad = \sup\{\mathrm{K.grade}_A(\mathfrak{b}, A \bowtie^f J) \mid \mathfrak{b} \in \Sigma\}\\
&\quad = \sup\{\min\{\mathrm{K.grade}_A(\mathfrak{b}, A), \mathrm{K.grade}_A(\mathfrak{b}, J)\} \mid \mathfrak{b} \in \Sigma\}\\
&\quad \leq \min\{\sup\{\mathrm{K.grade}_A(\mathfrak{b}, A) \mid \mathfrak{b} \in \Sigma\}, \sup\{\mathrm{K.grade}_A(\mathfrak{b}, J) \mid \mathfrak{b} \in \Sigma\}\}\\
&\quad = \min\{\mathrm{K.grade}_A(\mathfrak{a}, A), \mathrm{K.grade}_A(\mathfrak{a}, J)\}.
\end{aligned}$$

Again, using this in conjunction with [**2,** Proposition 2.2(iv)], we deduce that

$$\begin{aligned}\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A\bowtie^f J) &= \mathrm{K.grade}_A(\mathfrak{a}, A\bowtie^f J)\\ &\le \min\{\mathrm{K.grade}_A(\mathfrak{a}, A), \mathrm{K.grade}_A(\mathfrak{a}, J)\}. \square\end{aligned}$$

Lemma 3.2. *Assume that A is Cohen-Macaulay in the sense of (finitely generated) ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$ *for every (finitely generated) ideal $\mathfrak{a}$ of A. Then:*

$$\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A\bowtie^f J) = \mathrm{K.grade}_A(\mathfrak{a}, A) \le \mathrm{K.grade}_A(\mathfrak{a}, J)$$

for any (finitely generated) ideal $\mathfrak{a}$ of A.

Proof. Assume that $\mathfrak{a}$ is a (finitely generated) ideal of A, and let Σ be the class of all finitely generated subideals of $\mathfrak{a}$. Then, as in the proof of Lemma 3.1, again, using [**2,** Proposition 2.2(iv)], we have

$$\begin{aligned}&\mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A\bowtie^f J)\\ &\quad= \mathrm{K.grade}_A(\mathfrak{a}, A\bowtie^f J)\\ &\quad= \sup\{\mathrm{K.grade}_A(\mathfrak{b}, A\bowtie^f J) \mid \mathfrak{b}\in\Sigma\}\\ &\quad= \sup\{\min\{\mathrm{K.grade}_A(\mathfrak{b}, A), \mathrm{K.grade}_A(\mathfrak{b}, J)\} \mid \mathfrak{b}\in\Sigma\}\\ &\quad= \sup\{\mathrm{K.grade}_A(\mathfrak{b}, A) \mid \mathfrak{b}\in\Sigma\}\\ &\quad= \mathrm{K.grade}_A(\mathfrak{a}, A).\end{aligned}$$

The forth equality follows from [**2,** Lemma 3.2] and our assumption. This completes the proof. $\square$

The next lemma is a slight modification of [**2,** Lemma 3.2].

Lemma 3.3. *Let $\mathfrak{a}$ be an ideal of A and M an A-module.*

(i) *Let A be quasi-local with the maximal ideal $\mathfrak{m}$. If* $\mathrm{K.grade}_A(\mathfrak{m}, M) < \infty$, *then* $\mathrm{K.grade}_A(\mathfrak{m}, M) \le \dim A$.

(ii) *If, for every minimal prime ideal $\mathfrak{p}$ over $\mathfrak{a}$,* $\mathrm{K.grade}_A(\mathfrak{p}A_{\mathfrak{p}}, M_{\mathfrak{p}}) < \infty$, *e.g., when M is finitely generated, and $V(\mathfrak{a}) \subseteq \mathrm{Supp}_A(M)$, then* $\mathrm{K.grade}_A(\mathfrak{a}, M) \le \mathrm{ht}\,\mathfrak{a}$.

Proof.

(i) Using [**16,** Proposition 2.7], it is sufficient for us to show that $\check{\mathrm{C}}.\operatorname{grade}_A(\mathfrak{m}, M) \leq \dim A$. In order to prove this, assume that $\dim A < \infty$, and let $\mathbf{x}$ be a finite sequence of elements in $\mathfrak{m}$. It follows from [**16,** Proposition 2.4] that $\check{\mathrm{C}}.\operatorname{grade}_A(\mathbf{x}, M) \leq \dim A$. Therefore, $\check{\mathrm{C}}.\operatorname{grade}_A(\mathfrak{m}, M) \leq \dim A$.

(ii) Note, by [**2,** Proposition 2.2(iii)], that $\mathrm{K}.\operatorname{grade}_A(\mathfrak{a}, M) < \infty$. Then, by [**2,** Proposition 2.2(ii), (iii)], we may assume that A is quasi-local with the maximal ideal $\mathfrak{m}$. Now (i) completes the proof. □

4. Main results. Assume that A is Noetherian local, that J is contained in the Jacobson radical of B and it is a finitely generated A-module. Recall that a finitely generated module M over A is called a *maximal Cohen-Macaulay A-module* if $\operatorname{depth}_A M = \dim A$. Note that, in this circumstance, $\operatorname{depth}_A M$ equals the common length of the maximal M-regular sequences in the maximal ideal of A. In [**22,** Corollary 2.5], it is shown that $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is a maximal Cohen-Macaulay A-module. Our first main result improves this corollary by removing the Noetherian assumption.

The reader should be aware that, when we say $A \bowtie^f J$ is Cohen-Macaulay in the sense of a non-empty class of ideals, we mean $A \bowtie^f J$ is Cohen-Macaulay as a ring.

Theorem 4.1. *Assume that* $(A, \mathfrak{m})$ *is quasi-local such that* $\mathfrak{m}$ *is finitely generated. Assume that* J *is contained in the Jacobson radical of* B *and it is finitely generated as an* A*-module. Then,* $A \bowtie^f J$ *is Cohen-Macaulay* (*ring*) *in the sense of maximal ideals if and only if* A *is Cohen-Macaulay in the sense of maximal ideals and* $\mathrm{K}.\operatorname{grade}_A(\mathfrak{m}, J) = \dim A$.

Proof. Assume that $\mathfrak{m}$ is generated by the sequence $\mathbf{a} = a_1, \ldots, a_n$ and that J is generated by the sequence $\mathbf{b} = b_1, \ldots, b_m$. Hence, $\mathfrak{m}'^f = \mathfrak{m} \bowtie^f J$, the unique maximal ideal of $A \bowtie^f J$ [**9,** Corollary 2.7(3)], is generated by the sequence

$$\mathbf{c} = (a_1, f(a_1)), \ldots, (a_n, f(a_n)), (0, b_1), \ldots, (0, b_m).$$

Notice that, by [**9,** Corollary 3.2, Remark 3.3], we have

$$\sqrt{\iota_A(\mathbf{a})(A \bowtie^f J)} = \sqrt{\mathfrak{m}(A \bowtie^f J)} = \mathfrak{m}'^f = \mathbf{c}(A \bowtie^f J).$$

Therefore,

$$\begin{aligned}
\mathrm{K.\,grade}_{A\bowtie^f J}(\mathfrak{m}'^f, A \bowtie^f J) &= \check{\mathrm{C}}.\,\mathrm{grade}_{A\bowtie^f J}(\mathfrak{m}'^f, A \bowtie^f J) \\
&= \inf\{i \mid H^i_{\mathbf{c}}(A \bowtie^f J) \neq 0\} \\
&= \inf\{i \mid H^i_{\iota_A(\mathbf{a})}(A \bowtie^f J) \neq 0\} \\
&= \inf\{i \mid H^i_{\mathbf{a}}(A \bowtie^f J) \neq 0\} \\
&= \inf\{i \mid H^i_{\mathbf{a}}(A) \oplus H^i_{\mathbf{a}}(J) \neq 0\} \\
&= \min\{\check{\mathrm{C}}.\,\mathrm{grade}_A(\mathfrak{m}, A), \check{\mathrm{C}}.\,\mathrm{grade}_A(\mathfrak{m}, J)\} \\
&= \min\{\mathrm{K.\,grade}_A(\mathfrak{m}, A), \mathrm{K.\,grade}_A(\mathfrak{m}, J)\}.
\end{aligned}$$

The first equality is obtained by [**16,** Proposition 2.7], the third equality follows from [**16,** Proposition 2.1(e)] in conjunction with $\sqrt{\iota_A(\mathbf{a})(A \bowtie^f J)} = \mathbf{c}(A \bowtie^f J)$, the forth equality is deduced from [**16,** Proposition 2.1(f)] and the fifth equality holds as an A-module, $A \bowtie^f J \cong A \oplus J$ [**7,** Lemma 2.3(4)].

Consequently, the conclusion follows by the equality

$$\mathrm{K.\,grade}_{A\bowtie^f J}(\mathfrak{m}'^f, A \bowtie^f J) = \min\{\mathrm{K.\,grade}_A(\mathfrak{m}, A), \mathrm{K.\,grade}_A(\mathfrak{m}, J)\},$$

together with $\dim A \bowtie^f J = \dim A$. This last equality holds , since $A \bowtie^f J$ is integral over A (see [**8,** Proposition 4.2]). □

Corollary 4.2 ([**22,** Corollary 2.5])**.** *Assume that A is Noetherian local, that J is contained in the Jacobson radical of B and it is finitely generated as an A-module. Then, $A \bowtie^f J$ is a Cohen-Macaulay* (*ring*) *if and only if A is Cohen-Macaulay and J is a maximal Cohen-Macaulay A-module.*

The key to the next theorem is given by the following elementary lemmas. Their proofs are straightforward; thus, we omit them here. Recall from [**9,** Corollary 2.5] that the prime ideals of $A \bowtie^f J$ are of type $\overline{\mathfrak{q}}^f$ or $\mathfrak{p}'^f$, for $\mathfrak{q}$ varying in $\mathrm{Spec}(B)\backslash V(J)$ and $\mathfrak{p}$ in $\mathrm{Spec}(A)$, where

$$\mathfrak{p}'^f := \mathfrak{p} \bowtie^f J := \{(p, f(p) + j) \mid p \in \mathfrak{p}, j \in J\},$$

$$\overline{\mathfrak{q}}^f := \{(a, f(a)+j) \mid a \in A, j \in J, f(a)+j \in \mathfrak{q}\}.$$

Lemma 4.3. *Assume that* $\mathfrak{a}$ *is an ideal of* A*,* $\mathfrak{p}$ *is a prime ideal of* A *and that* $\mathfrak{q}$ *is a prime ideal of* B*. Then*:

(i) $\mathfrak{a}^e \subseteq \mathfrak{p}'^f$ *if and only if* $\mathfrak{a} \subseteq \mathfrak{p}$;
(ii) $\mathfrak{a}^e \subseteq \overline{\mathfrak{q}}^f$ *if and only if* $f(\mathfrak{a}) \subseteq \mathfrak{q}$.

In the sequel, we use $\mathrm{Nil}(B)$ to denote the nil radical of the ring B.

Lemma 4.4. *Assume that* $\mathfrak{a}$ *is an ideal of* A*,* $J \subseteq \mathrm{Nil}(B)$ *and that* $\mathfrak{p}$ *is a prime ideal of* A*. Then*:

(i) $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$ *if and only if* $\mathfrak{p}'^{f} \in \mathrm{Min}(\mathfrak{a}^e)$.
(ii) $\mathrm{ht}\,\mathfrak{a} = \mathrm{ht}\,\mathfrak{a}^e$.
(iii) $\mathrm{Min}(\mathfrak{p}^e) = \{\mathfrak{p}'^{f}\}$. *In particular,* $\mathrm{ht}\,\mathfrak{p}^e = \mathrm{ht}\,\mathfrak{p}'^{f}$.

Proposition 4.5. *Let* $\mathcal{A}$ *be a non-empty class of ideals of* A*. Assume that* $\mathrm{ht}\,\mathfrak{a}^e \geq \mathrm{ht}\,\mathfrak{a}$ *for each* $\mathfrak{a} \in \mathcal{A}$*. If* $A \bowtie^f J$ *is Cohen-Macaulay* (*ring*) *in the sense of* $\mathcal{A}^e := \{\mathfrak{a}^e \mid \mathfrak{a} \in \mathcal{A}\}$*, then* A *is Cohen-Macaulay in the sense of* $\mathcal{A}$ *and* $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$ *for each* $\mathfrak{a} \in \mathcal{A}$.

Proof. Assume that $\mathfrak{a} \in \mathcal{A}$. Then, by Lemma 3.1 (ii), we have

$$\begin{aligned}\mathrm{K.grade}_A(\mathfrak{a}, A) &\geq \mathrm{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A \bowtie^f J)\\ &= \mathrm{ht}\,\mathfrak{a}^e\\ &\geq \mathrm{ht}\,\mathfrak{a}\\ &\geq \mathrm{K.grade}_A(\mathfrak{a}, A).\end{aligned}$$

Thus, $\mathrm{K.grade}_A(\mathfrak{a}, A) = \mathrm{ht}\,\mathfrak{a}$. This means that A is Cohen-Macaulay in the sense of $\mathcal{A}$. Similarly, we obtain $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$. □

It is unclear whether, in general, the inequality $\mathrm{ht}\,\mathfrak{a}^e \geq \mathrm{ht}\,\mathfrak{a}$ holds for each $\mathfrak{a} \in \mathcal{A}$. However, under the assumption $J \subseteq \mathrm{Nil}(B)$, for each ideal $\mathfrak{a}$, we have the equality $\mathrm{ht}\,\mathfrak{a}^e = \mathrm{ht}\,\mathfrak{a}$ by Lemma 4.4.

The second main result of the paper is the following theorem.

Theorem 4.6. *Assume that* $J \subseteq \operatorname{Nil}(B)$. *Then,* $A \bowtie^f J$ *is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if* A *is Cohen-Macaulay in the sense of ideals and* $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht}\mathfrak{a}$ *for every ideal* $\mathfrak{a}$ *of* A.

Proof. One implication follows from Proposition 4.5 and Lemma 4.4 (ii). Then, to prove the converse, assume that A is Cohen-Macaulay in the sense of ideals and $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht}\mathfrak{a}$ for every ideal $\mathfrak{a}$ of A. Let $\mathfrak{a}$ be an ideal of A. First observe, by Lemmas 3.2 and 4.4 (ii), that

$$\begin{aligned}\operatorname{K.grade}_{A\bowtie^f J}(\mathfrak{a}^e, A \bowtie^f J) &= \operatorname{K.grade}_A(\mathfrak{a}, A)\\ &= \operatorname{ht}\mathfrak{a}\\ &= \operatorname{ht}\mathfrak{a}^e.\end{aligned}$$

Now, let I be an arbitrary proper ideal of $A \bowtie^f J$. Then, by [**20,** Chapter 5, Theorem 16], there exists a prime ideal $\mathcal{P}$ of $A \bowtie^f J$ containing I such that

$$\operatorname{K.grade}_{A\bowtie^f J}(I, A \bowtie^f J) = \operatorname{K.grade}_{A\bowtie^f J}(\mathcal{P}, A \bowtie^f J).$$

Note that $\mathcal{P} = \mathfrak{p}'^f$ for some prime ideal $\mathfrak{p}$ of A by [**9,** Corollaries 2.5, 2.7]. Hence, by Lemma 4.4 (iii), we have

$$\begin{aligned}\operatorname{ht} I &\geq \operatorname{K.grade}_{A\bowtie^f J}(I, A \bowtie^f J)\\ &= \operatorname{K.grade}_{A\bowtie^f J}(\mathfrak{p}'^f, A \bowtie^f J)\\ &\geq \operatorname{K.grade}_{A\bowtie^f J}(\mathfrak{p}^e, A \bowtie^f J)\\ &= \operatorname{ht}\mathfrak{p}^e\\ &= \operatorname{ht}\mathfrak{p}'^f\\ &\geq \operatorname{ht} I.\end{aligned}$$

Therefore, $A \bowtie^f J$ is Cohen-Macaulay in the sense of ideals. □

The next example shows that, if, in Theorem 4.6, the hypothesis $J \subseteq \operatorname{Nil}(B)$ is dropped, then the corresponding statement is no longer always true.

Example 4.7. Let k be a field and X, Y algebraically independent indeterminates over k. Set $A := k[[X]]$, $B := k[[X, Y]]$, and let

$J := (X, Y)$. Let $f : A \to B$ be the inclusion. Note that A is Cohen-Macaulay and $\mathrm{K.grade}_A(\mathfrak{a}, J) = \mathrm{ht}\,\mathfrak{a}$ for every ideal $\mathfrak{a}$ of A. Indeed, if $\mathfrak{a}$ is a non-zero proper ideal of A, and a is a non-zero element of $\mathfrak{a}$, then we have

$$1 \leq \mathrm{K.grade}_A(aA, J) \leq \mathrm{K.grade}_A(\mathfrak{a}, J) \leq \mathrm{ht}_J\,\mathfrak{a} \leq \mathrm{ht}\,\mathfrak{a} \leq 1.$$

The first and second inequalities follow from [**5,** Proposition 9.1.2(a),(f)], respectively, while the third inequality follows from Lemma 3.3 (ii), and the others are obvious. However, $A \bowtie^f J$, which is isomorphic to $k[[X, Y, Z]]/(Y, Z) \cap (X - Y)$, is not Cohen-Macaulay.

Let M be an A-module. Then, $A \ltimes M$ denotes the *trivial extension* of A by M. It should be noted that $0 \ltimes M$ is an ideal in $A \ltimes M$ and $(0 \ltimes M)^2 = 0$. As in [**7,** Example 2.8], if $B := A \ltimes M$, $J := 0 \ltimes M$, and $f : A \to B$ is the natural embedding, then $A \bowtie^f J \cong A \ltimes M$. Hence, the next result follows from Theorem 4.6. With it, we not only offer an application of Theorem 4.6, but we also provide a generalization of the well-known characterization of when the trivial extension is Cohen-Macaulay in the Noetherian (local) case, see [**1,** Corollary 4.14].

Corollary 4.8. *Let M be an A-module. Then, $A \ltimes M$ is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if A is Cohen-Macaulay in the sense of ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, M) \geq \mathrm{ht}\,\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A.*

Assume that A is Noetherian. In [**22,** Corollary 2.7], the authors showed that A is Cohen-Macaulay if $A \bowtie^f J$ is Cohen-Macaulay, provided that $f^{-1}(\mathfrak{q}) \neq \mathfrak{m}$ for each $\mathfrak{q} \in \mathrm{Spec}(B) \backslash V(J)$ and each $\mathfrak{m} \in \mathrm{Max}(A)$. In the following corollary, we improve the conclusion of the previously mentioned result in the circumstance that $J \subseteq \mathrm{Nil}(B)$.

Assume that A is Noetherian and M is a finitely generated A-module. It can be seen that

$$\mathrm{ht}\,\mathfrak{a} \leq \mathrm{grade}_A(\mathfrak{a}, M) (= \mathrm{K.grade}_A(\mathfrak{a}, M))$$

for every ideal $\mathfrak{a}$ of A if and only if $M_\mathfrak{p}$ is maximal Cohen-Macaulay for every prime ideal $\mathfrak{p} \in \mathrm{Supp}_A(M)$. Indeed, assume that $M_\mathfrak{p}$ is maximal Cohen-Macaulay for every prime ideal $\mathfrak{p} \in \mathrm{Supp}_A(M)$, and $\mathfrak{a}$ is an ideal of R. There is nothing to prove if $\mathfrak{a}M = M$ since, in this case,

$\operatorname{grade}_A(\mathfrak{a}, M) = \infty$. Thus, assume that $\mathfrak{a}M \neq M$. Then, using [**5,** Proposition 1.2.10(a)], there is a prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ such that $\operatorname{grade}_A(\mathfrak{a}, M) = \operatorname{depth} M_{\mathfrak{p}}$. Hence, by assumption, we have

$$\operatorname{grade}_A(\mathfrak{a}, M) = \operatorname{depth} M_{\mathfrak{p}} = \dim R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p} \geq \operatorname{ht} \mathfrak{a}.$$

In order to prove the converse, assume that $\mathfrak{p} \in \operatorname{Supp}_A(M)$. Then, again in view of [**5,** Proposition 1.2.10(a)], we have

$$\dim R_{\mathfrak{p}} = \operatorname{ht} \mathfrak{p} \leq \operatorname{grade}_A(\mathfrak{p}, M) \leq \operatorname{depth} M_{\mathfrak{p}}.$$

Thus, $M_{\mathfrak{p}}$ is maximal Cohen-Macaulay.

Corollary 4.9. *Assume that A is Noetherian, and that $J \subseteq \operatorname{Nil} B$ is finitely generated as an A-module. Then, $A \bowtie^f J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and $J_{\mathfrak{p}}$ is maximal Cohen-Macaulay for every prime ideal $\mathfrak{p} \in \operatorname{Supp}_A(J)$.*

The next proposition provides more sufficient and necessary conditions for $A \bowtie^f J$ to be Cohen-Macaulay in the sense of ideals.

Proposition 4.10. *With the notation and hypotheses from the beginning of Section* 3, *we have*:

(i) *Let $\mathcal{A}$ be a non-empty class of ideals of A. Assume that $\operatorname{ht} f^{-1}(\mathfrak{q}) \leq \operatorname{ht} \mathfrak{q}$ for every $\mathfrak{q} \in \operatorname{Spec}(B) \backslash V(J)$. If $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of $\mathcal{A}^e := \{\mathfrak{a}^e \mid \mathfrak{a} \in \mathcal{A}\}$, then A is Cohen-Macaulay in the sense of $\mathcal{A}$, and* $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht} \mathfrak{a}$ *for every* $\mathfrak{a} \in \mathcal{A}$.

(ii) *Assume that $\operatorname{ht} \mathcal{P} \leq \operatorname{ht} \mathcal{P}^c$ for every $\mathcal{P} \in \operatorname{Spec}(A \bowtie^f J)$, where the contraction $\mathcal{P}^c$ is given with respect to ι_A. If A is Cohen-Macaulay in the sense of ideals and* $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht} \mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A, then $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals.*

Proof.

(i) Assume that $A \bowtie^f J$ is a Cohen-Macaulay ring in the sense of $\mathcal{A}^e$. In order to prove the assertion, by Proposition 4.5, it is sufficient to show that $\operatorname{ht} \mathfrak{a}^e \geq \operatorname{ht} \mathfrak{a}$ for each ideal $\mathfrak{a} \in \mathcal{A}$. Towards this end, assume that $\mathfrak{a} \in \mathcal{A}$, and that $\mathcal{P}$ is a prime ideal of $A \bowtie^f J$ containing $\mathfrak{a}^e$. In view of [**9,** Corollaries 2.5, 2.7], we have the following three cases to consider.

Case 1. If $\mathcal{P} = \mathfrak{p}'^{f}$ for some prime ideal $\mathfrak{p}$ of A such that $f^{-1}(J) \nsubseteq \mathfrak{p}$, then

$$\operatorname{ht}\mathcal{P} = \operatorname{ht}\mathfrak{p}'^{f} = \dim(A \bowtie^{f} J)_{\mathfrak{p}'^{f}} = \dim A_{\mathfrak{p}} = \operatorname{ht}\mathfrak{p} \geq \operatorname{ht}\mathfrak{a},$$

by [**9,** Proposition 2.9] and Lemma 4.3 (i).

Case 2. If $\mathcal{P} = \mathfrak{p}'^{f}$ for some prime ideal $\mathfrak{p}$ of A such that $f^{-1}(J) \subseteq \mathfrak{p}$, then

$$\begin{aligned}\operatorname{ht}\mathcal{P} &= \operatorname{ht}\mathfrak{p}'^{f}\\ &= \dim(A \bowtie^{f} J)_{\mathfrak{p}'^{f}}\\ &= \dim(A_{\mathfrak{p}} \bowtie^{f_{\mathfrak{p}}} J_{S_{\mathfrak{p}}})\\ &= \max\{\dim A_{\mathfrak{p}}, \dim(f_{\mathfrak{p}}(A_{\mathfrak{p}}) + J_{S_{\mathfrak{p}}})\}\\ &\geq \dim A_{\mathfrak{p}}\\ &= \operatorname{ht}\mathfrak{p}\\ &\geq \operatorname{ht}\mathfrak{a},\end{aligned}$$

by [**8,** Proposition 4.1], [**9,** Proposition 2.9] and Lemma 4.3 (i), where $S_{\mathfrak{p}} := f(A \setminus \mathfrak{p}) + J$.

Case 3. If $\mathcal{P} = \overline{\mathfrak{q}}^{f}$ for some prime ideal $\mathfrak{q}$ of B, then

$$\begin{aligned}\operatorname{ht}\mathcal{P} &= \operatorname{ht}\overline{\mathfrak{q}}^{f}\\ &= \dim(A \bowtie^{f} J)_{\overline{\mathfrak{q}}^{f}}\\ &= \dim B_{\mathfrak{q}}\\ &= \operatorname{ht}\mathfrak{q}\\ &\geq \operatorname{ht} f^{-1}(\mathfrak{q})\\ &\geq \operatorname{ht}\mathfrak{a}.\end{aligned}$$

The third equality follows from [**9,** Proposition 2.9], the first inequality holds by assumption, and the second one follows from Lemma 4.3. This completes the proof of the first assertion.

(ii) Assume that A is Cohen-Macaulay in the sense of ideals and that $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht}\mathfrak{a}$ for every ideal $\mathfrak{a}$ of A. As indicated by [**2,** Theorem 3.3], it is sufficient to show that

$$\operatorname{K.grade}_{A\bowtie^{f}J}(\mathcal{P}, A \bowtie^{f} J) = \operatorname{ht}\mathcal{P}$$

for every prime ideal $\mathcal{P}$ of $A \bowtie^f J$. Let $\mathcal{P}$ be a prime ideal of $A \bowtie^f J$. Then,

$$\begin{aligned}\operatorname{ht}\mathcal{P} &\leq \operatorname{ht}\mathcal{P}^c\\ &= \operatorname{K.grade}_A(\mathcal{P}^c, A)\\ &= \operatorname{K.grade}_{A\bowtie^f J}(\mathcal{P}^{ce}, A\bowtie^f J)\\ &\leq \operatorname{K.grade}_{A\bowtie^f J}(\mathcal{P}, A\bowtie^f J)\\ &\leq \operatorname{ht}\mathcal{P}.\end{aligned}$$

The first inequality holds by assumption, the second inequality is from [**5,** Proposition 9.1.2(f)], and the last one is from Lemma 3.3 (ii), and the second equality follows from Lemma 3.2. □

We are now in a position to present our third main result.

Theorem 4.11. *With the notation and hypotheses as at the beginning of Section* 3, *the following statements hold*:

(i) *Let $\mathcal{A}$ be a non-empty class of ideals of A. Assume that the homomorphism $f : A \to B$ satisfies the going-down property. If $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of $\mathcal{A}^e := \{\mathfrak{a}^e \mid \mathfrak{a} \in \mathcal{A}\}$, then A is Cohen-Macaulay in the sense of $\mathcal{A}$, and* $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht}\mathfrak{a}$ *for every $\mathfrak{a} \in \mathcal{A}$.*

(ii) *Assume that $\iota_A : A \to A \bowtie^f J$ is an integral ring extension. If A is Cohen-Macaulay in the sense of ideals and* $\operatorname{K.grade}_A(\mathfrak{a}, J) \geq \operatorname{ht}\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A, then $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals.*

Proof. It is well known that $\operatorname{ht} f^{-1}(\mathfrak{q}) \leq \operatorname{ht}\mathfrak{q}$ for every $\mathfrak{q} \in \operatorname{Spec}(B)$ if the homomorphism $f : A \to B$ satisfies the going-down property by [**18,** Exercise 9.9]. In light of Proposition 4.10, this proves (i). To prove (ii), keeping in mind Proposition 4.10, note that, for every $\mathcal{P} \in \operatorname{Spec}(A \bowtie^f J)$, the inequality $\operatorname{ht}\mathcal{P} \leq \operatorname{ht}\mathcal{P}^c$ holds since

$$\iota_A : A \longrightarrow A \bowtie^f J$$

is an integral ring extension [**18,** Exercise 9.8], where the contraction $\mathcal{P}^c$ is given with respect to ι_A. □

Note that Example 4.7 also shows that we cannot neglect the integral assumption in Theorem 4.11 (ii).

Example 4.12.

(1) Assume that A is an integral domain with $\dim A \leq 1$ and that B is an integral domain containing A. Assume that J is an ideal of B which is a finitely generated A-module. Hence, as in Example 4.7, we have $\mathrm{K.grade}_A(\mathfrak{a}, J) = \mathrm{ht}\,\mathfrak{a}$ for every proper ideal $\mathfrak{a}$ of A. Note that A is Cohen-Macaulay in the sense of ideals by **[2,** page 2305]. Therefore, we obtain that $A \bowtie^f J$ is Cohen-Macaulay in the sense of ideals by Theorem 4.11.

(2) To construct a concrete example for (1), set

$$A := \mathbb{Q} + X\mathbb{R}[X],$$

where $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ is the field of real numbers and X is an indeterminate over $\mathbb{R}$. It is easy to see that A is a one-dimensional non integrally closed domain. Put $B := A[\sqrt{2}]$, which is finitely generated as an A-module. Let J be a finitely generated ideal of B. Consequently, from (1), $A \bowtie^f J$ is Cohen-Macaulay in the sense of ideals.

(3) Assume that A is a valuation domain, B an arbitrary integral domain containing A and that J is an ideal of B. Then, by **[11,** Corollary 4] and **[12,** Theorem 1], the inclusion homomorphism $f : A \hookrightarrow B$ satisfies the going-down property. Also note that, by **[2,** Proposition 3.12], A is Cohen-Macaulay in the sense of ideals if and only if $\dim A \leq 1$. Further, assume that $\dim A > 1$. Then, $A \bowtie^f J$ can never be Cohen-Macaulay in the sense of ideals by Theorem 4.11. In particular, the composite ring extensions $A + XB[X]$ and $A + XB[[X]]$ can never be Cohen-Macaulay in the sense of ideals.

Note that, if J is finitely generated as an A-module, then

$$\iota_A : A \longrightarrow A \bowtie^f J$$

is an integral ring extension, and that, in this case, $\mathrm{K.grade}_A(\mathfrak{a}, J) \leq \mathrm{ht}\,\mathfrak{a}$ for every ideal $\mathfrak{a}$ of A by Lemma 3.3. Hence, this immediately yields the following corollaries.

Corollary 4.13. *Assume that the homomorphism $f : A \to B$ satisfies the going-down property and that J is finitely generated as an A-module. Then, $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if A is Cohen-Macaulay in the sense of ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, J) = \mathrm{ht}\,\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A.*

Corollary 4.14. *Assume that $f : A \to B$ is a monomorphism of integral domains, A is integrally closed and that B is integral over A. Then, $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if A is Cohen-Macaulay in the sense of ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A.*

Proof. From [**18,** Theorem 9.4], $f : A \to B$ satisfies the going-down property. In addition, $\iota_A : A \to A \bowtie^f J$ is an integral ring extension by assumption and [**8,** Lemma 3.6]. $\square$

Corollary 4.15. *Assume that $f : A \to B$ is a flat and integral homomorphism. Then, $A \bowtie^f J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if A is Cohen-Macaulay in the sense of ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A.*

Proof. From [**18,** Theorem 9.5], $f : A \to B$ satisfies the going-down property. Also, $\iota_A : A \to A \bowtie^f J$ is an integral ring extension by assumption and [**8,** Lemma 3.6]. $\square$

In conclusion, we apply Corollary 4.15 on amalgamated duplication. Recall that, if $f := \mathrm{id}_A$ is the identity homomorphism on A, and J is an ideal of A, then

$$A \bowtie J := A \bowtie^{\mathrm{id}_A} J$$

is called the amalgamated duplication of A along J. Assume that $(A, \mathfrak{m})$ is Noetherian local. In [**6,** Discussion 10], assuming that A is Cohen-Macaulay, D'Anna showed that $A \bowtie J$ is Cohen-Macaulay if and only if J is maximal Cohen-Macaulay. Next, in [**21,** Corollary 2.7], the authors improved D'Anna's result as $A \bowtie J$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and J is maximal Cohen-Macaulay. Our final corollary generalizes these results.

Corollary 4.16. *Let J be an ideal of A. Then, $A \bowtie J$ is Cohen-Macaulay* (*ring*) *in the sense of ideals if and only if A is Cohen-Macaulay in the sense of ideals and* $\mathrm{K.grade}_A(\mathfrak{a}, J) \geq \mathrm{ht}\,\mathfrak{a}$ *for every ideal $\mathfrak{a}$ of A.*

Proof. This immediately follows from Corollary 4.15 since $f = \mathrm{id}_A : A \to A$ is flat and integral. $\square$

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