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ON HILBERT COEFFICIENTS OF PARAMETER IDEALS AND COHEN-MACAULAYNESS

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ABSTRACT. Let $(R, \mathfrak{m})$ be an unmixed Noetherian local ring, Q a parameter ideal and K an $\mathfrak{m}$-primary ideal of R containing Q. We give a necessary and sufficient condition for R to be Cohen-Macaulay in terms of $g_0(Q)$ and $g_1(Q)$, the Hilbert coefficients of Q with respect to K. As a consequence, we obtain a result of Ghezzi, et al., which settles the negativity conjecture of Vasconcelos **[15]** in unmixed local rings.

1. Introduction. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q an $\mathfrak{m}$-primary ideal of R. Let K be an ideal such that $Q \subseteq K$. Let

$$G(Q) = \bigoplus_{n \geq 0} Q^n/Q^{n+1}$$

be the associated graded ring of Q. The fiber cone of Q with respect to K is the standard graded algebra

$$F_K(Q) = \bigoplus_{n \geq 0} Q^n/KQ^n.$$

Let $\ell(M)$ denote the length of an R-module M. The *Hilbert function* of the fiber cone $F_K(Q)$ is given by $H(F, n) = \ell(Q^n/KQ^n)$. It is well known that $H(F, n)$ agrees with a polynomial $P(F, n)$ of degree $d-1$ for all $n \gg 0$, called the *Hilbert polynomial* of $F_K(Q)$. We can write $P(F, n)$ in the following way:

$$P(F, n) = \sum_{i=0}^{d-1} (-1)^i f_i(Q) \binom{n+d-i-1}{d-1-i},$$

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where the coefficients $f_i(Q)$ are integers and are referred to as the *fiber coefficients* of Q with respect to K.

The *Hilbert-Samuel function* of Q is the function $H(Q,n) = \ell(R/Q^n)$. We recall the notion of Hilbert function of Q with respect to K from [**9**]. It is the function $H_K(Q,-) : \mathbb{Z} \to \mathbb{N}$, defined as

$$H_K(Q,n) = \ell(R/KQ^n) \quad \text{for } n \in \mathbb{Z}.$$

It is known that, for $n \gg 0$, $H(Q,n)$ (respectively, $H_K(Q,n)$) agrees with a polynomial $P(Q,n)$ (respectively, $P_K(Q,n)$) of degree d. We can write these polynomials in the following manner:

$$P(Q,n) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-i-1}{d-i} \tag{1.1}$$

$$P_K(Q,n) = \sum_{i=0}^{d} (-1)^i g_i(Q) \binom{n+d-i-1}{d-i} \tag{1.2}$$

for unique integers $e_i(Q)$ (respectively, $g_i(Q)$), known as the *Hilbert coefficients* of Q (respectively, *Hilbert coefficients* of Q with respect to K).

In this paper, we relate the properties of the Hilbert coefficients $g_i(Q)$ with Cohen-Macaulayness of R. Jayanthan and Verma [**9**] have developed the basic tools for studying these coefficients which include the theory of $F_K(Q)$-superficial and regular elements, a version of Sally's machine for $F_K(Q)$, etc. Using these techniques, many interesting properties of the polynomial $P_K(Q,n)$ and coefficients $g_i(Q)$ have been discussed in [**2, 7, 8, 9, 16**] as a generalization of analogous properties of $P(Q,n)$ and $e_i(Q)$.

This work is inspired by the negativity conjecture posed by Vasconcelos [**15,** Conjecture 1] at the conference in Yokohama in 2008. He conjectured that, for every parameter ideal Q in a Noetherian local ring R, $e_1(Q) < 0$ if and only if R is not Cohen-Macaulay. Partially solving it, Mandal, Singh and Verma [**10**] proved that, if $\operatorname{depth} R = d-1$, then $e_1(Q) < 0$ for every parameter ideal Q. They also showed that $e_1(Q) \le 0$ in a Noetherian local ring. Ghezzi, Hong and Vasconcelos [**5**] proved the conjecture if R is an integral domain which is a homomorphic image of a Cohen-Macaulay ring. The conjecture is settled (more generally for modules) by Ghezzi, et al., [**3, 4**] for unmixed local rings. Recall that R is said to be *unmixed* if $\dim \widehat{R}/p = \dim R$ for all $p \in \operatorname{Ass}(\widehat{R})$. The precise result is the following:

Theorem 1.1 ([**3,** Theorem 2.1])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Then, the following statements are equivalent*:

(a) *R is Cohen-Macaulay*;
(b) *R is unmixed and $e_1(Q) = 0$*;
(c) *R is unmixed and $e_1(Q) \geq 0$.*

By using the methods of [**3**], we obtain following characterization of Cohen-Macaulayness in terms of $g_0(Q)$ and $g_1(Q)$. As a consequence of Theorem 1.2, we recover Theorem 1.1.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Then, the following statements are equivalent*:

(a) *R is Cohen-Macaulay*;
(b) *R is unmixed and $g_0(Q) + g_1(Q) = -\ell(R/K) + \ell(R/Q)$*;
(c) *R is unmixed and $g_0(Q) + g_1(Q) \geq -\ell(R/K) + \ell(R/Q)$*;
(d) *R is unmixed and $f_0(Q) \leq \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$*;
(e) *R is unmixed and $f_0(Q) = \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$.*

This paper is organized as follows. Section 2 is devoted to some preliminary results on $F_K(Q)$-superficial elements. In Section 3, we prove Theorem 1.2 and discuss its consequences. Throughout this paper, $(R, \mathfrak{m})$ denotes a Noetherian local ring, and $H^i_{\mathfrak{m}}(*)$ denotes the ith local cohomology functor with support in the maximal ideal $\mathfrak{m}$.

The reader is referred to [**1**] and [**11**] for undefined terms.

2. Preliminaries. In this section, we recall and prove basic properties of superficial elements and superficial sequences in $G(Q)$ and $F_K(Q)$. The theory of superficial elements in $G(Q)$ is an effective method for the study of Hilbert coefficients, $e_i(Q)$, as it allows for application of induction on the dimension of R. See [**13,** Proposition

1.2]. In [**9,** Section 2], the authors developed the theory of superficial elements in $F_K(Q)$.

Let Q be an ideal of R and K an ideal with $Q \subseteq K$. For an element $0 \neq x \in R$, let x^* (respectively, x^0) denote the *initial form* of x in $G(Q)$ (respectively, $F_K(Q)$), i.e., the image of x in $G(Q)_i$ (respectively, $F_K(Q)_i$), where i is the unique integer such that $x \in Q^i \setminus Q^{i+1}$ (respectively, $x \in Q^i \setminus KQ^i$).

Definition 2.1.

(1) For an element $x \in Q$ such that $x^* \neq 0$ in $G(Q)$, x^* is said to be $G(Q)$-*superficial* if there exists an integer $c > 0$ such that $(Q^n : x) \cap Q^c = Q^{n-1}$ for all $n > c$.

(2) For an element $x \in Q$ such that $x^0 \neq 0$ in $F_K(Q)$, x^0 is said to be $F_K(Q)$-*superficial* if there exists an integer $c > 0$ such that $(0 : x^0) \cap F_K(Q)_n = 0$ for all $n > c$.

(3) For a sequence $x_1, \ldots, x_k \in Q$, $x_1^*, \ldots, x_k^*$ (respectively, $x_1^0, \ldots, x_k^0$) is said to be $G(Q)$ (*respectively,* $F_K(Q)$)-*superficial* if, for all $i = 1, \ldots, k$, $(x_i')^*$ (respectively, $(x_i')^0$) is $G(QR_{i-1})$ (respectively, $F_{KR_{i-1}}(QR_{i-1})$)-superficial, where $R_{i-1} = R/(x_1, \ldots, x_{i-1})$, and x_i' denotes the image of x_i in R_{i-1}.

See [**8,** Proposition 2.1] and [**9,** Section 2] for the existence and basic properties of $F_K(Q)$-superficial elements. We recall the following lemma from [**9**] which provides a useful characterization of $F_K(Q)$-superficial elements.

Lemma 2.2 ([**9,** Lemma 2.3])**.** *Let R be a Noetherian local ring of dimension $d > 0$. Let Q be an ideal of R and K an $\mathfrak{m}$-primary ideal of R such that $Q \subseteq K$. Then, the following statements hold.*

- (a) *If there exists an integer $c > 0$ such that $(KQ^n : x) \cap Q^c = KQ^{n-1}$ for all $n > c$, then x^0 is $F_K(Q)$-superficial.*
- (b) *If x^0 is $F_K(Q)$-superficial and x^* is $G(Q)$-superficial, then there exists an integer $c > 0$ such that $(KQ^n : x) \cap Q^c = KQ^{n-1}$ for all $n > c$. Moreover, if x is regular on R, then $(KQ^n : x) = KQ^{n-1}$ for all $n \gg 0$.*

Existence of $F_K(Q)$-superficial elements is guaranteed by [**9,** Proposition 2.2] when K is an $\mathfrak{m}$-primary ideal. Indeed, we can choose $x \in Q$ such that x^0 is $F_K(Q)$-superficial as well as x^* is $G(Q)$-superficial by [**9,** Proposition 2.2]. Hence, we use the characterization given by Lemma 2.2 (b) for $F_K(Q)$-superficial elements in this paper.

The next lemma guarantees the existence of $F_K(Q)$-superficial elements avoiding a finite set of ideals not containing Q. This result is well known in the case of $G(Q)$-superficial elements, see [**14,** Corollary 8.5.9].

Lemma 2.3. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d>0$. Let Q be an ideal of R and K an $\mathfrak{m}$-primary ideal of R such that $Q \subseteq K$. Let $I_1,\dots,I_r$ be ideals in R not containing Q. Then, there exists an element $x \in Q \setminus \mathfrak{m}Q$ that is not contained in any I_i such that x^* is $G(Q)$-superficial and x^0 is $F_K(Q)$-superficial.*

In particular, if Q contains a nonzerodivisor, then there exists an element $x \in Q \setminus \mathfrak{m}Q$ such that x^* is $G(Q)$-superficial, x^0 is $F_K(Q)$-superficial and x is a nonzerodivisor.

Proof. The proof is similar to the proof of [**14,** Corollary 8.5.9]. $\square$

The Hilbert coefficients $g_i(Q)$ behave nicely on reducing modulo $F_K(Q)$-superficial element. We need a refined version of [**9,** Lemma 3.5].

Lemma 2.4. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d>0$. Let Q be an $\mathfrak{m}$-primary ideal of R and K an ideal of R such that $Q \subseteq K$. Let $x \in Q$ be such that x^* is $G(Q)$-superficial and x^0 is $F_K(Q)$-superficial. Let $g_i(\overline{Q})$ denote the coefficients of the polynomial $P_{\overline{K}}(\overline{Q},n)$, where $\overline{Q}$ (respectively, $\overline{K}$) denotes the image of an ideal Q (respectively, K) in $R/(x)$. Then,*

$$g_i(\overline{Q}) = \begin{cases} g_i(Q) & \text{for } 0 \le i \le d-2 \\ g_{d-1}(Q) + (-1)^{d-1}\ell(0:x) & \text{for } i = d-1. \end{cases}$$

Proof. For $n \in \mathbb{Z}$, the exact sequence

$$0 \longrightarrow \frac{KQ^n : x}{KQ^{n-1}} \longrightarrow R/KQ^{n-1} \xrightarrow{x} R/KQ^n \longrightarrow R/(KQ^n, x) \longrightarrow 0$$

yields that $\ell(R/\overline{KQ}^n) = \ell(R/KQ^n) - \ell(R/KQ^{n-1}) + \ell((KQ^n : x)/KQ^{n-1})$ for all $n \in \mathbb{Z}$.

Claim 2.5. *$KQ^n \cap (0:x) = 0$ and $(KQ^n : x) = KQ^{n-1} + (0:x)$ for $n \gg 0$.*

From Lemma 2.2 (b), there exists an integer $c > 0$ such that $(KQ^n : x) \cap Q^c = KQ^{n-1}$ for all $n > c$. Thus, for all $n > c$, $KQ^n \cap (0:x) \subseteq \cap_{m \geq n}(KQ^m : x) \cap Q^c = \cap_{m\geq n} KQ^{m-1} = 0$. By the Artin-Rees lemma, there exists an integer k such that $KQ^n \cap (x) \subseteq (x)Q^{n-k}$ for all $n > k$. Let $n > k + c$ and $y \in (KQ^n : x)$. Then, $yx \in KQ^n \cap (x) \subseteq xQ^{n-k}$. Suppose $yx = zx$ for some $z \in Q^{n-k}$. Hence, $z \in (KQ^n : x) \cap Q^c = KQ^{n-1}$. Therefore, $y = z + (y-z) \in KQ^{n-1} + (0:x)$. Thus, $\ell(R/\overline{KQ}^n) = \ell(R/KQ^n) - \ell(R/KQ^{n-1}) + \ell(0:x)$ for all $n \gg 0$. Hence, for $n \in \mathbb{Z}$,

$$P_{\overline{K}}(\overline{Q}, n) = P_K(Q, n) - P_K(Q, n-1) + \ell(0:x). \tag{2.1}$$

Now, the result follows by comparing the coefficients of both sides of equation (2.1). □

Remark 2.6. If $x \in Q$ is a regular element, then we obtain [**9,** Lemma 3.5].

We recall the following proposition from [**2**], which determines the value of $g_i(Q)$ for parameter ideals in a Cohen-Macaulay local ring. There is a misprint in the statement of [**2,** Theorem 7.2(2)]. However, the following version of the statement follows from the proof of [**2,** Theorem 7.2]. We provide an example below with $g_i(Q) = (-1)^i \ell(R/K)$ for $1 \leq i \leq d$.

Proposition 2.7 ([**2,** Theorem 7.2(2)])**.** *Let R be a Cohen-Macaulay local ring of dimension $d > 0$ and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Then, $g_i(Q) = (-1)^i \ell(R/K)$ for $1 \leq i \leq d$.*

Proof. Follows from comparing the coefficients of [**2,** Equation 24]. □

Example 2.8. Let $R = k[[x, y]]$ be a power series ring. Let $Q = (x^3, x^2y, y^3)$ and $K = \mathfrak{m}^2$. Then, $J = (x^3, y^3)$ is a minimal reduction of Q. We use Huneke's fundamental lemma [**9,** Corollary 3.3] to compute $g_0(Q)$ and $g_1(Q)$. Recall that

$$g_1(Q) = \sum_{n\geq 1} v_n \quad \text{and} \quad g_2(Q) = \sum_{n\geq 1}(n-1)v_n + \ell(R/K)$$

where

$$v_n = \begin{cases} e_0(Q) & \text{if } n = 0, \\ e_0(Q) - \ell(R/KQ) + \ell(R/K) & \text{if } n = 1, \\ \ell(KQ^n/KJQ^{n-1}) - \ell((KQ^{n-1} : J)/KQ^{n-2}) & \text{if } n \geq 2. \end{cases}$$

Therefore,

$$P_K(Q,n) = 9\binom{n+1}{2} + 3n + 3.$$

Hence, we see that $\ell(R/K) = 3 = (-1)^i g_i(Q)$ for $i = 1, 2$.

Remark 2.9.

1. Since $\ell(R/KQ^n) = \ell(R/Q^n) + \ell(Q^n/KQ^n)$ for all integers n, we have $P_K(Q,n) = P(Q,n) + P(F,n)$ for all integers n. Thus, comparing the coefficients of both sides, we obtain:

$$g_0(Q) = e_0(Q) \tag{2.2}$$

and

$$f_i(Q) = e_{i+1}(Q) - g_{i+1}(Q) + e_i(Q) - g_i(Q) \text{ for } 0 \leq i \leq d-1. \tag{2.3}$$

2. By putting $K = Q$ in (1.2), we see that $P_K(Q,n) = \ell(R/Q^{n+1}) = P(Q, n+1)$ for all $n \gg 0$. Comparing the coefficients, we get, for $0 \leq i \leq d$:

$$g_i(Q) = e_i(Q) - e_{i-1}(Q) + \cdots + (-1)^i e_0(Q). \tag{2.4}$$

3. A characterization of Cohen-Macaulayness. In this section, we first obtain a formula for the first Hilbert coefficient $g_1(Q)$ in a one-dimensional Noetherian local ring. Then, we give a characterization of Cohen-Macaulayness of an unmixed local ring in terms of the coefficients $g_0(Q)$ and $g_1(Q)$. This generalizes, in some sense, a result of [**3,** Theorem 2.1]. We further discuss the results reminiscent of those in [**3,** Section 2].

Definition 3.1. Let $(0) = \cap_{p \in \mathrm{Ass}(R)} Q(p)$ be a primary decomposition of (0) in R, and let $\mathrm{Assh}(R) = \{p \in \mathrm{Ass}(R) \mid \dim R/p = d\}$. The ideal $U_R(0) = \cap_{p \in \mathrm{Assh}(R)} Q(p)$ is called the *unmixed component* of (0) in R. A ring R is called *unmixed* if $\mathrm{Ass}(\widehat{R}) = \mathrm{Assh}(\widehat{R})$ where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R.

In [**15,** Conjecture 1], Vasconcelos conjectured that $e_1(Q) < 0$ for every ideal Q that is generated by a system of parameters if and only if R is not Cohen-Macaulay. In [**3,** Theorem 2.1] and [**4,** Theorem 3.1], Ghezzi, et al., proved that, if R is unmixed and $e_1(Q) \geq 0$ for some parameter ideal Q, then R is Cohen-Macaulay. Motivated by this, we ask:

Question 3.2. *If R is unmixed of dimension $d > 0$ and $g_1(Q) \geq -\ell(R/K)$ for some parameter ideal Q, then is R Cohen-Macaulay?*

The following example, which is worked out in [**12,** Example 3.8], shows that the answer to Question 3.2 can be negative.

Example 3.3 ([**12,** Example 3.8])**.** Let $R = k[[x^5, xy^4, x^4y, y^5]] \cong k[[t_1, t_2, t_3, t_4]]/J$, where

$$J = (t_2t_3 - t_1t_4,\ t_2^4 - t_3t_4^3,\ t_1t_2^3 - t_3^2t_4^2,\ t_1^2t_2^2 - t_3^3t_4,\ t_1^3t_2 - t_3^4,\ t_3^5 - t_1^4t_4).$$

Then, $\dim R = 2$. Let $Q = (x^5, y^5)$. We have

$$P(Q, n) = 5\binom{n+1}{2} + 2n.$$

Let $K = Q$. Then, by (2.4), $g_1(Q) = e_1(Q) - e_0(Q) = -7 \geq -\ell(R/K)$; however, $\operatorname{depth} R = 1$.

Remark 3.4. The answer to Question 3.2 is affirmative for $K = \mathfrak{m}$. Indeed, suppose that $K = \mathfrak{m}$ and $g_1(Q) \geq -\ell(R/\mathfrak{m}) = -1$ for some parameter ideal Q. From (2.2) and (2.3), we obtain that $e_1(Q) - g_1(Q) = f_0(Q) \geq 1$. Hence, $e_1(Q) \geq 0$, which implies that R is Cohen-Macaulay by [**3,** Theorem 2.1].

For an arbitrary K, we prove in Theorem 3.10 that an unmixed local ring is Cohen-Macaulay if $g_1(Q) \geq -\ell(R/K) + (\ell(R/Q) - g_0(Q))$ for some parameter ideal $Q \subseteq K$. Since $\ell(R/Q) - g_0(Q) \geq 0$, Theorem 3.10

is weaker than a complete solution of Question 3.2. We first discuss some results on $g_1(Q)$ for a parameter ideal Q in a one-dimensional ring.

For an $\mathfrak{m}$-primary ideal I in R and a finitely generated R-module M of dimension g, we write the *Hilbert-Samuel polynomial* of M with respect to I as

$$P_M(I,n) = \sum_{i=0}^{g} (-1)^i e_i(I,M) \binom{n+g-i-1}{g-i}.$$

Proposition 3.5. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension one and Q a parameter ideal of R. Let K be an $\mathfrak{m}$-primary ideal. Then, $g_1(Q) = -\ell(R/K) - \ell(H^0_{\mathfrak{m}}(K))$.*

Proof. The short exact sequence

$$0 \longrightarrow K \longrightarrow R \longrightarrow R/K \longrightarrow 0$$

induces a surjective map from $H^1_{\mathfrak{m}}(K)$ to $H^1_{\mathfrak{m}}(R)$. Since $H^1_{\mathfrak{m}}(R) \neq 0$, we obtain that $H^1_{\mathfrak{m}}(K) \neq 0$. Thus, K is an R-module of dimension one. For all integers n, we have

$$\ell(R/KQ^n) = \ell(R/K) + \ell(K/KQ^n).$$

Therefore,

$$g_1(Q) = -\ell(R/K) + e_1(Q,K) = -\ell(R/K) - \ell(H^0_{\mathfrak{m}}(K))$$

where the last equality follows from [**10,** Proposition 3.1]. □

Proposition 3.6. *Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension one and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Suppose that $g_0(Q) + g_1(Q) \geq -\ell(R/K) + \ell(R/Q)$. Then, R is Cohen-Macaulay.*

Proof. Set $W := H^0_{\mathfrak{m}}(R)$ and $\overline{R} := R/W$. Then $\overline{R}$ is a Cohen-Macaulay local ring of dimension one. Consider the following exact sequence

$$0 \longrightarrow W/(KQ^n \cap W) \longrightarrow R/KQ^n \longrightarrow \overline{R}/KQ^n\overline{R} \longrightarrow 0.$$

By the Artin-Rees lemma, there exists an integer k such that, for all $n \gg 0$, $KQ^n \cap W \subseteq Q^n \cap W \subseteq Q^{n-k}W = 0$. Hence, for all $n \gg 0$,

$$\ell(R/KQ^n) = \ell(\overline{R}/KQ^n\overline{R}) + \ell(W).$$

Thus,

$$P_K(Q, n) = P_{\overline{K}}(\overline{Q}, n) + \ell(W) \quad \text{for all integers } n. \tag{3.1}$$

This implies that

$$g_0(Q) = g_0(Q\overline{R})$$

and

$$g_1(Q) = g_1(Q\overline{R}) - \ell(W).$$

Therefore,

$$g_0(Q\overline{R}) + g_1(Q\overline{R}) \geq -\ell(R/K) + \ell(R/Q) + \ell(W). \tag{3.2}$$

Since $\overline{R}$ is Cohen-Macaulay, using (2.2), we get that $g_0(Q\overline{R}) = e_0(Q\overline{R}) = \ell(\overline{R}/Q\overline{R})$. From Proposition 2.7, $g_1(Q\overline{R}) = -\ell(\overline{R}/K\overline{R})$. Hence,

$$\begin{aligned} g_0(Q\overline{R}) + g_1(Q\overline{R}) &= \ell(\overline{R}/Q\overline{R}) - \ell(\overline{R}/K\overline{R})) \\ &= \ell(R/(W+Q)) - \ell(R/(W+K)) \\ &= \ell(K/(Q+W)\cap K)) \leq \ell(K/Q) \\ &= -\ell(R/K) + \ell(R/Q). \end{aligned} \tag{3.3}$$

Comparing equations (3.2) and (3.3), we obtain that $\ell(W) \leq 0$, which implies that $W = 0$. Thus, R is Cohen-Macaulay. □

In order to prove Theorem 3.10, we need an analog of [**3,** Lemma 2.3].

Lemma 3.7. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Suppose that $U = U_R(0) \neq 0$ and $S = R/U$. Then, the following assertions hold*:

(a) $\dim U < \dim R$.
(b) *We have* $g_0(Q) = g_0(QS)$ *and*

$$g_1(Q) = \begin{cases} g_1(QS) & \textit{if } \dim U \leq d-2, \\ g_1(QS) - s_0 & \textit{if } \dim U = d-1, \end{cases}$$

where s_0 *is the multiplicity of the module* $\oplus_{n\geq 0} U/(KQ^{n+1} \cap U)$.

(c) $g_1(Q) \leq g_1(QS)$ *with equality if and only if* $\dim U \leq d-2$.

Proof.

(a) Let $(0) = \cap_{p\in \mathrm{Ass}(R)} Q(p)$ be a primary decomposition of (0) and $U = U_R(0) = \cap_{p\in \mathrm{Assh}(R)} Q(p)$. For all $p \in \mathrm{Ass}(R)$ and $p' \in \mathrm{Assh}(R)$ with $p \neq p'$, we have $Q(p)R_{p'} = R_{p'}$. Hence, for all $p' \in \mathrm{Assh}(R)$, we obtain that $(0) = \cap_{p\in \mathrm{Ass}(R)} Q(p)R_{p'} = Q(p')R_{p'}$. Thus, $UR_{p'} = \cap_{p\in \mathrm{Assh}(R)} Q(p)R_{p'} = Q(p')R_{p'} = (0)$. Hence, $\mathrm{Assh}(R) \cap \mathrm{Supp}_R(U) = \phi$. Therefore, $\dim U < \dim R$.

(b) Considering the short exact sequence

$$0 \longrightarrow U/(KQ^n \cap U) \longrightarrow R/KQ^n \longrightarrow S/KQ^nS \longrightarrow 0,$$

we get that

$$\ell(R/KQ^n) = \ell(S/KQ^nS) + \ell(U/(KQ^n \cap U)) \quad \text{for all } n \in \mathbb{Z}. \tag{3.4}$$

Hence, $\ell(U/(KQ^n \cap U))$ agrees with a polynomial, say, $T(n)$ for $n \gg 0$. We write

$$T(n) = s_0\binom{n+t}{t} - s_1\binom{n+t-1}{t-1} + \ldots + (-1)^t s_t \tag{3.5}$$

for some $t \geq 0$ and $s_i \in \mathbb{Z}$ for $0 \leq i \leq t$. We claim that $t = \dim U$. By the Artin-Rees lemma, there exists an integer k such that, for all $n \gg 0$, $KQ^n \cap U \subseteq Q^n \cap U = Q^{n-k}(Q^k \cap U) \subseteq Q^{n-k}U$. Hence, $\ell(U/Q^{n-k}U) \leq \ell(U/(KQ^n \cap U))$ for all $n \gg 0$, which implies $t \geq \dim U$. On the other hand, since $\ell(U/KQ^nU) = \ell(U/Q^nU) + \ell(Q^nU/KQ^nU)$, we see that $\ell(U/KQ^nU)$ coincides with a polynomial of degree equals $\dim U$ for all $n \gg 0$. Therefore, $\ell(U/(KQ^n \cap U)) \leq \ell(U/KQ^nU)$ for all $n \in \mathbb{Z}$ implies that $t = \dim U$.

From (3.4), we obtain

$$P_K(Q, n) = P_{KS}(QS, n) + T(n) \quad \text{for all } n \in \mathbb{Z}. \tag{3.6}$$

By comparing the coefficients of both sides of (3.6) and using (3.5), we get the result.

(c) Follows from (b). □

The proof of Theorem 3.10 is based on the methods employed in **[3]**. We recall the following results from **[6]**, which are necessary for proving Theorem 3.10.

Lemma 3.8 (**[6,** Lemma 3.1]). *Let* $(R, \mathfrak{m})$ *be a complete local ring. Suppose* $\operatorname{Ass}(R) \subseteq \operatorname{Assh}(R) \cup \{\mathfrak{m}\}$. *Then,* $H^1_{\mathfrak{m}}(R)$ *has finite length.*

Using Lemma 2.3, the proof of **[6,** Proposition 3.3] shows that $x_1, \ldots, x_d$ in **[6,** Proposition 3.3] can be chosen such that $x_1^0, \ldots, x_d^0$ is $F_K(Q)$-superficial and $x_1^*, \ldots, x_d^*$ is $G(Q)$-superficial.

Proposition 3.9 (**[6,** Proposition 3.3]). *Let* $(R, \mathfrak{m})$ *be a homomorphic image of a Cohen-Macaulay local ring of dimension* d, *and assume that* $\operatorname{Ass}(R) \subseteq \operatorname{Assh}(R) \cup \{\mathfrak{m}\}$. *Let* Q *be a parameter ideal. Then, there exists a system of generators* $x_1, \ldots, x_d$ *of* Q *such that* $x_1^0, \ldots, x_d^0$ *is* $F_K(Q)$*-superficial,* $x_1^*, \ldots, x_d^*$ *is* $G(Q)$*-superficial and* $\operatorname{Ass}(R/Q_i) \subseteq \operatorname{Assh}(R/Q_i) \cup \{\mathfrak{m}\}$, *where* $Q_i = (x_1, \ldots, x_i)$ *for* $0 \leq i \leq d$.

We now prove the main theorem of this section.

Theorem 3.10. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$ *and* Q *a parameter ideal of* R. *Let* K *be an ideal such that* $Q \subseteq K$. *Then, the following statements are equivalent*:

(a) R *is Cohen-Macaulay*;
(b) R *is unmixed and* $g_0(Q) + g_1(Q) = -\ell(R/K) + \ell(R/Q)$;
(c) R *is unmixed and* $g_0(Q) + g_1(Q) \geq -\ell(R/K) + \ell(R/Q)$;
(d) R *is unmixed and* $f_0(Q) \leq \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$;
(e) R *is unmixed and* $f_0(Q) = \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$.

Proof. Using (2.2) and (2.3), we get that

$$\text{(e)} \Longleftrightarrow \text{(b)} \quad \text{and} \quad \text{(d)} \Longleftrightarrow \text{(c)}.$$

Hence, it suffices to prove

$$\text{(a)} \Longrightarrow \text{(b)} \Longrightarrow \text{(c)} \Longrightarrow \text{(a)}.$$

(a) $\Rightarrow$ (b). Follows from Proposition 2.7 and the fact that $g_0(Q) = e_0(Q) = \ell(R/Q)$.

(b) $\Rightarrow$ (c). Clear.

(c) $\Rightarrow$ (a). We prove this by induction on d. The result is clear for $d = 1$.

Let $d \geq 2$. We may assume that R is complete with infinite residue field. Suppose $d = 2$. Then, we may assume that $Q = (x_1, x_2)$ such that x_1^0 is $F_K(Q)$-superficial. Since R is unmixed, we can choose x_1 to be a nonzerodivisor on R. Let $S = R/(x_1)$. Then, $Q/(x_1)$ is a parameter ideal of S. By Lemma 2.4, $g_0(QS) = g_0(Q)$ and $g_1(QS) = g_1(Q)$. Hence,

$$g_0(QS) + g_1(QS) \geq -\ell(R/K) + \ell(R/Q) = -\ell(S/KS) + \ell(S/QS).$$

Therefore, by Proposition 3.6, S is Cohen-Macaulay which implies that R is Cohen-Macaulay.

Let $d \geq 3$. By Proposition 3.9, there exists a system of generators $x_1, \ldots, x_d$ of Q such that x_1^0 is $F_K(Q)$-superficial and

$$\operatorname{Ass}(R/(x_1)) \subseteq \operatorname{Assh}(R/(x_1)) \cup \{\mathfrak{m}\}.$$

Let $S = R/(x_1)$ and $\overline{S} = S/U_S(0)$. Then, $\overline{S}$ is an unmixed local ring of dimension $d-1$, and $Q\overline{S}$ is a parameter ideal contained in $K\overline{S}$. Since $\dim U_S(0) = 0$, by Lemma 3.7 (b), we get that $g_i(Q\overline{S}) = g_i(QS)$ for $i = 0, 1$. Therefore,

$$\begin{aligned} g_0(Q\overline{S}) + g_1(Q\overline{S}) &= g_0(Q) + g_1(Q) \geq -\ell(R/K) + \ell(R/Q) \\ &= -\ell(S/KS) + \ell(S/QS) \geq \ell(KS/((QS + U_S(0)) \cap KS)) \\ &= \ell((KS + U_S(0))/(QS + U_S(0))) \\ &= -\ell(S/(KS + U_S(0))) + \ell(S/(QS + U_S(0))) \\ &= -\ell(\overline{S}/K\overline{S}) + \ell(\overline{S}/Q\overline{S}). \end{aligned}$$

Hence, by the induction hypothesis, $\overline{S}$ is Cohen-Macaulay. Therefore, $H^i_{\mathfrak{m}}(\overline{S}) = 0$ for $0 \leq i \leq d-2$. The exact sequence

$$0 \longrightarrow U_S(0) \longrightarrow S \longrightarrow \overline{S} \longrightarrow 0$$

gives the following long exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{m}}(U_S(0)) \longrightarrow H^i_{\mathfrak{m}}(S) \longrightarrow H^i_{\mathfrak{m}}(\overline{S}) \longrightarrow \cdots .$$

Since $U_S(0)$ is Artinian, $H^0_{\mathfrak{m}}(U_S(0)) = U_S(0)$ and $H^i_{\mathfrak{m}}(U_S(0)) = 0$ for all $i \geq 1$. Therefore, $H^0_{\mathfrak{m}}(S) = U_S(0)$ and $H^i_{\mathfrak{m}}(S) = 0$ for $1 \leq i \leq d-2$.

Now, considering the exact sequence

$$0 \longrightarrow R \xrightarrow{x_1} R \longrightarrow S \longrightarrow 0,$$

we obtain the long exact sequence

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{m}}(S) \longrightarrow H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R) \longrightarrow H^i_{\mathfrak{m}}(S) \longrightarrow \cdots .$$

This implies that the map $H^1_{\mathfrak{m}}(R) \xrightarrow{x_1} H^1_{\mathfrak{m}}(R)$ is surjective, and $H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R)$ is injective, for $2 \leq i \leq d-1$. Thus, $H^1_{\mathfrak{m}}(R) = x_1 H^1_{\mathfrak{m}}(R)$. Since $H^1_{\mathfrak{m}}(R)$ is finitely generated by Lemma 3.8, using Nakayama's lemma yields that $H^1_{\mathfrak{m}}(R) = 0$. Since $H^i_{\mathfrak{m}}(R)$ is $\mathfrak{m}$-torsion, the injectivity of the map $H^i_{\mathfrak{m}}(R) \xrightarrow{x_1} H^i_{\mathfrak{m}}(R)$ gives that $H^i_{\mathfrak{m}}(R) = 0$ for $2 \leq i \leq d-1$. Therefore, R is Cohen-Macaulay. □

As a consequence, we recover the result of [**3,** Theorem 2.1].

Corollary 3.11. *Let R be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Then, the following statements are equivalent*:

(a) *R is Cohen-Macaulay*;
(b) *R is unmixed and $e_1(Q) = 0$*;
(c) *R is unmixed and $e_1(Q) \geq 0$.*

Proof. Let $K = Q$. From (2.4), $g_0(Q) + g_1(Q) = e_1(Q)$. Hence, the result follows from Theorem 3.10. □

In the case of $d = 1$, we have $g_1(Q) \leq -\ell(R/K)$ by Proposition 3.5. We obtain the following corollary in this direction for $d > 0$. In particular, for $K = Q$, we recover the non-positivity of the Chern number $e_1(Q)$ [**3,** Corollary 2.4(a)] in Corollary 3.13.

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Then*:

(1) (a) $g_0(Q) + g_1(Q) \leq -\ell(R/K) + \ell(R/Q)$.
    (b) $f_0(Q) \geq \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$.
(2) *Suppose* depth $R = d-1$. *Then,*
    (a) $g_0(Q) + g_1(Q) < -\ell(R/K) + \ell(R/Q)$.
    (b) $f_0(Q) > \ell(R/K) + e_1(Q) + e_0(Q) - \ell(R/Q)$.

Proof. In view of (2.2) and (2.3), it suffices to prove (1) (a) and (2) (a).

(1) (a) We may assume that R is complete. Let $d = 1$. Since $g_0(Q) = e_0(Q) \leq \ell(R/Q)$, using Proposition 3.5, we obtain that $g_0(Q)+g_1(Q) \leq -\ell(R/K)+\ell(R/Q)$. Suppose $d \geq 2$. Set $S = R/U_R(0)$. Then, S is an unmixed local ring and QS is a parameter ideal of S. Hence,

$$\begin{aligned} g_0(Q) + g_1(Q) &\leq g_0(QS) + g_1(QS) \quad \text{[by Lemma 3.7 (b)]} \\ &\leq -\ell(S/KS) + \ell(S/QS) \quad \text{[from Theorem 3.10]} \\ &\leq -\ell(R/K) + \ell(R/Q). \end{aligned}$$

(2) (a) Let $d = 1$. Then, $\operatorname{depth}(R) = 0$ implies that $g_0(Q) = e_0(Q) < \ell(R/Q)$. Hence, by Proposition 3.5, $g_0(Q) + g_1(Q) < -\ell(R/K) + \ell(R/Q)$. Suppose $d \geq 2$. Let $Q = (x_1, \ldots, x_d)$ be such that $x_1^0, \ldots, x_{d-1}^0$ is an $F_K(Q)$-superficial sequence. Since $\operatorname{depth}(R) = d - 1$, we may choose $x_1, \ldots, x_{d-1}$ to be an R-regular sequence. Let " " denote reduction modulo $x_1, \ldots, x_{d-1}$. Then, by Lemma 2.4, $g_0(Q) + g_1(Q) = g_0(Q\overline{R}) + g_1(Q\overline{R}) < -\ell(R/K) + \ell(R/Q)$ by the induction hypothesis. □

Corollary 3.13. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring of dimension* $d > 0$ *and* Q *a parameter ideal of* R. *Then*:

(a) $e_1(Q) \leq 0$;
(b) *if* $\operatorname{depth} R = d - 1$, *then* $e_1(Q) < 0$.

Proof. The proof follows from letting $K = Q$ in Corollary 3.12 and using (2.4). □

In the next example, we have the inequality as stated in part 1 (a) of Corollary 3.12.

Example 3.14. Let $S = k[[X, Y, Z]]$ be a power series ring over a field k and $I = (YZ, X^2Y, Y^3)$. Consider the ring $R = S/I = k[[x, y, z]]$. Then, $\dim R = 2$ and $\operatorname{depth} R = 0$. Let $Q = (x, z^2)$ and $K = (x, y, z^2)$. Note that $H^0_{\mathfrak{m}}(R) = (Y)/I$; hence, $\overline{R} = R/H^0_{\mathfrak{m}}(R) = k[[X, Z]]$, which is Cohen-Macaulay. From (3.1) and Proposition 2.7, we obtain

$g_0(Q) + g_1(Q) = g_0(\overline{Q}) + g_1(\overline{Q}) = \ell(\overline{R}/Q\overline{R}) - \ell(\overline{R}/K\overline{R}) = 2 - 2 = 0$, whereas $-\ell(R/K) + \ell(R/Q) = -2 + 4 = 2$.

The following two results can be seen as analogs of [**3,** Theorem 2.6, Corollary 2.7]. Corollary 3.16 gives a characterization of the Cohen-Macaulayness of R in terms of $g_i(Q)$.

Theorem 3.15. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$ and Q a parameter ideal of R. Let K be an ideal such that $Q \subseteq K$. Suppose R is a homomorphic image of a Cohen-Macaulay ring. Let $U = U_R(0)$. Then the following are equivalent*:

(a) $g_0(Q) + g_1(Q) = -\ell(R/(K + U)) + \ell(R/(Q + U))$;

(b) *R/U is Cohen-Macaulay and* $\dim U \leq d - 2$.

Proof.

(a) $\Rightarrow$ (b). Let $S = R/U$. Then, S is an unmixed local ring. If $U = 0$, then R is unmixed. Hence, $g_0(Q)+g_1(Q) = -\ell(R/K)+\ell(R/Q)$ implies that R is Cohen-Macaulay, by Theorem 3.10. Suppose $U \neq 0$. First, we show that $\dim U \leq d-2$. From Lemma 3.7 (a), $\dim U \leq d-1$. Suppose $\dim U = d - 1$. Then, $g_0(Q) = g_0(QS)$ and $g_1(Q) = g_1(QS) - s_0$ for some $s_0 \geq 1$, by Lemma 3.7 (b). Therefore,

$$\begin{aligned} g_0(Q) + g_1(Q) &< g_0(QS) + g_1(QS) \\ &\leq -\ell(S/KS) + \ell(S/QS) \quad \text{[by Corollary 3.12]} \\ &= -\ell(R/(K + U)) + \ell(R/(Q + U)), \end{aligned}$$

which is a contradiction. Hence, $\dim U \leq d - 2$. By Lemma 3.7 (b), $g_1(Q) = g_1(QS)$. Thus, $g_0(QS) + g_1(QS) = -\ell(S/KS) + \ell(S/QS)$. Therefore, by Theorem 3.10, $S = R/U$ is Cohen-Macaulay.

(b) $\Rightarrow$ (a). Let $S = R/U$. Since $\dim U \leq d - 2$, by Lemma 3.7 (b), $g_1(Q) = g_1(QS) = -\ell(S/KS)$, where the last equality holds by Proposition 2.7. Therefore,

$$\begin{aligned} g_0(Q) + g_1(Q) &= \ell(S/QS) - \ell(S/KS) \\ &= \ell(R/(Q + U)) - \ell(R/(K + U)). \end{aligned}$$ □

Corollary 3.16. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Suppose R is a homomorphic image of a Cohen-Macaulay ring. Let Q be a parameter ideal of R and K an ideal such that $Q \subseteq K$. Let*

$U = U_R(0)$. *Suppose*

$$g_i(Q) = (-1)^i(g_0(Q) - \ell(R/(Q+U)) + \ell(R/(K+U)))$$

for $1 \leq i \leq d$. *Then, R is Cohen-Macaulay.*

Proof. Since $g_1(Q) = -(g_0(Q) - \ell(R/(Q+U)) + \ell(R/(K+U)))$, by Theorem 3.15, R/U is Cohen-Macaulay and $\dim U \leq d-2$. Set $S = R/U$. By Lemma 3.7 (b), $g_0(Q) = g_0(QS) = \ell(S/QS) = \ell(R/(Q+U))$. From Proposition 2.7, we have

$$g_i(QS) = (-1)^i \ell(S/KS) \text{ for } 1 \leq i \leq d.$$

Hence,

$$g_i(Q) = (-1)^i \ell(R/(K+U)) = (-1)^i \ell(S/KS) \quad \text{for } 1 \leq i \leq d.$$

Therefore, for $n \gg 0$,

$$\begin{aligned}\ell(S/KQ^nS) = \ell(S/QS)\binom{n+d-1}{d} &+ \ell(S/KS)\binom{n+d-2}{d-1} + \cdots \\ &+ \ell(S/KS)\end{aligned}$$

and

$$\begin{aligned}\ell(R/KQ^n) = \ell(S/QS)\binom{n+d-1}{d} &+ \ell(S/KS)\binom{n+d-2}{d-1} + \cdots \\ &+ \ell(S/KS).\end{aligned}$$

Thus,

$$\ell(U/(KQ^n \cap U)) = \ell(R/KQ^n) - \ell(S/KQ^nS) = 0 \quad \text{for } n \gg 0.$$

This implies that $U = 0$, and hence, R is Cohen-Macaulay. □

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