JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 3, Fall 2018

A NOTE ON RESIDUAL COORDINATES OF POLYNOMIAL RINGS

M'HAMMED EL KAHOUI AND MUSTAPHA OUALI

ABSTRACT. A special case of the Dolgachev-Weisfeiler conjecture asserts that residual coordinates of the polynomial algebra $A = \mathbb{C}[x]^{[n]}$, $n \geq 3$, are coordinates. It is well known that such polynomials are stable coordinates; however, all the examples constructed thus far are actually 1-stable coordinates. In this paper, we show that all residual coordinates of A are 1-stable coordinates.

1. Introduction. Throughout, all rings considered are commutative with unity. Given a ring R and an R-algebra A, we will write $A = R^{[n]}$ to mean that A is isomorphic to the polynomial R-algebra in n variables (by convention $R^{[0]} = R$). A polynomial p in $A = R^{[n]}$ is said to be a *coordinate* if $A = R[p]^{[n-1]}$. It is said to be a *stable coordinate* if $A^{[m]} = R[p]^{[n+m-1]}$ for some $m \geq 1$. When m needs to be specified, we say that p is an m-stable coordinate. A polynomial p in A is said to be a *residual coordinate* if

$$K(\mathfrak{p}) \bigotimes_R A = \left(K(\mathfrak{p}) \bigotimes_R R[p] \right)^{[n-1]}$$

for every $\mathfrak{p} \in \operatorname{Spec} R$, where $K(\mathfrak{p})$ stands for the residue field of $\mathfrak{p}$.

The relationship between coordinates, residual coordinates and stable coordinates is well understood for $n = 2$. In particular, if R is a Noetherian ring containing $\mathbb{Q}$, it follows from the results of Bhatwadekar and Dutta **[3]** that the three notions are equivalent. The Noetherianity assumption can be dropped, as shown by van Rossum and van den Essen **[15]**. Recently, Das and Dutta **[6]** proved that, for any Noetherian domain R and any $n \geq 3$, residual coordinates of $A = R^{[n]}$ are stable coordinates.

2010 AMS *Mathematics subject classification.* Primary 13B25, 14R10.
Keywords and phrases. Residual coordinate, stable coordinate.
Received by the editors on August 20, 2014, and in revised form on April 23, 2016.
DOI:10.1216/JCA-2018-10-3-317

The classical example of Hochster [**8**] shows that residual coordinates of $A = R^{[n]}$, $n \geq 3$, need not be coordinates, in general. However, the question remains open when R is a polynomial ring over $\mathbb{C}$. The Vénéreau polynomial [**9, 16**] is the most well-known example of a residual coordinate over $\mathbb{C}[x]$ with an indefinite status; however, in fact, a very similar example was constructed earlier by Bhatwadekar and Dutta [**4,** Example 4.13].

As shown by Freudenburg [**7**], the Vénéreau polynomial is a 1-stable coordinate. Larger families of Vénéreau-type polynomials were constructed by Daigle and Freudenburg [**5**] and then in [**10**] by Lewis, who proved in [**11**] that all of the Vénéreau-type polynomials are 1-stable coordinates. In this paper we obtain the following generalization of the result of Lewis [**11**].

Theorem 1.1. *Let K be an algebraically closed field of characteristic zero. Then, every residual coordinate of $A = K[x]^{[n]}$, where $n \geq 3$, is a 1-stable coordinate.*

As already mentioned, given a Noetherian domain R, residual coordinates of $A = R^{[n]}$ are stable coordinates. When R has Krull dimension d and contains $\mathbb{Q}$, we obtain the following result.

Theorem 1.2. *Let R be a Noetherian d-dimensional ring containing $\mathbb{Q}$. Then, every residual coordinate of $A = R^{[n]}$ is a $(2^d - 1)n$-stable coordinate.*

Given a residual coordinate p of $A = R^{[n]}$, the above result shows that the number m of variables which must be added so that p becomes a coordinate in $B = A^{[m]}$ does not depend upon p, but only upon n and the Krull dimension of R. However, we do not know whether the bound we give is sharp.

2. Preliminaries. In this section, we fix notation and recall the main results that will be used.

2.1. Residual coordinates. Let R be a ring, and let p be a coordinate of $A = R^{[n]}$. It is a basic fact that p remains a coordinate over any localization and over any quotient of R. The same holds for residual coordinates and m-stable coordinates. Given a multiplicative set $S \subset R$ consisting of non zerodivisors, if p is a coordinate over the localization R_S, then there exists a $c \in S$ such that p is a coordinate over R_c. In the remainder of this paper, we will make use of these basic facts without further reference.

The following result was proven by Maubach [**12,** Theorem 4.5].

Theorem 2.1. *Let K be an algebraically closed field of characteristic zero, and let q be a polynomial of*

$$K[x][u,w] = K[x][u_1,\ldots,u_n,w]$$

of the form

$$q = c(x)w + p(x,u),$$

where $c(x) \in K[x]$ is non-constant. Assume that, for every root α of $c(x)$, the polynomial $p(\alpha,u)$ is a coordinate of $K[u]$. Then, q is a coordinate over $K[x]$.

To our knowledge, it is not known whether Theorem 2.1 holds without the assumption that K be algebraically closed.

The following result is due to Berson, Bikker and van den Essen [**1,** Proposition 5.3].

Theorem 2.2. *Let R be a ring, and let c be a non zerodivisor of R. Let $q(u,w)$ be a coordinate of*

$$R[u,w] = R[u_1,\ldots,u_n,w].$$

If $q(u,0)$ is a coordinate over R/cR, then $q(u,cw)$ is an $(n-1)$-stable coordinate of $R[u,w]$.

Consider a polynomial of $R[u,w]$ of the form $q = cw + p(u)$, where c is a non zerodivisor of R and $p \in R[u]$. A direct application of Theorem 2.2 shows that, if p is a coordinate over R/cR, then q is an $(n-1)$-stable coordinate of $R[u,w]$. More generally, if p is an m-stable coordinate over R/cR, then q is a $(2m+n-1)$-stable coordinate.

2.2. Automorphisms of polynomial rings. Given a ring R and $n \geq 1$, we denote by $\mathrm{GA}_n(R)$ the group of R-automorphisms of the polynomial R-algebra in n variables. The subgroup of $\mathrm{GA}_n(R)$ consisting of automorphisms of Jacobian determinant 1 is denoted by $\mathrm{SA}_n(R)$. Given a coordinate system u of $A = R^{[n]}$, an R-automorphism σ of A is said to be elementary in the coordinate system u if there exists an i such that $\sigma(u_j) = u_j$ for $j \neq i$ and $\sigma(u_i) = u_i + p(u)$, where p does not depend on u_i. We let $\mathrm{EA}_n(R, u)$ be the subgroup of $\mathrm{GA}_n(R)$ generated by the elementary automorphisms in the coordinate system u.

The following fundamental properties of automorphisms of polynomial algebras will be necessary for our purposes, see [**1,** Remark 2.2], [**2,** Proposition 2.7] and [**13,** Lemma 1.1.9]. Recall that $\mathrm{Nil}(R)$ denotes the nilradical of the ring R.

Proposition 2.3. *Let R be a ring, and let σ be an R-endomorphism of $A = R^{[n]}$. Then, the following properties hold*:

(i) *given an ideal $\mathfrak{a}$ contained in $\mathrm{Nil}(R)$, σ is an automorphism if and only if it is so over $R/\mathfrak{a}$. As a consequence, a polynomial p in A is a coordinate if and only if it is a coordinate over $R/\mathfrak{a}$*;

(ii) *if R is Noetherian, then σ is an automorphism if and only if it is so over every $R/\mathfrak{p}$, where $\mathfrak{p}$ ranges over the minimal prime ideals of R.*

Given two rings R and R_1, every ring homomorphism $\phi : R \to R_1$ naturally induces a group homomorphism

$$\phi^\star : \mathrm{GA}_n(R) \longrightarrow \mathrm{GA}_n(R_1)$$

that maps $\mathrm{SA}_n(R)$ to $\mathrm{SA}_n(R_1)$ and $\mathrm{EA}_n(R, u)$ to $\mathrm{EA}_n(R_1, u)$. When ϕ is surjective, the homomorphism $\phi^\star$ always maps $\mathrm{EA}_n(R, u)$ onto $\mathrm{EA}_n(R_1, u)$. The next result, due to van den Essen, Maubach and Vénéreau [**14**], shows that in some special situations the same holds for $\mathrm{SA}_n(R)$.

Theorem 2.4. *Let R be a ring containing $\mathbb{Q}$ and $m \in \mathbb{N}^\star$, and let $R_m = R[x]/x^m R[x]$. Then, the group homomorphism*

$$\mathrm{SA}_n(R[x]) \longrightarrow \mathrm{SA}_n(R_m),$$

induced by the canonical homomorphism $R[x] \longrightarrow R_m$, is surjective.

Given a ring R containing $\mathbb{Q}$, every coordinate q of $A = R^{[n]}$ is a component of an automorphism in $\mathrm{SA}_n(R)$, see [**1,** Theorem 4.3] and [**13,** Section 2.3]. By combining this fact with Proposition 2.3 and Theorem 2.4, we directly obtain the next result.

Corollary 2.5. *Let R be a ring containing $\mathbb{Q}$, and let q be a polynomial in $A = R[x]^{[n]}$. If q is a coordinate over $R[x]/xR[x]$, then, for every $m \in \mathbb{N}^\star$, there exists a coordinate system $v = v_1, \dots, v_n$ of A and $a(v) \in R[x][v]$ such that*

$$q = v_1 + x^m a(v).$$

2.3. Locally nilpotent derivations. Let R be a ring containing $\mathbb{Q}$, and let A be an R-algebra. An R-derivation ξ of A is said to be locally nilpotent if, for every $a \in A$, there exists an $m \in \mathbb{N}^\star$ such that $\xi^m(a) = 0$.

Let ξ be a locally nilpotent R-derivation of A and w an indeterminate over A. We extend ξ to $A[w]$ by setting $\xi(w) = 0$ and consider the map

$$\exp(w\xi) : A[w] \longrightarrow A[w],$$

defined for every $a \in A[w]$ by

$$\exp(w\xi).a = \sum \frac{\xi^m(a)}{m!} w^m.$$

A fundamental fact regarding locally nilpotent derivation theory is that $\exp(w\xi)$ is an $R[w]$-automorphism of $A[w]$, and its inverse is $\exp(-w\xi)$, see e.g., [**13,** Proposition 1.2.14].

3. The main results. The following lemma will be very useful in the sequel.

Lemma 3.1. *Let R be a ring containing $\mathbb{Q}$, and let c be a non zerodivisor of R. Let p be a polynomial in $A = R^{[n]}$, and assume that it is a coordinate over the localization R_c. Then, there exists a locally nilpotent R-derivation ξ of $R^{[n]}$ such that $\xi(p) = c^m$, for some $m \in \mathbb{N}$.*

Proof. Let $u = u_1, \dots, u_n$ be a coordinate system of A. Since p is a coordinate in $R_c[u]$, there exists a locally nilpotent R_c-derivation δ of $R_c[u]$ such that $\delta(p) = 1$. On the other hand, the assumption that c is a non zerodivisor of R implies that $R[u]$ is a subring of $R_c[u]$. Moreover, there exists an $m \in \mathbb{N}$ such that $c^m\delta(u_i) \in R[u]$ for every $i = 1, \ldots, n$. Thus, $\xi = c^m\delta$ is a locally nilpotent R-derivation of $R[u]$, and we have $\xi(p) = c^m$. $\square$

We can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let u be a coordinate system of $A = K[x]^{[n]}$, and let p be a residual coordinate of $K[x][u]$. Then, there exists a $c \in K[x] \setminus \{0\}$ such that p is a coordinate over the localization $K[x]_c$. By Lemma 3.1, there exists a locally nilpotent $K[x]$-derivation ξ of $K[x][u]$ such that $\xi(p) = c^m$ for some $m \in \mathbb{N}$.

Let w be an indeterminate over $K[x][u]$, and note that the $K[x]$-automorphism $\exp(w\xi)$ of $K[x][u, w]$ maps p to $c^m w + p$. Thus, p is a 1-stable coordinate if $c^m w + p$ is a coordinate of $K[x][u, w]$.

Clearly, if c is constant, then $c^m w + p$ is a coordinate, and we are done. Thus, assume in the rest of the proof that c is non-constant. Since p is a residual coordinate of $K[x][u]$ for every $\alpha \in K$, the polynomial $p(\alpha, u)$ is a coordinate of $K[u]$. This holds, in particular, for every root α of $c(x)$, and hence, by Theorem 2.1, $c^m w + p$ is a coordinate of $K[x][u, w]$. $\square$

Let R be a ring and $c \in R$ a non zerodivisor. Following the terminology in **[11]**, a polynomial p in $R^{[n]}$ is said to be a *c-strongly residual coordinate* if it is a coordinate over the localization R_c and also a coordinate over the quotient R/cR. The polynomial p is said to be a strongly residual coordinate if it is a c-strongly residual coordinate for some non zerodivisor c of R.

Two results are given below on strongly residual coordinates of the polynomial algebra $A = R^{[n]}$. The first one concerns the family of polynomials studied in **[10, 11]**.

Theorem 3.2. *Let R be a ring containing $\mathbb{Q}$, and let x be an indeterminate over R. Then, every x-strongly residual coordinate of $A = R[x]^{[n]}$ is a 1-stable coordinate.*

Proof. Let u be a coordinate system of A, and let p be an x-strongly residual coordinate of $R[x][u]$. Then, p is a coordinate of

$R[x]_x[u]$ and, by Lemma 3.1, there exists a locally nilpotent $R[x]$-derivation ξ of $R[x][u]$ such that $\xi(p) = x^m$ for some $m \in \mathbb{N}$. Let w be an indeterminate over $R[x][u]$, and consider the $R[x]$-automorphism $\exp(w\xi)$ of $R[x][u, w]$. Clearly, we have $\exp(w\xi)(p) = p + x^m w$, and thus, p is a 1-stable coordinate if $p + x^m w$ is a coordinate of $R[x][u, w]$.

Note that, if $m = 0$, then $p + x^m w = p + w$ is a coordinate of $R[x][u, w]$. Thus, we assume in the rest of the proof that $m \geq 1$. Since p is a coordinate over $R[x]/xR[x]$, there exists, by Corollary 2.5, a coordinate system $v = v_1, \ldots, v_n$ of $R[x][u]$ and $a(v) \in R[x][v]$ such that

$$p = v_1 + x^m a(v).$$

As a consequence, we have

$$p + x^m w = v_1 + x^m(w + a(v)).$$

Now, consider the elementary $R[x]$-automorphisms ϕ_1 and ϕ_2 of $R[x][v, w]$, defined by $\phi_1(v_1) = v_1 + x^m w$ and $\phi_2(w) = w + a(v)$. Then, $\phi_2\phi_1(v_1) = p + x^m w$, and thus, $p + x^m w$ is a coordinate of $R[x][v, w]$. □

As a consequence of Theorem 2.2 and Lemma 3.1, we obtain the next result.

Theorem 3.3. *Let R be a ring containing $\mathbb{Q}$. Then, every strongly residual coordinate of $A = R^{[n]}$ is an n-stable coordinate.*

Proof. Let p be a strongly residual coordinate of $A = R[u]$, and let c be a non zerodivisor of R such that p is a coordinate over the localization R_c and p is a coordinate over R/cR. From Lemma 3.1, there exists a locally nilpotent R-derivation ξ of $R[u]$ such that $\xi(p) = c^m$ for some $m \in \mathbb{N}$.

Given an indeterminate w over $R[u]$, the R-automorphism $\exp(w\xi)$ of $R[u, w]$ maps p to $p + c^m w$, and hence, p is an n-stable coordinate if $p + c^m w$ is an $(n-1)$-stable coordinate. The assumption that p is a coordinate over R/cR implies, by Proposition 2.3, that it is a coordinate over $R/c^m R$. From Theorem 2.2 it follows that $p + c^m w$ is an $(n-1)$-stable coordinate. □

The remainder of this paper is devoted to the proof of Theorem 1.2.

Lemma 3.4. *Let R be a Noetherian zero-dimensional ring. Then, every residual coordinate of $A = R^{[n]}$ is a coordinate.*

Proof. Let u be a coordinate system of A, and let p be a residual coordinate of $R[u]$. From Proposition 2.3, p is a coordinate if and only if it is so over $R/Nil(R)$. We can thus assume, without loss of generality, that R is reduced.

Since R is Noetherian, it has finitely many minimal prime ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$. The fact that R is zero-dimensional and reduced then implies that every $\mathfrak{m}_i$ is maximal and $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_r = (0)$. In particular, the canonical homomorphism

$$\pi : R[u] \longrightarrow \prod_i \left(R/\mathfrak{m}_i R\right)[u]$$

is an isomorphism. On the other hand, since p is a residual coordinate and $\mathfrak{m}_i$ is maximal, p is a coordinate over $R/\mathfrak{m}_i$, and hence, there exist $p_{i,2}, \ldots, p_{i,n} \in (R/\mathfrak{m}_i R)[u]$ that complete p into a coordinate system of $(R/\mathfrak{m}_i R)[u]$.

For every $j = 2, \ldots, n$ let p_j be the unique polynomial in $R[u]$ such that

$$\pi(p_j) = (p_{1,j}, \ldots, p_{r,j}),$$

and let σ be the R-endomorphism of $R[u]$ defined, for $j = 1, \ldots, n$, by $\sigma(u_j) = p_j$, where $p_1 = p$. For every $i = 1, \ldots, r$, we have

$$(p, p_2, \ldots, p_n) = (p, p_{i,2}, \ldots, p_{i,n}) \mod \mathfrak{m}_i,$$

and thus, σ, viewed over $R/\mathfrak{m}_i$, is an $R/\mathfrak{m}_i$-automorphism. By Proposition 2.3, it follows that σ is an R-automorphism, and hence, p is a coordinate of $R[u]$. □

Lemma 3.5. *Let R be a reduced Noetherian ring, and let p be a residual coordinate of $A = R^{[n]}$. Then, there exists a non zerodivisor c of R such that p is a coordinate over the localization R_c.*

Proof. Let $S \subset R$ be the multiplicative set of non zerodivisors. Since R is Noetherian and reduced, the localization R_S is zero-dimensional. From Lemma 3.4, it follows that p is a coordinate over R_S, and hence, there exists a $c \in S$ such that p is a coordinate over R_c. □

We now have enough material to prove Theorem 1.2.

Proof of Theorem 1.2. For $d = 0$, the result follows from Lemma 3.4. Hence, let $d \geq 1$, and assume that the result holds in dimension at most $d-1$. Let R be a Noetherian d-dimensional ring containing $\mathbb{Q}$, and let $p \in A = R[u]$ be a residual coordinate. By Proposition 2.3, p is a $(2^d-1)n$-stable coordinate if and only if it is so over $R/\operatorname{Nil}(R)$. Thus, we can assume, without loss of generality, that R is reduced.

From Lemma 3.5, there exists a non zerodivisor c of R such that p is a coordinate over the localization R_c. It then follows from Lemma 3.1 that there exists a locally nilpotent R-derivation ξ of $R[u]$ such that $\xi(p) = c^m$ for some $m \in \mathbb{N}$. Given an indeterminate w over $R[u]$, the R-automorphism $\exp(w\xi)$ of $R[u,w]$ maps p to $c^m w + p(u)$. In particular, if $c^m w + p(u)$ is a $((2^d-1)n-1)$-stable coordinate, then p is a $(2^d-1)n$-stable coordinate.

If c is a unit, then $c^m w + p(u)$ is a coordinate, and we are done. Thus, we assume in the rest of the proof that c is not a unit. Since, moreover, c is a non zerodivisor and R is Noetherian, it follows from Krull's principal ideal theorem that R/cR has dimension at most $d-1$. On the other hand, p is a residual coordinate over R/cR. It then follows from Proposition 2.3 and the induction hypothesis that p, viewed over R/c^mR, is an r stable coordinate, where $r = (2^{d-1}-1)n$. As a consequence of Theorem 2.2, the polynomial $c^m w+p(u)$ is a $(2r+n-1)$-stable coordinate, and hence, p is a $(2^d-1)n$-stable coordinate. $\square$

REFERENCES

1. J. Berson, J.W. Bikker and A. van den Essen, *Adapting coordinates*, J. Pure Appl. Alg. **184** (2003), 165–174.

2. J. Berson, A. van den Essen and S. Maubach, *Derivations having divergence zero on $R[X,Y]$*, Israel J. Math. **124** (2001), 115–124.

3. S.M. Bhatwadekar and A.K. Dutta, *On residual variables and stably polynomial algebras*, Comm. Alg. **21** (1993), 635–645.

4. ———, *On affine fibrations*, World Scientific Publishers, River Edge, NJ, 1994.

5. D. Daigle and G. Freudenburg, *Families of affine fibrations*, Birkhäuser, Boston, 2010.

6. P. Das and A.K. Dutta, *A note on residual variables of an affine fibration*, J. Pure Appl. Alg. **218** (2014), 1792–1799.

7. G. Freudenburg, *The Vénéreau polynomials relative to* $\mathbb{C}^\star$*-fibrations and stable coordinates*, in *Affine algebraic geometry*, Osaka University Press, Osaka, 2007.

8. M. Hochster, *Nonuniqueness of coefficient rings in a polynomial ring*, Proc. Amer. Math. Soc. **34** (1972), 81–82.

9. S. Kaliman, S. Vénéreau and M. Zaidenberg, *Simple birational extensions of the polynomial algebra* $\mathbb{C}^{[3]}$, Trans. Amer. Math. Soc. **356** (2004), 509–555 (electronic).

10. D. Lewis, *Vénéreau-type polynomials as potential counterexamples*, J. Pure Appl. Alg. **217** (2013), 946–957.

11. ———, *Strongly residual coordinates over* $A[x]$, in *Automorphisms in birational and affine geometry*, Springer Proc. Math. Stat. **79** (2014), 407–430.

12. S. Maubach, *The commuting derivations conjecture*, J. Pure Appl. Alg. **179** (2003), 159–168.

13. A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progr. Math. **190** (2000).

14. A. van den Essen, S. Maubach and S. Vénéreau, *The special automorphism group of* $R[t]/(t^m)[x_1,\dots,x_n]$ *and coordinates of a subring of* $R[t][x_1,\dots,x_n]$, J. Pure Appl. Alg. **210** (2007), 141–146.

15. A. van den Essen and P. van Rossum, *Coordinates in two variables over a* $\mathbb{Q}$*-algebra*, Trans. Amer. Math. Soc. **356** (2004), 1691–1703.

16. S. Vénéreau, *Automorphismes et variables de l'anneau de polynomes* $A[y_1,\dots,y_n]$, Ph.D. dissertation, Université de Grenoble I, Institut Fourier, Grenoble, 2001.

Cadi Ayyad University, Department of Mathematics, P.O. Box 2390, Marrakesh, Morocco
Email address: elkahoui@uca.ma

Cadi Ayyad University, Department of Mathematics, P.O. Box 2390, Marrakesh, Morocco
Email address: mustouali@gmail.com