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TOPICS ON SEQUENTIALLY COHEN-MACAULAY MODULES

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ABSTRACT. In this paper, we study the two different topics related to sequentially Cohen-Macaulay modules. Questions arise as to when the sequentially Cohen-Macaulay property preserves the localization and the module-finite extension of rings.

1. Introduction. Throughout this paper, unless otherwise specified, let R be a commutative Noetherian ring, $M \neq (0)$ a finitely generated R-module of dimension d. Then, there exists the largest R-submodule M_n of M with $\dim_R M_n \leq n$ for every $n \in \mathbb{Z}$ (here $\dim_R(0) = -\infty$, for convention). Let

$$\begin{aligned}\mathcal{S}(M) &= \{\dim_R N \mid N \text{ is an } R\text{-submodule of } M, N \neq (0)\}\\ &= \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M\}.\end{aligned}$$

Set $\ell = \sharp\mathcal{S}(M)$, and write $\mathcal{S}(M) = \{d_1 < d_2 < \cdots < d_\ell = d\}$. Then, the *dimension filtration* of M is a chain

$$\mathcal{D} : D_0 := (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = M$$

of R-submodules of M, where $D_i = M_{d_i}$ for every $1 \leq i \leq \ell$. Note that the notion of dimension filtration is due to Goto, Horiuchi and Sakurai [**4**] and is a little different from that of the original one given by Schenzel

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[**6**]. However, we adopt the above definition throughout this paper. We say that M is a *sequentially Cohen-Macaulay R-module* if the quotient module $C_i = D_i/D_{i-1}$ of D_i is Cohen-Macaulay for every $1 \leq i \leq \ell$. In particular, the ring R is called a *sequentially Cohen-Macaulay ring* if $\dim R < \infty$, and R is a sequentially Cohen-Macaulay module over itself.

The aim of this paper is to investigate the following questions. The first one is whether the sequentially Cohen-Macaulay property is inherited from localizations, which has already been studied by Cuong et al. [**2**] and Cuong and Nhan [**3**] in the case of local rings. In Section 2, we shall probe into a possible generalization of their results ([**2,** Proposition 2.6], [**3,** Proposition 4.7]). More precisely, we prove the following.

Theorem 1.1. *Suppose that* $\dim R/\mathfrak{p} = \dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}$ *for every* $\mathfrak{p} \in \operatorname{Ass}_R M$ *and for every maximal ideal* $\mathfrak{m}$ *of* R *such that* $\mathfrak{p} \subseteq \mathfrak{m}$. *Then, the following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M_P is a sequentially Cohen-Macaulay R_P-module for every* $P \in \operatorname{Supp}_R M$.

Another is the question of whether or not the sequentially Cohen-Macaulay property preserves the module-finite extension of rings. In Section 3, in particular, we will give the characterization of sequentially Cohen-Macaulay local rings obtained by the idealization, that is, trivial extension, which is stated as follows.

Theorem 1.2 (Corollary 3.9). *Suppose that R is a local ring. Let $R \ltimes M$ denote the idealization of M over R. Then the following conditions are equivalent.*

(1) *$R \ltimes M$ is a sequentially Cohen-Macaulay local ring.*
(2) *$R \ltimes M$ is a sequentially Cohen-Macaulay R-module.*
(3) *R is a sequentially Cohen-Macaulay local ring, and M is a sequentially Cohen-Macaulay R-module.*

2. Localization of sequentially Cohen-Macaulay modules.

The purpose of this section is mainly to prove Theorem 1.1.

Proof of Theorem 1.1.

(1) ⇒ (2). We may assume that $\ell > 1$, and the assertion holds for $\ell - 1$. Due to [**2,** Proposition 2.6], it is sufficient to show the case where P is a maximal ideal of R. Then, we obtain the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow C \longrightarrow 0$$

of R-modules where $N = D_{\ell-1}$ and $C = C_\ell$. We may also assume $N_P \neq (0)$, $C_P \neq (0)$ and $\dim_{R_P} N_P = \dim_{R_P} M_P$, since N_P is sequentially Cohen-Macaulay and C_P is Cohen-Macaulay. By using the hypothesis, we get $\dim_{R_P} M_P = \dim_{R_P} C_P$ (see also [**6,** Corollary 2.3]).

Let
$$E_0 = (0) \subsetneq E_1 \subsetneq \cdots \subsetneq E_{t-1} \subsetneq E_t = N_P$$

be the dimension filtration of N_P. Then, M_P/E_{t-1} is a Cohen-Macaulay R_P-module of dimension $\dim_{R_P} M_P$ since N_P/E_{t-1} and M_P/N_P are Cohen-Macaulay R_P-modules of the same dimension $\dim_{R_P} M_P$. Hence, M_P is a sequentially Cohen-Macaulay R_P-module.

(2) ⇒ (1). Suppose that C_i is not Cohen-Macaulay for some $1 \leq i \leq \ell$. Then, there exists a maximal ideal P of R such that $[C_i]_P \neq (0)$, and $[C_i]_P$ is not a Cohen-Macaulay R_P-module. Let $\alpha = \dim_{R_P}[C_i]_P$. We choose $\mathfrak{p} \in \operatorname{Ass}_R C_i$ such that $\mathfrak{p} \subseteq P$, $\alpha = \dim R_P/\mathfrak{p}R_P$. Then,

$$\alpha = \dim R_P/\mathfrak{p}R_P = \dim R/\mathfrak{p} = d_i$$

by using the hypothesis and $\mathfrak{p} \in \operatorname{Ass}_R C_i$. Therefore, $d_i \in \mathcal{S}(M_P) \subseteq \mathcal{S}(M)$ since

$$\mathcal{S}(M_P) = \{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M,\ \mathfrak{p} \subseteq P\} \subseteq \mathcal{S}(M).$$

Let $q = \sharp \mathcal{S}(M_P)$. We write $\mathcal{S}(M_P) = \{n_1 < n_2 < \cdots < n_q\}$. Then, $d_i = n_j$ for some $1 \leq j \leq q$. Let

$$(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass}_R M} M(\mathfrak{p})$$

be a primary decomposition of (0) in M. In this case,

$$(0) = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Ass}_R M \\ \mathfrak{p} \subseteq P}} [M(\mathfrak{p})]_P$$

forms a primary decomposition of (0) in M_P.

Claim 2.1. *The following assertions hold.*

(1) $[D_i]_P = D_j(M_P)$,
(2) $[D_{i-1}]_P = D_{j-1}(M_P)$

where $\{D_j(M_P)\}_{0 \le j \le q}$ *stands for the dimension filtration of* M_P*.*

Proof of Claim 2.1.

(1) We may assume that $i < \ell$. Then,

$$D_i = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R M \\ \dim R/\mathfrak{p} > d_i}} M(\mathfrak{p}),$$

so that

$$[D_i]_P = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R M \\ \dim R/\mathfrak{p} > n_j \\ \mathfrak{p} \subseteq P}} [M(\mathfrak{p})]_P.$$

We now assume that $\mathfrak{p} \not\subseteq P$ for every $\mathfrak{p} \in \mathrm{Ass}_R M$ with $\dim R/\mathfrak{p} > d_i$. Then, $M(\mathfrak{p})_P = M_P$ so that $[D_i]_P = M_P$. Then, $\dim_{R_P} M_P = d_i$ since $\dim_{R_P} M_P \le d_i \in \mathcal{S}(M_P)$. Therefore, $d_i = n_q$, $j = q$ and

$$[D_i]_P = M_P = D_q(M_P) = D_j(M_P).$$

Thus, we may assume that $\mathfrak{p} \subseteq P$ for some $\mathfrak{p} \in \mathrm{Ass}_R M$ with $\dim R/\mathfrak{p} > d_i$. Therefore,

$$n_q = \dim_{R_P} M_P \ge \dim_{R_P} R_P/\mathfrak{p}R_P = \dim R/\mathfrak{p} > d_i = n_j,$$

whence $1 \le j < q$. Hence,

$$[D_i]_P = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R M \\ \dim R_P/\mathfrak{p}R_P > n_j \\ \mathfrak{p} \subseteq P}} [M(\mathfrak{p})]_P = D_j(M_P).$$

(2) We obtain

$$[D_{i-1}]_P = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Ass}_R M \\ \dim R/\mathfrak{p} \ge d_i \\ \mathfrak{p} \subseteq P}} [M(\mathfrak{p})]_P$$

since

$$D_{i-1} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Ass}_R M \\ \dim R/\mathfrak{p} \geq d_i}} M(\mathfrak{p}).$$

We may assume that $j > 1$. Then, $d_i = n_j > n_{j-1}$ so that

$$[D_{i-1}]_P = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Ass}_R M \\ \dim R/\mathfrak{p} \geq d_i \\ \mathfrak{p} \subseteq P}} [M(\mathfrak{p})]_P = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Ass}_R M \\ \dim R_P/\mathfrak{p}R_P > n_{j-1}}} [M(\mathfrak{p})]_P = D_{j-1}(M_P). \quad \square$$

Hence, $C_j(M_P) = D_j(M_P)/D_{j-1}(M_P)$ is Cohen-Macaulay since M_P is sequentially Cohen-Macaulay, whence $[C_i]_P$ is a Cohen-Macaulay R_P-module, a contradiction. $\square$

If the base ring R is a finitely generated algebra over a field, then the assumption of Theorem 1.1 is automatically satisfied, and we obtain the following.

Corollary 2.2. *Let R be a finitely generated algebra over a field, $M \neq (0)$ a finitely generated R-module. Then, the following conditions are equivalent.*

(1) *M is a sequentially Cohen-Macaulay R-module.*
(2) *M_P is a sequentially Cohen-Macaulay R_P-module for every $P \in \operatorname{Supp}_R M$.*

From now on, we explore the localization property for graded rings. Let $R = \sum_{n \in \mathbb{Z}} R_n$ be a Noetherian $\mathbb{Z}$-graded ring, and assume that R is an H-local ring with an H-maximal ideal P of R in the sense of Goto and Watanabe, see [**5,** Definitions 1.1.5, 1.1.6]. Let $M \neq (0)$ be a finitely generated graded R-module of dimension d. In addition, let $\{D_i\}_{0 \leq i \leq \ell}$ be the dimension filtration of M. We set $q = \dim R/P$. For an arbitrary ideal I of a graded ring, I^* stands for the ideal generated by every homogeneous elements in I.

Lemma 2.3. $\dim_R M = \dim_{R_P} M_P + q$. *Therefore,* $\dim_R M = \dim_{R_\mathfrak{m}} M_\mathfrak{m}$ *for every maximal ideal $\mathfrak{m}$ of R with $\mathfrak{m} \supseteq P$.*

Proof. We may assume $q > 0$ (thus, $q = 1$). Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{m} \supseteq P$ and $\dim R_\mathfrak{m}/PR_\mathfrak{m} = 1$. Let $\mathfrak{p} \in \mathrm{Ass}_R M$ be such that $P \supseteq \mathfrak{p}$ and $\dim R_P/\mathfrak{p}R_P = \dim_{R_P} M_P$. Then, we have $\dim_R M \geq \dim_{R_P} M_P + 1$. Conversely, we choose $\mathfrak{p} \in \mathrm{Ass}_R M$ and a maximal ideal $\mathfrak{m}$ of R such that $\mathfrak{p} \subseteq \mathfrak{m}$ and $\dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} = d$. Since $\mathfrak{p}$ is a graded ideal of R, $\mathfrak{p} \subseteq \mathfrak{m}^*$. Note that $\mathfrak{m}$ is not a graded ideal of R since $q = 1$. Hence, we obtain $\mathfrak{m}^* \subseteq P$ and

$$\begin{aligned} d &= \dim R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m} = \dim R_{\mathfrak{m}^*}/\mathfrak{p}R_{\mathfrak{m}^*} + 1 \\ &\leq \dim R_P/\mathfrak{p}R_P + 1 \leq \dim_{R_P} M_P + 1. \end{aligned}$$ □

Apply [**4,** Theorem 2.3] and Lemma 2.3 to obtain the following.

Corollary 2.4. *The following assertions hold true.*

(1) $[D_0]_\mathfrak{m} = (0) \subsetneq [D_1]_\mathfrak{m} \subsetneq \cdots \subsetneq [D_\ell]_\mathfrak{m} = M_\mathfrak{m}$ *is the dimension filtration of* $M_\mathfrak{m}$ *for every maximal ideal* $\mathfrak{m}$ *of* R *such that* $\mathfrak{m} \supseteq P$.
(2) $[D_0]_P = (0) \subsetneq [D_1]_P \subsetneq \cdots \subsetneq [D_\ell]_P = M_P$ *is the dimension filtration of* M_P *so that* M *is a sequentially Cohen-Macaulay* R*-module if and only if* M_P *is a sequentially Cohen-Macaulay* R_P*-module.*

Finally, we reach the goal of this section.

Theorem 2.5. *The following conditions are equivalent.*

(1) M *is a sequentially Cohen-Macaulay* R*-module.*
(2) M_P *is a sequentially Cohen-Macaulay* R_P*-module.*

When this is the case, $M_\mathfrak{p}$ *is a sequentially Cohen-Macaulay* $R_\mathfrak{p}$*-module for every* $\mathfrak{p} \in \mathrm{Supp}_R M$.

Proof. The equivalence of conditions (1) and (2) follows from Corollary 2.4. We verify the last assertion. Note that $\mathfrak{p}^* \subseteq P$ for any $\mathfrak{p} \in \mathrm{Supp}_R M$. Due to [**2,** Proposition 2.6], $M_{\mathfrak{p}^*}$ is a sequentially Cohen-Macaulay $R_{\mathfrak{p}^*}$-module. Since $\mathfrak{p}^* R_{(\mathfrak{p})}$ is an H-maximal ideal of the homogeneous localization $R_{(\mathfrak{p})}$ of R, $M_{(\mathfrak{p})}$ is a sequentially Cohen-Macaulay $R_{(\mathfrak{p})}$-module. We may assume that $\mathfrak{p}$ is not a graded ideal of R so that $\mathfrak{p}R_{(\mathfrak{p})}$ is a maximal ideal of $R_{(\mathfrak{p})}$. Therefore, $M_\mathfrak{p}$ is a sequentially Cohen-Macaulay $R_\mathfrak{p}$-module. □

3. Module-finite extension of sequentially Cohen-Macaulay modules. In this section, we assume that R is a local ring with maximal ideal $\mathfrak{m}$. Recall that M_n stands for the largest R-submodule of M with $\dim_R M_n \leq n$ for each $n \in \mathbb{Z}$.

We note the following.

Lemma 3.1. *Let M and N be finitely generated R-modules. Then, $[M \oplus N]_n = M_n \oplus N_n$ for every $n \in \mathbb{Z}$.*

Proof. We have $[M \oplus N]_n \supseteq M_n \oplus N_n$ since $\dim_R(M_n \oplus N_n) = \max\{\dim_R M_n, \dim_R N_n\} \leq n$. Let

$$p : L = M \oplus N \longrightarrow M, (x, y) \longmapsto x$$

be the first projection. Then, $p(L_n) \subseteq M_n$ since $\dim_R p(L_n) \leq \dim_R L_n \leq n$. We similarly have $q(L_n) \subseteq N_n$, where

$$q : M \oplus N \longrightarrow N, (x, y) \longmapsto y$$

denotes the second projection. Hence, $[M \oplus N]_n \subseteq M_n \oplus N_n$, as claimed. $\square$

The following proposition includes the result [**3,** Proposition 4.5].

Proposition 3.2. *Let M and N $(M, N \neq (0))$ be finitely generated R-modules. Then, $M \oplus N$ is a sequentially Cohen-Macaulay R-module if and only if both M and N are sequentially Cohen-Macaulay R-modules.*

Proof. We set $L = M \oplus N$ and $\ell = \sharp\mathcal{S}(L)$. Then, $\mathcal{S}(L) = \mathcal{S}(M) \cup \mathcal{S}(N)$ since $\mathrm{Ass}_R L = \mathrm{Ass}_R M \cup \mathrm{Ass}_R N$. Hence, if $\ell = 1$, then $\mathcal{S}(L) = \mathcal{S}(M) = \mathcal{S}(N)$ and $\dim_R L = \dim_R M = \dim_R N$. Therefore, when $\ell = 1$, L is a sequentially Cohen-Macaulay R-module if and only if L is a Cohen-Macaulay R-module, and the second condition is equivalent to saying that the R-modules M and N are Cohen-Macaulay, that is, M and N are sequentially Cohen-Macaulay R-modules. Suppose that $\ell > 1$ and that our assertion holds true for $\ell - 1$. Let

$$D_0 = (0) \subsetneq D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_\ell = L$$

be the dimension filtration of $L = M \oplus N$, where $\mathcal{S}(L) = \{d_1 < d_2 < \cdots < d_\ell\}$. Then, $\{D_i/D_1\}_{1 \leq i \leq \ell}$ is the dimension filtration of L/D_1, and hence, L is a sequentially Cohen-Macaulay R-module if and only if D_1 is a Cohen-Macaulay R-module and L/D_1 is a sequentially Cohen-Macaulay R-module. Since

$$D_1 = \begin{cases} M_{d_1} \oplus (0) & (d_1 \in \mathcal{S}(M) \setminus \mathcal{S}(N)), \\ M_{d_1} \oplus N_{d_1} & (d_1 \in \mathcal{S}(M) \cap \mathcal{S}(N)), \\ (0) \oplus N_{d_1} & (d_1 \in \mathcal{S}(N) \setminus \mathcal{S}(M)), \end{cases}$$

by Lemma 3.1, the hypothesis on ℓ readily shows the second condition is equivalent to saying that the R-modules M and N are sequentially Cohen-Macaulay. □

In what follows, let A be a Noetherian local ring, and assume that A is a module-finite algebra over R.

The main result of this section is stated as follows.

Theorem 3.3. *Let $M \neq (0)$ be a finitely generated A-module. Then, the following assertions hold true.*

(1) *M_n is the largest A-submodule of M with $\dim_A M_n \leq n$ for every $n \in \mathbb{Z}$.*
(2) *The dimension filtration of M as an A-module coincides with that of M as an R-module.*
(3) *M is a sequentially Cohen-Macaulay A-module if and only if M is a sequentially Cohen-Macaulay R-module.*

Proof. Let $n \in \mathbb{Z}$ and X denote the largest A-submodule of M with $\dim_A X \leq n$. Then, $X \subseteq M_n$ since $\dim_R X = \dim_A X \leq n$. Let $Y = AM_n$. Then, $\dim_A Y \leq n$. In fact, let $\mathfrak{p} \in \mathrm{Ass}_R Y$. Then, since $[M_n]_\mathfrak{p} \subseteq Y_\mathfrak{p} = A_\mathfrak{p} {\cdot} [M_n]_\mathfrak{p} \subseteq M_\mathfrak{p}$, we see that $[M_n]_\mathfrak{p} \neq (0)$ so that $\mathfrak{p} \in \mathrm{Supp}_R M_n$. Hence, $\dim R/\mathfrak{p} \leq \dim_R M_n \leq n$. Thus, $\dim_A Y = \dim_R Y \leq n$, whence $M_n \subseteq Y \subseteq X$, which shows $X = M_n$. Therefore, assertions (1) and (2) follow. Since

$$\dim_A L = \dim_R L \quad \text{and} \quad \mathrm{depth}_A L = \mathrm{depth}_R L$$

for every finitely generated A-module L, we obtain assertion (3). □

Some conclusions are now summarized.

Corollary 3.4. *A is a sequentially Cohen-Macaulay local ring if and only if A is a sequentially Cohen-Macaulay R-module.*

Corollary 3.5. *Let M be a finitely generated A-module. Suppose that R is a direct summand of M as an R-module. If M is a sequentially Cohen-Macaulay A-module, then R is a sequentially Cohen-Macaulay local ring.*

Proof. We write $M = R \oplus N$, where N is an R-submodule of M. Since M is a sequentially Cohen-Macaulay A-module, by Theorem 3.3, it is a sequentially Cohen-Macaulay R-module as well so that, by Proposition 3.2, R is a sequentially Cohen-Macaulay local ring. □

Corollary 3.6. *Suppose that R is a direct summand of A as an R-module. If A is a sequentially Cohen-Macaulay local ring, then R is a sequentially Cohen-Macaulay local ring.*

Next, we consider the invariant subring $R = A^G$.

Corollary 3.7. *Let A be a Noetherian local ring, G a finite subgroup of* $\mathrm{Aut}\, A$. *Suppose that the order of G is invertible in A. If A is a sequentially Cohen-Macaulay local ring, then the invariant subring $R = A^G$ of A is a sequentially Cohen-Macaulay local ring.*

Proof. Since the order of G is invertible in A, A is a module-finite extension of $R = A^G$ such that R is a direct summand of A (see **[1]** and reduce to the case where A is a reduced ring). Hence, the assertion follows from Corollary 3.6. □

Remark 3.8. In the setting of Corollary 3.7, let $\{D_i\}_{0 \le i \le \ell}$ be the dimension filtration of A. Then, every D_i is a G-stable ideal of A (compare with Theorem 3.3 (1)) and the dimension filtration of R is given by a refinement of $\{D_i^G\}_{0 \le i \le \ell}$.

The goal of this section is the following.

Corollary 3.9. *Let R be a Noetherian local ring, $M \neq (0)$ a finitely generated R-module. We set $A = R \ltimes M$ as the idealization of M over R. Then, the following conditions are equivalent.*

(1) *$A = R \ltimes M$ is a sequentially Cohen-Macaulay local ring.*
(2) *$A = R \ltimes M$ is a sequentially Cohen-Macaulay R-module.*
(3) *R is a sequentially Cohen-Macaulay local ring, and M is a sequentially Cohen-Macaulay R-module.*

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REFERENCES

1. J.W. Brewer and E.A. Rutter, *Must R be Noetherian if R^G is Noetherian?*, Comm. Alg. **5** (1977), 969–979.

2. N.T. Cuong, S. Goto and H.L. Truong, *The equality $I^2 = \mathfrak{q}I$ in sequentially Cohen-Macaulay rings*, J. Algebra **379** (2013), 50–79.

3. N.T. Cuong and L.T. Nhan, *Pseudo Cohen-Macaulay and pseudo generalized Cohen-Macaulay modules*, J. Algebra **267** (2003), 156–177.

4. S. Goto, Y. Horiuchi and H. Sakurai, *Sequentially Cohen-Macaulayness versus parametric decomposition of powers of parameter ideals*, J. Comm. Alg. **2** (2010), 37–54.

5. S. Goto and K.-i. Watanabe, *On graded ring*s II (*$\mathbf{Z}^n$-graded rings*), Tokyo J. Math. **1** (1978), 237–261.

6. P. Schenzel, *On the dimension filtration and Cohen-Macaulay filtered modules*, in Proc. Ferrara Meeting in honour of Mario Fiorentini, University of Antwerp, Wilrijk, Belgium, 1998, 245–264.

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