JOURNAL OF COMMUTATIVE ALGEBRA
Volume 10, Number 1, Spring 2018

VALUATIVE AND GEOMETRIC CHARACTERIZATIONS OF COX SHEAVES

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ABSTRACT. We give an intrinsic characterization of Cox sheaves on Krull schemes in terms of their valuative algebraic properties. We also provide a geometric characterization of their graded relative spectra in terms of good quotients of graded schemes, extending the existing theory on relative spectra of Cox sheaves on normal varieties. Moreover, we obtain an irredundant characterization of Cox rings which, in turn, produces a normality criterion for certain graded rings.

Introduction. Cox sheaves on normal (pre-)varieties X currently are an active field of research with the focus on questions of finite generation and explicit calculation of their ring $\mathcal{R}(X)$ of global sections (called the *Cox ring*) and quotient constructions describing X in terms of a Cox ring and combinatorics [**4, 8, 9, 11, 12, 18, 19, 20, 27**]. Known properties of Cox rings include triviality of homogeneous units and graded factoriality [**2, 3, 7**], i.e., factoriality of the multiplicative monoid of non-zero homogeneous elements. In the case of a free class group $\mathrm{Cl}(X)$ the latter is equivalent to genuine factoriality of the Cox ring $\mathcal{R}(X)$ by a result from [**1**].

Our present purpose is to investigate and characterize Cox sheaves in the more general setting of Krull schemes, i.e., integral schemes with a finite cover by the spectra of Krull rings, compare [**23**]. A Cox sheaf on X is a $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}$ with homogeneous components $\mathcal{O}_X(D)$ for all $[D] \in \mathrm{Cl}(X)$, equipped with a *natural* multiplication. This translates into the more formal requirement that there exists a morphism of graded $\mathcal{O}_X$-algebras from the *divisorial $\mathcal{O}_X$-algebra* associated to $\mathrm{WDiv}(X)$:

$$\pi\colon \mathcal{O}_X(\mathrm{WDiv}(X)) := \bigoplus_{D\in \mathrm{WDiv}(X)} \mathcal{O}_X(D)\cdot \chi^D \longrightarrow \mathcal{R}$$

2010 AMS *Mathematics subject classification.* Primary 13A02, 13A18, 14A20.
Keywords and phrases. Cox rings, Krull schemes, graded schemes.
Received by the editors on December 18, 2013.
DOI:10.1216/JCA-2018-10-1-1

such that each restriction $\pi\colon \mathcal{O}_X(D) \to \mathcal{R}_{[D]}$ is an isomorphism, see Section 3 for more motivation of this definition.

All Cox sheaves on X are linked via the maps from $\mathcal{O}_X(\mathrm{WDiv}(X))$, and thus, their monoids of homogeneous elements are isomorphic modulo units, although the Cox sheaves themselves need not be isomorphic. Furthermore, one Cox sheaf has locally or globally finitely generated sections if and only if all Cox sheaves on X do, see Proposition 3.6.

Our first main result is an intrinsic characterization of Cox sheaves in terms of their valuative algebraic properties. For rings or sheaves thereof graded by an abelian group G an upper index $+$ marks the non-zero homogeneous elements and $\deg_G$ denotes the degree map. Recall that, on a Krull scheme X, each prime divisor Y with generic point η defines a discrete valuation $\nu_{Y,X}\colon \mathcal{K}(X)^* \to \mathbb{Z}$ with valuation ring $\mathcal{K}(X)_{\nu_{Y,X}} = \mathcal{O}_{X,\eta}$. The ring $\mathcal{O}_X(U)$ is then the intersection over all $\mathcal{O}_{X,\eta}$ with $\eta \in U$. In terms of sheaves, each Y defines a *discrete valuation* $\nu_Y\colon \mathcal{K}^* \to \mathbb{Z}^{(Y)}$ from the constant sheaf $\mathcal{K}^*$ onto the skyscraper sheaf at the generic point of Y, and $\mathcal{O}_X$ is the intersection

$$\mathcal{O}_X = \bigcap_{Y \text{prime}} \mathcal{K}_{\nu_Y} \subseteq \mathcal{K}$$

over the corresponding *discrete valuation sheaves* $\mathcal{K}_{\nu_Y}$. Thus, $\mathcal{O}_X$ is a *Krull sheaf*. The sum over all ν_Y then defines a homomorphism of presheaves of abelian groups

$$\mathrm{div} := \sum_Y \nu_Y\colon \mathcal{K}^* \longrightarrow \mathrm{WDiv} := \bigoplus_Y \mathbb{Z}^{(Y)}$$

to the presheaf of *Weil divisors* WDiv, whose image and cokernel presheaves are known as the presheaves of *principal divisors* PDiv, respectively, *class groups* Cl. We will show that Cox sheaves admit a similar description and may even be characterized in such terms. Let $G := \mathrm{Cl}(X)$, and let $\mathcal{R}$ be a Cox sheaf. The role as an ambient sheaf of $\mathcal{R}$ is taken by the constant G-graded sheaf $\mathcal{S}$ assigning the stalk $\mathcal{R}_\xi$ at the generic point. Every homogeneous component of $\mathcal{S}$ is of type $\mathcal{K}$, and hence, $\mathcal{S}$ is *G-simple*, i.e., every homogeneous section is invertible. Each prime divisor Y on X now defines a *discrete G-valuation* $\mu_Y\colon \mathcal{S}^+ \to \mathbb{Z}^{(Y)}$ on the subsheaf $\mathcal{S}^+ \subseteq \mathcal{S}$ of G-homogeneous non-zero elements and a corresponding *discrete G-valuation sheaf* $\mathcal{S}_{\mu_Y} \subseteq \mathcal{S}$, whose sections over U are generated by all G-homogeneous elements that are valuated non-negatively by $\mu_{Y,U}$. Then, $\mathcal{R}$ is the intersection

$$\mathcal{R} = \bigcap_{Y \text{prime}} \mathcal{S}_{\mu_Y} \subseteq \mathcal{S}$$

and is thereby a *G-Krull sheaf*. The sum over all μ_Y defines a homomorphism of presheaves

$$\operatorname{div}_G := \sum_Y \mu_Y : \mathcal{S}^+ \longrightarrow \operatorname{WDiv}.$$

In this terminology, whose precise definitions are given in Section 2, our result is the following.

Theorem 0.1. *Let X be a Krull scheme with generic point ξ, fraction field $\mathcal{K}$, essential valuations ν_Y for the prime divisors $Y = \overline{\{\eta\}}$ on X, and let G be an abelian group and $\mathcal{R}$ a G-graded sheaf of $\mathcal{O}_X$-algebras. Then, $\mathcal{R}$ is isomorphic to a Cox sheaf if and only if the following hold:*

(i) *$\mathcal{R}$ is a $\mathcal{O}_X$-subalgebra of a G-simple $\mathcal{O}_X$-algebra $\mathcal{S}$ with $\mathcal{S}_0 = \mathcal{K}$ such that $\mathcal{R}$ is a G-Krull sheaf in $\mathcal{S}$ defined by discrete G-valuations $\{\mu_Y\}_Y$ which restrict to $\{\nu_Y\}_Y$ on $\mathcal{K}^*$;*

(ii) $\deg_G(\mathcal{S}(X)^+) = G$, $\operatorname{div}_G := \sum_Y \mu_Y$ *is surjective, and* $\operatorname{div}_{G,X}$ *has kernel* $\mathcal{R}(X)^{+,*} = \mathcal{R}(X)^*_0$.

If the above conditions are satisfied, then the isomorphism of grading groups is

$$G \longrightarrow \operatorname{Cl}(X), \qquad \deg_G(f) \longmapsto [\operatorname{div}_{G,X}(f)].$$

Furthermore, we have $\mathcal{R}_0 = \mathcal{O}_X$, and $\mathcal{S}$ is the constant sheaf assigning $\mathcal{R}_\xi$. Each stalk $\mathcal{R}_\eta$ is a discrete G-valuation ring whose corresponding G-valuation on $Q^+(\mathcal{R}_\eta) = \mathcal{S}(X)$ is $\mu_{Y,X}$. For any $f \in \mathcal{S}(U)^+$, we then have $\mu_{Y,U}(f) = \mu_{Y,X}(f)$ if $\eta \in U$ and $\mu_{Y,U}(f) = 0$ otherwise.

If $\mathcal{R}$ is a Cox sheaf with the required $\operatorname{Cl}(X)$-grading, then the above map is the identity on $\operatorname{Cl}(X)$. This intrinsic characterization of Cox sheaves underlines the fact that they form a natural class of graded sheaves. Condition (i) consists of direct graded analoga of the properties of the structure sheaf; they occur in various graded $\mathcal{O}_X$-algebras, e.g., in *divisorial $\mathcal{O}_X$-algebras* $\mathcal{O}_X(L)$ of subgroups $L \leq \operatorname{WDiv}(X)$. Property (ii) ensures the correct $\operatorname{Cl}(X)$-grading, the second part being equivalent to surjectivity and the third to injectivity of the canonical map $G \to \mathrm{Cl}(X)$. The reference for the equation $\mathcal{R}(X)^{+,*} = \mathcal{O}(X)^*$ is [**2**].

In Theorem 0.2 below, we give further details on Cox sheaves. We briefly explain the graded properties and invariants which occur. A *G-integral* ring R (i.e., a ring without G-homogeneous zero divisors) is *G-factorial* if the monoid R^+ of non-zero homogeneous elements is factorial. The *homogeneous fraction ring* $Q^+(R)$ is the localization of R by R^+. A *G-Krull ring* is the graded analogon of a Krull ring. Its *essential G-valuations* form the minimal family of G-valuations defining R in $Q^+(R)$. They correspond bijectively to the *G-prime divisors*, i.e, the minimal non-zero G-prime ideals $\mathfrak{p}$ of R. The *G-valuation ring* of an essential G-valuation $\nu_\mathfrak{p}$ is the graded localization $R_\mathfrak{p}$. More detailed information on G-Krull rings is found in Section 1. The essential G-valuations of a G-Krull sheaf $\mathcal{R}$ are defined in terms of the G-Krull rings $\mathcal{R}(U)$ for all affine U, see Section 2.

Theorem 0.2. *Let $\mathcal{R}$ be a Cox sheaf on a Krull scheme X, and let $\mathcal{S}$ be the constant sheaf assigning $\mathcal{R}_\xi$. Then, $\mathcal{R}$ is quasi-coherent and has the following properties*:

(i) *For each open U, the ring $\mathcal{R}(U)$ is G-factorial, and* $\deg_G(\mathcal{R}(U)^+)$ *generates G. If U is affine, then $\mathcal{S}(U) = Q^+(\mathcal{R}(U))$, and* $\deg_G(\mathcal{R}(U)^+)$ *equals G.*

(ii) *The defining family $\{\mu_Y\}_Y$ from Theorem* 0.1 (i) *are the essential G-valuations of $\mathcal{R}$. $\mathcal{S}_{\mu_Y}(U)$ equals $\mathcal{R}_\eta$ if the generic point η of Y belongs to U and $\mathcal{S}(X)$ otherwise*; *in particular,*

$$\mathcal{R}(U) = \bigcap_{\eta \in U} \mathcal{R}_\eta \subseteq \mathcal{S}(X).$$

The stalk at $x \in X$ is the G-local G-Krull ring

$$\mathcal{R}_x = \bigcap_{x \in Y} \mathcal{S}(X)_{\mu_{Y,X}} = \bigcap_{x \in Y} \mathcal{R}_\eta \subseteq \mathcal{S}(X).$$

The homogeneous elements of its G-maximal ideal $\mathfrak{a}_x$, respectively, the homogeneous units of $\mathcal{R}_x$ are

$$\mathfrak{a}_x \cap \mathcal{R}_x^+ = \{f \in \mathcal{S}(X)^+;\ \textit{there exists a } U \ni x : f \in \mathcal{R}(U)^+, x \in |\mathrm{div}_{G,U}(f)|\}$$

$\mathcal{R}_x^{+,*} = \{f \in \mathcal{S}(X)^+$; *there exists a* $U \ni x : f \in \mathcal{R}(U)^+$, $\operatorname{div}_{G,U}(f) = 0\}$,

and $\deg_G(\mathcal{R}_x^{+,*}) \subseteq \operatorname{Cl}(X)$ *is the subgroup of classes* $[D]$ *represented by a divisor* D *which is principal near* x.

(iii) *For a prime divisor* Y *with generic point* η, *each generator of the maximal ideal of* $\mathcal{O}_{X,\eta} = (\mathcal{R}_\eta)_0$ *also generates* $\mathfrak{a}_\eta$; *in particular,* $\mathcal{R}_\eta$ *has units in every degree.*

G-factoriality of the rings $\mathcal{R}(U)$ is due to surjectivity of $\operatorname{div}_{G,U}$ which is essentially the argument from **[3]**. The first proof of G-factoriality of Cox rings is due to **[7]**. It is valid for Cox sheaves of finite type on normal prevarieties. Cox sheaves on affine Krull schemes also allow the following description.

Corollary 0.3. *A* G*-graded sheaf* $\mathcal{R}$ *on an affine Krull scheme* X *is a Cox sheaf if and only if it is the sheaf* $\mathcal{R} = \widetilde{R}$ *associated to a* G*-graded* $\mathcal{O}(X)$*-algebra* R *such that*:

(i) R *is* G*-factorial* (*in particular,* R *is a* G*-Krull ring*) *and* $R^{+,*} = R_0^*$;
(ii) $R_0 = \mathcal{O}(X)$, $Q^+(R) = (R_0 \setminus 0)^{-1}R$ *and* $\deg_G(Q^+(R)^+) = G$;
(iii) *the essential* G*-valuations of* R *restrict bijectively on* $Q(R_0)$ *to the essential valuations of* R_0.

The canonical choice for a geometric realization of a quasi-coherent G-graded $\mathcal{O}_X$-algebra $\mathcal{F}$ is its *graded relative spectrum* $\operatorname{Spec}_{G,X}(\mathcal{F})$ which is glued from the G-spectra (i.e., sets of G-prime ideals) of $\mathcal{F}(U)$ for all affine open $U \subseteq X$. This object belongs to the category of graded schemes (which contains the category of schemes, i.e., 0-graded schemes, as a full subcategory), wherein structure sheaves are graded and morphisms between affine graded schemes are comorphisms of maps of graded rings, see Section 4. From the perspective of **[13]** the category of graded schemes is situated between the categories of $\mathbb{F}_1$-schemes and classical schemes within their common parent category of sesquiad schemes, see Remark 4.4. Graded algebraic properties of $\mathcal{F}$ naturally correspond to geometric properties of $\operatorname{Spec}_{G,X}(\mathcal{F})$. The $\operatorname{Cl}(X)$-graded relative spectrum of a Cox sheaf $\mathcal{R}$ on X, together with the canonical morphism $q\colon \operatorname{Spec}_{\operatorname{Cl}(X),X}(\mathcal{R}) \to X$, is called its *graded characteristic space*. Since a Cox sheaf is a G-Krull sheaf, its graded characteristic space is a *G-Krull scheme*, which is the generalization of Krull schemes in the category of graded schemes. For a G-Krull scheme $\widehat{X}$, the constant sheaf assigning $\mathcal{O}_{\widehat{X},\widehat{\xi}}$ is G-simple and denoted $\mathcal{K}_{\widehat{X}}$ or $\mathcal{K}$ if no confusion can arise. A *G-prime divisor* on $\widehat{X}$ is an irreducible closed subset $\widehat{Y}$ of $\widehat{X}$; its generic point is denoted $\widehat{\eta}$. Each G-prime divisor $\widehat{Y}$ defines a discrete G-valuation $\nu_{\widehat{Y}} \colon \mathcal{K}^+ \to \mathbb{Z}^{(\widehat{Y})}$ to the skyscraper sheaf at the generic point of $\widehat{Y}$. The sum

$$\operatorname{div}^G := \sum_{\widehat{Y}\ G\text{-prime}} \nu_{\widehat{Y}} \colon \mathcal{K}_{\widehat{X}} \longrightarrow \operatorname{WDiv}^G := \bigoplus_{\widehat{Y}\ G\text{-prime}} \mathbb{Z}^{(\widehat{Y})}$$

defines a morphism of presheaves to the presheaf of *G-Weil divisors* on $\widehat{X}$. Its image and cokernel are the presheaves of *G-principal divisors*, respectively, *G-class groups*.

Our second main result is the following geometric characterization of graded characteristic spaces.

Theorem 0.4. *Let $q \colon \widehat{X} \to X$ be a morphism from a G-graded scheme to a scheme. Then, X is a Krull scheme and q is a graded characteristic space if and only if the following hold*:

(i) *$\widehat{X}$ is a G-graded G-Krull scheme*;
(ii) *q is a good quotient and induces a commutative diagram of presheaves*

$$\begin{tikzcd} \mathcal{K}_X^* \arrow[r, "\operatorname{div}"] \arrow[d, "q^*"', "\cong"] & \operatorname{WDiv} \\ (q_*\mathcal{K}_{\widehat{X}})_0^* \arrow[r, "q_*\operatorname{div}^G"] & q_*\operatorname{WDiv}^G \arrow[u, "\cong", "\widehat{Y}\mapsto q(\widehat{Y})"'] \end{tikzcd}$$

(iii) $\deg_G(\mathcal{K}(\widehat{X})^+) = G$, $\operatorname{Cl}^G(\widehat{X}) = 0$, *and* $\mathcal{O}(\widehat{X})^{+,*} = \mathcal{O}(\widehat{X})_0^*$.

If $\widehat{X} = \operatorname{Spec}_{G,X}(\mathcal{R})$ with a Cox sheaf $\mathcal{R}$, then, with $\operatorname{div}_G := \sum_Y \mu_Y$, the following commutative diagram extends the diagram of (ii):

$$\begin{tikzcd} \mathcal{S}^+ \arrow[r, two heads, "\operatorname{div}_G"] \arrow[d, "q^*"', "\cong"] & \operatorname{WDiv} \\ q_*\mathcal{K}_{\widehat{X}}^+ \arrow[r, two heads, "q_*\operatorname{div}^G"] & q_*\operatorname{WDiv}^G \arrow[u, "\cong", "\widehat{Y}\mapsto q(\widehat{Y})"'] \end{tikzcd}$$

For each prime divisor Y with generic point η, the preimage $q^{-1}(\eta)$ consists of the generic point $\widehat{\eta}$ of a single G-prime divisor $\widehat{Y}$. If $\widehat{x} \in \widehat{X}$ is the unique point contained in all closures of points mapped to $x \in X$, then $\widehat{x} \in \widehat{Y}$ if and only if $x \in Y$. In particular, $\mathcal{O}_{\widehat{X},\widehat{x}} = \mathcal{R}_x$.

This result extends the geometric characterization of relative spectra of Cox sheaves of finite type on normal prevarieties given in **[2]**; indeed, with respect to normal prevarieties, Theorem 0.4 allows a translation into terms of good quotients by quasi-torus actions, see Theorem 6.5. In the following theorem, we also generalize their characterization of Cox rings.

Theorem 0.5. *If X is a Krull scheme with class group G and Cox ring R, then the following hold:*

(i) *R is G-factorial;*

(ii) $R^{+,*} = R_0^*$;

(iii) $\deg_G(R^+)$ *generates G.*

If X has a cover by affine complements of divisors (e.g., X is separated or of affine intersection), then

(iv) *each localization $R_\mathfrak{p}$ at a G-prime divisor has units in every degree.*

Conversely, if G is finitely generated and R (is of finite type over $\mathbb{K}$ and) satisfies (i)–(iv)*, then there exists a Krull scheme (a $\mathbb{K}$-prevariety) X of affine intersection with class group G and Cox ring $\mathcal{R}(X) = R$.*

Here, property (iv) implies property (iii). The additional assumption is needed in order to translate the fact that the stalks $\mathcal{R}_\eta$ have units in every degree into a property of the global ring $\mathcal{R}(X)$. In the case that R is finitely generated over an algebraically closed base field $\mathbb{K}$, property (iv) translates into freeness of the action of $H = \mathrm{Spec}_{\max}(\mathbb{K}[G])$ on a big open subset of $\mathrm{Spec}_{\max}(R)$, see Remark 6.4, which is the property featured in the characterization of finitely generated Cox rings of normal prevarieties of affine intersection given by **[2]**. Our characterization of Cox rings of Krull schemes with cover by affine divisor complements and finitely generated class groups by conditions (i), (ii) and (iv) is irredundant, see Remark 5.2. Together with normality of Cox rings of normal prevarieties over $\mathbb{K}$ **[2,** Thm. I.5.1.1], we obtain:

Corollary 0.6. *Let G be a finitely generated abelian group, $\mathbb{K}$ an algebraically closed field and R a G-graded affine $\mathbb{K}$-algebra satisfying the above properties* (i)–(iv)*. Then, R is normal.*

The paper is organized as follows. Section 1 lays the algebraic foundations for the later parts, introducing G-Krull rings and providing the key preparation for the calculation of the essential G-valuations of a Cox sheaf $\mathcal{R}$, and hence, the G-prime divisors of $\mathrm{Spec}_{G,X}(\mathcal{R})$ in Theorem 1.10. In Section 2, we introduce G-Krull sheaves and discuss the example of divisorial $\mathcal{O}_X$-algebras $\mathcal{O}_X(L)$ associated to subgroups $L \leq \mathrm{WDiv}(X)$. In Section 3, we give background on the definition of Cox sheaves and prove Theorem 0.1. Section 4 offers a first general introduction to graded schemes and good quotients thereof as well as G-Krull schemes and their G-Weil divisors and class groups. The example of graded spectra of monoid algebras (Example 4.3) relates graded schemes to $\mathbb{F}_1$-schemes, i.e., schemes over the field with one element as treated in **[14]**. We also indicate how graded schemes fit into the more general framework of sesquiad schemes from **[13]**, see Remark 4.4. In Section 5, we prove Theorems 0.4 and 0.5 using Theorem 0.1. In Section 6, we point out some aspects of the connection between graded schemes of finite type and diagonalizable actions on prevarieties. In particular, we reformulate Theorem 0.4 in this more familiar setting. This requires the concept of invariant structure sheaves whose stalks naturally encode generic isotropy groups of the action, see Remark 6.4. Furthermore, we provide details on the connection between orbit closures, graded schemes and combinatorics (Remark 6.3), and go on to show that the toric graded scheme corresponding to a toric variety is canonically identified with the defining polyhedral fan (Remark 6.6).

1. *G*-Krull rings. We start by recalling some generalities and notation from graded algebra. All rings are taken to be commutative with unit. All abelian groups used to grade rings are written additively. A *G-graded ring* is a ring with a decomposition

$$R = \bigoplus_{w \in G} R_w$$

into abelian groups such that $R_w R_{w'} \subseteq R_{w+w'}$. R_w is called the *w-homogeneous component* of R, and its non-zero elements are called w-

homogeneous. The sets of *G-homogeneous elements*, with and without zero, and the group of G-homogeneous units are denoted $R^{+,0}$, R^+ and $R^{+,*}$, respectively.

A *morphism* of graded rings is a homomorphism $\phi\colon R \to R'$ of rings together with an accompanying group homomorphism $\psi\colon G \to G'$ such that ϕ restricts to group homomorphisms $R_w \to R'_{\psi(w)}$. A morphism of graded rings is called *degree-preserving* if the accompanying map is the identity. For any fixed G, the category of graded rings has a subcategory of G-graded rings with degree-preserving morphisms, and this subcategory has direct and inverse limits.

A *G-graded module* M over a G-graded ring R is an R-module with a decomposition $M = \bigoplus_{w \in G} M_w$ into abelian groups such that $R_w M_{w'} \subseteq M_{w+w'}$, where the elements of $\bigcup_{w \in G} M_w$ are called homogeneous elements. A *G-graded submodule* of M is a submodule of the form $N = \bigoplus_{w \in G} N \cap M_w$, i.e., a submodule generated by homogeneous elements.

Remark 1.1. If $B \to R$ is a morphism of graded rings, then R is also called a *graded algebra* over B. The graded algebras over B form a category with the obvious morphisms. This category has coproducts: If $\phi_R\colon B \to R$ and $\phi_S\colon B \to S$ are morphisms accompanied by $\psi_L\colon G \to L$ and $\psi_M\colon G \to M$, then $R \otimes_B S$ is naturally graded by $L \times M/\mathrm{im}(\psi_L \times -\psi_M)$ and the canonical maps $R \to R \otimes_B S$ and $S \to R \otimes_B S$ are morphisms of graded algebras over B. This statement is used to define fiber products of graded schemes.

Classical algebraic properties of rings and their elements and ideals have graded analoga which are obtained by restricting the defining axioms to homogeneous elements, respectively, graded ideals. In particular, there are natural concepts of graded divisibility theory, i.e., *G-integrality*, *G-prime* and *G-irreducible* elements, *G-factoriality*, as well as *G-prime* and *G-maximal* ideals, *G-locality*, *G-Noetherianity*, etc. Several authors have studied such properties: **[21]** treats G-prime ideals and invariants of graded modules over G-Noetherian rings. Graded divisibility theory was introduced and interpreted geometrically by **[3, 18]** who showed that G-factoriality is a natural property of Cox rings of normal prevarieties. Graded integral closures and their behavior under coarsening have been studied in **[26]**. The localization of R by a G-prime ideal $\mathfrak{p}$ is denoted $R_{\mathfrak{p}} := (R^+ \setminus \mathfrak{p})^{-1} R$. By localizing a G-integral ring R by R^+ we obtain $Q^+(R)$, the *G-homogeneous frac-*

tion ring in which every homogeneous element is invertible, making it *G-simple*. In general, a G-graded ring R is G-simple if and only if R_0 is a field and $\deg_G(R^+)$ is a group, and in that case, $rR_0 = R_{\deg_G(r)}$ holds for every $r \in R^+$. A G-integral ring R is *G-normal* if each homogeneous fraction that is integral over R (i.e., over $R^{+,0}$) is an element of R.

In the following, major part, of this section, we take a slightly more detailed look at G-Krull rings, the graded equivalent of Krull rings. Proofs of their basic properties are available in **[6]**; they may be obtained from the proofs of the respective properties of Krull rings (found, e.g., in **[16, 22]**) by restricting the arguments to homogeneous elements, respectively, graded ideals. In Construction 1.9, we treat a canonical class of G-Krull rings which form the algebraic analogon of divisorial $\mathcal{O}_X$-algebras $\mathcal{O}_X(L)$ of subgroups of Weil divisors. Theorem 1.10 gives details on their G-divisors and essential G-valuations and thus provides the key for the calculation of the essential G-valuations of Cox sheaves in Section 3 and the G-Weil divisors of their graded characteristic spaces in Section 5.

Definition 1.2. Let S be a G-simple ring.

(i) A *discrete G-valuation* on S is a group epimorphism $\nu\colon S^+ \to \mathbb{Z}$ with $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ for all $w \in G$, $a, b \in S_w \backslash 0$ with $a+b \neq 0$. Its *discrete G-valuation ring* is the subring $R_\nu \subseteq S$ generated by the preimage of $\mathbb{Z}_{\geq 0}$ under ν.

(ii) A *G-Krull ring* is an intersection R of discrete G-valuation rings $R_{\nu_j} \subseteq S$, $j \in J$ such that for each $a \in R^+$ only finitely many $\nu_j(a)$ are non-zero.

Let R be a G-integral ring. For G-graded R-submodules $\mathfrak{a}, \mathfrak{b}$ of $Q^+(R)$, product $\mathfrak{a}\mathfrak{b}$ and quotient

$$[\mathfrak{a} : \mathfrak{b}] = \{f \in Q^+(R); f\mathfrak{b} \subseteq \mathfrak{a}\}$$

are again G-graded. A *G-fractional ideal* is a G-graded proper R-submodule $\mathfrak{a} \leq_R Q^+(R)$ with $[R : \mathfrak{a}] \neq 0$.

Construction 1.3. *Let R be a G-integral ring. A G-fractional ideal $\mathfrak{a}$ is called a* G-divisor *of R if $\mathfrak{a} = [R : [R : \mathfrak{a}]]$. The set* $\mathrm{Div}^G(R)$ *of G-divisors of R equipped with the operation sending $\mathfrak{a}$ and $\mathfrak{b}$ to*

$$\mathfrak{a}+\mathfrak{b} := [R : [R : \mathfrak{a}\mathfrak{b}]]$$

and the partial order defined by $\mathfrak{a} \leq \mathfrak{b} :\Leftrightarrow \mathfrak{a} \supseteq \mathfrak{b}$ *is a partially ordered semi group with neutral element* R *in which each two elements have infimum and supremum. There is a canonical homomorphism*

$$\operatorname{div}^G \colon Q^+(R)^+ \longrightarrow \operatorname{Div}^G(R), \qquad f \longmapsto Rf$$

with kernel $R^{+,*}$ *whose image* $P\operatorname{Div}^G(R)$ *is called the group of* G-principal divisors. *The cokernel* $\operatorname{Cl}^G(R)$ *is called the* G-class semi-group *of* R.

The semi-group of G-divisors characterizes R as follows:

Theorem 1.4. *Let* R *a* G*-integral ring.*

(i) R *is a* G*-Krull ring if and only if* $\operatorname{Div}^G(R)$ *is a group and every non-empty set of positive elements in* $\operatorname{Div}^G(R)$ *has a minimal element.*

(ii) R *is* G*-factorial if and only if* R *is a* G*-Krull ring with* $\operatorname{Cl}^G(R) = 0$.

Remark 1.5. Let $\{\nu_j\}_{j\in J}$ be a defining family of the G-Krull ring R. For a G-fractional ideal $\mathfrak{a}$ and $j \in J$, we set

$$\nu(\mathfrak{a}_j) := \max\{-\nu_j(f);\, f \in [R : \mathfrak{a}] \cap Q^+(R)^+\} \in \mathbb{Z}.$$

This notion is well defined and satisfies $\nu_j(\mathfrak{a}) = \nu([R : [R : \mathfrak{a}]])$. Furthermore, there is a monomorphism of ordered groups

$$\operatorname{Div}^G(R) \longrightarrow \bigoplus_{j\in J} \mathbb{Z}, \qquad \mathfrak{a} \longmapsto \{\nu_j(\mathfrak{a})\}_{j\in J}.$$

Proposition 1.6. *Let* R *be a* G*-Krull ring. Then the following hold*:

(i) *the minimal positive* G*-divisors are those that are* G*-prime as ideals in* R*, and these are the minimal non-zero* G*-prime ideals of* R*; they are called the* G-prime divisors *of* R *and form a* $\mathbb{Z}$*-basis of* $\operatorname{Div}^G(R)$;

(ii) *for each* G*-prime divisor* $\mathfrak{p}$ *the map* $\nu_{\mathfrak{p}}$ *assigning to* $a \in Q^+(R)$ *the coefficient with which* $\mathfrak{p}$ *occurs in* $\operatorname{div}^G(a)$ *is a* G*-valuation on* $Q^+(R)$*. Its* G*-valuation ring is* $R_{\mathfrak{p}}$*, and we have* $\mathfrak{p} \cap R^+ = \nu_{\mathfrak{p}}^{-1}(\mathbb{Z}_{>0}) \cap R^+$*. The coefficient of a* G*-divisor* $\mathfrak{a}$ *at* $\mathfrak{p}$ *is*

$$\nu_{\mathfrak{p}}(\mathfrak{a}) = \min\{\nu_{\mathfrak{p}}(a);\ a \in \mathfrak{a} \cap Q^+(R)^+\}.$$

The family $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}}$ is minimal among all families defining R in $Q^+(R)$, it is called the family of essential G-valuations *of R.*

Remark 1.7. Assertion (i) is, in particular, an existence statement: A G-Krull ring has G-prime divisors if and only if it is not G-simple. A general G-integral ring need not have G-prime divisors (i.e., minimal non-zero G-prime ideals), even if it is not G-simple.

Proposition 1.8. *A G-Noetherian G-integral ring is a G-Krull ring if and only if it is G-normal.*

Next, we treat the algebraic construction that lies beneath divisorial $\mathcal{O}_X$-algebras of subgroups L of Weil divisors.

Construction 1.9. *Let A be a Krull ring with essential valuations $\{\nu_{\mathfrak{p}}\}_{\mathfrak{p}}$, and let $\phi\colon G \to Div(A)$ be a homomorphism of abelian groups. The group algebra $S := Q(A)[G]$ is G-simple and*

$$\begin{aligned} \mu_{\mathfrak{p}}\colon S^+ &\longrightarrow \mathbb{Z} \\ a\chi^w &\longmapsto \nu_{\mathfrak{p}}(a) + \nu_{\mathfrak{p}}(\phi(w)) = \nu_{\mathfrak{p}}(\operatorname{div}(a) + \phi(w)) \end{aligned}$$

defines a G-valuation on S for every prime divisor $\mathfrak{p}$. The ring $R = \bigcap_{\mathfrak{p}} S_{\mu_{\mathfrak{p}}}$ is a G-Krull ring with homogeneous components

$$\begin{aligned} R_w &= \{a \in Q(A);\ a = 0 \text{ or } \operatorname{div}(a) + \phi(w) \geq 0\} \cdot \chi^w \\ &= \{a \in Q(A);\ Aa \subseteq \phi(-w)\} \cdot \chi^w = \phi(-w) \cdot \chi^w \end{aligned}$$

for $w \in G$.

Theorem 1.10. *In the above notation, the ring R has the following properties*:

(i) *we have $R_0 = A$ and the universal property of localization induces isomorphisms $(R_0 \setminus 0)^{-1}R \cong Q^+(R) \cong Q(A)[G]$;*

(ii) *$\{\mu_{\mathfrak{p}}\}_{\mathfrak{p}}$ are the essential G-valuations of R, and there are mutually inverse isomorphisms*

$$\begin{aligned} \operatorname{Div}^G(R) &\longrightarrow \operatorname{Div}(A) \\ \alpha\colon \mathfrak{b} &\longmapsto \sum_{\mathfrak{p}} \mu_{\mathfrak{p}}(\mathfrak{b})\mathfrak{p} \end{aligned}$$

$$[R : [R : R\mathfrak{a}]] \longleftarrow\!\shortmid \mathfrak{a} \colon \beta$$

which restrict to mutually inverse bijections

$$\begin{aligned} \{G\text{-}\textit{prime divisors of } R\} &\longrightarrow \{\textit{prime divisors of } A\} \\ \mathfrak{q} &\longmapsto \mathfrak{q} \cap A \\ \langle \mu_{\mathfrak{p}}^{-1}(\mathbb{Z}_{>0}) \cap R \rangle &\longleftarrow\!\shortmid \mathfrak{p} \end{aligned}$$

and induce an isomorphism $\mathrm{Cl}_G(R) \cong \mathrm{Cl}(A)/\overline{\mathrm{im}(\phi)}$;

(iii) *the localization* $R_{\mathfrak{q}}$ *by a* G*-prime divisor* $\mathfrak{q}$ *of* R *has units in every degree with* $(R_{\mathfrak{q}})_0 = A_{\mathfrak{p}}$ *where* $\mathfrak{p} = \mathfrak{q} \cap A$;

(iv) R *has homogeneous components of every* G*-degree, i.e.,* $\deg_G(R^+) = G$.

Lemma 1.11. *In the above situation, let* $\mathfrak{a} \subset Q(A)$ *be a fractional ideal and* $\mathfrak{p}$ *a prime divisor of* A*. Then,*

$$\mu_{\mathfrak{p}}(R\mathfrak{a}) = \nu_{\mathfrak{p}}(\mathfrak{a}),$$

in particular, $\mu_{\mathfrak{p}}(R\mathfrak{p}') = \delta_{\mathfrak{p},\mathfrak{p}'}$ *holds for any prime divisor* $\mathfrak{p}'$ *of* A.

Proof. We calculate

$$\begin{aligned} \mu_{\mathfrak{p}}(R\mathfrak{a}) &\geq \max\{\mu_{\mathfrak{p}}(a);\, a \in Q(A)^*,\, \mathfrak{a} \subseteq Aa\} \\ &= \nu_{\mathfrak{p}}(\mathfrak{a}) = \min\{\mu_{\mathfrak{p}}(a);\, a \in \mathfrak{a}\} \\ &\geq \max\{\mu_{\mathfrak{p}}(r);\, r \in Q^+(R)^+,\, R\mathfrak{a} \subseteq Rr\} = \mu_{\mathfrak{p}}(R\mathfrak{a}). \qquad \square \end{aligned}$$

Proof of Theorem 1.10. Assertion (iv) follows from the fact that G-divisors contain non-zero elements. For (i), we observe that the canonical morphism $(R_0 \setminus 0)^{-1}R \to Q^+(R) \to Q(A)[G]$ is surjective. Indeed, for every $w \in G$, there exists an element $0 \neq a\chi^w \in R_w$ by (iv), and thus, $a\chi^w/a\chi^0$ is mapped to χ^w.

For (ii), we first observe that, by Remark 1.5, α is a monomorphism of partially ordered groups. Since the above lemma and Remark 1.5 give $\alpha(\beta(\mathfrak{a})) = \mathfrak{a}$, α is also surjective and β is its inverse map. In particular, they induce bijections between the sets of minimal positive elements of $\mathrm{Div}^G(R)$ and $\mathrm{Div}(A)$. This means that the divisorial ideals $\beta(\mathfrak{p})$ are the G-prime divisors of R. For any prime divisor $\mathfrak{p}$ of A and any G-prime divisor $\mathfrak{q}' = \beta(\mathfrak{p}')$, we have $\mu_{\mathfrak{p}}(\mathfrak{q}') = \delta_{\mathfrak{p},\mathfrak{p}'}$, and therefore, $\mu_{\mathfrak{p}} = \nu_{\beta(\mathfrak{p})}$ is the essential G-valuation corresponding to $\mathfrak{q} = \beta(\mathfrak{p})$, and we have $\mathfrak{q} = \langle \mu_{\mathfrak{p}}^{-1}(\mathbb{Z}_{>0}) \cap R \rangle$. Since $\nu_{\mathfrak{q}}$ restricts to $\nu_{\mathfrak{p}}$, this implies $\mathfrak{q} \cap A = \mathfrak{p}$.

For (iii), let $\mathfrak{q}$ be a G-prime divisor of R, and let $\mathfrak{p} = \mathfrak{q} \cap A$ be the corresponding prime divisor of A. First, we show $(R_{\mathfrak{q}})_0 = A_{\mathfrak{p}}$. Let $f/g \in (R_{\mathfrak{p}})_0$. Then, there are $a, b \in A$ with $f/g = a/b$, and $\nu_{\mathfrak{p}}(a/b) = \nu_{\mathfrak{q}}(f/g) \geq 0$, i.e., $a/b \in A_{\mathfrak{p}}$. Each generator t_0 of the maximal ideal of $A_{\mathfrak{p}}$ satisfies $\nu_{\mathfrak{q}}(t_0) = \nu_{\mathfrak{p}}(t_0) = 1$ and is thus a generator of the G-maximal ideal of $R_{\mathfrak{q}}$. This implies the assertion. □

Remark 1.12. From **[15]**, we observe that, if G is free, then for a G-prime divisor $\mathfrak{q}$ of R and $\mathfrak{p} = A \cap \mathfrak{q}$, each choice of a generator for the maximal ideal of $A_{\mathfrak{p}}$ gives an isomorphism $R_{\mathfrak{q}} \cong A_{\mathfrak{p}}[G]$. In particular, $R_{\mathfrak{q}}$ is a Krull ring, and it may be concluded that R is a Krull ring.

The following well-behaved class of graded morphisms will be used in Section 3 to describe the relation between the sections of $\mathcal{O}_X(\mathrm{WDiv}(X))$ and Cox sheaves. Their properties, some of which are listed in Proposition 1.13 below, ensure that Cox sheaves inherit all graded properties from $\mathcal{O}_X(\mathrm{WDiv}(X))$.

A *component-wise bijective epimorphism* (CBE) is an epimorphism of graded rings $\phi\colon R' \to R$ accompanied by an epimorphism $\psi\colon G' \to G$ such that each restriction $R'_w \to R_{\psi(w)}$ is bijective. If ψ is fixed, then, for each given R, one obtains R' and ϕ constructively and uniquely by setting $R'_{w'} := R_{\psi(w')}$ for $w' \in G'$. The functor from G-graded rings to G'-graded rings thus defined is right adjoint to the coarsening functor associated to ψ, see **[25]**. If R' and ψ are fixed, then each homomorphism $\chi\colon \ker(\psi) \to R'^{+,*}$ with $\chi(w') \in R'_{w'}$ defines a CBE

$$\phi\colon R' \longrightarrow R'/\langle 1 - \chi(w'); w' \in \ker(\psi) \rangle.$$

Proposition 1.13. *Let* $\phi\colon R' \to R$ *be a* CBE. *Then, the following hold*:

(i) $R^+ \cong R'^+/\phi^{-1}(1)$ *and there is a bijection of sets of graded ideals*

$$\begin{aligned} \{\mathfrak{a}' \trianglelefteq R'\} &\longrightarrow \{\mathfrak{a} \trianglelefteq R\} \\ \mathfrak{a}' &\longmapsto \phi(\mathfrak{a}') \\ \langle \phi^{-1}(\mathfrak{a}) \cap R'^+ \rangle &\longleftarrow\!\shortmid \mathfrak{a} \end{aligned}$$

respecting inclusions, products, quotients, sums and intersections.

(ii) *Likewise, there is a bijection between the graded R'-, respectively, R-modules of $Q^+(R')$ and $Q^+(R)$.*

(iii) *If $M \subseteq R^+$ is a submonoid and $M' := \phi^{-1}(M)$, then $M'^{-1}R' \to M^{-1}R$ is again a* CBE, *in particular, for every $f' \in R'^+$, the map $R'_{f'} \to R_{\phi(f')}$ is a* CBE.

(iv) *R' is G'-integral/-simple/-factorial, respectively, has units in every G'-degree if and only if R is G-integral/-simple/-factorial, respectively, has units in every G-degree.*

(v) *Let $\phi_S\colon S' \to S$ be a* CBE *with $R' \subseteq S'$ and $R \subseteq S$ extending the* CBE *$\phi\colon R' \to R$. Suppose that S' is G'-simple. Then, the following hold*:

(a) *$S' = Q^+(R')$ if and only if $S = Q^+(R)$.*

(b) *Each G'-valuation ν on S' with* $\ker(\phi) \subseteq \ker(\nu)$ *induces a G-valuation $\overline{\nu}$ on S and vice versa.*

(c) *R' is a G'-Krull ring defined by $\{\nu_j\}_{j\in J}$ if and only if R is a Krull ring defined by $\{\overline{\nu}_j\}_{j\in J}$. Here, $\{\nu_j\}_{j\in J}$ are the essential G'-valuations if and only if $\{\overline{\nu}_j\}_{j\in J}$ are the essential G-valuations.*

Thus, R' and R share all of the graded properties defined in terms of graded ideals and all properties of $R^+/R^{+,*}$.

2. Divisorial $\mathcal{O}_X$-algebras. Let G be an abelian group. We begin with the prerequisites on graded sheaves needed for the definition of G-Krull sheaves–the sheaf-theoretic analogon of G-Krull rings. Recall that a G-graded (pre-)sheaf of rings $\mathcal{F}$ on a topological space X is a (pre-)sheaf of G-graded rings with degree-preserving restriction maps. $\mathcal{F}$ is also called a graded (pre-)sheaf with grading group $G = \mathrm{gr}(\mathcal{F})$. As a presheaf, $\mathcal{F}$ then equals $\bigoplus_{w\in G} \mathcal{F}_w$, where $\mathcal{F}_w \subseteq \mathcal{F}$ is the (pre-)sheaf of abelian groups assigning $\mathcal{F}(U)_w$ to U. The monoid of G-homogeneous elements of $\mathcal{F}$ is the sheaf of monoids

$$\mathcal{F}^{+,0} := \bigcup_{w\in G} \mathcal{F}_w.$$

If all $\mathcal{F}(U)$ are G-integral and all restrictions are injective, then $\mathcal{F}^+$ denotes the sheaf of monoids given by $\mathcal{F}^{+,0}(U) \setminus \{0\}$. A morphism $\phi\colon \mathcal{G} \to \mathcal{F}$ of graded (pre-)sheaves comes with a group homomorphism $\psi\colon\ \mathrm{gr}(\mathcal{G}) \to \mathrm{gr}(\mathcal{F})$ such that each of the morphisms of graded rings ϕ_U is accompanied by ψ. $\mathcal{F}$ is also called a graded $\mathcal{G}$-algebra.

Definition 2.1. Let X be a topological space.

(i) A *discrete value sheaf* is a sheaf of abelian groups $\mathcal{Z}$ with values in $\{0, \mathbb{Z}\}$ such that $\mathcal{Z}_{\geq 0}(U) := \mathcal{Z}(U)_{\geq 0}$ defines a subsheaf of $\mathcal{Z}$.

(ii) Let G be an abelian group. Let $\mathcal{S}$ be a sheaf of G-simple rings on X. A *discrete G-valuation* on $\mathcal{S}$ is a morphism $\nu \colon \mathcal{S}^+ \to \mathcal{Z}$ to a discrete value sheaf such that each ν_U is surjective and either a discrete G-valuation or zero. The associated *G-valuation sheaf* is the graded subsheaf $\mathcal{S}_\nu \subseteq \mathcal{S}$ of rings generated by $\nu^{-1}(\mathcal{Z}_{\geq 0})$.

(iii) A *G-Krull sheaf* in $\mathcal{S}$ is an intersection

$$\mathcal{R} = \bigcap_{j \in J} \mathcal{S}_{\nu_j}$$

of G-valuation sheaves such that, for every U and every $f \in \mathcal{R}^+(U)$, only finitely many $\nu_{j,U}(f)$ are non-zero.

Definition 2.2. Let X be a scheme (or a graded scheme, see Section 4), and let $\mathcal{R}$ be a G-Krull sheaf on X given in a G-simple sheaf $\mathcal{S}$ by a family of discrete G-valuations $\{\nu_j\}_{j \in J}$. Suppose that $\mathcal{S}$ is constant and, for each affine open $U \subseteq X$, we have $Q^+(\mathcal{R}(U)) = \mathcal{S}(U)$. Then, the family $\{\nu_j\}_{j \in J}$ is called the family of *essential* G-valuations if, for each affine $U \subseteq X$, the family $\{\nu_{j,U};\, j \in J,\, \nu_{j,U} \not\equiv 0\}$ is the family of essential G-valuations of the G-Krull ring $\mathcal{R}(U)$.

For $G = 0$, we usually omit the prefix G. A *Krull scheme* is an integral scheme which has a finite cover by spectra of Krull rings, cf., **[23]**. A prime divisor is a closed irreducible subset $Y \subseteq X$. Its generic point is denoted $\eta \in X$. Sums and intersections with the subscript Y are taken over all prime divisors Y of X unless specified otherwise.

Example 2.3. Let X be a Krull scheme and $G = 0$. Each prime divisor defines a valuation $\nu_Y : \mathcal{K}^* \to \mathbb{Z}^{(Y)}$ to the skyscraper sheaf $\mathbb{Z}^{(Y)}$ at the generic point of η. The sections of its valuation sheaf $\mathcal{K}_{\nu_Y}$ on U are $\mathcal{O}_{X,\eta}$ if $\eta \in U$ and $\mathcal{K}$ otherwise. This turns the structure sheaf

$$\mathcal{O}_X = \bigcap_Y \mathcal{K}_{\nu_Y}$$

into a Krull sheaf with essential valuations ν_Y.

Remark 2.4. A quasi-compact scheme X is a Krull scheme if and only if $\mathcal{O}_X$ is a Krull sheaf.

For the remainder of this section, X is a Krull scheme. Recall that the presheaf of Weil divisors is $\mathrm{WDiv} := \bigoplus_Y \mathbb{Z}^{(Y)}$, and there is a morphism

$$\mathrm{div} := \sum_Y \nu_Y : \mathcal{K}^* \longrightarrow \mathrm{WDiv}\,.$$

Its image PDiv is the presheaf of principal divisors of X, and its cokernel is the presheaf Cl of divisor class groups. For each prime divisor Y there is a natural projection $pr_Y\colon\ \mathrm{WDiv} \to \mathbb{Z}^{(Y)}$. The support $|D|$ of a Weil divisor $D \in \mathrm{WDiv}(U)$ is the intersection of U with the union over all prime divisors occurring with non-zero coefficient in D.

Construction 2.5. *For a subgroup $L \leq \mathrm{WDiv}(X)$, the constant sheaf of L-graded group algebras*

$$\mathcal{S} := \mathcal{K}[L] = \bigoplus_{D\in L} \mathcal{K}\cdot\chi^D$$

is a sheaf of L-simple rings. Each prime divisor $\eta \in X$ defines an L-valuation

$$\mu_Y : \mathcal{S}^+ \longrightarrow \mathbb{Z}^{(Y)}$$

$$\mathcal{S}^+(U) \ni f\chi^D \longmapsto \mu_{Y,U}(f\chi^D) := \nu_{Y,U}(f) + pr_{Y,U}(D_{|U}).$$

Then, $\mathcal{O}_X(L) := \bigcap_Y \mathcal{S}_{\mu_Y}$ is an L-Krull sheaf on X, called the divisorial $\mathcal{O}_X$-algebra associated to L. Its homogeneous parts have sections $\mathcal{O}_X(L)_D(U) = \mathcal{O}_X(D)(U)\cdot\chi^D$, where $\mathcal{O}_X(D)$ is the $\mathcal{O}_X$-submodule of $\mathcal{K}$ associated to D with sections

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}(U); f = 0 \quad \textit{or} \quad \mathrm{div}_U(f) + D_{|U} \geq 0\},$$

in particular, $\mathcal{O}_X(L)_0 = \mathcal{O}_X$. The sum over all μ_Y defines a morphism

$$\mathrm{div}_L := \sum_Y \mu_Y : \mathcal{S}^+ \longrightarrow \mathrm{WDiv}$$
$$\mathcal{S}^+(U) \ni f\chi^D \longmapsto \mathrm{div}_{L,U}(f\chi^D) = \mathrm{div}_U(f) + D_{|U}$$

with kernel $\mathcal{O}_X(L)^{+,*}$. In particular, $\mathcal{O}_X(L)(X)^{+,*}$ is the set of all elements $f\chi^D$ with $\mathrm{div}_X(f) = -D$.

Proposition 2.6. *In the above notation, the divisorial algebra $\mathcal{R} := \mathcal{O}_X(L)$ has the following properties:*

(i) *there is a canonical isomorphism* $\mathcal{S}(X) \cong \mathcal{R}_\xi$, *and, for affine* U, *we have* $\mathcal{S}(U) \cong Q^+(\mathcal{R}(U))$ *and* $\deg_L(\mathcal{R}(U)^+) = L$;

(ii) $\{\mu_Y\}_Y$ *are the essential* L*-valuations of* $\mathcal{R}$;

(iii) *the sections of* $\mathcal{S}_{\mu_Y}$ *on* U *equal* $\mathcal{R}_\eta$ *if* $\eta \in U$ *and* $\mathcal{S}$ *otherwise*;

(iv) *the stalk at* $x \in X$ *is the* L*-local* L*-Krull ring*

$$\mathcal{R}_x = \bigcap_{x \in Y} \mathcal{S}(X)_{\mu_{Y,X}} \subseteq \mathcal{S}(X)$$

whose L*-maximal ideal* $\mathfrak{a}_x$ *has homogeneous elements*

$$\begin{aligned} &\mathfrak{a}_x \cap \mathcal{S}(X)^+ \\ &\quad = \{g \in \mathcal{R}_x^+;\ \textit{there is a } U \ni x \textit{ with } g \in \mathcal{R}(U),\ x \in |\mathrm{div}_{L,U}(g)|\}. \end{aligned}$$

Its homogeneous units are

$$\begin{aligned} \mathcal{R}_x^{+,*} &= \{g \in \mathcal{S}(X)^+;\ \textit{there is a } U \ni x \textit{ with } \mathrm{div}_{L,U}(g) = 0\} \\ &= \bigcap_{x \in Y} \ker(\mu_{Y,X}), \end{aligned}$$

and $\deg_L(\mathcal{R}_x^{+,*})$ *is the subgroup of Weil divisors in* L *that are principal near* x. *The stalk at the generic point* η *of a prime divisor* Y *has units in every degree, and* $\mathfrak{a}_\eta$ *has a generator in* $(\mathcal{R}_\eta)_0 = \mathcal{O}_{X,\eta}$.

(v) *The image of* $\mathrm{div}_{L,U}$ (*respectively,* $\mathrm{div}_{L,U|\mathcal{R}(U)^+}$) *consists of* (*the non-negative elements in*) *the union over all* $\mathrm{Cl}(U)$*-classes of divisors in* $L_{|U}$, *in particular, if* $L_{|U}$ *maps onto* $\mathrm{Cl}(U)$, *then* $\mathcal{R}(U)$ *is* L*-factorial.*

Remark 2.7 ([**2,** Remark I.3.1.6])**.** For an open set $U \subset X$, each $g \in \mathcal{O}_X(L)(U)^+$ defines an open subset $U_g := U \setminus |\mathrm{div}_{L,U}(g)|$ and a canonical isomorphism

$$\mathcal{O}_X(L)(U)_g \cong \mathcal{O}_X(L)(U_g).$$

In particular, $\mathcal{O}_X(L)$ is quasi-coherent.

Remark 2.8. As a subsheaf of a constant sheaf, $\mathcal{O}_X(L)$ has injective restriction maps, and therefore, all the canonical maps $\mathcal{O}_X(L)(U) \to \mathcal{O}_X(L)_x$ for $x \in U$ and $\mathcal{O}_X(L)_x \to \mathcal{O}_X(L)_{x'}$ for $x \in \overline{x'}$ are injective as well.

Proof of Proposition 2.6. For (i), note that the inclusions $\imath_V\colon \mathcal{R}(V) \subseteq \mathcal{S}(X)$ induce an injection

$$\imath_\xi\colon \mathcal{R}_\xi \longrightarrow \mathcal{S}(X).$$

For affine open $U \subseteq X$, we consider the canonical monomorphism

$$\alpha_U\colon Q^+(\mathcal{R}(U)) \longrightarrow \mathcal{S}(X), \qquad g/h \longmapsto \imath_U(g)\imath_U(h)^{-1}.$$

For surjectivity of $\imath_\xi$ and α_U, let $f\chi^D \in S(X)_D$, and set $W := X \setminus |\mathrm{div}_X(f) + D|$. Then, we have $f_{|W}\chi^D \in \mathcal{R}(W)_D$ and $\imath_\xi((f_{|W}\chi^D)_\xi) = f\chi^D$. Furthermore, there exists an $h \in \mathcal{O}_X(U) = \mathcal{R}(U)_0$ with $U_h \subseteq W \cap U$, and, by Remark 2.7, there are $m > 0$ and $g\chi^D \in \mathcal{R}(U)_D$ such that $g\chi^D(h\chi^0)^{-m} = f_{|U_h}\chi^D$. Hence, $\alpha_U(g\chi^D/(h\chi^0)^m) = f\chi^D$. For the supplement and assertion (ii), we invoke Theorem 1.10 with $A := \mathcal{O}(U)$, $\phi\colon L \to L_{|U}$ and $R = \mathcal{R}(U)$.

For (iii), note that, if $\eta \notin U$, then $\mu_{Y,U} = 0$, and therefore, $\mathcal{S}(U)_{\mu_{Y,U}} = \mathcal{S}(U)$. If $\eta \in U$, then (iv) gives $\mathcal{R}_\eta = \mathcal{S}(X)_{\mu_{Y,X}} = \mathcal{S}(U)_{\mu_{Y,U}}$.

For (iv), first note that, in $\mathcal{S}(X)$, we have

$$\mathcal{R}_x^+ = \{f\chi^D \in \mathcal{S}(X)^+;\ \text{there is a } U \ni x \text{ with } f \in \mathcal{O}_X(D)(U)\}.$$

If $f \in \mathcal{O}_X(D)(U)$ and $x \in U$, then, for every prime divisor Y containing x in its closure, we have $\eta \in U$, and thus, $\mu_{Y,X}(f\chi^D) = \mu_{Y,U}(f\chi^D) \geq 0$. Conversely, if $f\chi^D \in \mathcal{S}(X)^+$ satisfies $\mu_{Y,X}(f\chi^D) \geq 0$ for all prime divisors Y with $x \in Y$, then, for the complement W of all prime divisors Y' with $\mu_{Y',X}(f\chi^D) < 0$, we have $f \in \mathcal{O}_X(D)(W)$ and $x \in W$. This establishes that $\mathcal{R}_x$ is the L-Krull ring in $\mathcal{S}(X)$ defined by all $\mu_{Y,X}$ with $x \in Y$. Its homogeneous units are therefore obtained as the intersection of the kernels of the defining L-valuations.

For the second representation, let $g \in \mathcal{R}_x^{+,*} \subseteq \mathcal{S}(X)^+$,and let W be the complement of all prime divisors Y' with $\mu_{Y',X}(g) \neq 0$. None of these Y' contain x in their closure; therefore, $x \in W$. The equation $\mathrm{div}_{L,W}(g) = 0$ holds by definition of W. Conversely, if $g \in \mathcal{S}(X)^+$ satisfies $\mathrm{div}_{L,U}(g) = 0$ for some U containing x, then g is invertible in $\mathcal{R}(U)$, and hence, in $\mathcal{R}_x$. In particular, $\deg_L(\mathcal{R}_x^{+,*})$ is contained in the subgroup of Weil divisors in L that are principal near x. For the converse inclusion, let $D_{|U} = \mathrm{div}_U(f)$. Then, $f^{-1}\chi^D \in \mathcal{R}(U)_D$ is a unit, and thus, $(f^{-1}\chi^D)_x$ is a unit. Let $\mathfrak{a}_x$ be the ideal generated by the set

$$\{g \in \mathcal{R}_x^+;\ \text{there is a } U \ni x \text{ with } g \in \mathcal{R}(U),\ x \in |\mathrm{div}_{L,U}(g)|\}$$

Due to Lemma 5.1, this set is closed under addition of elements of the same degree and therefore coincides with $\mathfrak{a}_x \cap \mathcal{R}_x^+$. Its complement in $\mathcal{R}_x^+$ is $\mathcal{R}_x^{+,*}$, and thus, $\mathfrak{a}_x$ is the only L-maximal ideal of $\mathcal{R}_x$.

For the supplement on the stalks at generic points of prime divisors, first note that $(\mathcal{R}_\eta)_0 = \mathcal{O}_{X,\eta}$ since taking stalks commutes with direct sums. Now, let $U \subseteq X$ be affine with $\eta \in U$ and $D \in L$. By the Approximation theorem for Krull rings, there exists an $f \in \mathcal{O}_X(D)(U)$ with $\mathrm{pr}_Y(\mathrm{div}_U(f)) = 0$ and $\mathrm{pr}_{Y'}(\mathrm{div}_U(f)) = \mathrm{pr}_{Y'}(D_{|U})$ for all $Y' \in |D_{|U}|$ and $\mathrm{pr}_{Y''}(\mathrm{div}_U(f)) \geq 0$ for all other prime divisors. Then, $(f\chi^D)_Y$ is a unit of degree D in R_Y.

In (v), the statements on the image of $\mathrm{div}_{L,U}$ follow from the definition of div_L. If $\mathrm{div}_{L,U}$ is surjective, then $\mathcal{R}(U)^+/\mathcal{R}(U)^{+,*} \cong \mathrm{WDiv}_{\geq 0}(U)$ is a factorial monoid, and thus, also $\mathcal{R}(U)^+$ is a factorial monoid, meaning that $\mathcal{R}(U)$ is L-factorial. $\square$

Remark 2.9. By arguments from [**15**], each $\mathcal{R}(U)$ is a Krull ring. Thus, if $L_{|U}$ maps onto $\mathrm{Cl}(U)$, then $\mathcal{R}(U)$ is a factorial ring by [**1**]. However, the sections of Cox sheaves will in general not be factorial. Integrality and normality for the sections of Cox sheaves are proven in the case of normal (pre-)varieties, see [**2,** Section I.5.1] or [**7**], but neither proof seems to be applicable in the more general setting of Krull schemes.

3. Characterization of Cox sheaves. Intuitively, a Cox sheaf should be a $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}$ whose $[D]$-homogeneous parts are of type $\mathcal{O}_X(D)$, i.e., there should exist isomorphisms of $\mathcal{O}_X$-modules $\pi_D \colon \mathcal{O}_X(D) \to \mathcal{R}_{[D]}$ for all Weil divisors $D \in \mathrm{WDiv}(X)$. This requirement fixes the $\mathcal{O}_X$-module structure of Cox sheaves. There is, however, no canonical way to equip such an $\mathcal{O}_X$-module with an $\mathcal{O}_X$-algebra structure. But, we can and do require that the multiplication in $\mathcal{R}$ be *natural* in the sense that, up to the isomorphisms π_D, it is given by the multiplication in $\mathcal{K}$, meaning that to multiply homogeneous sections of degree $[D]$ and $[D']$ in $\mathcal{R}$ is the same as to apply π_D^{-1}, respectively, $\pi_{D'}^{-1}$, multiply the resulting sections of $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ in $\mathcal{K}$ and then apply $\pi_{D+D'}$. This translates into the condition that the morphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_X(\mathrm{WDiv}(X)) = \bigoplus_{D \in \mathrm{WDiv}(X)} \mathcal{O}_X(D) \cdot \chi^D \xrightarrow{\ \pi\ } \mathcal{R} = \bigoplus_{[D] \in \mathrm{Cl}(X)} \mathcal{R}_{[D]},$$

defined by the sum of the π_D is a morphism of graded $\mathcal{O}_X$-algebras. Summing up, a Cox sheaf is defined as a $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}$ possessing a graded morphism from $\mathcal{O}_X(\mathrm{WDiv}(X))$ to $\mathcal{R}$ which restricts to isomorphisms of the homogeneous parts. Such a type of morphism between two graded sheaves has useful properties and thus justifies the following definition.

Definition 3.1. A morphism $\pi\colon \mathcal{F} \to \mathcal{G}$ of graded presheaves of rings with the accompanying map $\psi\colon L \to G$ of abelian groups is called a *component-wise bijective epimorphism* (CBE) if ψ is an epimorphism and the restriction $\pi_{|\mathcal{F}_w}$ is bijective for every $w \in L$. Equivalently, every pair (π_U, ψ) is a CBE of rings.

Remark 3.2. Let $\psi\colon L \to G$ be an epimorphism of abelian groups. Let $\pi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of graded presheaves accompanied by ψ.

(i) π is a CBE if and only if, for each open $U \subseteq X$ and each $v \in \ker(\psi)$, there exists precisely one preimage $\chi_U(v)$ of $1_{\mathcal{G}(U)}$ in $\mathcal{F}(U)_v$ and π_U induces isomorphisms

$$\mathcal{F}(U)^{+,0}/\mathrm{im}(\chi_U) \cong \mathcal{G}(U)^{+,0},$$
$$\mathcal{G}(U) \cong \mathcal{F}(U)/\langle 1_{\mathcal{F}(U)} - \chi_U(v);\ v \in \ker(\psi)\rangle.$$

Note that the map $\chi_U\colon \ker(\psi) \to \mathcal{F}(U)^{+,*}$ is automatically a group homomorphism and defines a homomorphism $\chi\colon \ker(\psi) \to \mathcal{F}^{+,*}$ of presheaves, called the *kernel character* of π, where $\ker(\psi)$ is considered as a constant presheaf.

(ii) If π is a CBE, then $\mathcal{F}$ is a sheaf if and only if $\mathcal{G}$ and $\ker(\pi)$ are sheaves.

(iii) A morphism $\pi\colon \mathcal{F} \to \mathcal{G}$ of graded sheaves of rings is a CBE of sheaves if and only if every $\pi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ is a CBE of graded rings.

Definition 3.3. Let X be a scheme. A CBE between graded $\mathcal{O}_X$-algebras which is also a morphism of $\mathcal{O}_X$-algebras is a *CBE of $\mathcal{O}_X$-algebras.*

In this terminology, the precise definition of Cox sheaves is the following.

Definition 3.4. Let X be a Krull scheme. A Cox sheaf on X is a $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}$ such that there exists a CBE $\pi\colon \mathcal{O}_X(\mathrm{WDiv}(X)) \to \mathcal{R}$ of $\mathcal{O}_X$-algebras that is accompanied by the canonical map $\mathrm{WDiv}(X) \to \mathrm{Cl}(X)$.

Remark 3.5. For a $\mathrm{Cl}(X)$-graded $\mathcal{O}_X$-algebra $\mathcal{R}$ on a Krull scheme X, the following are equivalent:

(i) $\mathcal{R}$ is a Cox sheaf;

(ii) for every subgroup $L \leq \mathrm{WDiv}(X)$ mapping onto $\mathrm{Cl}(X)$, there exists a CBE $\pi\colon \mathcal{O}_X(L) \to \mathcal{R}$;

(iii) for some subgroup $L \leq \mathrm{WDiv}(X)$ mapping onto $\mathrm{Cl}(X)$, there exists a CBE $\pi\colon \mathcal{O}_X(L) \to \mathcal{R}$.

Proof. Assume that (iii) holds. Let

$$\pi\colon \mathcal{O}_X(L) \longrightarrow \mathcal{R}$$

be a CBE. Let D_j, $j \in J$, be a basis of $\mathrm{WDiv}(X)$. Then, there exist $D'_j \in L$, $j \in J$ and $f_j \in \mathcal{O}_X(D'_j - D_j)(X)$ with $\mathrm{div}(f_j) + D'_j = D_j$, and the isomorphisms

$$\mathcal{O}_X(D_j) \xrightarrow{\cdot f_j} \mathcal{O}_X(D'_j)$$

fit together to a homomorphism $\Phi\colon \mathcal{O}_X(\mathrm{WDiv}(X)) \to \mathcal{O}_X(L)$ with accompanying homomorphism $\phi\colon\ \mathrm{WDiv}(X) \to L, D_j \mapsto D'_j$. The composition $\pi \circ \Phi\colon \mathcal{O}_X(L) \to \mathcal{R}$ is the epimorphism requested in assertion (i). $\square$

Existence of Cox sheaves follows from Remark 3.2 because, for any L mapping onto $\mathrm{Cl}(X)$, a suitable map χ_X is defined by assigning arbitrary $f_j \in \mathcal{O}_X(D_j)(X)$ with $\mathrm{div}_X(f_j) = -D_j$ to the elements of a basis $\{D_j\}_{j \in J}$ of $L \cap \mathrm{PDiv}(X)$, and thus, the presheaf $\mathcal{R}$ defined by

$$\mathcal{R}(U) = \mathcal{O}_X(L)(U)/\langle 1_{\mathcal{O}_X(L)(U)} - \chi_X(D)_{|U}; D \in L \cap \mathrm{PDiv}(X)\rangle$$

is a Cox sheaf, compare [**2**]. As this discussion shows, our definition is the axiomatic version of the constructive approach of Hausen and Arzhantsev.

Uniqueness is a matter of caution. In the case that $\mathrm{Cl}(X)$ is free, it is well known that all Cox sheaves are isomorphic.

A further condition enforcing uniqueness in the case of prevarieties over an algebraically closed field $\mathbb{K}$ is $\mathcal{O}(X)^* = \mathbb{K}^*$, which holds, for example, if X is projective, see [**2,** Section I.4.3]. In general, Cox sheaves are only *weakly unique* in the following sense.

Proposition 3.6. *Let X be a Krull scheme, let $\mathcal{R}$ and $\mathcal{R}'$ be Cox sheaves on X and let $U \subseteq X$ be open. Then, the following hold*:

(i) $\mathcal{R}^+/\mathcal{R}^{+,*} \cong \mathcal{O}_X(\mathrm{WDiv}(X))^+/\mathcal{O}_X(\mathrm{WDiv}(X))^{+,*} \cong \mathcal{R}'^+/\mathcal{R}'^{+,*}$.

(ii) *There are bijections respecting sums, intersections, inclusions, products and ideal quotients between the sets of graded ideals of $\mathcal{R}(U)$ and $\mathcal{R}'(U)$.*

(iii) *$\mathcal{R}(U)$ is finitely generated as a $\mathcal{O}_X(U)$-algebra if and only if $\mathcal{R}'(U)$ is so. If X is a scheme over $S = \mathrm{Spec}(B)$, then $\mathcal{R}(U)$ is finitely generated over B if and only if $\mathcal{R}'(U)$ is so.*

Proof. Everything but the last assertion directly follows from Proposition 1.13. Suppose that $\mathcal{R}(U)$ is finitely generated by homogeneous sections $g_1, \dots, g_m$. Then, $\mathrm{Cl}(X)$ is finitely generated by Theorem 0.1. Let $L \leq \mathrm{WDiv}(X)$ be a finitely generated subgroup mapping onto $\mathrm{Cl}(X)$, and let

$$\pi\colon \mathcal{O}_X(L) \longrightarrow \mathcal{R}$$

and

$$\pi'\colon \mathcal{O}_X(L) \longrightarrow \mathcal{R}'$$

be CBEs. Let χ be the kernel character of π, and let $D_1, \dots, D_n$ be a basis of $L \cap \mathrm{PDiv}(X)$. Then, $\mathcal{O}_X(L)(U)$ is generated by $\chi_U(\pm D_1), \dots, \chi_U(\pm D_n)$ and any choice of homogeneous preimages under π_U $f_1, \dots, f_m$ for $g_1, \dots, g_m$. Thus, $\mathcal{R}'(U)$ is generated by their images under π'_U. $\square$

The above shows that the question of uniqueness is of little practical consequence since all Cox sheaves on a given X behave in the same manner. We now proceed with the proof of Theorem 0.1. The general ideal for showing that Cox sheaves have the asserted properties is to show that they are inherited from $\mathcal{O}_X(\mathrm{WDiv}(X))$ since CBEs preserve most graded properties (even in both directions). The second part of the proof adapts the arguments of [**2,** Thm. I.6.4.3 and Prop. I.6.4.5].

Proof of Theorem 0.1. Let $\mathcal{R}$ be a Cox sheaf on X, and let $\pi\colon \mathcal{O}_X(L) \to \mathcal{R}$ be a CBE of $\mathcal{O}_X$-algebras. Then, the map $\mathcal{O}_X \to \mathcal{R}_0$ is an isomorphism since it is the composition of the isomorphisms $\mathcal{O}_X \cong \mathcal{O}_X(L)_0$ and $\mathcal{O}_X(L)_0 \cong \mathcal{R}_0$.

For $x \in U$, respectively, $x \in \overline{\{x'\}}$, we have commutative diagrams whose downward arrows are CBEs of rings:

$$\text{(a)}\quad \begin{tikzcd} \mathcal{O}_X(L)(U) \arrow[r] \arrow[d, "\pi_U"'] & \mathcal{O}_X(L)_x \arrow[d, "\pi_x"] \\ \mathcal{R}(U) \arrow[r] & \mathcal{R}_x \end{tikzcd} \qquad \text{(b)}\quad \begin{tikzcd} \mathcal{O}_X(L)_x \arrow[r] \arrow[d, "\pi_x"'] & \mathcal{O}_X(L)_{x'} \arrow[d, "\pi_{x'}"] \\ \mathcal{R}_x \arrow[r] & \mathcal{R}_{x'}, \end{tikzcd}$$

and the lower arrows inherit injectivity from the upper ones. Considering diagram (a) for the generic point ξ, we see that $\mathcal{R}_\xi$ is G-simple with degree zero part $\mathcal{O}_{X,\xi}$ and $\deg_G(\mathcal{R}_\xi^+) = G$ since $\mathcal{O}_X(L)_\xi$ is L-simple with degree zero part $\mathcal{O}_{X,\xi}$ and $\deg_L(\mathcal{O}_X(L)_\xi^+) = L$. Denote by $\mathcal{S}_L$ the constant sheaf assigning $\mathcal{O}_X(L)_\xi$ and by $\mathcal{S}$ the constant sheaf assigning $\mathcal{R}_\xi$. Then, π_ξ defines a CBE of sheaves $\pi'\colon \mathcal{S}_L \to \mathcal{S}$. $\mathcal{R}$ is a subsheaf of $\mathcal{S}$ and hence G-integral.

For (i), note that π' extends π and both have the same kernel character χ. Since $\mathrm{im}(\chi_X)$ is contained in $\mathcal{O}_X(L)(X)^{+,*}$, its elements are trivially valuated by all $\mu_{Y,X}$. Consequently, $\mathrm{im}(\chi_U)$ is trivially valuated by all $\mu_{Y,U}$. Thus, each μ_Y induces a G-valuation $\overline{\mu}_Y : \mathcal{S}^+ \to \mathbb{Z}_Y$, which also restricts to ν_Y on $\mathcal{K}^*$. Since $\mathcal{O}_X(L)$ is defined in $\mathcal{S}_L$ by the family $\{\mu_Y\}_Y$, the equality $\mathcal{R} = \bigcap_Y \mathcal{S}_{\overline{\mu}_Y}$ now follows by applying Proposition 1.13(v) to the sections over arbitrary open U, and assertion (i) is proven.

The first part of assertion (ii) is due to the fact that μ_Y and $\overline{\mu}_Y$ have the same image. For the second part, consider an element $\pi_X(f\chi^D)$ of the kernel of $\mathrm{div}_{G,X} = \sum_Y \overline{\mu}_Y$, i.e., a global homogeneous unit. Then, $\mathrm{div}_{L,X}(f\chi^D) = 0$, i.e., $D = -\mathrm{div}_X(f)$ is a principal divisor; hence, $\pi_X(f\chi^D)$ has degree $[D] = [0]$. Concerning the supplement, we calculate

$$\begin{aligned}[D] &= \deg_G(\pi_\xi(f\chi^D)) = [\deg_L(f\chi^D)] \\ &= [\mathrm{div}_{L,X}(f\chi^D)] = [\mathrm{div}_{G,X}(\pi_\xi(f\chi^D))].\end{aligned}$$

It remains to show that a G-graded sheaf $\mathcal{R}$ satisfying (i) and (ii) is a Cox sheaf. Recall the notation $\mathrm{div}_G := \sum_Y \overline{\mu}_Y \colon \mathcal{S}^+ \to \mathrm{WDiv}$ and note that by (i) we have $\mathrm{div}_{G|\mathcal{K}^*} = \mathrm{div}$. In order to show that G is canonically isomorphic to $\mathrm{Cl}(X)$, we first note that by (i) the degree map $\deg_G$ induces an isomorphism $\mathcal{S}(X)^+/\mathcal{S}(X)_0^* \cong G$. The homomorphism $\delta\colon \mathcal{S}(X)^+ \to \mathrm{Cl}(X), f \mapsto [\mathrm{div}_{G,X}(f)]$ thus induces the map

$$\overline{\delta}\colon\ \deg_G(\mathcal{S}(R)^+) \longrightarrow \mathrm{Cl}(X), \quad \deg_G(f) \mapsto [\mathrm{div}_{G,X}(f)]$$

which has cokernel $\mathrm{Cl}(X)/\overline{\mathrm{im}(\mathrm{div}_{G,X})}$ and kernel $\deg_G(\mathcal{R}(X)^{+,*})$. Thus, condition (ii) precisely states that $\overline{\delta}$ is an isomorphism from G to $\mathrm{Cl}(X)$.

Next, we show that the isomorphism $\mathcal{K} = \mathcal{S}_0 \xrightarrow{\lambda_f} \mathcal{S}_{\deg_G(f)}$ given by multiplication with $f \in \mathcal{S}(X)^+$ restricts to an isomorphism $\lambda_f\colon \mathcal{O}_X(\mathrm{div}_{X,K}(f)) \to \mathcal{R}_{\deg_G(f)}$. Indeed, for a non-zero $g \in \mathcal{O}_X(\mathrm{div}_{G,X}(f))(U)$, we calculate

$$\mathrm{div}_{U,K}(f_{|U}g) = \mathrm{div}_U(g) + (\mathrm{div}_{X,K}(f))_{|U} \geq 0,$$

i.e., $f_{|U}g \in \mathcal{R}(U)^+$. Conversely, each non-zero $h \in \mathcal{R}(U)_{\deg_G(f)}$ satisfies

$$\mathrm{div}_U((f_{|U})^{-1}h) + \mathrm{div}_{X,K}(f)_{|U} = \mathrm{div}_{G,U}(h) \geq 0.$$

Now, let $L \leq \mathrm{WDiv}(X)$ be any subgroup mapping onto $\mathrm{Cl}(X)$, and let $\{D_j\}_{j\in J}$ be a basis of L. Then, there exist $f_j \in S^+$, $j \in J$, with $\mathrm{div}_{G,X}(f_j) = D_j$. We set

$$f_D := \prod_{j\in J} f_j^{m_j}$$

for

$$D = \sum_{j\in J} m_j D_j.$$

By our first claim, we know that $\overline{\delta}(\deg_G(f_D)) = [D]$. From our second claim, every f_D defines an isomorphism

$$\lambda_{f_D}\colon \mathcal{O}_X(D) \to \mathcal{R}_{[D]}.$$

By construction, the isomorphisms respect the multiplication of homogeneous components in $\mathcal{O}_X(L)$ and thus define a graded epimorphism

$\mathcal{O}_X(L) \to \mathcal{R}$. Hence, $\mathcal{R}$ is a Cox sheaf. The supplements will be proven separately after the proof of Theorem 0.2. $\square$

Proof of Theorem 0.2. Quasi-coherence of $\mathcal{R}$ follows from the more general observation that, for every non-zero $g \in \mathcal{R}(U)^+$, there is a canonical isomorphism

$$\mathcal{R}(U)_g \cong \mathcal{R}(U_g),$$

where $U_g := U \setminus |\mathrm{div}_{U,K}(g)|$. This, in turn, follows directly from the corresponding observation on $\mathcal{O}_X(L)$ using that π_U and π_{U_g} are CBEs and Proposition 1.13 (iii).

For (i), note that G-factoriality of $\mathcal{R}(U)$ follows via Proposition 1.13 (iv) from L-factoriality of $\mathcal{O}_X(U)$, which was proven in Proposition 2.6 (v). Now, consider any Weil divisor $D = D^+ - D^-$ on X, written as a difference of effective divisors. Choose a subgroup L which contains D^+ and D^- mapping onto $\mathrm{Cl}(X)$ and a componentwise isomorphic epimorphism $\pi\colon \mathcal{O}_X(L) \to \mathcal{R}$. Then, for all open U we have

$$[D] = \deg_G(\pi_U(\chi^{D^+})) - \deg_G(\pi_U(\chi^{D^-})) \in \langle \deg_G(\mathcal{R}(U)^+)\rangle.$$

For the case that U is affine, observe that, in the diagram of CBEs:

$$\begin{tikzcd}
\mathcal{O}_X(L)(U) \arrow[r] \arrow[d] & Q^+(\mathcal{O}_X(L)(U)) \arrow[r] \arrow[d] & \mathcal{O}_X(L)_\xi \arrow[d] \\
\mathcal{R}(U) \arrow[r] & Q^+(\mathcal{R}(U)) \arrow[r] & \mathcal{R}_\xi
\end{tikzcd}$$

the lower right arrow is a graded isomorphism since the upper right arrow is one. Furthermore,

$$\deg_G(\mathcal{R}(U)^+) = \overline{\deg_L(\mathcal{O}_X(L)(U)^+)} = \overline{L} = G.$$

The first part of assertion (ii) follows directly from Proposition 2.6 (ii) and Proposition 1.13 (v). The second statement is obvious for $\eta \notin U$ and follows from (iii) otherwise.

For assertion (iii), consider the diagram (b) of inclusions and CBEs from the beginning of the proof of Theorem 0.1 with $x' = \xi$. Since $\mathcal{O}_X(L)_x$ is the intersection over all $\mathcal{K}[L](X)_{\mu_{Y,X}}$ with $x \in Y$, Proposition 1.13 (v) implies that $\mathcal{R}_x$ is the intersection over all $\mathcal{S}(X)_{\overline{\mu}_{Y,X}}$ with $x \in Y$. Moreover, $\mathcal{R}_x^{+,*}$ is the image of $\mathcal{O}_X(L)_x^{+,*}$ and thus has the requested description. Furthermore, the unique L-maximal ideal of $\mathcal{O}_X(L)_x$ is mapped onto a unique G-maximal ideal $\mathfrak{a}_x$ by Proposition 1.13 (i). Its homogeneous elements, which were calculated in Proposition 2.6 (iv), are mapped onto the homogeneous elements of $\mathfrak{a}_x$, which establishes the desired description.

For (iv), let $t \in \mathcal{O}_{X,\eta} = (\mathcal{O}_X(L)_\eta)_0$ be a uniformizer. Then,

$$1 = \nu_{Y,X}(t) = \mu_{Y,X}(t) = \overline{\mu}_{Y,X}(\pi_\xi(t)) = \overline{\mu}_{Y,X}(t),$$

which means that t is a homogeneous uniformizer of $\mathcal{R}_\eta$. □

Proof of Theorem 0.1*, supplements.* Suppose that $\mathcal{R}$ is a $\mathcal{O}_X$-algebra of G-Krull type defined in some K-simple $\mathcal{O}_X$-algebra $\mathcal{S}$ such that conditions (i) and (ii) from Theorem 0.1 are satisfied.

First of all, we show that $\mathcal{S}$ is constant. For a non-zero $f \in \mathcal{S}(X)_w$, we have $\mathcal{S}(U)_w = \mathcal{S}(U)_0 f_{|U} = \mathcal{K}(X) f_{|U}$, which shows that the map $\mathcal{S}(X)_w \to \mathcal{S}(U)_w$ is bijective. We claim that the canonical monomorphism $\mathcal{R}_\xi \to \mathcal{S}_\xi$ is surjective. Let $f \in \mathcal{S}(U)^+$ be a representative of $f_\xi \in \mathcal{S}_\xi^+$, and let $V \subseteq U$ be the complement of those prime divisors Y with $\mu_{Y,U}(f) < 0$. Then, $f_{|V} \in \mathcal{R}(V)^+$, and $(f_{|V})_\xi$ is mapped to f_ξ. This defines an isomorphism from the constant sheaf $\mathcal{K}_\mathcal{R}$ assigning $\mathcal{R}_\xi$ to $\mathcal{S}$.

Let Y be a prime divisor with generic point η. Since $\mathcal{R}$ is isomorphic to a Cox sheaf, $\mathcal{R}_\eta$ is a discrete G-valuation ring, and we have $Q^+(\mathcal{R}_\eta) = \mathcal{R}_\xi$ and $(\mathcal{R}_\eta)_0 = \mathcal{O}_{X,\eta}$. Let $t \in \mathcal{O}_{X,\eta}$ be a generator of the maximal ideal of $\mathcal{O}_{X,\eta}$, i.e., an element with $\nu_{Y,X}(t) = 1$. Let $\mu'_{Y,X}$ be the discrete G-valuation on $\mathcal{R}_\xi = \mathcal{S}(X)$ with $\mathcal{R}_\eta = \mathcal{S}(X)_{\mu'_{Y,X}}$. Since $\mathcal{R}_\eta$ has units in every degree t also generates the G-maximal ideal of $\mathcal{R}_\eta$, and thus $\mu'_{Y,X}(t) = \nu_{Y,X}(t) = \mu_{Y,X}(t)$, which implies $\mu'_{Y,X} = \mu_{Y,X}$.

For $f \in \mathcal{S}(X)^+$, we have $\mu_{Y,U}(f_{|U}) = (\mu_{Y,X}(f))_{|U}$. By definition of $\mathbb{Z}^{(Y)}$, this term is zero if η is not an element of U since, in this case, $\mathbb{Z}^{(Y)}(U) = 0$. Otherwise, $\mathbb{Z}^{(Y)}(U) = \mathbb{Z}$, and the restriction map $\mathbb{Z}^{(Y)}(X) \to \mathbb{Z}^{(Y)}(U)$ is the identity, i.e., $\mu_{Y,U}(f_{|U}) = \mu_{Y,X}(f)$. □

Remark 3.7. One of the starting points for the present considerations on the valuative structure of Cox sheaves was **[2,** Section I.5]. The $[D]$-divisor of a non-zero $f \in \mathcal{R}(X)_{[D]}$ defined there is $\mathrm{div}_{\mathrm{Cl}(X),X}(f)$ in our notation, and it is shown that the assignment $f \mapsto \mathrm{div}_{[D]}(f)$ is homomorphic and encodes the divisibility relation in $\mathcal{R}(X)$, which in our setting is due to the definition of $\operatorname{div}_{\mathrm{Cl}(X),X}$ as the sum of all $\mu_{Y,X}$.

4. Graded schemes. Graded schemes are implicitly already well known from the proj construction, see Example 4.12. Related examples of graded schemes in the context of toric good quotients have been studied in [**24**]. More generally, the categories of G-graded, respectively, Noetherian $\mathbb{Z}$-graded schemes with degree-preserving morphisms have been discussed in [**10, 28**]. The category of graded schemes introduced below includes the aforementioned and has more morphisms, in particular good quotients, which are affine morphisms from G-graded to 0-graded schemes satisfying a natural condition on their structure sheaves, see Definition 4.10. Good quotients behave very naturally in that they respect intersections of closed sets and are surjective with distinguished points in each fibre, see Proposition 4.11. We also introduce the other concepts needed for Theorem 0.4, namely, G-Krull schemes which are the most general objects with well-behaved notions of Weil and principal divisors.

For a G-graded ring R and an ideal $\mathfrak{a}$ of R, we denote by $\mathfrak{a}^+$ the G-graded ideal generated by $\mathfrak{a} \cap R^+$. If $\mathfrak{a}$ is prime, then $\mathfrak{a}^+$ is G-prime.

Definition 4.1. The *G-spectrum* of a G-graded ring R is the set $X := \operatorname{Spec}_G(R)$ of G-prime ideals of R, endowed with the topology whose closed sets are of the form $V(\mathfrak{a}) = \{\mathfrak{p} \in X; \mathfrak{a} \subseteq \mathfrak{p}\}$ with G-graded ideals $\mathfrak{a} \trianglelefteq R$. Its G-graded structure sheaf $\mathcal{O}_X$ (with G-local stalks) is defined on the basis $X_f := X \setminus V(\langle f \rangle)$ of principal open sets for $f \in R^{+,0}$ by $\mathcal{O}_X(X_f) := R_f$ and on arbitrary open $U \subseteq X$ by

$$\mathcal{O}_X(U) := \varprojlim_{X_f \subseteq U} \mathcal{O}_X(X_f),$$

where the limit is taken in the category of G-graded rings. The pair $(\operatorname{Spec}_G(R), \mathcal{O}_{\operatorname{Spec}_G(R)})$ is the *affine G-graded scheme* corresponding to R.

A *graded scheme* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a graded sheaf of rings $\mathcal{O}_X$ that has a cover by affine $\operatorname{gr}(\mathcal{O}_X)$-graded schemes $(U, \mathcal{O}_{X|U})$. $(X, \mathcal{O}_X)$ is also called a $\operatorname{gr}(\mathcal{O}_X)$-graded scheme. A *morphism* of the graded schemes $(X, \mathcal{O}_X)$ and $(X', \mathcal{O}_{X'})$ is a continuous map $\phi\colon X \to X'$, together with a morphism of graded sheaves $\phi^*\colon \mathcal{O}_{X'} \to \phi_*\mathcal{O}_X$ such that, for each $x \in X$, the induced graded homomorphism $\phi_x^*\colon \mathcal{O}_{X',\phi(x)} \to \mathcal{O}_{X,x}$ satisfies $(\phi_x^*)^{-1}(\mathfrak{m}_x)^+ = \mathfrak{m}_{\phi(x)}$, where $\mathfrak{m}_x$ and $\mathfrak{m}_{\phi(x)}$ are the respective unique $\mathrm{gr}(\mathcal{O}_X)$-$/gr(\mathcal{O}_{X'})$-maximal ideals.

Schemes are the same as 0-graded schemes; they form a full subcategory of the category of graded schemes. We will often only write X for the graded scheme $(X, \mathcal{O}_X)$. When talking about a morphism of graded schemes, we will also write the continuous map $\phi\colon X \to X'$ of the underlying topological spaces in place of the pair (ϕ, ϕ^*).

Remark 4.2. For an affine G-graded scheme $X = \mathrm{Spec}_G(R)$, the stalk at $\mathfrak{p} \in X$ is the graded localization $R_{\mathfrak{p}}$. Morphisms between affine graded schemes are given by maps of graded rings: if $\phi\colon X \to X'$ is a morphism of affine graded schemes, then $\phi^*\colon R' = \mathcal{O}(X') \to R = \mathcal{O}(X)$ is a graded morphism, and ϕ maps the point $\mathfrak{p} \in X$ to the point $(\phi^*)^{-1}(\mathfrak{p})^+$.

Example 4.3. Let M be an abelian monoid contained in an abelian group G, and let k be a field. Let $R := k[M]$ be the canonically G-graded monoid algebra of M over k. Recall that a *face* of an abelian monoid is a submonoid $\tau \subseteq M$ such that $w + w' \in \tau$ implies $w, w' \in \tau$ for all $w, w' \in M$. Then, there is an order reversing bijection

$$\begin{gathered}\mathrm{faces}(M) \longleftrightarrow \mathrm{Spec}_G(R) =: X \\ \tau \longmapsto \mathfrak{p}_\tau := \langle \chi^w; w \in M \setminus \tau \rangle \\ \deg_G(R^+ \setminus \mathfrak{p}) =: \tau_{\mathfrak{p}} \longleftarrow\!\shortmid \mathfrak{p}\end{gathered}$$

where $\mathrm{Spec}_G(R)$ is ordered by inclusion. Furthermore, $\mathcal{O}_X(X_{\chi^w}) = k[M - \mathbb{Z}_{\geq 0} w]$ and $\mathcal{O}_{X,\mathfrak{p}} = k[M - \tau_{\mathfrak{p}}]$. If $\psi\colon G' \to G$ is a group homomorphism mapping the submonoid M' into M, then it induces a graded map $\widetilde{\psi}\colon R' := k[M'] \to R$ and a map

$$\mathrm{faces}(M) \to \mathrm{faces}(M'), \tau \mapsto \psi^{-1}(\tau) \cap M'.$$

The corresponding map of graded schemes is

$$\phi\colon X \longrightarrow X' := \mathrm{Spec}_{G'}(R'), \qquad \mathfrak{p}_\tau \longmapsto \mathfrak{p}'_{\psi^{-1}(\tau)\cap M'}.$$

This consideration links graded schemes to combinatorics when applied to finitely generated monoids. On the other hand, we observe that $\tau \subseteq M$ is a face if and only if $(M \setminus \tau) \sqcup \{\infty\} \subseteq M \sqcup \{\infty\}$ is a prime ideal in the sense of **[14]**. Moreover, a subgroup $\mathfrak{p}$ of a G-graded ring R generated by homogeneous elements is a G-prime ideal if and only if

$R^{+,0} \setminus \mathfrak{p}$ is a face of the multiplicative monoid $R^{+,0}$. Thus, there is also a canonical homeomorphism between $\mathrm{Spec}_G(R)$ and the space of prime ideals of $M \sqcup \{\infty\}$, which is a scheme over the field $\mathbb{F}_1$ with one element in the sense of **[14]**. This line of thought is continued in Remark 6.6.

Remark 4.4. The category of graded rings is a subcategory of the category of *monoidal pairs* or *sesquiads* from **[13]** via the assignment sending a graded ring $R = \bigoplus_{w \in G} R_w$ to $(R^{+,0}, R)$. Graded ideals of R correspond to ideals of $(R^{+,0}, R)$, and graded localizations of R correspond to localizations of $(R^{+,0}, R)$. Therefore, the affine graded scheme $(\mathrm{Spec}_G(R), \mathcal{O}_{\mathrm{Spec}_G(R)})$ is naturally identified with the set of prime ideals of $(R^{+,0}, R)$, equipped with the structure sheaf induced by $(R^{+,0}, R)$, and the category of graded schemes becomes a subcategory of the category of *sesquiad Zariski schemes*, which is obtained by gluing prime spectra of sesquiads. Within this category (non-trivially) graded schemes take an intermediate position between schemes, whose affine charts are given by monoid pairs of the form (R, R), and $\mathbb{F}_1$-schemes, whose affine charts are given by pairs of the form $(M, \mathbb{Z}[M])$.

Remark 4.5. Each non-empty closed irreducible subset of a graded scheme has a (unique) generic point. Indeed, if $Y = V(\mathfrak{a}) \subseteq \mathrm{Spec}_G(R)$ is non-empty and irreducible, then

$$\sqrt{\mathfrak{a}}^{+} = \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p}$$

is the generic point of Y.

Open and closed graded *subschemes* and *embeddings* are defined by extending the corresponding notions for schemes in the obvious manner. Furthermore, *G-reduced*, respectively, *G-integral* G-graded schemes, are defined by the absence of homogeneous nilpotent elements, respectively, homogeneous zero divisors in all sections. For a G-integral G-graded scheme X, the constant sheaf $\mathcal{K}$ assigning the stalk $\mathcal{O}_{X,\xi}$ at the generic point is a sheaf of G-simple rings.

Remark 4.6. A graded scheme X is $\mathrm{gr}(X)$-integral if and only if it is irreducible and $\mathrm{gr}(X)$-reduced.

A quasi-compact G-graded scheme is *G-Noetherian* if the sections $\mathcal{O}_X(U)$ are G-Noetherian for all (or equivalently, some cover of X by) affine U. A *G-Krull scheme* is a G-integral scheme with a finite cover by graded spectra of G-Krull rings. For a G-Krull scheme X, a *G-prime divisor* is a closed irreducible subset Y of codimension one in X. Its generic point is denoted by η.

Remark 4.7. If X is a G-Krull scheme, then the stalks at G-prime divisors are G-valuation rings. Thus, each G-prime divisor Y defines a G-valuation $\nu_Y : \mathcal{K}^+ \to \mathbb{Z}^{(Y)}$, and these are the essential G-valuations of the G-Krull sheaf $\mathcal{O}_X = \bigcap_Y \mathcal{K}_{\nu_Y}$. Therefore, a G-graded scheme X is a G-Krull scheme if and only if it is quasi-compact and $\mathcal{O}_X$ is a G-Krull sheaf.

If X is a G-Krull scheme, then the direct sum over the skyscraper sheaves $\mathbb{Z}^{(Y)}$ is the presheaf WDiv^G of *G-Weil divisors*, and the direct sum over the G-valuations ν_Y defines a morphism $\mathrm{div}^G : \mathcal{K}^+ \to \mathrm{WDiv}^G$. Its image PDiv^G is the presheaf of *G-principal divisors*, and its cokernel is the presheaf of *G-divisor class groups* Cl^G. The *support* $|D|$ of a Weil divisor $D \in \mathrm{WDiv}(U)$ is the intersection of U with the union over all G-prime divisors Y occurring with non-zero coefficients in D.

Proposition 4.8. *Let X be a G-Krull scheme. Then, the stalk of $\mathcal{O}_X$ at x is*

$$\mathcal{O}_{X,x} = \bigcap_{x \in Y} (\mathcal{O}_{X,\xi})_{\nu_{Y,X}} \subseteq \mathcal{O}_{X,\xi},$$

where Y runs through all G-prime divisors Y containing x in their closure.

Since the category of graded rings over a fixed graded ring has the graded tensor product as a coproduct, the category of graded schemes has fiber products. A graded scheme X is called *separated*, respectively, of *affine intersection* if the diagonal morphism $\Delta_X : X \to X \times X$ is a closed embedding, respectively, affine. The latter property is equivalent to affineness of the intersection of any two affine opens subsets.

Proposition 4.9. *In a G-Krull scheme X of affine intersection the following hold*:

(i) *every open affine $U \subseteq X$ is the complement of a G-divisor on X.*
(ii) *If X is affine, then $V(f) = |\mathrm{div}^G(f)|$ for every $f \in \mathcal{O}(X)^+$.*

Next, we introduce good quotients of graded schemes.

Definition 4.10. A morphism from a G-graded scheme to a scheme is called *G-invariant*. A *good quotient by G* is an affine G-invariant morphism $q\colon X \to Y$ such that the pullback $\mathcal{O}_Y \to (q_*\mathcal{O}_X)_0$ is an isomorphism.

Proposition 4.11. *If $q\colon X \to Y$ is a good quotient, then the following hold:*

(i) *q is surjective and closed;*

(ii) *if $A_i \subseteq X$, $i \in I$, are closed then $q(\bigcap_i A_i) = \bigcap_i q(A_i)$;*

(iii) *every preimage $q^{-1}(y)$ contains a unique element which is contained in all closures of elements of $q^{-1}(y)$; and y is a closed point if and only if this element is a closed point.*

Proof. For closedness, we use the fact that $\langle \mathfrak{a}_0 \rangle_R \cap R_0 = \mathfrak{a}_0$ holds for $\mathfrak{a}_0 \trianglelefteq R_0$. For (ii), we use $(\sum_i \mathfrak{a}_i) \cap R_0 = \sum_i (\mathfrak{a}_i \cap R_0)$. Surjectivity and (iii) follow from the fact that, for a prime ideal $\mathfrak{q}$ of R_0, there exists a unique G-prime $\mathfrak{p}$ that is maximal with $\mathfrak{p} \cap R_0 = \mathfrak{q}$, and $\mathfrak{q}$ is maximal if and only if $\mathfrak{p}$ is G-maximal. Explicitly, the homogeneous elements of $\mathfrak{p}$ are those $r \in R^{+,0}$ with $\langle r \rangle \cap R_0 \subseteq \mathfrak{q}$. $\square$

For the $\mathbb{Z}$-graded case, properties (i) and (iii) were observed in [**10,** Lemma 1.1.2]. A good quotient that is bijective, i.e., a homeomorphism, is called *geometric*. A well-known example for a graded geometric quotient is the proj construction of a $\mathbb{Z}$-graded ring.

Example 4.12. Let R be a $\mathbb{Z}$-graded ring with $\deg_{\mathbb{Z}}(R^+) \subseteq \mathbb{Z}_{\geq 0}$, and consider the proper $\mathbb{Z}$-graded ideal

$$\mathfrak{a} = \bigoplus_{n>0} R_n.$$

Let $\overline{X} := \operatorname{Spec}_{\mathbb{Z}}(R)$, and set $\widehat{X} := \overline{X} \setminus V(\mathfrak{a})$. Then the quotients $\overline{X}_f \to \operatorname{Spec}((R_f)_0)$ for $f \in \mathfrak{a} \cap R^+$ glue to a good quotient $q\colon \widehat{X} \to \operatorname{Proj}(R)$ by $\mathbb{Z}$ which is even bijective, i.e., geometric. However, unless $R = R_0$, the structure sheaf of $\operatorname{Proj}(R)$ will differ from $q_*\mathcal{O}_{\widehat{X}}$.

Remark 4.13. If $q\colon \widehat{X} \to X$ is a good quotient by K and $\widehat{X}$ is K-integral, then X is integral.

G-graded schemes also occur naturally as *relative G-spectra* of a quasi-coherent G-graded sheaves on schemes:

Construction 4.14. *Let X be a scheme, and let $\mathcal{R}$ be a quasi-coherent G-graded $\mathcal{O}_X$-algebra. Then,* $\mathrm{Spec}_G(\mathcal{R}(U))$ *is open in* $\mathrm{Spec}_G(\mathcal{R}(V))$ *for any two affine open $U \subseteq V \subseteq X$, and hence, the G-prime spectra of $\mathcal{R}(U)$ for all affine U glue to a G-graded scheme, called the relative G-spectrum* $\mathrm{Spec}_{G,X}(\mathcal{R})$ *of $\mathcal{R}$, and there is a commutative diagram*

$$\begin{tikzcd} \mathrm{Spec}_X(\mathcal{R}) \arrow[rr] \arrow[dr] & & X \\ & \mathrm{Spec}_{G,X}(\mathcal{R}) \arrow[ur, "q"'] & \end{tikzcd}$$

and $q_*\mathcal{O}_{\mathrm{Spec}_{G,X}(\mathcal{R})} = \mathcal{R}$ *holds. If $\mathcal{R}_0 = \mathcal{O}_X$, then q is a good quotient.*

Remark 4.15. Let $L \leq \mathrm{WDiv}(X)$ be any subgroup mapping onto $\mathrm{Cl}(X)$, and let $\widetilde{X} := \mathrm{Spec}_{L,X}(\mathcal{O}_X(L))$. For any Cox sheaf $\mathcal{R}$, every CBE $\pi\colon \mathcal{O}_X(L) \to \mathcal{R}$ induces a graded homeomorphism $\widehat{X} := \mathrm{Spec}_{\mathrm{Cl}(X),X}(\mathcal{R}) \to \widetilde{X}$, which is an isomorphism if and only if L maps isomorphically onto $\mathrm{Cl}(X)$ (which in turn can only occur if $\mathrm{Cl}(X)$ is free).

5. Proofs of Theorems 0.4 and 0.5.

Proof of Theorem 0.4. First, suppose that conditions (i)–(iii) hold. Consider the cover of X by all affine open U. Then, $\widehat{X}$ is covered by their preimages $\widehat{U} = q^{-1}(U)$, and this cover has a finite subcover. Consequently, X has a finite affine cover and each $\mathcal{O}(U) = \mathcal{O}(\widehat{U})_0$ is a Krull ring since $\mathcal{O}(\widehat{U})$ is a G-Krull ring. Since integrality of X follows from graded integrality of $\widehat{X}$, we conclude that X is a Krull scheme.

We show that $\mathcal{R} := q_*\mathcal{O}_{\widehat{X}}$ is a Cox sheaf by applying Theorem 0.1. Set $\mathcal{S} := q_*\mathcal{K}_{\widehat{X}}$. Then, $\mathcal{S}$ is G-simple and $\mathcal{S}_0 = \mathcal{K}_X$ by condition (ii).

For the verification of Theorem 0.1 (i), consider a G-prime divisor $\widehat{Y}$ and its image $Y := q(\widehat{Y})$. Due to (ii), we have $\mathbb{Z}^{(Y)} = q_*\mathbb{Z}^{(\widehat{Y})}$ and $q_*\,\mathrm{WDiv}^G \cong \mathrm{WDiv}$, and

$$\mu_Y := q_*\nu_{\widehat{Y}} \colon \mathcal{S}^+ \longrightarrow \mathbb{Z}^{(Y)}$$

restricts to ν_Y on $\mathcal{K}^*$. By definition, the family $\{\mu_Y\}_Y$ defines $\mathcal{R}$ as a G-Krull sheaf in $\mathcal{S}$.

For Theorem 0.1 (iii), we first observe that $\mathrm{div}_{G,X} := \sum_Y \mu_{Y,X} = q_*\mathrm{div}_X^G$ is surjective because $\mathrm{div}_{\widehat{X}}^G$ is so, due to $\mathrm{Cl}^G(\widehat{X}) = 0$. Since $\mathcal{R}(X)^{+,*} = \mathcal{R}(X)_0^*$ and $\deg_G(\mathcal{S}(X)^+) = G$, Theorem 0.1 now implies that $\mathcal{R}$ is a Cox sheaf on X.

Now, suppose that $\mathcal{R}$ is a Cox sheaf and

$$q\colon \widehat{X} = \mathrm{Spec}_{G,X}(\mathcal{R}) \longrightarrow X$$

its characteristic space. Let $\mathcal{S}$ be the constant sheaf assigning $\mathcal{R}_\xi$. Since, for affine open $U \subseteq X$, the map $(\mathcal{O}(U)\setminus 0)^{-1}\mathcal{R}(U) \to Q^+(\mathcal{R}(U))$ is an isomorphism, we conclude that $\mathcal{R}_\xi \to \mathcal{O}_{\widehat{X},\widehat{\xi}}$ is an isomorphism. Hence, the canonical map $q^*\colon \mathcal{S} \to q_*\mathcal{K}_{\widehat{X}}$ is an isomorphism.

Let $X = U_1 \cup \ldots \cup U_d$ be an affine open cover. Then, $\widehat{X} = q^{-1}(U_1) \cup \cdots \cup q^{-1}(U_d)$ is an affine open cover by spectra of G-Krull rings, and the sets $q^{-1}(U_i)$ intersect pairwise non-trivially because the sets U_i do so. Thus, $\widehat{X}$ is irreducible. Since each $\mathcal{O}(q^{-1}(U_i))$ is G-integral, its localizations at G-prime ideals are G-integral and, in particular, G-reduced. Hence, $\widehat{X}$ is G-integral, and we have verified that $\widehat{X}$ is a G-Krull scheme.

Since $\{\mu_Y\}_Y$ are the essential G-valuations of $\mathcal{R}$ and they restrict to $\{\nu_Y\}_Y$, there are natural bijections

$$\alpha\colon \langle \mu_{Y,U}^{-1}(\mathbb{Z}_{>0}) \cap \mathcal{O}(\widehat{U})^+\rangle \longmapsto \langle \nu_{Y,U}^{-1}(\mathbb{Z}_{>0}) \cap \mathcal{O}(U)\rangle = Y$$

between the G-prime divisors of $\widehat{U} = \mathrm{Spec}_G(\mathcal{R}(U))$ and the prime divisors of U. Because q is affine, these glue to an isomorphism

$$\alpha\colon \mathrm{WDiv}^G(\widehat{X}) \longrightarrow \mathrm{WDiv}(X), \qquad \widehat{Y} \longmapsto q(\widehat{Y}).$$

Thus, for a G-prime divisor $\widehat{Y}$ with generic point $\widehat{\eta}$, we have $q(\widehat{\eta}) \in U$ if and only if $\widehat{\eta} \in q^{-1}(U)$ for all open $U \subseteq X$, meaning that $\mathbb{Z}^{(q(\widehat{Y})} = q_*\mathbb{Z}^{(\widehat{Y})}$. This induces an isomorphism of presheaves $q_*\colon Q_*\,\mathrm{WDiv}^G \cong \mathrm{WDiv}$. By construction, we have $q_*\nu_{\widehat{Y}} \circ q^* = \mu_{q(\widehat{Y})}$, which gives $q_* \circ q_*\mathrm{div}^G \circ q^* = \mathrm{div}_G$, the first supplement. In particular, $q_* \circ q_*\mathrm{div}^G \circ q^*_{|\mathcal{K}^*} = \mathrm{div}$ holds and assertions (i)–(iii) are verified.

For the second supplement, consider a G-prime divisor $\widehat{Y}$ with generic point $\widehat{\eta}$. Then, $Y := q(\widehat{Y})$ is a closed, irreducible proper subset of X which contains $\eta := q(\widehat{\eta})$. On the other hand, the closure of η has codimension $\dim(\mathcal{O}_{X,\eta}) = 1$, and thus, η is the generic point of Y.

For an $x \in X$, let $\widehat{x} \in \widehat{X}$ be the unique point in the preimage of x which is contained in the closures of all other points mapped to x. If $\widehat{x} \in \widehat{Y}$, then $x \in Y$. Conversely, if $x \in Y$, then $\widehat{Y}$ contains a point z with $q(z) = x$, and we conclude that $\widehat{x} \in \overline{\{z\}} \subseteq \widehat{Y}$.

Thus, we obtain

$$\mathcal{R}_x = \bigcap_{x \in Y} \mathcal{S}(X)_{\mu_{Y,X}} = \bigcap_{\widehat{x} \in \widehat{Y}} \mathcal{K}(\widehat{X})_{\nu_{\widehat{Y},\widehat{X}}} = \mathcal{O}_{\widehat{X},\widehat{x}}$$

in $\mathcal{R}_\xi = \mathcal{O}_{\widehat{X},\widehat{\xi}}$. In particular, for $x = \eta$, the point $\widehat{x}$ lies in the closure of $\{\widehat{\eta}\}$ and $\mathcal{O}_{\widehat{X},\widehat{x}} = \mathcal{R}_\eta = \mathcal{O}_{\widehat{X},\widehat{\eta}}$, which implies $\widehat{x} = \widehat{\eta}$. □

The following lemma is necessary to show graded locality of the stalks $\mathcal{O}_X(L)_x$ and $\mathcal{R}_x$ in the respective proofs.

Lemma 5.1. *Let $\mathcal{R}$ be a Cox sheaf on X, and let $f, f' \in \mathcal{R}(U)_{[D]} \setminus 0$ with $f + f' \neq 0$. Then,*

$$|\mathrm{div}_{G,U}(f)| \cap |\mathrm{div}_{G,U}(f')| \subseteq |\mathrm{div}_{G,U}(f + f')|.$$

Accordingly, for $g, g' \in \mathcal{O}_X(D)(U) \setminus 0$ with $g + g' \neq 0$, we have

$$|\mathrm{div}_U(g) + D| \cap |\mathrm{div}_U(g') + D| \subseteq |\mathrm{div}_U(g + g') + D|.$$

Proof. It suffices to consider the case that U is affine. Using Proposition 4.9, we calculate

$$\begin{aligned}
&|\mathrm{div}_{G,U}(f)| \cap |\mathrm{div}_{G,U}(f')| \\
&\quad = q(|\mathrm{div}^G_{q^{-1}(U)}(q^*(f))|) \cap q(|\mathrm{div}^G_{q^{-1}(U)}(q^*(f'))|) \\
&\quad = q(|\mathrm{div}^G_{q^{-1}(U)}(q^*(f))| \cap |\mathrm{div}^G_{q^{-1}(U)}(q^*(f'))|) \\
&\quad = q(V_{q^{-1}(U)}(q^*(f)) \cap V_{q^{-1}(U)}(q^*(f'))) \\
&\quad \subseteq q(V_{q^{-1}(U)}(q^*(f + f'))) = |\mathrm{div}_{G,U}(f + f')|.
\end{aligned}$$

□

Proof of Theorem 0.5. First, let $R = \mathcal{R}(X)$ be the Cox ring of X. Then (i), (ii) and (iii) follow from Theorem 0.1. For assertion (iv), we additionally suppose that X has an affine cover by complements of Weil divisors. Let $\mathfrak{p}$ be a G-prime divisor. By G-factoriality, we have

$\mathfrak{p} = \langle f \rangle$ with some G-prime $f \in R$. Then $Y := \operatorname{div}_{G,X}(f)$ is contained in some affine open set $U = X \setminus |D|$ with some effective Weil divisor D. Let $\mathcal{O}_X(L)$ map onto $\mathcal{R}$. Then, there exists a $g \in \mathcal{O}_X(L)(X)$ with $D = \operatorname{div}_{L,X}(g) = \operatorname{div}_{G,X}(\pi_X(g))$. Let $h := \pi_X(g)$. Since $\operatorname{div}_{G,X}(f) \nleq \operatorname{div}_{G,X}(h)$, f does not divide h, which means that $h \notin \mathfrak{p}$. Hence, $R_{\mathfrak{p}} = (R_h)_{\mathfrak{p}}$, and the second ring has units in every G-degree by Theorem 1.10 (iii) and Proposition 1.13 (iv).

Now, let G be finitely generated and R an algebraic Cox ring, i.e., a G-graded ring satisfying conditions (i), (ii) and (iv). We claim that there is a set of pairwise non-associated G-primes $f_1, \ldots, f_r$ such that $\langle \deg_G(f_j); j \neq k \rangle = G$ for every $k = 1, \ldots, r$. Since G is finitely generated, there are pairwise non-associated G-primes $f_1, \ldots, f_m$ with $\langle \deg_G(f_1), \ldots, \deg_G(f_m) \rangle = G$. The localization $R_{\langle f_1 \rangle}$ has units in every degree by (iv), so there are fractions $g_1/h_1, \ldots, g_n/h_n$, where none of the g_j, h_j are divisible by f_1, whose degrees together generate G. Decomposing gives $f_{m+1}, \ldots, f_t$ such that $f_1, \ldots, f_t$ are pairwise non-associated G-primes and $\langle \deg_G(f_2), \ldots, \deg_G(f_t) \rangle = G$. Proceeding in this way for $k = 2, \ldots, m$, we arrive at a set $f_1, \ldots, f_r$ with the requested properties.

For $j = 1, \ldots, r$, let R_j be the localization by the product of all f_k with $k \neq j$. Let $\widehat{X}$ be the union over the open affine subsets $\widehat{X}_j := \operatorname{Spec}_G(R_j)$ of $\overline{X} := \operatorname{Spec}_G(R)$. Then, by choice of $f_1, \ldots, f_r$, all $X_j = \operatorname{Spec}((R_j)_0)$ contain $X' = \operatorname{Spec}((R_{f_1 \cdots f_r})_0)$ as a principal open subset and thus glue to a scheme X. Since R_j has units in every G-degree, the maps $\widehat{X}_j \to X_j$ are geometric good quotients and they glue to a good quotient $q \colon \widehat{X} \to X$ by G. Since $X_i \cap X_j = X'$ for $i \neq j$, we obtain that the diagonal morphism of X is affine.

We verify that R is the Cox ring of X by showing that q is a graded characteristic space. $\widehat{X}$ is a G-Krull scheme since every R_j is a G-Krull ring (they are even G-factorial). Each $\widehat{X}_j$ contains all G-prime divisors of $\overline{X}$ except the G-principal divisors of those f_k with $k \neq j$. Thus, $\widehat{X}$ contains every G-prime divisor of $\overline{X}$, which implies $\mathcal{O}(\widehat{X}) = R$ and $\mathcal{O}(\widehat{X})^{+,*} = \mathcal{O}(\widehat{X})^*_0$ by (ii). Moreover, $\mathcal{K}^G(\widehat{X}) = \mathcal{K}^G(\widehat{X}_j) = Q^+(R)$, and hence, $\operatorname{Cl}^G(\widehat{X}) = \operatorname{Cl}(R) = 0$ by (i).

By (iv), each of the rings R_j has units of every G-degree. Firstly, this yields $Q^+(R_j)_0 = Q((R_j)_0)$ and $\deg_G(R_j) = G$. Hence, the map

$q^*\colon \mathcal{K}_X \to (q_*\mathcal{K}_{\widehat{X}})_0$ is an isomorphism, and we have $\deg_G(\mathcal{K}_{\widehat{X}}(\widehat{X})^+) = G$. Secondly, there are bijections respecting sums, products, intersections and inclusions between the G-graded ideals of R_j and the ideals of $(R_j)_0$. In particular, the assignment $\widehat{Y} \mapsto q(\widehat{Y})$ defines an isomorphism $\mathrm{WDiv}^G(\widehat{X}_j) \to \mathrm{WDiv}(X_j)$ and, for every G-prime divisor $\widehat{\eta} \in \widehat{X}_j$, the generator of the maximal ideal of $\mathcal{O}_{X_j,q(\widehat{Y})}$ also generates the G-maximal ideal of $\mathcal{O}_{\widehat{X}_j,\widehat{Y}}$. The induced isomorphism $\alpha\colon \mathrm{WDiv}^G(\widehat{X}) \to \mathrm{WDiv}(X)$ therefore satisfies $\alpha \circ \mathrm{div}^G_{\widehat{X}} \circ q^*_{|\mathcal{K}(X)^*} = \mathrm{div}_X$. Moreover, α induces an isomorphism $q_*\colon q_*\,\mathrm{WDiv}^G \to \mathrm{WDiv}$ of presheaves on X, which fits into a commutative diagram as required. Thus, Theorem 0.4 gives the assertion.

For the supplement, note that, if R is a finitely generated $\mathbb{K}$-algebra, then G-integrality implies that R is reduced. Hence, the localizations R_j are finitely generated over $\mathbb{K}$ and reduced. The ring $(R_j)_0$ is then the ring of invariants with respect to the action of $H = \mathrm{Spec}(\mathbb{K}[G])$ on $\widehat{X}_j$. Since H is reductive, $(R_j)_0$ is also finitely generated. Thus, X is reduced and of finite type over $\mathbb{K}$. □

Remark 5.2. Conditions (i), (ii) and (iv) irredundantly characterize Cox rings of Krull schemes with affine cover by divisor complements and finitely generated class group. Indeed, **[5]** offers examples of rings satisfying (i) and (ii) but not (iv). In addition, one may take any G-graded Cox ring R and trivially extend the grading to a $G \oplus \mathbb{Z}$-grading. The monoid R^+ stays the same and its units remain in degree 0; however, $\deg_G(R^+) = G \oplus 0 \neq G \oplus \mathbb{Z}$, so the units of $R_{\mathfrak{p}}$ cannot attain degrees outside of $G \oplus 0$.

Examples with (i) and (iv), but not (ii), are obtained in Construction 1.9 whenever $\mathrm{Cl}(A)$ is not free and $\phi := \mathrm{id}_{Div(A)}$ is chosen. Examples with (ii) and (iv), but not (i), are also obtained from Construction 1.9 whenever $\mathrm{Cl}(A)$ is free and ϕ is the inclusion of a subgroup $G \subset Div(A)$ which maps injectively but not surjectively to $\mathrm{Cl}(A)$.

6. Graded schemes and diagonalizable actions. The defining algebraic data of a graded scheme $(X, \mathcal{O}_X)$ also define a scheme: by equipping each $\mathcal{O}_X(U)$ with the trivial 0-grading, a 0-graded quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{O}_X^{(0)}$ may be obtained whose 0-graded relative spectrum $X^{(0)} := \mathrm{Spec}_X(\mathcal{O}_X^{(0)})$ is a scheme. The canonical affine morphism $X^{(0)} \to X$ is surjective and restricts to $\mathrm{Spec}(R) \to \mathrm{Spec}_G(R), \mathfrak{p} \mapsto \mathfrak{p}^+$ on affine charts.

In this section, let $\mathbb{K}$ be an algebraically closed field. We will indicate how the functor $\mathfrak{f}\colon X \mapsto X^{(0)}$ induces an equivalence between the category $\mathcal{A}$ of graded reduced schemes X of finite type over $\mathbb{K}$ with finitely generated grading groups $\mathrm{gr}(X)$ and the category $\mathcal{B}$ of prevarieties over $\mathbb{K}$ with quasi-torus actions admitting affine invariant covers. This allows us to translate the characterization of graded characteristic spaces into terms of invariant geometry of quasi-torus actions.

Definition 6.1. Let Z be a prevariety over $\mathbb{K}$ with the action of a diagonalizable group $H := \mathrm{Spec}_{\max}(\mathbb{K}[G])$. Then, Z has an induced *H-invariant topology* $\Omega_{Z,H}$ consisting of the H-invariant Zariski open sets. The G-graded sheaf of rings obtained by restricting $\mathcal{O}_Z$ to $\Omega_{Z,H}$ is denoted $\mathcal{O}_{Z,H}$ and called the *H-invariant structure sheaf.*

If Z is affine, then the $\Omega_{Z,H}$-closed (and $\Omega_{Z,H}$-irreducible) subsets are precisely the Zariski closed sets with G-homogeneous (G-prime) vanishing ideal. Equivalently, a subset Y of Z is $\Omega_{Z,H}$-closed and $\Omega_{Z,H}$-irreducible if it is Ω_Z-closed and H operates transitively on the set of Ω_Z-irreducible components of Y.

Proposition 6.2. *Let $\mathfrak{t}$ denote the functor sending a ringed space to its space of closed irreducible subsets with induced structure sheaf. Let $\mathfrak{g}$ be the equivalence from schemes of finite type over $\mathbb{K}$ to prevarieties over $\mathbb{K}$. Then, we have mutually essentially inverse equivalences of categories*

$$\mathcal{A} \longleftrightarrow \mathcal{B}$$
$$\mathfrak{r}\colon X \longmapsto \mathfrak{g}(\mathfrak{f}(X)) = \mathrm{Spec}_{\max,X}(\mathcal{O}_X^{(0)})$$
$$\mathfrak{t}(Z, \Omega_{Z,H}, \mathcal{O}_{Z,H}) \longleftarrow [H \times Z \to Z]\colon \mathfrak{s}.$$

Here, $\mathfrak{r}(X)$ comes with the action by $\mathrm{Spec}_{\max}(\mathbb{K}[gr(X)])$, which is induced on affine charts by the map $R \to \mathbb{K}[gr(X)] \otimes_{\mathbb{K}} R, f_w \mapsto \chi^w \otimes f_w$.

Remark 6.3. Let Z be an affine H-variety. Then, H-orbits correspond naturally to G-prime ideals of $R := \mathcal{O}(Z)$ of the form $\mathfrak{m}^+ = \langle \mathfrak{m} \cap R^+ \rangle$, where $\mathfrak{m}$ is a maximal ideal of R. Let $X := \mathrm{Spec}_G(R)$, and, for a point $z \in Z$, set $S_z := \deg_G(R^+ \setminus I(Hz))$. The set $V_X(I(Hz))$ of G-prime ideals containing $I(Hz)$ fits into the known correspondence (e.g., **[18,** Proposition 3.8]) between the orbits contained in $\overline{Hz} \cong \mathrm{Spec}_{\max}(\mathbb{K}[S_z])$

and the faces of S_z in the following manner:

$$\begin{array}{rcl} \mathrm{orbits}(\overline{Hz}) \longleftrightarrow & V_X(I(Hz)) & \longleftrightarrow \mathrm{faces}(S_z) \\ Hz_0 \longmapsto & I(Hz_0), \quad \mathfrak{p} & \longmapsto \deg_G(R^+ \setminus \mathfrak{p}) \\ O_{\mathfrak{p}} \longleftarrow\!\shortmid & \mathfrak{p}, \qquad\quad \mathfrak{p}_\tau & \longleftarrow\!\shortmid \tau \end{array}$$

where $O_{\mathfrak{p}} := V_Z(\mathfrak{p}) \setminus \bigcup_{\mathfrak{p} \neq \mathfrak{q} \in V_X(\mathfrak{p})} V_Z(\mathfrak{q})$ and $\mathfrak{p}_\tau := I(Hz) + \sum_{w \in S_z \setminus \tau} R_w$. The above bijections are mutually inverse and order-reversing, where the order on orbits is defined as $Hz_0 < Hz_1 :\Leftrightarrow Hz_0 \subseteq \overline{Hz_1}$ and the other sets are ordered by inclusion.

For an $\Omega_{Z,H}$-closed-irreducible subset $Y \subseteq Z$, the stalk of $\mathcal{O}_{Z,H}$ at Y is defined as

$$(\mathcal{O}_{Z,H})_Y := \varinjlim_{\substack{U \in \Omega_{Z,H} \\ U \cap Y \neq \emptyset}} \mathcal{O}_Z(U).$$

It coincides with the stalk of $\mathcal{O}_{\mathfrak{s}(Z)}$ at the point $\eta \in \mathfrak{s}(Z)$ corresponding to Y. If Z is $\Omega_{Z,H}$-irreducible, then we denote by $\mathcal{K}_H$ the constant sheaf assigning the stalk at Z.

Remark 6.4. $\Omega_{Z,H}$ has a basis of affine H-invariant open sets if and only if $Z \in \mathcal{B}$ (e.g., Z allows a good quotient by H). In this case, the following statements hold:

(i) the stalk $(\mathcal{O}_{Z,H})_Y$ at Y coincides with the graded localization $\mathcal{O}(U)_{I(Y)}$ for every affine invariant open set U meeting Y. In particular, if Z is $\Omega_{Z,H}$-irreducible, then $\mathcal{K}_H$ is a G-simple sheaf.

(ii) The generic isotropy group of an $\Omega_{Z,H}$-closed $\Omega_{Z,H}$-irreducible set Y is $\mathrm{Spec}_{\max}(\mathbb{K}[G/\deg_G((\mathcal{O}_{Z,H})_Y^{+,*})])$. In particular, H acts freely on a big open subset of Z if and only if H acts freely on non-empty open subsets of all H-prime divisors, i.e., if and only if the stalks $(\mathcal{O}_{Z,H})_Y$ at all H-prime divisors have units in every degree.

A graded scheme X in $\mathcal{A}$ is of finite type, in particular, G-Noetherian. Hence, a G-integral X is a G-Krull scheme if and only if it is *G-normal*, i.e., X has a cover by G-spectra of G-normal rings. Correspondingly, an H-prevariety $Z \in \mathcal{B}$ is called *H-normal* if and only if the sections of $\mathcal{O}_{Z,H}$ over affine invariant subsets are $\mathbb{X}(H)$-normal. A *H-prime divisor* is an $\Omega_{Z,H}$-closed $\Omega_{Z,H}$-irreducible subset Y, which is maximal among the proper subsets of Z with these properties. An equivalent condition is that Y is closed and the Ω_Z-irreducible components of Y are one-codimensional and are permuted by H, compare **[2,** Section I.6.4]. Each H-prime divisor Y on an H-normal prevariety Z defines a discrete value sheaf $\mathbb{Z}^{(Y)}$ on $\Omega_{Z,H}$, which takes values $\mathbb{Z}$ if U intersects Y and 0 otherwise, and a discrete G-valuation $\nu_Y : \mathcal{K}_H^+ \to \mathbb{Z}^{(Y)}$. These define the G-Krull sheaf $\mathcal{O}_{Z,H} = \bigcap_Y (\mathcal{K}_H)_{\nu_Y} \subseteq \mathcal{K}_H$. Their sum defines a morphism

$$\mathrm{div}^H := \sum_Y \nu_Y : \mathcal{K}_H^+ \longrightarrow \mathrm{WDiv}^H := \bigoplus_Y \mathbb{Z}^{(Y)}$$

to the presheaf of H-Weil divisors. The image and cokernel presheaf are the *H-principal divisors* PDiv^H, respectively, the *H-class group* Cl^H. In this terminology, Theorem 0.4 translates into the following characterization of characteristic spaces $\mathrm{Spec}_Z(\mathcal{R}) \to Z$ of Cox sheaves of finite type.

Theorem 6.5. *Let $q\colon \widehat{Z} \to Z$ be a H-invariant morphism of prevarieties. Then, Z is normal and q is a characteristic space if and only if the following hold:*

(i) *$\widehat{Z}$ is H-normal with* $\deg_G(\mathcal{K}_H(\widehat{Z})^+) = G$;

(ii) *q is a good quotient and induces a commutative diagram of presheaves:*

$$\begin{tikzcd} \mathcal{K}^* \arrow[r, "\mathrm{div}"] \arrow[d, "q^*"', "\cong"] & \mathrm{WDiv} \\ (q_*\mathcal{K}_H)_0^* \arrow[r, "q_*\mathrm{div}^H"] & q_*\,\mathrm{WDiv}^H \arrow[u, "\cong", "\widehat{Y} \mapsto q(\widehat{Y})"'] \end{tikzcd}$$

(iii) $\mathrm{Cl}^H(\widehat{Z}) = 0$, *and* $\mathcal{O}(\widehat{Z})^{+,*} = \mathcal{O}(\widehat{Z})_0^*$.

If $\widehat{Z} = \mathrm{Spec}_Z(\mathcal{R})$ with a Cox sheaf $\mathcal{R}$, then with $\mathrm{div}_G := \sum_Y \mu_Y$ the following commutative diagram extends the diagram of (ii)*:*

$$\begin{tikzcd} \mathcal{S}^+ \arrow[r, two heads, "\mathrm{div}_G"] \arrow[d, "q^*"', "\cong"] & \mathrm{WDiv} \\ q_*\mathcal{K}_H^+ \arrow[r, two heads, "q_*\mathrm{div}^H"] & q_*\,\mathrm{WDiv}^H \arrow[u, "\cong", "\widehat{Y} \mapsto q(\widehat{Y})"'] \end{tikzcd}$$

Each prime divisor Y is the image $q(\widehat{Y})$ of a unique H-prime divisor $\widehat{Y}$. If $H\widehat{z} \subseteq \widehat{Z}$ is the unique closed orbit in $q^{-1}(z)$, then $H\widehat{z} \subseteq \widehat{Y}$ if and only if $z \in Y$. In particular, $(\mathcal{O}_{\widehat{X},H})_{H\widehat{z}} = \mathcal{R}_z$.

This result only required those properties of normal prevarieties which they share with Krull schemes. Using the fact that X_{reg} is big in X, and $\mathrm{WDiv}(X_{\mathrm{reg}}) = \mathrm{CaDiv}(X_{\mathrm{reg}})$ combined with Noetherianity yields additional properties of characteristic spaces, e.g., irreducibility and normality, and the property that $q^{-1}(X_{\mathrm{reg}})$ is a big subset on which H acts freely and q is geometric, see [**2,** Sect. I.5, I.6] for proofs and further properties of characteristic spaces.

Remark 6.6. Recall that a lattice is a finitely generated free abelian group. By a separated *toric* graded scheme over $\mathbb{K}$, we mean a quasi-compact separated graded scheme X of finite type over $\mathbb{K}$ such that

(i) the grading group $M = \mathrm{gr}(X)$ is a lattice;

(ii) X is M-normal, i.e., $\mathcal{O}(U)$ is M-normal for every affine open $U \subseteq X$;

(iii) X contains $\mathrm{Spec}_M(\mathbb{K}[M])$ as a dense open subset;

(iv) X is *effectively graded*, i.e., $\langle \deg_M(\mathcal{O}(U)) \rangle = M$ for all affine open $U \subseteq X$.

$Z \in \mathcal{B}$ is a (separated) toric variety if and only if $X = \mathfrak{s}(Z) \in \mathcal{A}$ is a separated toric graded scheme. If Σ is the fan in $N = M^*$ describing Z where $M = \mathrm{gr}(X)$, then $\Omega_{Z,T}$ is finite and its basis consists of the affine invariant charts $\{Z_\sigma\}_{\sigma \in \Sigma}$. The $\Omega_{Z,T}$-irreducible $\Omega_{Z,T}$-closed subsets are the orbit closures $\{V(\sigma)\}_{\sigma \in \Sigma}$. For $\sigma \in \Sigma$, let $I_\sigma \trianglelefteq \mathbb{K}[M \cap \sigma^\vee]$ be the vanishing ideal of the closed orbit of Z_σ. Then, $\mathcal{O}_{X,I_\sigma} = \mathbb{K}[M \cap \sigma^\vee]_{I_\sigma} = \mathbb{K}[M \cap \sigma^\vee]$, and there is a natural bijection

$$\begin{aligned} \Sigma &\longrightarrow X \\ \sigma &\longmapsto I_\sigma \\ \deg_M(\mathcal{O}_{X,p})^\vee &\longleftarrow\!\!\shortmid\; p. \end{aligned}$$

Furthermore, applying Example 4.3 to each monoid $\sigma^\vee \cap M$ and its monoid algebra $\mathbb{K}[\sigma^\vee \cap M]$, we see that X is canonically homeomorphic to the $\mathbb{F}_1$-scheme A_Σ obtained by gluing the spectra of $(\sigma^\vee \cap M) \sqcup \{\infty\}$. For a cone $\sigma \in \Sigma$, we denote by p_σ the closed point $(\sigma^\vee \cap M) \sqcup \{\infty\} \setminus (\sigma^\perp \cap M)$ of $\mathrm{Spec}((\sigma^\vee \cap M) \sqcup \{\infty\})$. The canonical bijection between A_Σ and Σ is

$$\begin{aligned} \Sigma &\longrightarrow A_\Sigma \\ \sigma &\longmapsto p_\sigma \\ (\mathcal{O}_{A_\Sigma,p} \setminus \{\infty\})^\vee &\longleftarrow\!\!\shortmid\; p. \end{aligned}$$

Thus, we may view A_Σ as the orbit space of the toric variety Z_Σ. This observation complements the connection between the categories of $\mathbb{F}_1$-schemes of finite type and toric varieties established in **[14]**.

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