Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 421601, 10 pages
doi:10.1155/2011/421601

Research Article

Further Results on the Dilogarithm Integral

Biljana Jolevska-Tuneska[1] and Brian Fisher[2]

[1] *Faculty of Electrical Engineering and Informational Technologies, Ss. Cyril and Methodius University in Skopje, Karpos II bb, 1000 Skopje, Macedonia*
[2] *Department of Mathematics, University of Leicester, Leicester LE1 7RH, England, UK*

Correspondence should be addressed to Biljana Jolevska-Tuneska, biljanaj@feit.ukim.edu.mk

Received 5 May 2011; Accepted 4 July 2011

Academic Editor: Ch Tsitouras

The dilogarithm integral $\mathrm{Li}(x^s)$ and its associated functions $\mathrm{Li}_+(x^s)$ and $\mathrm{Li}_-(x^s)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of these functions and other functions are then found.

1. Introduction

The dilogarithm integral $\mathrm{Li}(x)$ is defined by

$$\mathrm{Li}(x) = -\int_0^x \frac{\ln|1-t|}{t}\,dt \tag{1.1}$$

(see [1]). More generally, we have

$$\mathrm{Li}(x^s) = -\int_0^{x^s} \frac{\ln|1-t|}{t}\,dt = -s\int_0^x \frac{\ln|1-t^s|}{t}\,dt \tag{1.2}$$

for $s = 1, 2, \ldots$.

The associated functions $\mathrm{Li}_+(x^s)$ and $\mathrm{Li}_-(x^s)$ are defined by

$$\mathrm{Li}_+(x^s) = H(x)\mathrm{Li}(x^s), \qquad \mathrm{Li}_-(x^s) = H(-x)\mathrm{Li}(x^s) = \mathrm{Li}(x^s) - \mathrm{Li}_+(x^s), \tag{1.3}$$

where $H(x)$ denotes Heaviside's function.

Next, we define the distribution $\ln|1-x^s|x^{-1}$ by

$$\ln|1-x^s|x^{-1} = -s^{-1}[\mathrm{Li}(x^s)]', \tag{1.4}$$

and its associated distributions $\ln|1-x^s|x_+^{-1}$ and $\ln|1-x^s|x_-^{-1}$ are defined by

$$\begin{aligned}\ln|1-x^s|x_+^{-1} &= H(x)\ln|1-x^s|x^{-1} = -s^{-1}[\mathrm{Li}_+(x^s)]',\\ \ln|1-x^s|x_-^{-1} &= H(-x)\ln|1-x^s|x^{-1} = -s^{-1}[\mathrm{Li}_-(x)]'.\end{aligned} \tag{1.5}$$

The classical definition of the convolution product of two functions f and g is as follows.

Definition 1.1. *Let f and g be functions. Then the convolution $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt \tag{1.6}$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f, \tag{1.7}$$

and if $(f * g)'$ and $f * g'$(or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or} \quad f' * g). \tag{1.8}$$

Definition 1.1 can be extended to define the convolution $f * g$ of two distributions f and g in $\mathfrak{D}'$ with the following definition; see Gel'fand and Shilov [2].

Definition 1.2. *Let f and g be distributions in $\mathfrak{D}'$. Then the convolution $f * g$ is defined by the equation*

$$\langle (f * g)(x), \varphi\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle \tag{1.9}$$

for arbitrary φ in $\mathfrak{D}$, provided f and g satisfy either of the following conditions:

(a) *either f or g has bounded support,*

(b) *the supports of f and g are bounded on the same side.*

It follows that if the convolution $f * g$ exists by this definition then (1.7) and (1.8) are satisfied.

In order to extend Definition 1.2 to distributions which do not satisfy conditions (a) or (b), let τ be a function in $\mathfrak{D}$, see [3], satisfying the conditions:

(i) $\tau(x) = \tau(-x)$,

(ii) $0 \leq \tau(x) \leq 1$,

(iii) $\tau(x) = 1 \quad \text{for} \quad |x| \leq 1/2$,

(iv) $\tau(x) = 0 \quad \text{for} \quad |x| \geq 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau\left(n^n x - n^{n+1}\right), & x > n, \\ \tau\left(n^n x + n^{n+1}\right), & x < -n, \end{cases} \tag{1.10}$$

for $n = 1, 2, \ldots$.

The following definition of the noncommutative neutrix convolution was given in [4].

Definition 1.3. *Let f and g be distributions in $\mathfrak{D}'$, and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the noncommutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that*

$$N - \lim_{n \to \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle \tag{1.11}$$

for all φ in $\mathfrak{D}$, where N is the neutrix, see van der Corput [5], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \ldots \tag{1.12}$$

and all functions which converge to zero in the normal sense as n tends to infinity.

In particular, if

$$\lim_{n \to \infty} \langle f_n * g, \varphi \rangle = \langle h, \varphi \rangle \tag{1.13}$$

exists, we say that the non-commutative convolution $f \circledast g$ exists.

It is easily seen that any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution. Note also that because of the lack of symmetry in the definition of $f \circledast g$ the neutrix convolution is in general non-commutative.

The following results proved in [4] hold, first showing that the neutrix convolution is a generalization of the convolution.

Theorem 1.4. *Let f and g be distributions in $\mathfrak{D}'$, satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g. \tag{1.14}$$

Theorem 1.5. *Let f and g be distributions in $\mathfrak{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. Then the neutrix convolution $f \circledast g'$ exists and*

$$(f \circledast g)' = f \circledast g'. \tag{1.15}$$

Note however that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$ but we do have the following theorem.

Theorem 1.6. *Let f and g be distributions in $\mathfrak{D}'$ and suppose that the neutrix convolution $f \circledast g$ exists. If $N-\lim_{n\to\infty}\langle (f\tau_n') * g, \varphi\rangle$ exists and equals $\langle h, \varphi\rangle$ for all φ in $\mathfrak{D}$, then $f' \circledast g$ exists and*

$$(f \circledast g)' = f' \circledast g + h. \tag{1.16}$$

2. Main Result

We define the function $I_{s,r}(x)$ by

$$I_{s,r}(x) = \int_0^x u^r \ln|1-u^s|\, du \tag{2.1}$$

for $r = 0,1,2,\ldots$ and $s = 1,2,\ldots$. In particular, we define the function $I_r(x)$ by

$$I_r(x) = I_{1,r}(x) \tag{2.2}$$

for $r = 0,1,2,\ldots$.

The following theorem was proved in [6].

Theorem 2.1. *The convolutions $Li_+(x) * x_+^r$ and $\ln|1-x|x_+^{-1} * x_+^r$ exist and*

$$\mathrm{Li}_+(x) * x_+^r = \frac{1}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}I_{r-i}(x)x_+^i + \frac{1}{r+1}x_+^{r+1}\mathrm{Li}_+(x) \tag{2.3}$$

for $r = 0,1,2,\ldots$ and

$$\ln|1-x|x_+^{-1} * x_+^r = \sum_{i=0}^{r-1}\binom{r}{i}(-1)^{r-i}I_{r-i-1}(x)x_+^i - \mathrm{Li}_+(x)x_+^r \tag{2.4}$$

for $r = 1,2,\ldots$.

We now prove the following generalization of Theorem 2.1.

Theorem 2.2. *The convolutions $\mathrm{Li}_+(x^s) * x_+^r$ and $\ln|1-x^s|x_+^{-1} * x_+^r$ exist and*

$$\mathrm{Li}_+(x^s) * x_+^r = \frac{s}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}I_{s,r-i}(x)x_+^i + \frac{1}{r+1}\mathrm{Li}_+(x^s)x_+^{r+1} \tag{2.5}$$

for $r = 0, 1, 2, \ldots, s = 1, 2, \ldots$ *and*

$$\ln|1 - x^s|x_+^{-1} * x_+^r = \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i} I_{s,r-i-1}(x) x_+^i - \frac{1}{s} \mathrm{Li}_+(x^s) x_+^r \tag{2.6}$$

for $r, s = 1, 2, \ldots$.

Proof. It is obvious that $\mathrm{Li}_+(x^s) * x_+^r = 0$ if $x < 0$.

When $x > 0$, we have

$$\begin{aligned}
\mathrm{Li}_+(x^s) * x_+^r &= -s \int_0^x (x-t)^r \int_0^t u^{-1} \ln|1 - u^s| du \, dt \\
&= -s \int_0^x u^{-1} \ln|1 - u^s| \int_u^x (x-t)^r dt \, du \\
&= \frac{s}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i} x^i \binom{r+1}{i} \int_0^x u^{r-i} \ln|1 - u^s| du \\
&= \frac{s}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i} x^i I_{s,r-i}(x) + \frac{1}{r+1} x^{r+1} \mathrm{Li}(x^s)
\end{aligned} \tag{2.7}$$

proving (2.5).

Next, using (1.8) and (2.5), we have

$$\begin{aligned}
-s \ln|1 - x^s|x_+^{-1} * x_+^r &= r \mathrm{Li}_+(x^s) * x_+^{r-1} \\
&= s \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i-1} I_{s,r-i-1}(x) x_+^i + \mathrm{Li}_+(x^s) x^r,
\end{aligned} \tag{2.8}$$

and (2.6) follows. □

Corollary 2.3. *The convolutions* $\mathrm{Li}_-(x^s) * x_-^r$ *and* $\ln|1 - x^s|x_-^{-1} * x_-^r$ *exist and*

$$\mathrm{Li}_-(x^s) * x_-^r = \frac{s}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} (-1)^{r-i+1} I_{s,r-i}(x) x_-^i + \frac{1}{r+1} \mathrm{Li}_-(x^s) x_-^{r+1} \tag{2.9}$$

for $r = 0, 1, 2, \ldots, s = 1, 2, \ldots$ *and*

$$\ln|1 - x^s|x_-^{-1} * x_-^r = \sum_{i=0}^{r-1} \binom{r}{i} (-1)^{r-i+1} I_{s,r-i-1}(x) x_-^i - \frac{1}{s} Li_-(x^s) x_-^r \tag{2.10}$$

for $r, s = 1, 2, \ldots$.

Proof. Equations (2.9) and (2.10) are obtained applying a similar procedure as used in obtaining (2.5) and (2.6).

The next two theorems were proved in [6] and to prove it, our set of negligible functions was extended to include finite linear sums of the functions $n^s \operatorname{Li}(n^r)$ for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$. □

Theorem 2.4. *The convolution* $\operatorname{Li}_+(x) \circledast x^r$ *exists and*

$$\operatorname{Li}_+(x) \circledast x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i \tag{2.11}$$

for $r = 0, 1, 2, \ldots$.

Theorem 2.5. *The convolution* $\ln|1-x| x_+^{-1} \circledast x^r$ *exists and*

$$\ln|1-x| x_+^{-1} \circledast x^r = \sum_{i=0}^{r-1} \binom{r}{i} \frac{(-1)^{r-i+1}}{(r-i)^2} x^i \tag{2.12}$$

for $r = 1, 2, \ldots$.

Before proving some further results, we need the following lemma.

Lemma 2.6. *If* $r + 1/s \in \mathbf{N}$ *for* $r, s = 1, 2, \ldots$, *then*

$$N - \lim_{n \to \infty} I_{s,r}(n) = -\frac{s}{(r+1)^2}. \tag{2.13}$$

Proof. Because

$$I_{s,r}(n) = \frac{1}{r+1}\left(n^{r+1} - 1\right) \ln|1 - n^s| - \frac{1}{r+1} \int_0^{n^s} \frac{1 - t^{(r+1)/s}}{1-t} dt \tag{2.14}$$

when $r + 1/s \in \mathbf{N}$, we have

$$\begin{aligned} I_{s,r}(n) &= \frac{1}{r+1}\left(n^{r+1} - 1\right) \ln|1 - n^s| - \frac{1}{r+1} \sum_{i=0}^{(r+1)/s-1} \frac{n^{si+s}}{i+1} \\ &= \frac{1}{r+1}\left(n^{r+1} - 1\right)\left(s \ln n + \ln\left|1 - n^{-s}\right|\right) - \frac{1}{r+1} \sum_{i=0}^{(r+1)/s-1} \frac{n^{si+s}}{i+1}, \end{aligned} \tag{2.15}$$

and (2.13) follows. □

We now prove the following generalization of Theorems 2.4 and 2.5.

Theorem 2.7. *The neutrix convolution* $\mathrm{Li}_+(x^s) \circledast x^r$ *exists when* $r+1/s \in \mathbf{N}$ *and*

$$\mathrm{Li}_+(x^s) \circledast x^r = \frac{s^2}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^{r-i}}{(r-i+1)^2}x_+^i \tag{2.16}$$

for $r, s = 1, 2, \ldots$.

Proof. We put $[\mathrm{Li}_+(x^s)]_n = \mathrm{Li}_+(x^s)\tau_n(x)$. Then the convolution $[\mathrm{Li}_+(x^s)]_n * x^r$ exists by Definition 1.1 and

$$\begin{aligned}[\mathrm{Li}_+(x^s)]_n * x^r &= \int_0^n \mathrm{Li}(t^s)(x-t)^r\, dt + \int_n^{n+n^{-n}} \tau_n(t)\mathrm{Li}(t^s)(x-t)^r dt \\ &= I_1 + I_2,\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}I_1 &= \int_0^n \mathrm{Li}(t^s)(x-t)^r dt \\ &= -s\int_0^n (x-t)^r \int_0^t \frac{\ln|1-u^s|}{u} du\, dt \\ &= -s\int_0^n \frac{\ln|1-u^s|}{u}\int_u^n (x-t)^r dt\, du \\ &= \frac{s}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i}x^i\binom{r+1}{i}\int_0^n u^{-1}\ln|1-u^s|\left\{u^{r-i+1}-n^{r-i+1}\right\}du \\ &= \frac{s}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}x^i I_{s,r-i}(n) + \frac{1}{r+1}x^{r+1}\mathrm{Li}(n^s) \\ &\quad + \frac{1}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}(-1)^{r-i}x^i\mathrm{Li}(n^s)n^{r-i+1}.\end{aligned} \tag{2.18}$$

Thus, using Lemma 2.6, we have

$$\mathrm{N}-\lim_{n\to\infty} I_1 = \frac{s^2}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^{r-i+1}}{(r-i+1)^2}x^i. \tag{2.19}$$

Further, it is easily seen that

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} \tau_n(t)\mathrm{Li}(t^s)(x-t)^r dt = 0, \tag{2.20}$$

and (2.16) follows from (2.17), (2.19), and (2.20), proving the theorem. □

Theorem 2.8. *The neutrix convolution* $\ln|1-x^s|x_+^{-1} \circledast x^r$ *exists when* $r+1/s \in \mathbf{N}$ *and*

$$\ln|1-x^s|x_+^{-1} \circledast x^r = s\sum_{i=0}^{r-1}\binom{r}{i}\frac{(-1)^{r-i}}{(r-i)^2}\, x_+^i \tag{2.21}$$

for $r, s = 1, 2, \ldots$.

Proof. Using Theorems 1.5 and 1.6, we have

$$-s\ln|1-x^s|x_+^{-1} \circledast x^r = r\mathrm{Li}_+(x^s) \circledast x^{r-1} + \underset{n\to\infty}{N-\lim}\left[\mathrm{Li}_+(x^s)\tau_n'(x)\right] * x^r, \tag{2.22}$$

where, on integration by parts, we have

$$\begin{aligned}\left[\mathrm{Li}_+(x^s)\tau_n'(x)\right] * x^r &= \int_n^{n+n^{-n}} \tau_n'(t)\mathrm{Li}(t^s)(x-t)^r dt \\ &= -\mathrm{Li}(n^s)(x-n)^r - s\int_n^{n+n^{-n}} \ln|1-t^s|t^{-1}(x-t)^r\tau_n(t)dt \\ &\quad + r\int_n^{n+n^{-n}} \mathrm{Li}(t^s)(x-t)^{r-1}\tau_n(t)dt.\end{aligned} \tag{2.23}$$

It is clear that

$$\begin{aligned}&\lim_{n\to\infty}\int_n^{n+n^{-n}} \ln|1-t^s|t^{-1}(x-t)^r\tau_n(t)dt = 0 \\ &\lim_{n\to\infty}\int_n^{n+n^{-n}} \mathrm{Li}(t^s)(x-t)^{r-1}\tau_n(t)dt = 0\ .\end{aligned} \tag{2.24}$$

It now follows from (2.23) and (2.24) that

$$\underset{n\to\infty}{N-\lim}\left[\mathrm{Li}_+(x^s)\tau_n'(x)\right] * x^r = 0. \tag{2.25}$$

Equation (2.21) now follows directly from (2.16) and (2.22), proving the theorem. □

Corollary 2.9. *The neutrix convolutions* $\mathrm{Li}_-(x^s) \circledast x^r$ *and* $\ln|1-x^s|x_-^{-1} \circledast x^r$ *exist when* $r+1/s \in \mathbf{N}$ *and*

$$\begin{aligned}\mathrm{Li}_-(x^s) \circledast x^r &= \frac{s^2}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^{r-i+1}}{(r-i+1)^2}x_-^i, \\ \ln|1-x^s|x_-^{-1} \circledast x^r &= s\sum_{i=0}^{r-1}\binom{r}{i}\frac{(-1)^{r-i+1}}{(r-i)^2}x_-^i\end{aligned} \tag{2.26}$$

for $r, s = 1, 2, \ldots$

Proof. Equations (2.26) are obtained applying a similar procedure as used in obtaining (2.16) and (2.21). □

Corollary 2.10. *The neutrix convolutions* $\mathrm{Li}_+(x^s) \circledast x_-^r$ *and* $\mathrm{Li}_-(x^s) \circledast x_+^r$ *exist when* $r + 1/s \in \mathbf{N}$ *and*

$$\begin{aligned}
\mathrm{Li}_+(x^s) \circledast x_-^r &= \frac{s}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^i}{(r-i+1)^2}\left[s-(r-i+1)^2 I_{s,r-i}(x)\right]x_+^i \\
&\quad + \frac{(-1)^{r+1}}{r+1}\mathrm{Li}_+(x^s)x^{r+1}, \\
\mathrm{Li}_-(x^s) \circledast x_+^r &= \frac{s}{r+1}\sum_{i=0}^{r}\binom{r+1}{i}\frac{(-1)^{i+1}}{(r-i+1)^2}\left[s-(r-i+1)^2 I_{s,r-i}(x)\right]x_-^i \\
&\quad + \frac{(-1)^{r+1}}{r+1}\mathrm{Li}_-(x^s)x^{r+1}
\end{aligned} \tag{2.27}$$

for $r, s = 1, 2, \ldots.$

Proof. Since the neutrix convolution product is distributive with respect to addition, we have

$$\mathrm{Li}_+(x^s) \circledast x^r = \mathrm{Li}_+(x^s) * x_+^r + (-1)^r \mathrm{Li}_+(x^s) \circledast x_-^r, \tag{2.28}$$

and (2.27) follows from (2.16) and (2.5). Equation (27) is obtained applying similar procedure as in the case of (2.27). □

Corollary 2.11. *The neutrix convolutions* $\ln|1-x^s|x_+^{-1} \circledast x_-^r$ *and* $\ln|1-x^s|x_-^{-1} \circledast x_+^r$ *exist when* $r + 1/s \in \mathbf{N}$ *and*

$$\begin{aligned}
\ln|1-x^s|x_+^{-1} \circledast x_-^r &= \sum_{i=0}^{r-1}\binom{r}{i}\frac{(-1)^i}{(r-i)^2}\left[s-(r-i)^2 I_{s,r-i-1}(x)\right]x_+^i \\
&\quad + \frac{(-1)^r}{s}Li_+(x^s)x^r, \\
\ln|1-x^s|x_-^{-1} \circledast x_+^r &= \sum_{i=0}^{r-1}\binom{r}{i}\frac{(-1)^{i+1}}{(r-i)^2}\left[s-(r-i)^2 I_{s,r-i-1}(x)\right]x_-^i \\
&\quad + \frac{(-1)^r}{s}\mathrm{Li}_-(x^s)x^r
\end{aligned} \tag{2.29}$$

for $r, s = 1, 2, \ldots$

Proof. Equation (2.29) follows from (2.21) and (2.6). Equation (29) is obtained applying similar procedure as in the case of (2.29). □

Acknowledgment

This research was supported by FEIT, University of Ss. Cyril and Methodius in Skopje, Republic of Macedonia, Project no. 08-3619/8.

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, NY, USA, 9th edition, 1972.
[2] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 1, chapter 1, Academic Press, New York, NY, USA, 1964.
[3] D. S. Jones, "The convolution of generalized functions," *The Quarterly Journal of Mathematics*, vol. 24, pp. 145–163, 1973.
[4] B. Fisher, "Neutrices and the convolution of distributions," *Univerzitet u Novom Sadu. Zbornik Radova Prirodno-Matematičkog Fakulteta. Serija za Matemati*, vol. 17, no. 1, pp. 119–135, 1987.
[5] J. G. van der Corput, "Introduction to the neutrix calculus," *Journal d'Analyse Mathématique*, vol. 7, pp. 291–398, 1959.
[6] B. Jolevska-Tuneska, B. Fisher, and E. Özçağ, "On the dilogarithm integral," *International Journal of Applied Mathematics*, vol. 24, no. 3, pp. 361–369, 2011.