A new approach for the solution of fourth-order Boundary Value Problems

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Abstract

Many of the results on second-order boundary value problems cannot be generalized to third- and fourth-order studies because they are difficult or impossible to solve. Therefore, solving higher-order boundary value problems is still an interesting topic. In this study, for the numerical solution of higher-order boundary value problems, we demonstrate an iteration method that includes the Green's function. We obtained the necessary conditions to demonstrate the validity of the method and proved the existence and uniqueness theorems. We solved a series of numerical examples to demonstrate considered method's reliability and accuracy and we compared our results with well-known existing methods in literature to prove its effectiveness. Numerical calculations were made using MATLAB.

Keywords: Fourth-order boundary value problem, numerical solutions, Green's function, convergence rate, iteration method.

1 Introduction

The first description of fixed point iteration of initial value problems was made by Liouville [12] in 1837. Following this work, [15] described fixed-point iteration for initial value problems (IVPs), which was modified into a new method by

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Mann [13] in 1953. These are methods that are still used today and referenced in many studies. These methods were generated by an arbitrary point $u_0 \in X$ and defined as follows:

$$\begin{aligned} u_{n+1} &= Tu_n = T^n u_0, \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n T u_n, \quad n \in Z_+ \end{aligned}$$

Here $T : X \to X$ is a mapping on a nonempty and convex subset X of a Banach space, and α_n is a parametric sequence in $(0, 1)$.

Various other methods were introduced in order to obtain approximate solutions of initial value problems and boundary value problems, such as the Jacobi-Gauss collocation method [4], class of hybrid collocation methods [3]. The most popular works are Na's [14] and Wei's[16]. Moreover, [5], [6], [7], [9], [11], [16], [14] and [16] are some other studies.

Consider the following fourth-order BVP,

$$L[v] = b_0(x)v^{''''}(x) + b_1(x)v^{'''}(x) + b_2(x)v^{''}(x) + b_3(x)v^{'}(x) + b_4(x)v(x) = g(x) \tag{1}$$

with the boundary conditions

$$\begin{aligned} B_{k_1}[v] &= \alpha_1 v(k_1) + \alpha_2 v^{'}(k_1) + \alpha_3 v^{''}(k_1) + \alpha_4 v^{'''}(k_1) = a_1 \\ B_{k_2}[v] &= \beta_1 v(k_2) + \beta_2 v^{'}(k_2) + \beta_3 v^{''}(k_2) + \beta_4 v^{'''}(k_2) = a_2 \\ B_{k_3}[v] &= \gamma_1 v(k_3) + \gamma_2 v^{'}(k_3) + \gamma_3 v^{''}(k_3) + \gamma_4 v^{'''}(k_3) = a_3 \\ B_{k_4}[v] &= \omega_1 v(k_4) + \omega_2 v^{'}(k_4) + \omega_3 v^{''}(k_4) + \omega_4 v^{'''}(k_4) = a_4 \end{aligned} \tag{2}$$

where $b_i(x), i = (1, \cdots, 4)$ are continuous functions, $x \in (a, b)$, $a_i, i = (1, \cdots 4)$ are constants and either $k_3 = k_1$ or $k_3 = k_2$ and either $k_4 = k_1$ or $k_4 = k_2$. The Green's function $G(x, s)$ corresponding to (1) is

$$G(x,s) = \begin{cases} c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4, & a < x < s \\ d_1 v_1 + d_2 v_2 + d_3 v_3 + d_4 v_4, & s < x < b \end{cases}, \tag{3}$$

where $x \neq s$, c_i, d_i are constants, and v_i are linearly independent for $(i = 1, \cdots, 4)$. We obtain the Green's function of (1) and (2) by using the following well-known five properties given in [8].

1. G assures the homogeneous boundary conditions:

$$B_{k_1}[G(x,s)] = B_{k_2}[G(x,s)] = B_{k_3}[G(x,s)] = B_{k_4}[G(x,s)] = 0. \tag{4}$$

2. G satisfies continuity at $x = s$:

$$c_1 v_1(s) + c_2 v_2(s) + c_3 v_3(s) + c_4 v_4(s) = d_1 v_1(s) + d_2 v_2(s) + d_3 v_3(s) + d_4 v_4(s). \tag{5}$$

3. $G^{'}$ satisfies continuity at $x = s$:

$$c_1 v_1^{'}(s) + c_2 v_2^{'}(s) + c_3 v_3^{'}(s) + c_4 v_4^{'}(s) = d_1 v_1^{'}(s) + d_2 v_2^{'}(s) + d_3 v_3^{'}(s) + d_4 v_4^{'}(s). \tag{6}$$

4. $G^{''}$ satisfies continuity at $x = s$:

$$c_1 v_1^{''}(s)+c_2 v_2^{''}(s)+c_3 v_3^{''}(s)+c_4 v_4^{''}(s) = d_1 v_1^{''}(s)+d_2 v_2^{''}(s)+d_3 v_3^{''}(s)+d_4 v_4^{''}(s). \quad (7)$$

5. Let $b_0 > 0$. Then $G^{'''}$ has jump discontinuity at $x = s$:

$$c_1 v_1^{'''}(s)+c_2 v_2^{'''}(s)+c_3 v_3^{'''}(s)+c_4 v_4^{'''}(s)+\frac{1}{b_0(s)} = d_1 v_1^{'''}(s)+d_2 v_2^{'''}(s)+d_3 v_3^{'''}(s)+d_4 v_4^{'''}(s). \quad (8)$$

A particular solution to $v^{''''} = f(t, v, v', v'', v''', v^{''''})$ is expressed in terms of G and is given by

$$v_p = \int_a^b G(x,s) g(s, v_p, v_p^{'}, v_p^{''}, v_p^{'''}) ds. \quad (9)$$

Although boundary value problems are important and most frequently used in fields such as physics and engineering, recently they are encountered in problems in statistics, economics, neural networks, computer science and various other fields. Wave equations, heat equations, optimization problems, heat conduction, gravity-based flows, boundary layer theory, elastic stability beam theory are some of the examples.

Boundary value problems (BVPs) contain ordinary differential equations that satisfy certain conditions and a certain class of differential equations can be solved analytically. Problems with higher order differential equations and complex boundary conditions, can only be solved by numerical methods and a closed solution can be obtained. Many of the results on second-order boundary value problems cannot be generalized to third- and fourth-order studies because they are difficult or impossible to solve. Although there are many studies in the literature on the solution of second-order boundary value problems and are still being developed, there are few studies on numerical solutions of higher-order boundary value problems solved by different methods. Therefore, the solution of higher order problems is still an interesting issue because the applicability of existing methods to higher order boundary value problems is not always possible. Moreover, the use of the iteration method including Green's method in the numerical solution of higher order boundary value problems is an extremely rare and new issue. In this study, Khan-Green's method will be generalized for the fourth order BVPs. This method, which has recently attracted attention in the literature [1], [2] and has been a reference for many studies due to the efficiency of the numerical results and its fast convergence to the real value, will be generalized for the fourth-order boundary value problems by determining the necessary conditions in order to be applied. This method includes both Picard's and Mann's methods. In the study, appropriate conditions were determined, the results were supported by theorems, and numerical examples were used to show that the convergence discussed was faster than other known methods.

The proposed iteration method is defined in the following form;

$$\begin{aligned} q_n &= (1-\alpha_n)p_n + \alpha_n T(p_n) \\ p_{n+1} &= T(q_n), \quad n \in Z_+ \end{aligned} \quad (10)$$

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where $p_0 \in X$.

2 Generalization of iteration method

Consider the following equation

$$v^{''''}(x) = g(x, v(x), v^{'}(x), v^{''}(x), v^{'''}(x)) \tag{11}$$

with the boundary conditions:

$$v(0) = 0, \quad v^{'}(0) = 0, \quad v(1) = 0, \quad v^{'}(1) = 0. \tag{12}$$

Let

$$L(v) + N(v) = g(x, v, v^{'}, v^{''}, v^{'''}). \tag{13}$$

Here $L(v)$ is a linear operator, $N(v)$ is a nonlinear operators and $g(x, v, v^{'}, v^{''}, v^{'''})$ is a nonlinear function.

Let the integral operator be defined as

$$T(v_p) = \int_a^b G(x, s) L(v_p) ds, \tag{14}$$

where v_p is the particular solution of Eq. (13) and $G(x, s)$ is the Green's function of $L(v)$. Let $v_p = v$. Then the precise form of the integral operator is

$$\begin{aligned} T(v) &= \int_a^b G(x,s)(L(v) + N(v) - g(s, v, v^{'}, v^{''}, v^{'''}) - N(v) + g(s, v, v^{'}, v^{''}, v^{'''}) ds \\ &= v + \int_a^b G(x,s)(L(v) + N(v) - g(s, v, v^{'}, v^{''}, v^{'''})) ds. \end{aligned} \tag{15}$$

Using Eq. (15) and Eq. (10), we obtain the general iterative form of Khan-Green's method.

$$\begin{aligned} q_n &= p_n + \alpha_n \int_a^b G(x,s)(L(p_n) + N(p_n) - g(s, p_n, p_n^{'}, p_n^{''}, p_n^{''''})) ds \\ p_{n+1} &= q_n + \int_a^b G(x,s)(L(q_n) + N(q_n) - g(s, q_n, q_n^{'}, q_n^{''}, q_n^{''''})) ds. \end{aligned} \tag{16}$$

Here $\{\alpha_n\}$ is a parametric sequence in $(0, 1)$, and the initial function p_0 that satistfies the boundary conditions is chosen to be the exact solution $L(v) = 0$.

3 Convergence analysis

Consider the fourth-order BVP given in (11)-(12). The adjoint of Green's function $G(x, s)$ of the defined BVP is

$$G^*(x,s) = -\begin{cases} (\frac{s^3}{3} - \frac{s^2}{2})(1-x)^3 + (\frac{-s^3}{2} + \frac{s^2}{2})(1-x)^2, & 0 < s < x, \\ \\ (\frac{-s^3}{3} + \frac{s^2}{2} - \frac{1}{6})x^3 + (\frac{s^3}{2} - s^2 + \frac{s}{2})x^2, & x < s < 1 \end{cases} \tag{17}$$

Implementing Khan-Green's iterative method, the following equation is obtained:

$$q_n = p_n - \alpha_n \int_0^x ((\tfrac{s^3}{3} - \tfrac{s^2}{2})(1-x)^3 + (\tfrac{-s^3}{2} + \tfrac{s^2}{2})(1-x)^2)$$

$$\times (p_n^{'''}(s) - g(s, p_n(s), p_n^{'}(s), p_n^{''}(s), p_n^{'''}(s)))ds$$

$$-\alpha_n \int_x^1 ((\tfrac{-s^3}{3} + \tfrac{s^2}{2} - \tfrac{1}{6})x^3 + (\tfrac{s^3}{2} - s^2 + \tfrac{s}{2})x^2)$$

$$\times (p_n^{'''}(s) - g(s, p_n(s), p_n^{'}(s), p_n^{''}(s), p_n^{'''}(s)))ds, \tag{18}$$

$$p_{n+1} = q_n - \int_0^x ((\tfrac{s^3}{3} - \tfrac{s^2}{2})(1-x)^3 + (\tfrac{-s^3}{2} + \tfrac{s^2}{2})(1-x)^2)$$

$$\times (q_n^{'''}(s) - g(s, q_n(s), q_n^{'}(s), q_n^{''}(s), q_n^{'''}(s)))ds$$

$$-\int_x^1 ((\tfrac{-s^3}{3} + \tfrac{s^2}{2} - \tfrac{1}{6})x^3 + (\tfrac{s^3}{2} - s^2 + \tfrac{s}{2})x^2)$$

$$\times (q_n^{'''}(s) - g(s, q_n(s), q_n^{'}(s), q_n^{''}(s), q_n^{'''}(s)))ds). \tag{19}$$

The rate of convergence for Eq. (11) - (12) can be found using the following integral operator.

$$T(v) = v + \int_0^1 G^*(x,s)(v^{''''}(s) - g(s, v(s), v^{'}(s), v^{''}(s), v^{'''}(s))ds. \tag{20}$$

Theorem 3.1 *Assume $g(x, v, v', v'', v''')$ be a continuous function, $g'(x, v, v', v'', v''')$ is bounded and*

$$\sup_{([0,1]\times R^5)} |\frac{\partial g}{\partial v}| < 40\sqrt{183}.$$

Then integral operator $T(v)$ given in Eq. (20) is a contraction on the Banach space.

Proof Let,

$$
\begin{aligned}
\|T(v)-T(u)\| &= \| \int_0^1 G^*(x,s) g(s,u,u^{'},u^{''},u^{'''})ds - \int_0^1 G^*(x,s) g(s,v,v^{'},v^{''},v^{'''})ds\| \\
&\leq \left(\int_0^1 |G^*(x,s)|^2 ds\right)^{(\frac{1}{2})} \left(\int_0^1 |g(s,u,u^{'},u^{''},u^{'''}) - g(s,v,v^{'},v^{''},v^{'''})|^2 ds\right)^{(\frac{1}{2})} \\
&= \frac{\sqrt{t^4(2t+1)^2(t-1)^8 + t^8(2t-3)^2(t-1)^4}}{24} \\
&\times \left(\int_0^1 |g(s,u,u^{'},u^{''},u^{'''}) - g(s,v,v^{'},v^{''},v^{'''})|^2 ds\right)^{\frac{1}{2}} \\
&\leq \frac{1}{40\sqrt{183}} \left(\int_0^1 |g(s,u,u^{'},u^{''},u^{'''}) - g(s,v,v^{'},v^{''},v^{'''})|^2 ds\right)^{\frac{1}{2}},
\end{aligned} \tag{21}
$$

From Eq.(21), by using the Mean Value Theorem, we obtain

$$
\|T(v)-T(u)\| \leq \frac{1}{40\sqrt{183}} \times \sup_{([0,1]\times R^5)} |\frac{\partial g}{\partial v}| \times \sup_{[0,1]} |u(t)-v(t)| < \sup_{[0,1]} |u(t)-v(t)| = \|u-v\|. \tag{22}
$$

is obtained. Thus, T is a Banach's contraction mapping.

Theorem 3.2 *Let the integral operator $T(y)$ given in (20) be a Banach contraction. Then the sequence p_n defined in (16) is convergent and this solution converges faster than the solutions in Picard Green's method (PGEM) and Mann Green's method (MGEM).*

Proof Let x^* be the solution of the problem (11)-(12). Then $T(x^*) = x^*$.
Let $p_0 \to x^*$ as $n \to \infty$. Then,

$$
\begin{aligned}
\|p_{n+1} - x^*\| &= \|T(q_n) - x^*\| \leq \delta \|q_n - x^*\| \\
&\leq \delta \|(1-\alpha_n)p_n + \alpha_n T(p_n) - x^*\| \\
&\leq \delta \left((1-\alpha_n)\|p_n - x^*\| + \delta\alpha_n \|p_n - x^*\|\right) \\
&= \delta(1-(1-\delta)\alpha_n)\|p_n - x^*\|.
\end{aligned} \tag{23}
$$

is obtained. Taking into account that $(1-(1-\delta)\alpha_n) < 1$ for $0 < \delta < 1$ and $\alpha_n \in (0,1)$, the following inequality holds:

$$
\|p_{n+1} - x^*\| \leq \delta \|p_n - x^*\| \leq \delta^{n+1} \|p_0 - x^*\|. \tag{24}
$$

Therefore, $\{p_n\}$ converges strongly to x^*. Proposition 1 in [10] can accomplish the remaining part of the proof.
Remark: The proof follows analogously with every set of boundary conditions.

4 Numerical Example

Example 4.1 *Consider the nonlinear boundary value problem*

$$v^{''''}(x) = \frac{-v^{'''}(x)}{24} + v(x)v^{'}(x) + \frac{v^{'}(x)}{8} + \frac{1}{2}, \tag{25}$$

with boundary conditions

$$v(0) = v^{'}(0) = v^{''}(1) = v^{'''}(1) = 0. \tag{26}$$

The Green's function for the given problem is

$$G(x,s) = \begin{cases} \frac{-x^3}{6} + \frac{sx^2}{2}, 0 < x < s \\ \frac{-s^3}{6} + \frac{xs^2}{2}, s < x < 1. \end{cases} \tag{27}$$

Applying Khan-Green's iteration method, we get

$$\begin{aligned} q_n &= p_n - \alpha_n \Bigg\{ \Bigg[\int_0^x (\frac{-s^3}{6} + \frac{xs^2}{2})(p_n^{''''}(s) + \frac{p_n^{'''}(s)}{24} - p_n(s)p_n^{'}(s) - \frac{p_n^{'}(s)}{8} - \frac{1}{2})ds \Bigg] \\ &+ \Bigg[\int_x^1 (\frac{-x^3}{6} + \frac{sx^2}{2})(p_n^{''''}(s) + \frac{p_n^{'''}(s)}{24} - p_n(s)p_n^{'}(s) - \frac{p_n^{'}(s)}{8} - \frac{1}{2})ds \Bigg] \Bigg\} \\ p_{n+1} &= q_n - \Bigg\{ \Bigg[\int_0^x (\frac{-s^3}{6} + \frac{xs^2}{2})(q_n^{''''}(s) + \frac{q_n^{'''}(s)}{24} - q_n(s)q_n^{'}(s) - \frac{q_n^{'}(s)}{8} - \frac{1}{2})ds \Bigg] \\ &+ \Bigg[\int_x^1 (\frac{-x^3}{6} + \frac{sx^2}{2})(q_n^{''''}(s) + \frac{q_n^{'''}(s)}{24} - q_n(s)q_n^{'}(s) - \frac{q_n^{'}(s)}{8} - \frac{1}{2})ds \Bigg] \Bigg\}. \end{aligned} \tag{28}$$

Here, $\alpha_n = 0.99$ and $p_0 = 0$.

We calculated numerical results with four different iterations for Khan, PGEM, and MGEM Methods (for $\alpha = 0.90$ and $\alpha = 0.75$ respectively) by using MATLAB. We calculated the absolute error

$$Error(n) = |v_{n+1} - v_n|. \tag{29}$$

Numerical results showing the absolute error values for the solution of Example 1 for different iterations are listed in Table 1. As seen in Table 1, no matter how many iterations are made to achieve high accuracy, the error value in the method we consider is much lower than other methods. This clearly shows that the method discussed is much more effective than other methods. Moreover, regarding Figure 1, it is obvious that Khan-Green's method has much better results than other methods when the error values are compared.

Example 4.2 *Consider the linear boundary value problem*

$$y^{''''}(x) = xy(x) + x^2y^{'}(x) + 2x + 1, \tag{30}$$

Table 1: Absolute errors obtained with different iteration methods for the problem in Example 1

t	Khan Err (4)	Picard Err (4)	Mann($\alpha = 0,90$) Err (4)	Mann($\alpha = 0,75$) Err (4)
0	0	0	0	0
0.2	$1.1349E-11$	$1.5246E-07$	$8.7290E-06$	$2.2034E-04$
0.4	$3.8614E-11$	$5.1883E-07$	$3.0099E-05$	$7.6466E-04$
0.6	$7.4020E-11$	$9.9475E-07$	$5.8328E-05$	$1.4901E-03$
0.8	$1.1276E-10$	$1.5151E-06$	$8.9515E-05$	$2.2965E-03$
1.0	$1.5232E-10$	$2.0467E-06$	$1.2149E-04$	$3.1253E-03$

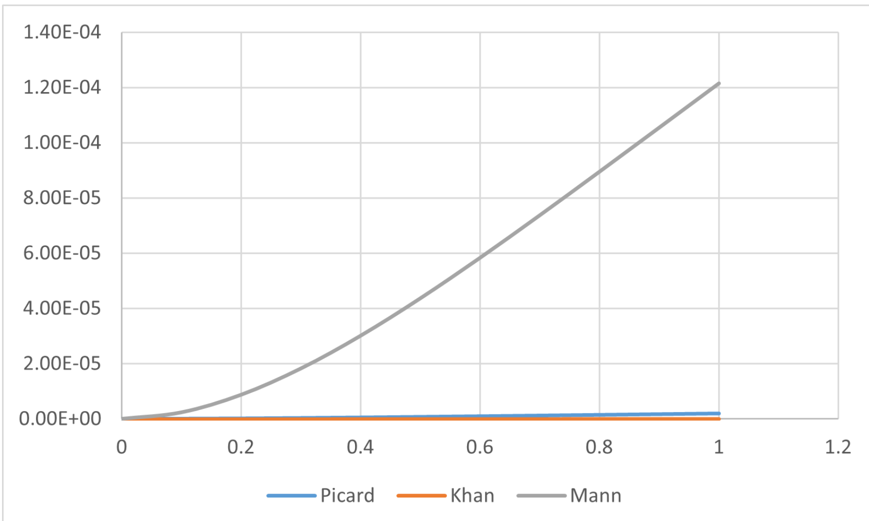

Figure 1: Comparisons of absolute errors obtained for the problem in Example 1

with boundary conditions

$$y(0) = y'(0) = y'(1) = y''(1) = 0. \tag{31}$$

The Green's function of (30) - (31) *is*

$$G(x,s) = \begin{cases} \frac{(s^2-1)x^3}{6} - \frac{(-s^2+s)x^2}{2}, & 0 < x < s \\ \frac{-s^2(1-x)^3}{6} + \frac{-s^3+s^2}{6}, & s < x < 1. \end{cases} \tag{32}$$

Then we have

$$q_n = p_n - \alpha_n \int_0^x (\frac{-s^2(1-x)^3}{6} + \frac{-s^3+s^2}{6})(p_n^{''''} - sp_n - s^2(p_n^{'})^2 - 2s - 1)ds \tag{33}$$

$$- \alpha_n \int_x^1 (\frac{(s^2-1)x^3}{6} - \frac{(-s^2+s)x^2}{2})(p_n^{''''} - sp_n - s^2(p_n^{'})^2 - 2s - 1)ds$$

$$p_{n+1} = q_n - \int_0^x (\frac{-s^2(1-x)^3}{6} + \frac{-s^3+s^2}{6})(q_n^{''''} - sq_n - s^2(q_n^{'})^2 - 2s - 1)ds$$

$$- \int_x^1 (\frac{(s^2-1)x^3}{6} - \frac{(-s^2+s)x^2}{2})(q_n^{''''} - sq_n - s^2(q_n^{'})^2 - 2s - 1)ds,$$

where $\alpha_n = 0.99$.*Here the initial value* $p_0 = 0$.

Table 2: Absolute errors obtained with different iteration methods for the problem in Example 2

t	Khan Err (5)	Picard Err (5)	Mann($\alpha = 0,99$) Err (5)	Mann($\alpha = 0,80$) Err(5)
0	0	0	0	0
0.2	$6.6325E-11$	$2.2596E-07$	$1.3694E-06$	$1.7425E-04$
0.4	$2.0855E-10$	$7.1052E-07$	$4.2213E-06$	$5.2108E-04$
0.6	$3.4732E-10$	$1.1834E-06$	$6.9238E-06$	$8.3418E-04$
0.8	$4.2700E-10$	$1.4549E-06$	$8.4389E-06$	$1.0024E-03$
1.0	$4.4256E-10$	$1.5079E-06$	$8.7297E-06$	$1.0337E-03$

Table (2) *compares the errors obtained by Khan-Green's iteration method with the approximation algorithms of PGEM and MGEM. In Khan Green's method we take* $\alpha = 0.99$ *and for MGEM the values* $\alpha = 0.99$ *and* $\alpha = 0.80$ *are taken respectively. It is clear from Table* (2) *that the results obtained by the generalized Khan-Green's fixed point iteration method converge to the values obtained by the real numerical solution faster than other known iteration methods.*

5 Conclusion

In this paper, we generalize the Khan-Green method, a new approach based on embedding the Green's function in a fixed-point iterative procedure, for fourth-order BVPs. We successfully applied the method for the numerical solution of a large family of fourth-order BVPs, achieving a faster approximation than well-known iteration methods in the literature. We used MATLAB for numerical calculations. Considering other studies in the literature and the numerical calculations made for different values of α_n throughout our study, we reached the best result for $\alpha_n = 0.99$. Moreover, we prove that the proposed methodology is

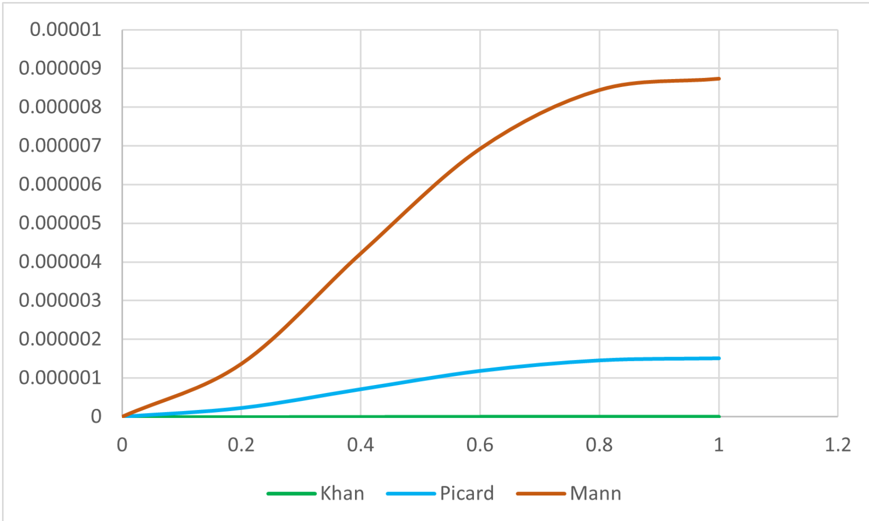

Figure 2: Comparisons of absolute errors obtained for the problem in Example 2

a better approach with minimal error. This new strategy improves the solutions of fourth-order boundary value problems, which are rare in the literature. The proposed method also increases the convergence rate of other existing methods based on Picard-Green and Mann-Green's iteration methods.

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