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THE ANSWER TO AN OPEN QUESTION IN $\mathbb{R}^m$-b-METRIC SPACES AND APPLICATION TO INTEGRAL EQUATIONS

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ABSTRACT. In this paper, we give an affirmative answer to an open question in $\mathbb{R}^m$-b-metric spaces by using the subordinate property of the matrix norm to the ℓ^∞-norm on $\mathbb{K}^m$. As applications, we get Perov's fixed point theorem and Matkowski's fixed point theorem in $\mathbb{R}^m$-b-metric spaces. We also show that the fixed point theorem in $\mathbb{R}^m$-b-metric spaces can be applied to prove the existence and uniqueness of the solution to an integral equation but fixed point theorems in metric spaces may not be.

1. Introduction and preliminaries

Many authors have established generalised metric spaces and studied fixed point theorems on such spaces, see [2], [10], [19], [24]. In 1964, Perov [30] defined a $\mathbb{R}^m$-*metric space* by replacing $\mathbb{R}_+$ by $\mathbb{R}^m_+$ in the definition of a metric space. In 1971, Coifman and Guzmán [14] defined a *quasi-metric space* by replacing the triangle inequality by

$$d(x,y) \leq s[d(x,z)+d(z,y)]$$

where $s \geq 1$. This notion was then reintroduced by the name *b-metric space* in [6], [15], [16]. For the developments in fixed point theory on b-metric spaces, the reader may refer to [9], [13], and [36]. There are several types of integral equations have been solved by using fixed point theorems in b-metric spaces, see [3, Theorem 4.1], [20, Example 2.3], [22, Theorem 6], [25, Theorem 5.1], [29, Theorem 5.1], [34, Theorem 4.1], [33, Theorem 3.1], [35, Theorem 3.1] and many others. However, as on [36, page 47], these integral equations may be solved by certain fixed point result in metric spaces instead of that in b-metric spaces. Then, the problem of finding an application of fixed point theorems in b-metric spaces but not in metric spaces is still open.

In 2009, Boriceanu [11] extended b-metric spaces to $\mathbb{R}^m$-b-metric spaces and presented some fixed point results for generalised single-valued and multi-valued contractions in such spaces. In 2017, Miculescu and Mihail [28] indicated a way to generalise a series of fixed point results in the framework of b-metric. In 2023, Bota *et al.* [12] proved the existence and stability results for cyclic graphical contractions in complete b-metric spaces are given. An application to a coupled fixed point problem is also derived. Bota *et al.* also asked to prove a similar result to [28, Lemma 2.2] for the case of $\mathbb{R}^m$-b-metric spaces, see Question 1.16 below.

First, we recall the following definitions and properties which will be used latter.

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Definition 1.1. (1) Let $\mathfrak{M}_{m,n}(\mathbb{K})$ be the set of all matrices of size $m\times n$ with entries belonging to $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$) and $A=(a_{ij})\in\mathfrak{M}_{m,n}(\mathbb{K})$. Then $|A|=(|a_{ij}|)$, where $|a_{ij}|$ is the modulus of a_{ij}.

(2) Let $\mathfrak{M}_{m,n}(\mathbb{R}_+)$ be the set of all matrices of size $m\times n$ with entries belonging to $\mathbb{R}_+$ and $A,B\in\mathfrak{M}_{m,n}(\mathbb{R}_+)$. Then
  (a) The matrix $\Theta\preceq A$ if all $0\le a_{ij}$, where $A=(a_{ij})\in\mathfrak{M}_{m,n}(\mathbb{R}_+)$, and $\Theta\in\mathfrak{M}_{m,n}(\mathbb{R}_+)$ is the zero matrix.
  (b) $A\preceq B$ if $\Theta\preceq B-A$.

(3) The norm $\|.\|$ is called *monotone with respect to the partial ordering* $\preceq$ in $\mathfrak{M}_{m,1}(\mathbb{R}_+)$ if for all $x,y\in\mathfrak{M}_{m,1}(\mathbb{R}_+), x\preceq y$, then $\|x\|\le\|y\|$.

In $\mathfrak{M}_{m,n}(\mathbb{K})$, we consider the following norms for all $A=(a_{ij})\in\mathfrak{M}_{m,n}(\mathbb{K})$.

(1) The Frobenius (or Schur or Euclidean) norm $||A||_F=(\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$.

(2) The ℓ^p-norm $||A||_p=(\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}|a_{ij}|^p)^{1/p}$ for $p\ge 1$.

(3) The ℓ^∞-norm $||A||_\infty=\max\limits_{i=1,\dots,m,j=1,\dots,n}|a_{ij}|$.

The above norms are monotone concerning the partial ordering $\preceq$ in Definition 1.1 in the case $\mathbb{K}=\mathbb{R}_+$. In several papers, the vector spaces $\mathfrak{M}_{m,1}(\mathbb{K})$ and $\mathbb{K}^m$ are identical.

Definition 1.2 ([5], Definition 3.1.2)**.** Let $|||.|||:\mathfrak{M}_{m,m}(\mathbb{K})\to\mathbb{R}_+$ be a map such that for all $A,B\in\mathfrak{M}_{m,m}(\mathbb{K}),\lambda\in\mathbb{K}$.

(1) $|||A|||=0$ if and only if $A=0$.
(2) $|||\lambda A|||=|\lambda|.|||A|||$.
(3) $|||A+B|||\le|||A|||+|||B|||$.
(4) $|||AB|||\le|||A|||.|||B|||$.

Then $|||.|||$ is called a *matrix norm on* $\mathfrak{M}_{m,m}(\mathbb{K})$.

Definition 1.3 ([5], Definition 3.1.3)**.** Let $\|.\|$ be a norm on $\mathfrak{M}_{m,1}(\mathbb{K})$ and for all $A\in\mathfrak{M}_{m,m}(\mathbb{K})$,

$$|||A|||=\sup_{x\in\mathfrak{M}_{m,1}(\mathbb{K}),x\neq\Theta}\frac{||Ax||}{||x||}. \tag{1.1}$$

Then $|||.|||$ is a matrix norm on $\mathfrak{M}_{m,m}(\mathbb{K})$ and is called *subordinate to the norm* $\|.\|$.

Remark 1.4. (1) On $\mathfrak{M}_{m,m}(\mathbb{K})$, ℓ^p-norm and ℓ^∞-norm are matrix norms and there exists a norm on $\mathfrak{M}_{m,m}(\mathbb{K})$ which is not a matrix norm, for example $||A||=\max\limits_{1\le i,j\le m}|a_{ij}|$ [5, Example 3.1.2].

(2) For the matrix norm $|||.|||$ on $\mathfrak{M}_{m,m}(\mathbb{K})$ which is subordinate to the norm $||.||$ on $\mathbb{K}^m$, then $||Ax||\le|||A|||.||x||$ for all $A\in\mathfrak{M}_{m,m}(\mathbb{K})$ and $x\in\mathfrak{M}_{m,1}(\mathbb{K})$.

(3) Let $\{A_n\}$ be a sequence of the matrices in $\mathfrak{M}_{m,m}(\mathbb{R}_+)$, where $a_{ij}^{(n)}$ is the entry in row i and column j of the matrix A_n. Then the sequence of the matrices $\{A_n\}$ is called *convergent to a matrix* $A=(a_{ij})$, written that $\lim\limits_{n\to\infty}A_n=A$, if $\lim\limits_{n\to\infty}a_{ij}^{(n)}=a_{ij}$ for all $i,j=1,\dots,m$.

Moreover, let the matrix norm $|||.|||$ be subordinate to the norm $||.||$ in $\mathfrak{M}_{m,1}(\mathbb{R}_+)$. Then $\lim\limits_{n\to\infty} A_n = A$ if and only if $\lim\limits_{n\to\infty} |||A_n - A||| = 0$ [37, page 12].

The following theorems present the relation between a matrix norm and a norm.

Theorem 1.5 ([5], Proposition 3.1.1)**.** *Assume that the matrix norm $|||.|||$ on $\mathfrak{M}_{m,m}(\mathbb{K})$ is subordinate to the norm $||.||$ on $\mathfrak{M}_{m,1}(\mathbb{K})$. Then we have*

(1) For all $A \in \mathfrak{M}_{m,m}(\mathbb{K})$ and $x \in \mathfrak{M}_{m,1}(\mathbb{K})$,

$$|||A||| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|\le 1} \|Ax\|. \tag{1.2}$$

(2) There exists $x_A \in \mathfrak{M}_{m,1}(\mathbb{K})$, $x_A \neq \Theta$ satisfying

$$|||A||| = \frac{\|Ax_A\|}{\|x_A\|}.$$

In particular, sup *can be replaced by* max *in* (1.1) *and* (1.2)*.*

(3) For all $A, B \in \mathfrak{M}_{m,m}(\mathbb{K})$, we have

$$|||AB||| \le |||A|||.|||B|||.$$

Theorem 1.6 ([5], Proposition 3.1.2)**.** *Assume that for all $A = (a_{ij}) \in \mathfrak{M}_{m,m}(\mathbb{K})$. Then*

(1) The matrix norm $|||.|||_1$ defined by

$$|||A|||_1 = \max_{1\le j\le m} \sum_{i=1}^{m} |a_{ij}|$$

is subordinate to the ℓ^1-norm on $\mathfrak{M}_{m,1}(\mathbb{K})$.

(2) The matrix norm $|||.|||_\infty$ defined by

$$|||A|||_\infty = \max_{1\le i\le m} \sum_{j=1}^{m} |a_{ij}|$$

is subordinate to the ℓ^∞-norm on $\mathfrak{M}_{m,1}(\mathbb{K})$.

Definition 1.7 ([18], page 149)**.** Let $A \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$. Then

(1) $\lambda \in \mathbb{C}$ satisfies $\det(A - \lambda I) = 0$ is called an *eigenvalue* of A, where I is the identity matrix in $\mathfrak{M}_{m,m}(\mathbb{R}_+)$.

(2) $\sigma(A) = \{\lambda : \lambda$ is the eigenvalue of $A\}$ is called the *spectrum* of A.

(3) $r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ is called the *spectral radius* of A.

The following lemma gives properties related to matrices that are convergent to zero.

Lemma 1.8 ([32], Lemma 2)**.** *Assume that $A \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$. Then the following statements are equivalent.*

(1) A is convergent to zero, that is, $\lim\limits_{n\to\infty} A^n = \Theta$, where $A^n = \underbrace{A.A\ldots A}_{n \text{ times}}$.

(2) $r(A) < 1$.

(3) $I - A$ is non-singular and $(I - A)^{-1}$ has non-negative elements.

The following lemma gives bounds on the size of the entries of the matrix A^k for all $k \in \mathbb{N}$.

Lemma 1.9 ([21], Corollary 5.6.13)**.** *Assume that $A \in \mathfrak{M}_{m,m}(\mathbb{C})$ and $\varepsilon > 0$. Then there exists a constant $c = c(A,\varepsilon)$ such that for all $k \in \mathbb{N}$ and all $i, j = 1,\dots,m$*

$$|(A^k)_{ij}| \leq c(r(A)+\varepsilon)^k$$

where $|(A^k)_{ij}|$ is the module of the entry in row i and column j of the matrix A^k.

Definition 1.10. Let X be a non-empty set, $s \geq 1$ and a map $d : X \times X \to \mathbb{R}$ satisfy for all $x,y,z \in X$,

(1) $d(x,y) = 0$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) \leq s(d(x,z)+d(z,y))$.

Then we have

(1) d is called a *b-metric* and (X,d,s) is called a *b-metric space* [16].
(2) If the condition (1) is replaced by $d(x,x) = 0$, then d is called a *pseudo-b-metric* [1].

Perov in [30] established a fixed point theorem in $\mathbb{R}^m$-metric spaces by replacing the contraction constant in $[0,1)$ in the Banach contraction principle by a matrix with the spectral radius in $[0,1)$.

Theorem 1.11 ([30], Perov's fixed point theorem in $\mathbb{R}^m$-metric spaces)**.** *Assume that*

(1) (X,d) is a complete $\mathbb{R}^m$-metric space and $f : X \to X$ is a map.
(2) There exists a matrix $A \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$ such that
   (a) $r(A) < 1$.
   (b) For all $x,y \in X$,

$$d(fx,fy) \preceq Ad(x,y).$$

Then f has a unique fixed point $x^* \in X$ and for all $x \in X, x^* = \lim_{n\to\infty} f^n x$.

Definition 1.12 ([11], Definition 2.1)**.** Let X be a non-empty set, $s \geq 1$ and a map $d : X \times X \to \mathfrak{M}_{m,1}(\mathbb{R}_+)$ satisfy for all $x,y,z \in X$,

(1) $d(x,y) = \Theta$ if and only if $x = y$.
(2) $d(x,y) = d(y,x)$.
(3) $d(x,y) \preceq s(d(x,z)+d(z,y))$.

Then d is called a $\mathbb{R}^m$-*b-metric* and (X,d,s) is called a $\mathbb{R}^m$-*b-metric space*.

Remark 1.13. (1) A $\mathbb{R}^m$-b-metric space is also called a *generalised b-metric space* [7, Definition 2.2].
(2) If we replace the coefficient $s \geq 1$ by the matrix $S \in \mathfrak{M}_{m,m}(\mathbb{R}_+), I \preceq S$ in the definition of the $\mathbb{R}^m$-b-metric space, then it is called a *generalised b-metric space* [31, Definition 2.1] or a *Czerwik generalised metric space* [4, Definition 2.1] with the additional condition "S is a diagonal matrix". Moreover, in [13, page 140], the authors introduced the notion of a *generalised b-metric*, where the generalised b-metric may take the value $+\infty$.
(3) For $m = 1$, $\mathbb{R}$-b-metric space (X,d,s) is a b-metric space in the sense of Czerwik [16].
(4) For $s = 1$, $\mathbb{R}^m$-b-metric space $(X,d,1)$ is a $\mathbb{R}^m$-metric space in the sense of Perov [30].

(5) The convergence, Cauchy sequence and completeness in $\mathbb{R}^m$-b-metric spaces are defined similarly as in b-metric spaces.
(6) $\{d_1,\dots,d_m\}$ is a *separating family of pseudo b-metrics* if for all $i=1,\dots,m$, d_i is pseudo b-metric and for all $x\neq y\in X$, then $d_i(x,y)>0$ for some $i=1,\dots,m$.

Lemma 1.14 ([12], Lemma 2.5)**.** *Let (X,d,s) be a $\mathbb{R}^m$-b-metric space and $\{x_n\}$ be a sequence in X. Then we have*

$$d(x_0,x_m)\preceq s^n\sum_{i=0}^{m-1}d(x_i,x_{i+1})$$

for all $n\in\mathbb{N}$ and $m=1,\dots,2^n$.

Bazine *et al.* [8] established the conditions so that a sequence $\{x_n\}$ in $\mathbb{R}^m$-b-metric spaces is Cauchy. However, these conditions required the matrix sA to be convergent to zero.

Lemma 1.15 ([8], Lemma 2)**.** *Assume that*

(1) (X,d,s) is a $\mathbb{R}^m$-b-metric space and $\{x_n\}$ is a sequence in X.
(2) There exists $A\in\mathfrak{M}_{m,m}(\mathbb{R}_+)$ such that
   (a) sA is convergent to zero.
   (b) For every $n\in\mathbb{N}$,

$$d(x_n,x_{n+1})\preceq Ad(x_{n-1},x_n).$$

Then the sequence $\{x_n\}$ is Cauchy in (X,d,s).

An open problem was raised in [12] by Bota *et al.* as follows.

Question 1.16 ([12], Conjecture on page 92)**.** *Is the condition "the matrix sA is convergent to zero" replaced by the condition "the matrix A is convergent to zero" in Lemma 1.15?*

In this paper, we give an affirmative answer to an open question in $\mathbb{R}^m$-b-metric spaces by using the subordinate property of the matrix norm to the ℓ^∞-norm on $\mathbb{K}^m$. As applications, we get Perov's fixed point theorem and Matkowski's fixed point theorem in $\mathbb{R}^m$-b-metric spaces. We also show that the fixed point theorem in $\mathbb{R}^m$-b-metric spaces is applicable to prove the existence and uniqueness of the solution to an integral equation but fixed point theorems in metric spaces may be not.

2. Main results

Firstly, we prove the following theorem to give an affirmative answer to Question 1.16. The following theorem only needs the condition "the matrix A is convergent to zero", that means "$r(A)<1$", while Lemma 1.15 needs the condition "$r(A)<\frac{1}{s}$". The novel technique here is to use the subordinate property of the matrix norm to the ℓ^∞-norm on $\mathbb{K}^m$ mentioned in Theorem 1.6.

Theorem 2.1. *Assume that*

(1) (X,d,s) is a $\mathbb{R}^m$-b-metric space and $\{x_n\}$ is a sequence in X.
(2) There exists $A\in\mathfrak{M}_{m,m}(\mathbb{R}_+)$ such that
   (a) A is convergent to zero.

(b) For every $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \preceq A d(x_{n-1}, x_n). \tag{2.1}$$

Then the sequence $\{x_n\}$ is Cauchy in (X, d, s).

Proof. Since $r(A) < 1$, we can choose $\varepsilon = \frac{1-r(A)}{2} > 0$. Putting $\gamma = r(A) + \varepsilon$, then $\gamma < 1$. From Lemma 1.9, we have any entry of the matrix A^n is less than or equal $c\gamma^n$ for all $n \in \mathbb{N}$. Therefore, we have

$$A^n \preceq c\gamma^n.B \tag{2.2}$$

where $B \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$, $B = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{pmatrix}$.

From (2.1), we have for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \preceq A^n d(x_0, x_1). \tag{2.3}$$

For all $l, k \in \mathbb{N}$, we denote $p = [\log_2 k]$ which is the integer part of $\log_2 k$. We get

$$d(x_{l+1}, x_{l+k}) \preceq \sum_{n=1}^{p} s^n d(x_{l+2^{n-1}}, x_{l+2^n}) + s^{p+1} d(x_{l+2^p}, x_{l+k}) \tag{2.4}$$

where $\sum_{n=1}^{p} s^n d(x_{l+2^{n-1}}, x_{l+2^n})$ is assumed to be Θ if $p = 0$.

Note that $k \leq 2^{p+1}$. From (2.4) and Lemma 1.14, we obtain

$$\begin{aligned} d(x_{l+1}, x_{l+k}) &\preceq \sum_{n=1}^{p} s^{2n} \sum_{i=l}^{l+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) + s^{2(p+1)} \sum_{i=l}^{l+k-2^p-1} d(x_{2^p+i}, x_{2^p+i+1}) \\ &\preceq \sum_{n=1}^{p} s^{2n} \sum_{i=l}^{l+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}) + s^{2(p+1)} \sum_{i=l}^{l+2^p-1} d(x_{2^p+i}, x_{2^p+i+1}) \\ &= \sum_{n=1}^{p+1} s^{2n} \sum_{i=l}^{l+2^{n-1}-1} d(x_{2^{n-1}+i}, x_{2^{n-1}+i+1}). \end{aligned} \tag{2.5}$$

From (2.2), (2.3), and (2.5), we get

$$\begin{aligned} d(x_{l+1}, x_{l+k}) &\preceq \sum_{n=1}^{p+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} A^{l+2^{n-1}+i} d(x_0, x_1) \\ &\preceq c \sum_{n=1}^{p+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} \gamma^{l+2^{n-1}+i} B d(x_0, x_1). \end{aligned} \tag{2.6}$$

By Theorem 1.6, we have the matrix norm $|||.|||_\infty$ on $\mathfrak{M}_{m,m}(\mathbb{R}_+)$ is subordinate to the norm $||.||_\infty$ on $\mathfrak{M}_{m,1}(\mathbb{R}_+)$. It follows from (2.6) and Remark 1.4 that

$$\begin{aligned} ||d(x_{l+1},x_{l+k})||_\infty &\leq ||c\sum_{n=1}^{p+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} \gamma^{l+2^{n-1}+i} Bd(x_0,x_1)||_\infty \\ &\leq c\sum_{n=1}^{p+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} \gamma^{l+2^{n-1}+i} ||Bd(x_0,x_1)||_\infty \\ &\leq c\sum_{n=1}^{p+1} s^{2n} \sum_{i=0}^{2^{n-1}-1} \gamma^{l+2^{n-1}+i} |||B|||_\infty ||d(x_0,x_1)||_\infty \\ &= c.m.\gamma^l ||d(x_0,x_1)||_\infty \sum_{n=1}^{p+1} s^{2n}\gamma^{2^{n-1}} \sum_{i=0}^{2^{n-1}-1} \gamma^i \\ &\leq c.m.\gamma^l \frac{||d(x_0,x_1)||_\infty}{1-\gamma} \sum_{n=1}^{p+1} s^{2n}\gamma^{2^{n-1}} \\ &= c.m.\gamma^l \frac{||d(x_0,x_1)||_\infty}{1-\gamma} \sum_{n=1}^{p+1} \gamma^{2^{n-1}+2n\log_\gamma s}. \end{aligned} \tag{2.7}$$

We find that $\lim_{n\to\infty}(2^{n-1}+2n\log_\gamma s - n) = \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $2^{n-1}+2n\log_\gamma s - n \geq 1$. Then $\gamma^{2^{n-1}+2n\log_\gamma s} \leq \gamma^{n+1}$. Therefore the series $\sum_{n=1}^{\infty} \gamma^{2^{n-1}+2n\log_\gamma s}$ is convergent. Combining with (2.7), we have

$$||d(x_{l+1},x_{l+k})||_\infty \leq c.m.\gamma^l \frac{||d(x_0,x_1)||_\infty}{1-\gamma} S \tag{2.8}$$

for all $l,k \in \mathbb{N}$ and $S = \sum_{n=1}^{\infty} \gamma^{2^{n-1}+2^n\log_\gamma s}$. Letting the limit as $l \to \infty$ in (2.8), we have

$$\lim_{l\to\infty} d(x_{l+1},x_{l+k}) = \Theta.$$

This proves that $\{x_n\}$ is Cauchy in (X,d,s). □

From Theorem 2.1, we deduce Perov's fixed point theorem in $\mathbb{R}^m$-b-metric spaces as follows.

Corollary 2.2 (Perov's fixed point theorem in $\mathbb{R}^m$-b-metric spaces)**.** *Assume that*

(1) (X,d,s) is a complete $\mathbb{R}^m$-b-metric space and $f: X \to X$ is a map.

(2) There exists a matrix $A \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$ such that

(a) $r(A) < 1$.

(b) For all $x,y \in X$,

$$d(fx,fy) \preceq Ad(x,y). \tag{2.9}$$

Then f has a unique fixed point $x^* \in X$ and for all $x \in X$, $x^* = \lim_{n\to\infty} f^n x$.

Proof. Let $x_0 \in X$ and $x_n = f^n x_0 = f x_{n-1}$ for all $n \in \mathbb{N}$. We get

$$d(x_n, x_{n+1}) = d(f x_{n-1}, f x_n) \preceq A d(x_{n-1}, x_n). \tag{2.10}$$

From (2.10), by Lemma 1.8 and Theorem 2.1, we infer $\{x_n\}$ is Cauchy. Since (X, d, s) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{2.11}$$

By (2.9), we also have

$$\begin{aligned} d(fx^*, x^*) &\preceq s(d(fx^*, fx_n) + d(fx_n, x^*)) \\ &= s(d(fx^*, fx_n) + d(x_{n+1}, x^*)) \\ &\preceq s(Ad(x^*, x_n) + d(x_{n+1}, x^*)). \end{aligned} \tag{2.12}$$

By putting $q = \max\limits_{1\le i\le m} \sum\limits_{j=1}^{m} a_{ij}$ with $A = (a_{ij})$ and using Remark 1.4, we have

$$\|Ad(x^*, x_n)\|_\infty \le |||A|||_\infty . \|d(x^*, x_n)\|_\infty \le q\|d(x^*, x_n)\|_\infty. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\begin{aligned} \|d(fx^*, x^*)\|_\infty &\le \|s(Ad(x^*, x_n) + d(x_{n+1}, x^*))\|_\infty \\ &\le \|s.Ad(x^*, x_n)\|_\infty + \|s.d(x_{n+1}, x^*)\|_\infty \\ &\le sq\|d(x^*, x_n)\|_\infty + s\|d(x_{n+1}, x^*))\|_\infty. \end{aligned} \tag{2.14}$$

Taking the limit as $n \to \infty$ in (2.14) and using (2.11), we have $\|d(fx^*, x^*)\|_\infty = 0$. That means $d(fx^*, x^*) = \Theta$. Therefore $fx^* = x^*$, that is, x^* is a fixed point of f.

Now, we show that x^* is a unique fixed point of f. Indeed, let z^* be also a fixed point of f. We have $x^* = fx^* = f^2x^* = \ldots = f^n x^*$ and $z^* = fz^* = f^2 z^* = \ldots = f^n z^*$ for all $n \in \mathbb{N}$. Hence, we obtain

$$\begin{aligned} d(x^*, z^*) &= d(f^n x^*, f^n z^*) \\ &\preceq Ad(f^{n-1}x^*, f^{n-1}z^*) \\ &\ldots \\ &\preceq A^n d(x^*, z^*). \end{aligned} \tag{2.15}$$

Since $r(A) < 1$, by Remark 1.4.(3), we have

$$\lim_{n\to\infty} |||A^n|||_\infty = 0. \tag{2.16}$$

Taking the norm $||.||_\infty$ in (2.15) and then taking the limit as $n \to \infty$ and using (2.16), we have

$$0 \le ||d(x^*, z^*)||_\infty \le \lim_{n\to\infty} |||A^n|||_\infty ||d(x^*, z^*)||_\infty = 0.||d(x^*, z^*)||_\infty = 0.$$

Then $||d(x^*, z^*)||_\infty = 0$, that is, $x^* = z^*$. Therefore, x^* is a unique fixed point of f. Moreover, since x_0 is arbitrary, by (2.11), we have $x^* = \lim\limits_{n\to\infty} f^n x$ for all $x \in X$. □

Next, by using the definitions directly, we give the following lemma to characterise a $\mathbb{R}^m$-b-metric by a separating family of pseudo b-metrics.

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Lemma 2.3. *Assume that*

(1) *X is a non-empty set and $d_i : X \times X \to \mathbb{R}$ for all $i = 1, \ldots, m$ are given functions.*
(2) *A function $d : X \times X \to \mathfrak{M}_{m,1}(\mathbb{R}_+)$ is defined by $d = (d_1, \ldots, d_m)$.*

Then d is a $\mathbb{R}^m$-b-metric on X if and only if $\{d_1, \ldots, d_m\}$ is a separating family of pseudo b-metrics on X.

In particular, $d_1, \ldots, d_m$ are called pseudo b-metrics associated with d.

The next lemma shows the equivalence of convergence, Cauchy sequence, and completeness between the $\mathbb{R}^m$-b-metric d and pseudo b-metrics $d_1, \ldots, d_m$ associated with d.

Lemma 2.4. *Assume that*

(1) *(X, d, s) is a $\mathbb{R}^m$-b-metric space.*
(2) *$d_1, \ldots, d_m$ are pseudo b-metrics associated with d.*

Then the following statements hold.

(1) *$\lim_{n\to\infty} d(x_n, x) = \Theta$ if and only if $\lim_{n\to\infty} d_i(x_n, x) = 0$ for all $i = 1, \ldots, m$.*
(2) *$\lim_{n,k\to\infty} d(x_n, x_k) = \Theta$ if and only if $\lim_{n,k\to\infty} d_i(x_n, x_k) = 0$ for all $i = 1, \ldots, m$.*
(3) *(X, d, s) is complete if and only if (X, d_i, s) is complete for all $i = 1, \ldots, m$.*

Remark 2.5. Assume that

(1) (X, d, s) is a $\mathbb{R}^m$-b-metric space.
(2) $d^{(l)}(x, y) = \|d(x, y)\|_l$ for all $x, y \in X$ and $l = 1, 2, \infty$.

Then we have

(1) $d^{(1)}, d^{(2)}, d^{(\infty)}$ are b-metrics associated with d.
(2) If one of the spaces (X, d, s), $(X, d^{(l)}, s)$ is complete for $l = 1, 2, \infty$, then all of them are complete.

Matkowski's theorem for self-maps of the Cartesian product of metric spaces was proved in [27]. After that, Jachymski and Klima [23] showed the relation between this theorem and Perov's fixed point theorem in $\mathbb{R}^m$-metric spaces. Next, we use Corollary 2.2 to prove Matkowski's fixed point theorem for self-maps on the Cartesian product of b-metric spaces.

Corollary 2.6 (Matkowski's fixed point theorem)**.** *Assume that $m \in \mathbb{N}$ and for all $i = 1, \ldots, m$,*

(1) *(X_i, d_i, s_i) are complete b-metric spaces and $X = X_1 \times \ldots \times X_m$.*
(2) *$f_i : X \to X_i$ are given maps and $f = (f_1, \ldots, f_m)$.*
(3) *There exists a matrix $A = (a_{ij}) \in \mathfrak{M}_{m,m}(\mathbb{R}_+)$ with $r(A) < 1$.*
(4) *For all $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in X$,*

$$d_i(f_i(x_1, \ldots, x_m), f_i(y_1, \ldots, y_m)) \le \sum_{j=1}^{m} a_{ij} d_j(x_j, y_j). \tag{2.17}$$

Then f has a unique fixed point $x^* = (x_1^*, \ldots, x_m^*) \in X$ and for all $x \in X$, $\lim_{n\to\infty} f^n x = x^*$.

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Proof. For all $x=(x_1,\dots,x_m), y=(y_1,\dots,y_m)\in X$, and $i=1,\dots,m$, we denote

$$D_i(x,y)=d_i(x_i,y_i) \tag{2.18}$$

and

$$D(x,y)=(D_1(x,y),\dots,D_m(x,y)).$$

Then $\{D_1,\dots,D_m\}$ is a separating family of pseudo b-metrics on X. By Lemma 2.3, (X,D,s) is a $\mathbb{R}^m$-b-metric space, where $s=\max\limits_{i=1,\dots,m} s_i$. By putting $\rho(x,y)=\sum\limits_{i=1}^{m} d_i(x_i,y_i)$ for all $x,y\in X$ and using Remark 2.5, we infer that (X,ρ,s) is a b-metric space. Since (X_i,d_i,s_i) is complete for all $i=1,\dots,m$, (X,ρ,s) is complete. Using Remark 2.5 and since (X,ρ,s) is complete for all $i=1,\dots,m$, we infer that (X,D,s) is complete. Combining (2.17) with (2.18), we obtain

$$\begin{aligned} D_i(fx,fy) &= d_i(f_i(x_1,x_2,\dots,x_m), f_i(y_1,y_2,\dots,y_m)) \\ &\le \sum_{j=1}^{m} a_{ij} d_j(x_j,y_j) \\ &= \sum_{j=1}^{m} a_{ij} D_j(x,y). \end{aligned} \tag{2.19}$$

By (2.19), we have

$$D(fx,fy)\preceq AD(x,y).$$

So, the assumptions of Corollary 2.2 are satisfied. Then f has a unique fixed point $x^*=(x_1^*,\dots,x_m^*)\in X$ and for all $x\in X$, $\lim\limits_{n\to\infty} f^n x = x^*$. □

There are several types of integral equations have been solved by using fixed point theorems in b-metric spaces, see [3, Theorem 4.1], [20, Example 2.3], [22, Theorem 6], [25, Theorem 5.1], [29, Theorem 5.1], [34, Theorem 4.1], [33, Theorem 3.1], [35, Theorem 3.1] and many others. However, as on [36, page 47], these integral equations may be solved just using a certain fixed point result in metric spaces, and without using any fixed point results in b-metric spaces.

Now, we apply Corollary 2.2 to prove the existence and the uniqueness of an integral equation which may not be solved by using fixed point theorems in metric spaces as mentioned on [36, page 47].

Indeed, in the following proof we have

$$|(Tx)(t)-(Ty)(t)|^{\frac{1}{p}} \le L\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} |x(s)-y(s)|^{\frac{1}{p}} ds.$$

It is equivalent to

$$|(Tx)(t)-(Ty)(t)| \le \Big(L\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} |x(s)-y(s)|^{\frac{1}{p}} ds\Big)^p.$$

However, we can not take the power p under the integral sign to get

$$|(Tx)(t)-(Ty)(t)| \le L^{\frac{1}{p}}\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} |x(s)-y(s)| ds.$$

Then the integral equation (2.21) may not be solved by using fixed point theorems in the metric space $C[0,1]$ as mentioned on [36, page 47].

Theorem 2.7. *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a function such that for all $x\in C[0,1]$, for some $L>0$ and $0<p\le 1$,*

$$|f(t,u)-f(t,v)|\le L|u-v|^{\frac{1}{p}} \tag{2.20}$$

for any $t\in[0,1]$ and $u,v\in\mathbb{R}$, and $f(s,x(s))$ is integrable with respect to s on $[0,1]$ for any $x\in C[0,1]$. Then the equation

$$x^{\frac{1}{p}}(t)=\int_0^t f(s,x(s))ds,\ \ t\in[0,1] \tag{2.21}$$

has a unique solution $x^*\in C[0,1]$.

Proof. We put

$$(Tx)(t)=\Big(\int_0^t f(s,x(s))ds\Big)^p$$

for all $t\in[0,1]$ and $x\in C[0,1]$. Then there exist $m\in\mathbb{N}$ and $t_j\in[0,1]$, $j=0,1,\dots,m$ such that $0=t_0<t_1<\dots<t_m=1$ and

$$\max_{1\le j\le m}(t_j-t_{j-1})\le\frac{p}{L}. \tag{2.22}$$

For all $j=1,\dots,m$ and $x,y\in C[0,1]$, we define

$$d_j(x,y)=\max_{t_{j-1}\le t\le t_j}|x(t)-y(t)|^{\frac{1}{p}}. \tag{2.23}$$

Then d_j is a pseudo b-metric in $C[0,1]$. Therefore, $\{d_1,\dots,d_m\}$ is a separating family of pseudo b-metrics in $C[0,1]$. Put $d=(d_1,\dots,d_m)$. By Lemma 2.3, d is a $\mathbb{R}^m$-b-metric in $C[0,\lambda]$ coefficient $s=2^{\frac{1}{p}-1}$. By Remark 2.5, we find that $d^{(\infty)}$ defined by $d^{(\infty)}(x,y)=\|d(x,y)\|_\infty$ for all $x,y\in C[0,1]$ is a b-metric in $C[0,1]$ with coefficient $s=2^{\frac{1}{p}-1}$. By Remark 2.5 and since $(C[0,1],d^\infty,2^{\frac{1}{p}-1})$ is complete, we have $(C[0,1],d,2^{\frac{1}{p}-1})$ is complete.

Now, for each $j=1,\dots,m,x,y\in C[0,1]$, $t\in[t_{j-1},t_j]$, by (2.20), (2.22), we get

$$
\begin{aligned}
|(Tx)(t)-(Ty)(t)|^{\frac{1}{p}} &= \left|\left(\int_0^t f(s,x(s))ds\right)^p - \left(\int_0^t f(s,y(s))ds\right)^p\right|^{\frac{1}{p}} \\
&\leq \int_0^t |f(s,x(s))-f(s,y(s))|ds \\
&\leq \int_0^{t_j} |f(s,x(s))-f(s,y(s))|ds \\
&= \sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} |f(s,x(s))-f(s,y(s))|ds \\
&\leq L\sum_{i=1}^{j}\int_{t_{i-1}}^{t_i} |x(s)-y(s)|^{\frac{1}{p}}ds \\
&\leq L\sum_{i=1}^{j} d_i(x,y)(t_i-t_{i-1}) \\
&\leq p\sum_{i=1}^{j} d_i(x,y).
\end{aligned}
$$

By (2.23), we infer that

$$d_j(Tx,Ty) \leq p\sum_{i=1}^{j} d_i(x,y).$$

For $i,j=1,\dots,m$, we put

$$a_{ji} = \begin{cases} p & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

and define $A=(a_{ji})$. Then $d(Tx,Ty) \preceq Ad(x,y)$ and A is a triangular matrix with p on the diagonal. Since p is the only eigenvalue of A, we get $r(A)=p<1$.

By the above arguments and using Corollary 2.2, T has a unique fixed point $x_* \in C[0,1]$. That is, for all $t\in[0,1]$, we have

$$\left(\int_0^t f(s,x_*(s))ds\right)^p = x_*(t).$$

It is equivalent to

$$x_*^{\frac{1}{p}}(t) = \int_0^t f(s,x_*(s))ds.$$

It implies that the equation (2.21) has a unique solution $x_* \in C[0,1]$. □

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