An asymptotic expansion for perturbations in the velocity field of Stokes system due to the presence of thin interfaces

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Abstract

We consider the Stokes system for a viscous medium consisting of a thin inclusion having no uniform thickness merged in consistent background medium. Based on layer potential methods we rigorously derive an asymptotic expansion for two-dimensional velocity field associated with thin inclusion. We extend these techniques to determine a relationship between Stokes solutions measurements and the shape of the thin impurity.

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1 Introduction and statement of main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^{2,\eta}$ boundary for some $\eta > 0$. Let D be an open subset of Ω such that

$$dist(\partial\Omega, D) \geq d > 0,$$

representing an inclusion made of a different Newtonian fluid material.We assume that ∂D is of classe $\mathcal{C}^{2,\eta}$. In this case, ∂D can be parametrized by a vector-valued function $t \to X(t)$, that is, $\partial D := \{x = X(t), t \in [a,b] \; with \; a < b\}$, where X is a $\mathcal{C}^{2,\eta}$ function satisfying $|X^{'}(t)| = 1$ for all $t \in [a,b]$, and $X(a) = X(b)$.
Let D_δ be an $\delta-$perturbation of $D, i.e.$, there is $h \in \mathcal{C}^1(\partial D)$ such that ∂D_δ is given by

$$\partial D_\delta := \Big\{ \tilde{x} : \tilde{x} = x + \delta h(x)\mathbf{n}(x), x \in \partial D \Big\},$$

where $\mathbf{n}(x)$ is the outward normal to D. We assume that $h(x) \geq C > 0$ for all $x \in \partial D$. Suppose that the thin layer $D_\delta \backslash \overline{D}$ lies inside Ω. We denote by σ_0, σ_1 and σ_2 the stress tensor fields in of $\Omega \backslash \overline{D}_\delta$, D, and $D_\delta \backslash \overline{D}$, respectively. We assume that $\Omega \backslash \overline{D}_\delta$, D, and $D_\delta \backslash \overline{D}$ and D are occupied by isotropic and homogeneous Newtonian fluids, μ_0, μ_1 and μ_2 are the viscosity constants of the flow in $\Omega \backslash \overline{D}_\delta$, D, and $D_\delta \backslash \overline{D}$ and D respectively,

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then the tensors σ_0, σ_1 and σ_2 can be given by

$$(\sigma_s)_{ijlk} = \mu_s(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{lj}),\ for\ i,j,k,l = 1,2\ and\ for\ s = 0,1,2.$$

where μ_0 , μ_1 and μ_2 are the viscosity constants of the flow in $\Omega \backslash \overline{D}_\delta$, D, and $D_\delta \backslash \overline{D}$ respectively. Given two (2×2) matrices A and B we denote by A:B the contraction, i.e., A:B $= \sum_{ij} a_{ij} b_{ij}$ The strain rate tensor D is given by :

$$\mathbf{D}(u) = (\frac{1}{2}\nabla u + (\nabla u)^T)$$

We define

$$\sigma_\delta := \sigma_0 \chi_{\mathbb{R}^2 \backslash \overline{D}_\delta} + \sigma_2 \chi_{D_\delta \backslash \overline{D}} + \sigma_1 \chi_D, \qquad \sigma := \sigma_0 \chi_{\mathbb{R}^2 \backslash \overline{D}} + \sigma_1 \chi_D,$$

where χ_D is the indicator function of D.

Let $(\mathbf{u}_\delta, p_\delta)$ the solution field in the presence of the thin $D_\delta \backslash \overline{D}$ to

$$\begin{cases} -div\big(\sigma \mathbf{D}(\mathbf{u}_\delta) - p_\delta Id\big) = 0 & \text{in } \Omega, \\ \nabla.\mathbf{u}_\delta = 0 & \text{in } \Omega, \\ \mathbf{u}_\delta = \mathbf{F} & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\mathbf{u}_\delta$ denotes the velocity perturbed field while the scalar function p_δ is the perturbed pressure, $\mathbf{F}$ is a given vector valued function. Moreover, $(\mathbf{u}_\delta, p_\delta) \in (H^2(\Omega))^2 \times H^1(\Omega)$ and Id means the identity.

The corresponding conormal derivative $\dfrac{\partial \mathbf{w}}{\partial \nu_s}$ associated to μ_s is defined by

$$\frac{\partial \mathbf{w}}{\partial \nu_s} := \mu_s \frac{\partial \mathbf{w}}{\partial \mathbf{n}} - p\mathbf{n} \tag{1.2}$$

The problem (1.1) is equivalent to the following problem

$$\begin{cases} -div\big(\sigma_0 \mathbf{D}(\mathbf{u}_\delta) - p_\delta Id\big) = 0 & \text{in } \Omega \backslash \overline{D}_\delta, \\ -div\big(\sigma_2 \mathbf{D}(\mathbf{u}_\delta) - p_\delta Id\big) = 0 & \text{in } D_\delta \backslash \overline{D}, \\ -div\big(\sigma_1 \mathbf{D}(\mathbf{u}_\delta) - p_\delta Id\big) = 0 & \text{in } D, \\ \mathbf{u}_\delta\big|_- = \mathbf{u}_\delta\big|_+ \quad \text{on } \partial D, \quad \mathbf{u}_\delta\big|_- = \mathbf{u}_\delta\big|_+ \quad \text{on } \partial D_\delta, \\ \dfrac{\partial \mathbf{u}_\delta}{\partial \nu_1}\Big|_- = \dfrac{\partial \mathbf{u}_\delta}{\partial \nu_2}\Big|_+ \quad \text{on } \partial D, \quad \dfrac{\partial \mathbf{u}_\delta}{\partial \nu_2}\Big|_- = \dfrac{\partial \mathbf{u}_\delta}{\partial \nu_0}\Big|_+ \quad \text{on } \partial D_\delta, \\ \nabla.\mathbf{u}_\delta = 0 & \text{in } \Omega, \\ \mathbf{u}_\delta(x) = \mathbf{F}(x) & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

The notation $\mathbf{u}_\delta|_\pm$ on ∂D denote the limits from outside and inside of D, respectively. The first main result of this paper is the derivation of the leading-order term in the asymptotic expansion of $(u_\delta - u)$ as $\delta \to 0$, where Ω is a bounded region outside the inclusion D, and away from both ∂D and $\partial\Omega$.The methods and results developed in this paper can be generalized to higher dimension thin interface problems and can be extended to other PDEs systems, Maxwell.

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Theorem 1.1 *Let* $(\mathbf{u}_\delta, p_\delta)$ *be the solution to* (1.3). *For* $x \in \Omega$, *the following pointwise asymptotic expansion holds:*

$$\begin{aligned} \mathbf{u}_\delta(x) &= \mathbf{u}(x) + \delta \mathbf{u}_1(x) + o(\delta), \\ p_\delta(x) &= p(x) + \delta p_1(x) + o(\delta), \end{aligned} \tag{1.4}$$

where the remainder $o(\delta)$ *depends only on* μ_j *for j=0,1,2, the* $\mathcal{C}^2$*-norm of* X, *the* $\mathcal{C}^1$*-norm of* h, *and* $dist(\Omega, \partial D)$, $(\mathbf{u}, p)$ *is the unique solution to*

$$\begin{cases} -div\big(\sigma \boldsymbol{D}(\mathbf{u}) - pId\big) = 0 & in\ \Omega, \\ \nabla.\mathbf{u} = 0 & in\ \Omega, \\ \mathbf{u}(x) = \mathbf{F}(x) & on\ \partial\Omega \end{cases} \tag{1.5}$$

and $(\mathbf{u}_1, p_1)$ *is the unique solution of the following transmission problem:*

$$\begin{cases} -div\big(\sigma_0 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ \Omega\backslash\overline{D}_\delta, \\ -div\big(\sigma_2 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ D_\delta\backslash\overline{D}, \\ -div\big(\sigma_1 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ D, \\ \mathbf{u}_1\big|_- - \mathbf{u}_1\big|_+ = 0 & on\ \partial D, \\ \dfrac{\partial \mathbf{u}_1}{\partial \nu_1}\Big|_- - \dfrac{\partial \mathbf{u}_1}{\partial \nu_0}\Big|_+ = \dfrac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big[(\mathbb{M}_{2,1} - \mathbb{M}_{0,1})D(v^i)\big]\boldsymbol{\tau}\Big) & on\ \partial D, \\ \nabla.\mathbf{u}_1 = 0 & in\ \Omega, \\ \mathbf{u}_1(x) = \mathbf{F}(x) & on\ \partial\Omega \end{cases} \tag{1.6}$$

with $\boldsymbol{\tau}$ *is the tangential vector to* ∂D,

$$\mathbb{M}_{l,k} := 2\mu_k \mathbb{I} + 2(\mu_l - \mu_k)\mathbf{I} \otimes (\boldsymbol{\tau} \otimes \boldsymbol{\tau})$$

Here $\mathbb{I}$ means the identity four-tensor,$\mathbf{I}$ the identity in $\mathbb{R}^2$

Let $(\mathbf{v}, q)$ be the solution of the following problem:

$$\begin{cases} -div\big(\sigma_0 \mathbf{D}(\mathbf{v}) - qId\big) = 0 & \text{in } \Omega\backslash\overline{D}, \\ -div\big(\sigma_1 \mathbf{D}(\mathbf{v}) - qId\big) = 0 & \text{in } D, \\ \mathbf{v}|_- = \mathbf{v}|_+ & \text{on } \partial D, \\ \dfrac{\partial \mathbf{v}}{\partial \nu_1}\Big|_- = \dfrac{\partial \mathbf{v}}{\partial \nu_0}\Big|_+ & \text{on } \partial D, \\ \nabla.\mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{v}(x) = \mathbf{G}(x) & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

As a consequence of the theorem 1.1 we obtain the following relationship between velocity measurements and the deformation h.

Our expansion is valid when $\mu_2 = \mu_1$, we can check that the asymptotic expansion in (1.4) coincides with the asymptotic expansion of the velocity field resulting from small perturbations of the shape of an impurity already derived in [15]

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Theorem 1.2 *Let* $(\mathbf{u}_\delta, p_\delta)$, $(\mathbf{u}, p)$, *and* $(\mathbf{v}, q)$ *be the solutions to* (1.1), (1.5), *and* (1.7), *respectively. Let* S *be a Lipschitz closed curve enclosing* D *away from* ∂D. *The following asymptotic expansion holds:*

$$\int_S (\mathbf{u}_\delta - \mathbf{u}) \cdot \frac{\partial \mathbf{v}}{\partial \nu_0} d\sigma - \int_S \Big(\frac{\partial \mathbf{u}_\delta}{\partial \nu_0} - \frac{\partial \mathbf{u}}{\partial \nu_0}\Big) \cdot \mathbf{G} d\sigma$$

$$= \delta \int_{\partial D} h \Bigg(([\mathbb{M}_{0,1} - \mathbb{M}_{2,1}] \boldsymbol{D}(\mathbf{u}^i)\boldsymbol{\tau} \cdot \boldsymbol{D}(\mathbf{v}^i)\boldsymbol{\tau} \Bigg) d\sigma + o(\delta), \tag{1.8}$$

where the remainder $o(\delta)$ *depends only on* μ_j *for j=0,1,2, the* $\mathcal{C}^2$*-norm of* X*, the* $\mathcal{C}^1$*-norm of* h*, and* $dist(S, \partial D)$*. The dot denotes the scalar product in* $\mathbb{R}^2$.

The asymptotic expansion in (1.8) can be used to design algorithms to identify certain properties of thin impurity like location and thickness based on Stokes solutions measurements, it have the big advantage to derive higher order terms in the asymptotic formulae and allow a generalization to 3-dimensional thin interface problems by using [6, 14].

The asymptotic results of this work represent a powerful tool to solve the inverse problem of identifying small thin inclusions (see [1], [4]).

Remark : To illustrate, we can consider D the disk centered with radius ρ, then the formula (1.8) implies that the Fourier coefficients h_p related to h can be determined using a finite number of measurments provided that the order of magnitude of h_p is much larger then δ(for more details see [9] and [14]).

This paper is organized as follows. In Section 2, we review some basic facts on the layer potentials of the Stokes system and derive a representation formula for the solution of the problem (1.1). In Section 3, we derive asymptotic expansions of layer potentials. In Section 4, based on layer potentials techniques, we rigorously derive the asymptotic expansion for perturbations in the velocity field and find the relationship between stokes solution measurement and the deformation h (Theorem 1.1 and Theorem 1.2).

2 Representation of solutions

Let $(\mathbf{\Gamma}, F)$ the fundamental solution for stokes system $(\mathbf{\Gamma}, F)$ in $\mathbb{R}^2$ (see for instance [22]) is given by

$$\begin{cases} \mathbf{\Gamma}_{ij}(x) = -\dfrac{1}{2\pi}\big(\delta_{ij} e_1(\sqrt{\lambda}|x|) + \dfrac{x_i x_j}{|x|^2} e_2(\sqrt{\lambda}|x|)\big), \\ F_j(x) = -\dfrac{x_i}{2\pi |x|^2} \end{cases} \tag{2.1}$$

with

$$\begin{cases} e_1(\kappa) = K_0(\kappa) + \kappa^{-1} K_1(\kappa) - \kappa^{-2} \\ e_2(\kappa) = -K_0(\kappa) - 2\kappa^{-1} K_1(\kappa) + 2\kappa^{-2} \end{cases}$$

where $K_n (n \in \mathbb{N}_0)$ denotes the modified Bessel function of order n and and δ_{ij} is the Kronecker symbol.

The single and double layer potentials of the density function $\boldsymbol{\phi}$ on $(L^2(\partial D))^2$ associated to

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the viscosity constant μ_s are defined by

$$\boldsymbol{\mathcal{S}}_{s,D}(\lambda)[\boldsymbol{\phi}](x) = \int_{\partial D} \boldsymbol{\Gamma}_s(\lambda, |x-y|)\boldsymbol{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^2, \tag{2.2}$$

$$\begin{aligned}\boldsymbol{\mathcal{D}}_{s,D}(\lambda)[\boldsymbol{\phi}](x) &= \int_{\partial D} \Big(\big[\nabla_y \boldsymbol{\Gamma}_s(\lambda, |x-y|)\mathbf{n}(y)\big]^T + \nabla_y \boldsymbol{\Gamma}_s^T(\lambda, |x-y|)\mathbf{n}(y) \Big) \boldsymbol{\phi}(y)d\sigma(y) \\ &:= \int_{\partial D} \mathbb{K}_s(x-y)\boldsymbol{\phi}(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D. \end{aligned} \tag{2.3}$$

The followings are the well-known properties of both single and double layer potentials due to [2]. Let D be a Lipschitz bounded domain in $\mathbb{R}^2$. Then we have

$$\boldsymbol{\mathcal{S}}_{s,D}(\lambda)[\boldsymbol{\phi}]_+ = \boldsymbol{\mathcal{S}}_{s,D}(\lambda)[\boldsymbol{\phi}]_- = \boldsymbol{\mathcal{S}}_{s,D}(\lambda)[\boldsymbol{\phi}] \quad \text{a.e. } x \in \partial D \tag{2.4}$$

$$\frac{\partial \boldsymbol{\mathcal{S}}_{s,D}(\lambda)[\boldsymbol{\phi}]}{\partial \nu_s}\Big|_{\pm}(x) = \Big(\pm \frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_{s,D}^* \Big)[\boldsymbol{\phi}](x) \quad \text{a.e. } x \in \partial D, \tag{2.5}$$

$$\boldsymbol{\mathcal{D}}_{s,D}(\lambda)[\boldsymbol{\phi}]\big|_{\pm}(x) = \Big(\mp \frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_{s,D} \Big)[\boldsymbol{\phi}](x) \quad \text{a.e. } x \in \partial D, \tag{2.6}$$

where $\boldsymbol{\mathcal{K}}_{s,D}$ is defined by

$$\boldsymbol{\mathcal{K}}_{s,D}[\boldsymbol{\phi}](x) = p.v. \int_{\partial D} \mathbb{K}_s(x-y)\boldsymbol{\phi}(y)d\sigma(y) \quad \text{a.e. } x \in \partial D,$$

and $\boldsymbol{\mathcal{K}}_{s,D}^*$ is the adjoint operator of $\boldsymbol{\mathcal{K}}_{s,D}$, that is,

$$\boldsymbol{\mathcal{K}}_{s,D}^*[\boldsymbol{\phi}](x) = p.v. \int_{\partial D} \mathbb{K}_s^T(x-y)\boldsymbol{\phi}(y)d\sigma(y) \quad \text{a.e. } x \in \partial D.$$

Here $p.v.$ denotes the Cauchy principal value.

Note that we drop the $p.v.$ in this stage; this is because ∂D is $\mathcal{C}^{2,\eta}$.

Denote by

$$\mathcal{X}(\partial D) := (L^2(\partial D))^2, \quad \mathcal{X}_0(\partial D) := L^2(\partial D) \times L_0^2(\partial D), \quad \mathcal{Y}(\partial D) := W_1^2(\partial D) \times L^2(\partial D).$$

where $W_1^2(\partial D)$ is the first L^2-Sobolev of space of order 1 on ∂D and $L_0^2(\partial D) = \{\mathbf{f} \in L^2(\partial D)$ *such that* $\int_{\partial D} \mathbf{f} d\sigma = 0\}$

We have the following solvability result for more details see [[4], [5] ,[10], [16]].

Theorem 2.1 *For any given* $(\boldsymbol{\phi}_1, \boldsymbol{\psi}_1) \in W_1^2(\partial D) \times L^2(\partial D)$, *there exists a unique pair* $(\boldsymbol{\phi}, \boldsymbol{\psi}) \in L^2(\partial D) \times L^2(\partial D)$ *such that*

$$\begin{cases} \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}]\big|_- - \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]\big|_+ = \boldsymbol{\phi}_1 & \text{on } \partial D, \\ \big(-\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_{1,D}^*\big)[\boldsymbol{\phi}] - \big(\frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{K}}_{0,D}^*\big)[\boldsymbol{\psi}] = \boldsymbol{\psi}_1 & \text{on } \partial D, \end{cases} \tag{2.7}$$

Furthermore, there exists a constant $C > 0$ *depending only on* μ_0, μ_1, *and the Lipschitz character of* D *such that*

$$\|\boldsymbol{\phi}\|_{L^2(\partial D)} + \|\boldsymbol{\psi}\|_{L^2(\partial D)} \leq C\Big(\|\boldsymbol{\phi}_1\|_{W_1^2(\partial D)} + \|\boldsymbol{\psi}_1\|_{L^2(\partial D)} \Big). \tag{2.8}$$

Moreover, if $\boldsymbol{\psi}_1 \in L_0^2(\partial D)$, *then* $\boldsymbol{\psi} \in L_0^2(\partial D)$.

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The following theorem is important to us to establish our representation (see [17]) formula.

Theorem 2.2 *For each* $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mathbf{F}_4) \in \mathcal{Y}(\partial D) \times \mathcal{Y}(\partial D_\delta)$*, there exists a unique solution* $\Theta := (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \widetilde{\boldsymbol{\Psi}}_2, \widetilde{\boldsymbol{\phi}}_0) \in \mathcal{X}(\partial D) \times \mathcal{X}(\partial D_\delta)$ *to the system of integral equations*

$$\begin{cases} \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]\big|_- - \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]\big|_+ - \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2] = \mathbf{F}_1 & \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]}{\partial \nu_1}\Big|_- - \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]}{\partial \nu_2}\Big|_+ - \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]}{\partial \nu_2} = \mathbf{F}_2 & \text{on } \partial D, \\ \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2] + \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]\big|_- - \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0]\big|_+ = \mathbf{F}_3 & \text{on } \partial D_\delta, \\ \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]}{\partial \nu_2} + \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]}{\partial \nu_2}\Big|_- - \dfrac{\partial \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0]}{\partial \nu_0}\Big|_+ = \mathbf{F}_4 & \text{on } \partial D_\delta. \end{cases} \tag{2.9}$$

Moreover, if $(\mathbf{F}_2, \mathbf{F}_4) \in L^2_0(\partial D) \times L^2_0(\partial D)$*, then* $(\boldsymbol{\phi}_2, \widetilde{\boldsymbol{\phi}}_0) \in L^2_0(\partial D) \times L^2_0(\partial D)$.

We now prove a representation theorem for the solution of the transmission problem (1.3) which will be the main ingredient in deriving the asymptotic expansion in Theorem 1.1.

Theorem 2.3 *The solution* $\mathbf{u}_\delta$ *to the problem* (1.3) *is represented by*

$$\mathbf{u}_\delta(x) = \begin{cases} \mathbf{H}(x) + \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0](x), & x \in \mathbb{R}^2 \setminus \overline{D}_\delta, \\ \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2](x) + \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2](x), & x \in D_\delta \setminus \overline{D}, \\ \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1](x), & x \in D, \end{cases} \tag{2.10}$$

$$p_\delta(x) = \begin{cases} \vartheta_{D_\delta}[\widetilde{\boldsymbol{\phi}}_0](x), & x \in \mathbb{R}^2 \setminus \overline{D}_\delta, \\ \vartheta_D[\boldsymbol{\phi}_2](x) + \vartheta_{D_\delta}[\widetilde{\boldsymbol{\Psi}}_2](x), & x \in D_\delta \setminus \overline{D}, \\ \vartheta_D[\boldsymbol{\phi}_1](x), & x \in D, \end{cases} \tag{2.11}$$

where $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \widetilde{\boldsymbol{\Psi}}_2, \widetilde{\boldsymbol{\phi}}_0) \in \mathcal{X}_0(\partial D) \times \mathcal{X}_0(\partial D_\delta)$ *is the unique solution to the system of integral equations*

$$\begin{cases} \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]\big|_- - \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]\big|_+ - \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2] = 0 & \text{on } \partial D, \\ \dfrac{\partial \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]}{\partial \nu_1}\Big|_- - \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]}{\partial \nu_2}\Big|_+ - \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]}{\partial \nu_2} = 0 & \text{on } \partial D, \\ \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2] + \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]\big|_- - \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0]\big|_+ = \mathbf{H} & \text{on } \partial D_\delta, \\ \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}_2]}{\partial \nu_2} + \dfrac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\Psi}}_2]}{\partial \nu_2}\Big|_- - \dfrac{\partial \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0]}{\partial \nu_0}\Big|_+ = \dfrac{\partial \mathbf{H}}{\partial \nu_0} & \text{on } \partial D_\delta. \end{cases} \tag{2.12}$$

Proof. Let $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \widetilde{\boldsymbol{\Psi}}_2, \widetilde{\boldsymbol{\phi}}_0) \in \mathcal{X}_0(\partial D) \times \mathcal{X}_0(\partial D_\delta)$ be the unique solution of (2.12), then it clearly $\mathbf{u}_\delta$ defined by (2.10) satisfies the transmission conditions (the conditions on the fourth and fifth lines in (1.3)). This end the proof of the theorem.

Now let Φ_δ be the diffeomorphism from ∂D onto ∂D_ϵ given by $\tilde{x} = \Phi_\delta(x) = x + \delta h(x)\mathbf{n}(x)$,

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where $x = X(t) \in \partial D$. Define the operators $\mathcal{T}_\delta$ and $\mathcal{W}_\delta$ from $L^2(\partial D) \times L^2(\partial D_\delta)$ into $\mathcal{Y}(\partial D)$ by

$$\mathcal{T}_\delta(\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) := \Big(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}]\big|_- - \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\psi}}] \circ \Phi_\delta\big|_+, \frac{\partial \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}]}{\partial \nu_1}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\psi}}]}{\partial \nu_0} \circ \Phi_\delta\Big|_+ \Big), \tag{2.13}$$

$$\mathcal{W}_\delta(\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) := \Big(\boldsymbol{\mathcal{S}}_{2,D}[\phi] \circ \Phi_\delta - \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]\big|_+ + \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\psi}}] \circ \Phi_\delta\big|_- - \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\psi}}],$$

$$\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2} \circ \Phi_\delta - \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2}\Big|_+ + \frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\psi}}]}{\partial \nu_2} \circ \Phi_\delta\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\psi}}]}{\partial \nu_2} \Big), \tag{2.14}$$

and the matrix-valued function $\mathcal{H}_\delta$ on ∂D by

$$\mathcal{H}_\delta(x) := \Big(\mathbf{H}\big(x + \delta h(x)\mathbf{n}(x)\big), \frac{\partial \mathbf{H}}{\partial \nu_0}\big(x + \delta h(x)\mathbf{n}(x)\big) \Big). \tag{2.15}$$

The following lemma holds.

Lemma 2.4 *Let $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \widetilde{\boldsymbol{\psi}}_2, \widetilde{\boldsymbol{\phi}}_0) \in \mathcal{X}_0(\partial D) \times \mathcal{X}_0(\partial D_\delta)$ be the unique solution of* (2.12)*; then $(\boldsymbol{\phi}_1, \widetilde{\boldsymbol{\phi}}_0)$ and $(\boldsymbol{\phi}_2, \widetilde{\boldsymbol{\psi}}_2)$ satisfy the following system of integral equations:*

$$\mathcal{T}_\delta(\boldsymbol{\phi}_1, \widetilde{\boldsymbol{\phi}}_0) = \mathcal{H}_\delta - \mathcal{W}_\delta(\boldsymbol{\phi}_2, \widetilde{\boldsymbol{\psi}}_2) \quad \text{on } \partial D. \tag{2.16}$$

In the next section, we will derive the asymptotic expansions of the layer potentials , which are appeared in the system of integral equations (2.16) with $(\boldsymbol{\phi}_1, \widetilde{\phi}_0) \in L^2(\partial D) \times L^2(\partial D_\delta)$ and $(\boldsymbol{\phi}_2, \widetilde{\boldsymbol{\Psi}}_2) \in \mathcal{C}^{1,\eta}(\partial D) \times \mathcal{C}^{1,\eta}(\partial D_\delta)$. These asymptotic expansions will help us to derive the asymptotic expansion of the displacement field $\mathbf{u}_\delta$.

3 Asymptotic expansions of layer potentials

Let $\tilde{x} = x + \delta h(x)\mathbf{n}(x) \in \partial D_\delta$ for $x \in \partial D$. The following asymptotic expansions of $\mathbf{n}(\tilde{x})$ and the length element $d\sigma_\epsilon(\tilde{x})$ hold (see [14]):

$$\mathbf{n}(\tilde{x}) = \mathbf{n}(x) - \delta h'(t)\boldsymbol{\tau}(x) + O(\delta^2), \tag{3.1}$$

and

$$d\sigma_\epsilon(\tilde{x}) = \big(1 - \delta\kappa(x)h(x) + O(\delta^2)\big)d\sigma(x). \tag{3.2}$$

Here, the remainder term $O(\delta^2)$ is bounded by $C\delta^2$ for some constant C which depends only on $\mathcal{C}^2$-norm of ∂D and $\mathcal{C}^1$-norm of h.

Let $\boldsymbol{\phi}(x)$ and $\phi(x)$ be a vector function and scalar function, respectively, which belong to $\mathcal{C}^2([a,b])$ for $x = X(\cdot) \in \partial D$. By d/dt, we denote the tangential derivative in the direction of $\boldsymbol{\tau}(x) = X'(t)$. We have

$$\frac{d}{dt}\big(\boldsymbol{\phi}(x)\big) = \nabla\boldsymbol{\phi}(x)X'(t) = \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{\tau}}(x), \qquad \frac{d}{dt}\big(\phi(x)\big) = \nabla\phi(x) \cdot X'(t) = \frac{\partial \phi}{\partial \boldsymbol{\tau}}(x).$$

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The restriction of stokes system in D to a neighborhood of ∂D can be expressed as follows (see [15]):

$$div(\sigma_j \mathbf{D}(\boldsymbol{\phi}(x))) = \frac{1}{2}\mu_j\big(\nabla\nabla\boldsymbol{\phi}(x)\mathbf{n}(x)\mathbf{n}(x) + \nabla(\nabla\boldsymbol{\phi})^T(x)\mathbf{n}(x)\mathbf{n}(x)\big) - \kappa(x)(\sigma_j\mathbf{D}(\boldsymbol{\phi}))\mathbf{n}(x) + \frac{d}{dt}\Big(\big(\sigma_j\mathbf{D}(\boldsymbol{\phi}(x))\big)\boldsymbol{\tau}(x)\Big), \quad x \in \partial D. \tag{3.3}$$

We have from the following lemma form [15]

Lemma 3.1 *Let $\widetilde{\boldsymbol{\phi}} \in L^2(\partial D_\delta)$, we denote by $\boldsymbol{\phi} := \widetilde{\boldsymbol{\phi}} \circ \Phi_\delta$. For $s = 0, 2$, the following asymptotic expansions hold:*

$$\boldsymbol{\mathcal{S}}_{s,D_\delta}[\widetilde{\boldsymbol{\phi}}] \circ \Phi_\delta\big|_\pm = \boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}] - \delta\boldsymbol{\mathcal{S}}_{s,D}[\kappa h\boldsymbol{\phi}] + \delta\Big(h\frac{\partial\boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}]}{\partial\mathbf{n}} + \boldsymbol{\mathcal{D}}^\sharp_{s,D}[h\boldsymbol{\phi}]\Big)\Big|_\pm + O_1(\delta^2) \quad on\ \partial D, \tag{3.4}$$

$$\frac{\partial\boldsymbol{\mathcal{S}}_{s,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial\nu_s} \circ \Phi_\delta\Big|_\pm = \frac{\partial\boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}]}{\partial\nu_s}\Big|_\pm + \delta\Big(h div(\sigma_s\boldsymbol{D}(\boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}]) + \kappa h(\sigma_s\boldsymbol{D}(\boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}]))\mathbf{n} - \frac{\partial\boldsymbol{\mathcal{S}}_{s,D}[\kappa h\boldsymbol{\phi}]}{\partial\nu_s}\Big)\Big|_\pm + \delta\Big(\frac{\partial\boldsymbol{\mathcal{D}}^\sharp_{s,D}[h\boldsymbol{\phi}]}{\partial\nu_s} - \frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\sigma_j\boldsymbol{D}(\boldsymbol{\mathcal{S}}_{s,D}[\boldsymbol{\phi}])\big)\boldsymbol{\tau}\Big)\Big)\Big|_\pm + O_2(\delta^2) \quad on\ \partial D, \tag{3.5}$$

where $\|O_1(\delta^2)\|_{W_1^2(\partial D)}, \|O_2(\delta^2)\|_{L^2(\partial D)} \leq C\delta^2$ for some constant C depends only on μ_s, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h.

Now we are going to derive $\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](\tilde{x})$ for $\boldsymbol{\phi} \in \mathcal{C}^{1,\eta}(\partial D)$, $\tilde{x} = x + \delta h(x)\mathbf{n}(x) \in \partial D_\delta$, and $x \in \partial D$. Since ∂D is $C^{2,\eta}$, $\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]$ is $\mathcal{C}^{2,\eta}(\mathbb{R}^2 \backslash D)$ then we have

$$\Big|\nabla\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](\tilde{x}) - \nabla\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x)\big|_+ - \delta h(x)\nabla^2\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x)\mathbf{n}(x)\big|_+\Big| \leq C\delta^{1+\eta}\big\|\boldsymbol{\phi}\big\|_{\mathcal{C}^{1,\eta}(\partial D)}. \tag{3.6}$$

Thus

$$\begin{aligned}\frac{\partial\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial\boldsymbol{\tau}}(\tilde{x}) &= \Big(\nabla\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x)\big|_+ + \delta h(x)\nabla^2\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x)\mathbf{n}(x)\big|_+ + O(\delta^{1+\eta})\Big) \times \Big(\boldsymbol{\tau}(x) + \delta h'(t)\mathbf{n}(x) + O(\delta^2)\Big) \\ &= \frac{\partial}{\partial\boldsymbol{\tau}}\Big(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x) + \delta h(x)\frac{\partial\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial\mathbf{n}}\Big|_+(x)\Big) + O(\delta^{1+\eta}), \quad x \in \partial D, \end{aligned} \tag{3.7}$$

Similarly to (3.6), we get

$$\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](\tilde{x}) = \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x) + \delta h(x)\frac{\partial\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial\mathbf{n}}\Big|_+(x) + O(\delta^2), \quad x \in \partial D, \tag{3.8}$$

then we get from (3.7) and (3.8) that

$$\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](\tilde{x}) = \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x) + \delta h(x)\frac{\partial\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial\mathbf{n}}\Big|_+(x) + O(\delta^{1+\eta}), \quad x \in \partial D, \tag{3.9}$$

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where $\|O(\delta^{1+\eta})\|_{W_1^2(\partial D)}$ is bounded by $C\delta^{1+\eta}\|\boldsymbol{\phi}\|_{\mathcal{C}^{1,\eta}(\partial D)}$.

we have the following taylor expansion (see [15]) , for $\tilde{x} \in \partial D$,

$$\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2}(\tilde{x}) = \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2}\Big|_+(x) + \delta\kappa(x)h(x)(\sigma_0 \mathbf{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}])\mathbf{n}(x)\Big|_+(x)$$
$$+ \delta h(x)(div(\sigma_0 \mathbf{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]))\Big|_+ - \delta\frac{d}{dt}\Big(h(x)\Big(\sigma_0 \mathbf{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}](x))\Big)\tau(x)\Big)\Big|_+ + O(\delta^{1+\eta}), \quad \mathbf{x} \in \partial D, \tag{3.10}$$

where $\|O(\delta^{1+\eta})\|_{L^2(\partial D)}$ is bounded by $C\delta^{1+\eta}\|\boldsymbol{\phi}\|_{\mathcal{C}^{1,\eta}(\partial D)}$.

We now expand $\boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}](x)$ and $\partial\boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]/\partial\nu_2(x)$ for $x \in \partial D$ when $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^{1,\eta}(\partial D_\delta)$. Let $\mathbf{f}$ be a $\mathcal{C}^{1,\eta}$ vector function on ∂D and let $\mathbf{v}$ be the solution to $-div(\sigma_2 D(\mathbf{v}) - q_2 Id) = 0$ in D satisfying $\mathbf{v} = \mathbf{f}$ on ∂D. Then, we get

$$\int_{\partial D} \frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial \nu_2}(x) \cdot \mathbf{f}(x) d\sigma(x) = \int_{\partial D} \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}](x) \cdot \frac{\partial \mathbf{v}}{\partial \nu_2}(x) d\sigma(x)$$
$$= \int_{\partial D_\delta} \widetilde{\boldsymbol{\phi}}(\tilde{x}) \cdot \boldsymbol{\mathcal{S}}_{2,D}\Big[\frac{\partial \mathbf{v}}{\partial \nu_2}\Big](\tilde{x}) d\sigma(\tilde{x}). \tag{3.11}$$

Define $\boldsymbol{\phi} := \widetilde{\boldsymbol{\phi}} \circ \Phi_\delta$. By using (3.2), we get

$$\int_{\partial D} \frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial \nu_2} \cdot \mathbf{f} d\sigma = \int_{\partial D} \boldsymbol{\phi} \cdot \Big(\boldsymbol{\mathcal{S}}_{2,D}\Big[\frac{\partial \mathbf{v}}{\partial \nu_2}\Big] + \delta h \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}}{\partial \mathbf{n}}\Big[\frac{\partial \mathbf{v}}{\partial \nu_2}\Big]\Big|_+ + O(\delta^{1+\eta})\Big)$$
$$\times \Big(1 - \delta\kappa h + O(\delta^2)\Big) d\sigma$$
$$= \int_{\partial D} \Big(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}] + \delta \boldsymbol{\mathcal{D}}^\sharp_{2,D}[h\boldsymbol{\phi}]\big|_- - \delta \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}]\Big) \cdot \frac{\partial \mathbf{v}}{\partial \nu_2} d\sigma + O(\delta^{1+\eta})$$
$$= \int_{\partial D} \Big(\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2}\Big|_- + \delta \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_{2,D}[h\boldsymbol{\phi}]}{\partial \nu_2}\Big|_- - \delta \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}]}{\partial \nu_2}\Big|_-\Big) \cdot \mathbf{f} d\sigma$$
$$+ O(\delta^{1+\eta}).$$

Therefore, the following asymptotic expansion holds:

$$\frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial \nu_2} = \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \nu_2}\Big|_- + \delta \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_{2,D}[h\boldsymbol{\phi}]}{\partial \nu_2}\Big|_- - \delta \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}]}{\partial \nu_2}\Big|_- + O(\delta^{1+\eta}) \quad \text{on } \partial D, \tag{3.12}$$

where the remainder term $O(\delta^{1+\eta})$ is in $L^2(\partial D)$.

Let $\boldsymbol{\phi} = \widetilde{\boldsymbol{\phi}} \circ \Phi_\delta$ for $\widetilde{\boldsymbol{\phi}} \in \mathcal{C}^{1,\eta}(\partial D_\delta)$. Let $\mathbf{f}$ be a $\mathcal{C}^{1,\eta}$ vector function on ∂D. Similarly to

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(3.11), we have

$$\int_{\partial D} \frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial \boldsymbol{\tau}} \cdot \mathbf{f} d\sigma = - \int_{\partial D} \boldsymbol{\phi} \cdot \left(\boldsymbol{\mathcal{S}}_{2,D}\Big[\frac{\partial \mathbf{f}}{\partial \boldsymbol{\tau}}\Big] + \delta h \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}}{\partial \mathbf{n}}\Big[\frac{\partial \mathbf{f}}{\partial \boldsymbol{\tau}}\Big]\Big|_+ + O(\delta^{1+\eta}) \right) \times \Big(1 - \delta\kappa h + O(\delta^2)\Big) d\sigma$$

$$= - \int_{\partial D} \Big(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}] + \delta \boldsymbol{\mathcal{D}}^{\sharp}_{2,D}[h\boldsymbol{\phi}]\big|_- - \delta \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}] \Big) \cdot \frac{\partial \mathbf{f}}{\partial \boldsymbol{\tau}} d\sigma + O(\delta^{1+\eta})$$

$$= \int_{\partial D} \left(\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} + \delta \frac{\partial \boldsymbol{\mathcal{D}}^{\sharp}_{2,D}[h\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}}\Big|_- - \delta \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} \right) \cdot \mathbf{f} d\sigma + O(\delta^{1+\eta}).$$

Thus

$$\frac{\partial \boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}]}{\partial \boldsymbol{\tau}} = \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} + \delta \frac{\partial \boldsymbol{\mathcal{D}}^{\sharp}_{2,D}[h\boldsymbol{\phi}]}{\partial \boldsymbol{\tau}}\Big|_- - \delta \frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}]}{\partial \boldsymbol{\tau}} + O(\delta^{1+\eta}) \quad \text{on } \partial D, \tag{3.13}$$

Similarely, we have

$$\boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}] = \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}] + \delta \boldsymbol{\mathcal{D}}^{\sharp}_{2,D}[h\boldsymbol{\phi}]\big|_- - \delta \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}] + O(\delta^2) \quad \text{on } \partial D, \tag{3.14}$$

Then it follows from (3.13) and (3.14) that

$$\boldsymbol{\mathcal{S}}_{2,D_\delta}[\widetilde{\boldsymbol{\phi}}] = \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}] + \delta \boldsymbol{\mathcal{D}}^{\sharp}_{2,D}[h\boldsymbol{\phi}]\big|_- - \delta \boldsymbol{\mathcal{S}}_{2,D}[\kappa h \boldsymbol{\phi}] + O(\delta^{1+\eta}) \quad \text{on } \partial D, \tag{3.15}$$

where the remainder term $O(\delta^{1+\eta})$ is in $W^2_1(\partial D)$.

The following proposition is a direct consequence of (2.13), (2.14), (3.4), (3.5), (3.9), (3.10), (3.12), and (3.15).

Proposition 3.2 *The following expansions hold on ∂D:*

$$\mathcal{T}_\delta(\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) = \mathcal{T}_0(\boldsymbol{\phi}, \boldsymbol{\psi}) - \delta \mathcal{T}_1(\boldsymbol{\psi}) + O(\delta^2) \quad \textit{for } (\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) \in L^2(\partial D) \times L^2(\partial D_\delta),$$

$$\mathcal{W}_\delta(\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) = \delta \mathcal{W}_1(\boldsymbol{\phi}, \boldsymbol{\psi}) + o(\delta) \quad \textit{for } (\boldsymbol{\phi}, \widetilde{\boldsymbol{\psi}}) \in \mathcal{C}^{1,\eta}(\partial D) \times \mathcal{C}^{1,\eta}(\partial D_\delta),$$

where $\boldsymbol{\psi} = \widetilde{\boldsymbol{\psi}} \circ \Phi_\delta$, the remainder terms $O(\delta^2)$ and $o(\delta)$ are in $W^2_1(\partial D) \times L^2(\partial D)$, and the operators $\mathcal{T}_0 : \mathcal{X}(\partial D) \to \mathcal{Y}(\partial D)$, $\mathcal{T}_1 : L^2(\partial D) \to \mathcal{Y}(\partial D)$, and $\mathcal{W}_1 : \mathcal{C}^{1,\eta}(\partial D) \times \mathcal{C}^{1,\eta}(\partial D) \to$

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$\mathcal{Y}(\partial D)$ *are defined by*

$$\mathcal{T}_0(\boldsymbol{\phi},\boldsymbol{\psi}) = \Big(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}] - \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}], \frac{\partial \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\varphi}]}{\partial \nu_1}\Big|_- - \frac{\partial \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]}{\partial \nu_0}\Big|_+\Big), \tag{3.16}$$

$$\mathcal{T}_1(\boldsymbol{\psi}) = \Big(- \boldsymbol{\mathcal{S}}_{0,D}[\kappa h \boldsymbol{\psi}] + h\frac{\partial \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]}{\partial \mathbf{n}}\Big|_+ + \boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\psi}]\Big|_+, h div(\sigma_0 \boldsymbol{D}((\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]))\Big|_+$$
$$+ h\kappa(\sigma_0 \boldsymbol{D}((\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]))\mathbf{n}\Big|_+ - \frac{\partial \boldsymbol{\mathcal{S}}_{0,D}[\kappa\ h\boldsymbol{\psi}]}{\partial \nu_0}\Big|_+ + \frac{\partial \boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\psi}]}{\partial \nu_0}\Big|_+ - \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0 \widehat{\nabla} \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\psi}]\big)\boldsymbol{\tau}\Big)\Big|_+\Big), \tag{3.17}$$

$$\mathcal{W}_1(\boldsymbol{\phi},\boldsymbol{\psi}) = \Big(h\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]}{\partial \mathbf{n}}\Big|_+ + h\frac{\partial \boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\psi}]}{\partial \mathbf{n}}\Big|_-, h div(\sigma_2 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]))\Big|_+ + h\kappa(\sigma_2 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]))\mathbf{n}\Big|_+$$
$$+ h div(\sigma_2 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\psi}]))\Big|_- + h\kappa(\sigma_2 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\psi}]))\mathbf{n}\Big|_-$$
$$- \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\sigma_2 \boldsymbol{D}\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\phi}]\big)\big)\boldsymbol{\tau}\Big)\Big|_+ \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\sigma_2 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{2,D}[\boldsymbol{\psi}])\big)\boldsymbol{\tau}\Big)\Big|_-\Big). \tag{3.18}$$

The following proposition holds (see [17])

Proposition 3.3 *Let* $(\boldsymbol{\phi}_1, \widetilde{\boldsymbol{\phi}}_0) \in L^2(\partial D) \times L^2(\partial D_\delta)$ *be the solution of* (2.12). *Then the following asymptotic expansion holds:*

$$\mathcal{T}_0(\boldsymbol{\phi}_1,\boldsymbol{\phi}_0) - \delta\big[\mathcal{T}_1(\boldsymbol{\phi}_0) - \mathcal{G}(\boldsymbol{\phi}_1)\big] = \mathcal{H}_\delta + o(\delta) \quad \text{on } \partial D, \tag{3.19}$$

where $\phi_0 = \widetilde{\boldsymbol{\phi}}_0 \circ \Phi_\delta$*, the remainder term* $o(\delta)$ *is in* $W^2_1(\partial D) \times L^2(\partial D)$*,* $\mathcal{H}_\delta$ *is defined by* (2.15)*,* $\mathcal{T}_0$ *and* $\mathcal{T}_1$ *are defined in* (3.16) *and* (3.17)*, respectively, and the operator* $\mathcal{G}$ *is defined from* $L^2(\partial D)$ *into* $\mathcal{Y}(\partial D)$ *by*

$$\mathcal{G}(\boldsymbol{\phi}_1) := \Big(h\frac{\partial \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]}{\partial \mathbf{n}}\Big|_-, h div(\sigma_1 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1])) + h\kappa(\sigma_1 \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1]))\mathbf{n}\Big|_-$$
$$- \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{M}_{2,1} \boldsymbol{D}(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1])\big)\boldsymbol{\tau}\Big)\Big|_-\Big). \tag{3.20}$$

with

$$\mathbb{M}_{2,1} = 2\mu_1 \mathbb{I} + (\mu_2 - \mu_1)\mathbf{I} \otimes (\boldsymbol{\tau} \otimes \boldsymbol{\tau})$$

4 Asymptotic expansion of the displacement field

The following lemma is important for us.

Lemma 4.1 *For any given* $(\boldsymbol{\varphi}, \boldsymbol{\Psi}) \in \mathcal{Y}(\partial D)$*, there exists a unique pair* $(\boldsymbol{\phi},\boldsymbol{\psi}) \in \mathcal{X}(\partial D)$ *such that*

$$\mathcal{T}_0(\boldsymbol{\phi},\boldsymbol{\psi}) - \delta\big[\mathcal{T}_1(\boldsymbol{\phi}) - \mathcal{G}(\boldsymbol{\psi})\big] = \big(\boldsymbol{\varphi}, \boldsymbol{\Psi}\big). \tag{4.1}$$

Furthermore, there exists a constant C *depending only on* μ_0, μ_1*, and the Lipschitz character of* D *such that*

$$\|\boldsymbol{\phi}\|_{L^2(\partial D)} + \|\boldsymbol{\psi}\|_{L^2(\partial D)} \leq C\Big(\|\boldsymbol{\varphi}\|_{W^2_1(\partial D)} + \|\boldsymbol{\Psi}\|_{L^2(\partial D)}\Big). \tag{4.2}$$

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Proof. The operator $\mathcal{T} : \mathcal{X}(\partial D) \to \mathcal{Y}(\partial D)$ defined by $\mathcal{T}(\boldsymbol{\phi}, \boldsymbol{\psi}) = \mathcal{T}_1(\boldsymbol{\phi}) - \mathcal{G}(\boldsymbol{\psi})$ is bounded on $\mathcal{X}(\partial D)$. By Theorem 2.1, $\mathcal{T}_0 : \mathcal{X}(\partial D) \to \mathcal{Y}(\partial D)$ is invertible. For δ small enough, it follows from [13], that the operator $\mathcal{T}_0 - \delta\mathcal{T}$ is invertible. This completes the proof of solvability of (4.1). The estimate (4.2) is a consequence of solvability and the closed graph theorem.

Theorem 4.2 *Let* $(\mathbf{u}_\delta, p_\delta)$ *be the solution to* (1.3)*. Let* Ω *be a bounded region away from* ∂D*. For* $x \in \Omega$*, the following pointwise asymptotic expansion holds:*

$$\mathbf{u}_\delta(x) = \mathbf{u}(x) + \delta \mathbf{u}_1(x) + o(\delta), \tag{4.3}$$

$$p_\delta(x) = p(x) + \delta p_1(x) + o(\delta), \tag{4.4}$$

where the remainder $o(\delta)$ *depends only on* μ_j *for j=0,1,2, the* $\mathcal{C}^2$*-norm of* X*, the* $\mathcal{C}^1$*-norm of* h*, and* $dist(\Omega, \partial D)$*,* $(\mathbf{u}_\delta, p_\delta)$ *is the unique solution to*

$$\begin{cases} -div\big(\sigma \boldsymbol{D}(\mathbf{u}) - pId\big) = 0 & in\ \Omega, \\ \nabla.u = 0 & in\ \Omega, \\ \mathbf{u} = \mathbf{F}(x) & on\ \partial\Omega \end{cases} \tag{4.5}$$

and $(\mathbf{u}_1, p_\delta)$ *is the unique solution of the following transmission problem:*

$$\begin{cases} -div\big(\sigma_0 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ \Omega\backslash\overline{D}_\delta, \\ -div\big(\sigma_2 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ D_\delta\backslash\overline{D}, \\ -div\big(\sigma_1 \boldsymbol{D}(\mathbf{u}_1) - p_1 Id\big) = 0 & in\ D, \\ \mathbf{u}_1\big|_- - \mathbf{u}_1\big|_+ = 0 & on\ \partial D, \\ \dfrac{\partial \mathbf{u}_1}{\partial \nu_1}\Big|_- - \dfrac{\partial \mathbf{u}_1}{\partial \nu_0}\Big|_+ = \dfrac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big[(\mathbb{M}_{2,1} - \mathbb{M}_{0,1})D(v^i)\big]\boldsymbol{\tau}\Big) & on\ \partial D, \\ \nabla.\mathbf{u}_1 = 0 & in\ \Omega, \\ \mathbf{u}_1 = \mathbf{F}(x) & on\ \partial\Omega \end{cases} \tag{4.6}$$

with $\boldsymbol{\tau}$ *is the tangential vector to* ∂D*,*

$$\mathbb{M}_{l,k} := 2\mu_k \mathbb{I} + 2(\mu_l - \mu_k)\mathbf{I} \otimes (\boldsymbol{\tau} \otimes \boldsymbol{\tau})$$

By taking $\mu_2 = \mu_1$, we reduce our problem to that proposed in [15]. So it is obvious to obtain the asymptotic expansion of the displacement field resulting from small perturbations of the shape of impurety.

4.1 Proof of the theorem 4.2

We have the following Taylor expansion for $\tilde{x} = x + \delta h(x)\mathbf{n}(x) \in \partial D_\delta$:

$$\mathbf{H}(\tilde{x}) = \mathbf{H}(x) + \delta h(x)\frac{\partial \mathbf{H}}{\partial \mathbf{n}}(x) + O(\delta^2), \quad x \in \partial D. \tag{4.7}$$

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using the Taylor expansion, (3.1) and (3.3), we have

$$\frac{\partial \mathbf{H}}{\partial \nu_0}(\tilde{x}) = \frac{\partial \mathbf{H}}{\partial \nu_0}(x) + \delta h(x) div(\sigma_0 \mathbf{D}(\mathbf{H})) + h\kappa(\sigma_0 \mathbf{D}(\mathbf{H}))\mathbf{n}_0(x) - \delta \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big(\mathbb{C}_0 \widehat{\nabla} \mathbf{H}\big)\boldsymbol{\tau}\Big)(x) + O(\delta^2), \quad x \in \partial D. \tag{4.8}$$

It follows from (2.15), (4.7), and (4.8) that

$$\begin{aligned}\mathcal{H}_\delta &= \Big(\mathbf{H}, \frac{\partial \mathbf{H}}{\partial \nu_0}\Big) + \delta\Big(h\frac{\partial \mathbf{H}}{\partial \mathbf{n}}, h(x) div(\sigma_0 \mathbf{D}(\mathbf{H})) + h\kappa(\sigma_0 \mathbf{D}(\mathbf{H}))\mathbf{n}_0(x) - \frac{\partial}{\partial \boldsymbol{\tau}}\big(h\big[\mathbb{C}_0 \widehat{\nabla} \mathbf{H}\big]\boldsymbol{\tau}\big)\Big) \\ &\quad + O(\delta^2) \\ &:= \mathcal{H}_0 + \delta \mathcal{H}_1 + O(\delta^2) \quad \text{ on } \partial D. \end{aligned} \tag{4.9}$$

We now introduce $(\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_0^0)$ and $(\boldsymbol{\phi}_1^1, \boldsymbol{\phi}_0^1)$ by the following recursive relations

$$\mathcal{T}_0(\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_0^0) = \mathcal{H}_0, \tag{4.10}$$

$$\mathcal{T}_0(\boldsymbol{\phi}_1^1, \boldsymbol{\phi}_0^1) = \mathcal{H}_1 + \mathcal{T}_1(\boldsymbol{\phi}_0^0) - \mathcal{G}(\boldsymbol{\phi}_1^0), \tag{4.11}$$

where $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{G}$ are defined in (3.16), (3.17), and (3.20), respectively. One can see the existence and uniqueness of $(\boldsymbol{\phi}_1^n, \boldsymbol{\phi}_0^n)$ for $n = 0, 1$, by using [10].

Let $(\boldsymbol{\phi}_1, \widetilde{\boldsymbol{\phi}}_0)$ be the solution of (2.12). It follows from (4.10) and (4.11) that

$$\begin{aligned}&\mathcal{T}_0(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_1^0 - \delta\boldsymbol{\phi}_1^1, \widetilde{\boldsymbol{\phi}}_0 \circ \Phi_\delta - \phi_0^0 - \delta\boldsymbol{\phi}_0^1) - \delta\big[\mathcal{T}_1(\widetilde{\boldsymbol{\phi}}_0 \circ \Phi_\delta - \boldsymbol{\phi}_0^0 - \delta\boldsymbol{\phi}_0^1) - \mathcal{G}(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_1^0 - \delta\boldsymbol{\phi}_1^1)\big] \\ &\quad = \mathcal{H}_\delta - \mathcal{H}_0 - \delta\mathcal{H}_1 + o(\delta) \quad \text{ on } \partial D, \end{aligned} \tag{4.12}$$

where $\|o(\delta)\|_{W_1^2(\partial D)\times L^2(\partial D)} \leq C\delta^{1+\eta}$ for some $\eta > 0$ and $(\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_0^0)$ and $(\boldsymbol{\phi}_1^1, \boldsymbol{\phi}_0^1)$ are the solutions to (4.10) and (4.11), respectively.

The following lemma holds immediately from (4.9), (4.12) and the estimate in (4.2).

Lemma 4.3 *Let $(\boldsymbol{\phi}_1, \widetilde{\boldsymbol{\phi}}_0)$ be the solution of* (2.12). *For δ small enough, there exists C depending only on μ_s for s=0,1,2, the $\mathcal{C}^2$-norm of X, and the $\mathcal{C}^1$-norm of h such that*

$$\Big\|\boldsymbol{\phi}_1 - \boldsymbol{\phi}_1^0 - \delta\boldsymbol{\phi}_1^1\Big\|_{L^2(\partial D)} + \Big\|\widetilde{\boldsymbol{\phi}}_0 \circ \Phi_\delta - \boldsymbol{\phi}_0^0 - \delta\boldsymbol{\phi}_0^1\Big\|_{L^2(\partial D)} \leq C\delta^{1+\eta} \tag{4.13}$$

for some $\eta > 0$, where $(\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_0^0)$ and $(\boldsymbol{\phi}_1^1, \boldsymbol{\phi}_0^1)$ are the solutions to (4.10) *and* (4.11), *respectively.*

Recall that the domain D is separated apart from Ω, then

$$\sup_{x\in\Omega, y\in\partial D}\Big|\partial^i \boldsymbol{\Gamma}_0(x-y)\Big| \leq C, \quad i \in \mathbb{N}^2,$$

for some constant C depending on $dist(D, \Omega)$. After the change of variables $\tilde{y} = \Phi_\delta(y)$, we get from (3.2), (4.13), and the Taylor expansion of $\boldsymbol{\Gamma}_0(x - \tilde{y})$ in $y \in \partial D$ for each fixed $x \in \Omega$

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that

$$\boldsymbol{\mathcal{S}}_{0,D_\delta}[\widetilde{\boldsymbol{\phi}}_0](x) = \int_{\partial D_\delta} \boldsymbol{\Gamma}_0(x-\tilde{y})\widetilde{\boldsymbol{\phi}}_0(\tilde{y})d\sigma(\tilde{y})$$

$$= \int_{\partial D} \Big(\boldsymbol{\Gamma}_0(x-y) + \delta h(y)\nabla\boldsymbol{\Gamma}_0(x-y)\mathbf{n}(y)\Big)\Big(\boldsymbol{\phi}_0^0(y) + \delta\boldsymbol{\phi}_0^1(y)\Big)$$

$$\times \Big(1 - \delta\kappa(y)h(y)\Big)d\sigma(y) + o(\delta)$$

$$= \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0](x) + \delta\Big(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1](x) - \boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0](x) + \boldsymbol{\mathcal{D}}_{0,D}^\sharp[h\boldsymbol{\phi}_0^0](x)\Big) + o(\delta),$$

Therefore, we obtain from (2.10) that for $x \in \Omega$,

$$\mathbf{u}_\delta(x) = \mathbf{H}(x) + \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0](x) + \delta\Big(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1](x) - \boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0](x) + \boldsymbol{\mathcal{D}}_{0,D}^\sharp[h\boldsymbol{\phi}_0^0](x)\Big) + o(\delta). \tag{4.14}$$

According to [2] (see also [23]), the solution $(\mathbf{u}, p)$ to (4.5) is represented as

$$\mathbf{u}(x) = \begin{cases} \mathbf{H}(x) + \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0](x), & x \in \Omega\backslash\overline{D}, \\ \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^0](x), & x \in D, \end{cases} \tag{4.15}$$

where $(\boldsymbol{\phi}_1^0, \boldsymbol{\phi}_0^0)$ is the unique solution of (4.10).

$$p(x) = \begin{cases} \vartheta_D[\boldsymbol{\phi}_0^0](x), & x \in \Omega\backslash\overline{D}, \\ \vartheta_D[\boldsymbol{\phi}_1^0](x), & x \in D, \end{cases} \tag{4.16}$$

The following theorem follows immediately from (4.14) and (4.15).

Theorem 4.4 *For δ small enough. The following pointwise expansion holds for $x \in \Omega$*

$$\mathbf{u}_\delta(x) = \mathbf{u}(x) + \delta\Big(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1](x) - \boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0](x) + \boldsymbol{\mathcal{D}}_{0,D}^\sharp[h\boldsymbol{\phi}_0^0](x)\Big) + o(\delta), \tag{4.17}$$

where $\boldsymbol{\phi}_0^0$ and $\boldsymbol{\phi}_0^1$ are defined by (4.10) and (4.11), respectively. The remainder $o(\delta)$ depends only on μ_s for s=0,1,2, the $\mathcal{C}^2$-norm of X, the $\mathcal{C}^1$-norm of h, and dist(Ω, D).

We now prove a representation theorem for the solution of the transmission problem (4.6) which will help us to derive the theorem 4.2.

Theorem 4.5 *The solution $\mathbf{u}_1$ of (4.6) is represented by*

$$\mathbf{u}_1(x) = \begin{cases} \boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1](x) - \boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0](x) + \boldsymbol{\mathcal{D}}_{0,D}^\sharp[h\boldsymbol{\phi}_0^0](x), & x \in \Omega\backslash\overline{D}, \\ \boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^1](x), & x \in D, \end{cases} \tag{4.18}$$

where $\boldsymbol{\phi}_0^0$ and $(\boldsymbol{\phi}_1^1, \boldsymbol{\phi}_0^1)$ are defined by (4.10) and (4.11), respectively.

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Proof. One can easily see that

$$-div\big(\sigma_0\mathbf{D}(\mathbf{u}_1)-p_1Id\big)=0 \quad \text{in } \Omega\backslash\overline{D}, \qquad -div\big(\sigma_2\mathbf{D}(\mathbf{u}_1)-p_1Id\big)=0 \quad \text{in } D.$$

It follows from (4.11), (4.15) (4.18) that

$$\begin{aligned}
\mathbf{u}_1^i-\mathbf{u}_1^e =&\Big(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^1]-\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1]\Big)+\boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0]-\boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\phi}_0^0]\big|_+\\
=&\Big[\mathcal{H}_1+\mathcal{Q}_1(\boldsymbol{\phi}_0^0)-\mathcal{Z}(\boldsymbol{\phi}_1^0)\Big]_1+\boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0]-\boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\phi}_0^0]\big|_+\\
=&h\Big(\frac{\partial\mathbf{H}}{\partial\mathbf{n}}+\frac{\partial\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0]}{\partial\mathbf{n}}\Big|_+-\frac{\partial\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^0]}{\partial\mathbf{n}}\Big|_-\Big)\\
=&h\big(\nabla\mathbf{u}^e\mathbf{n}-\nabla\mathbf{u}^i\mathbf{n}\big)\\
=&0 \quad \text{on } \partial D.
\end{aligned}$$

Using (4.11), we get

$$\begin{aligned}
\frac{\partial\mathbf{u}_1}{\partial\nu_1}\Big|_--\frac{\partial\mathbf{u}_1}{\partial\nu_0}\Big|_+=&\Big(\frac{\partial\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^1]}{\partial\nu_1}\Big|_--\frac{\partial\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^1]}{\partial\nu_0}\Big|_+\Big)+\frac{\partial\boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0]}{\partial\nu_0}\Big|_+-\frac{\partial\boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\phi}_0^0]}{\partial\nu_0}\Big|_+\\
=&\Big[\mathcal{H}_1+\mathcal{Q}_1(\boldsymbol{\phi}_0^0)-\mathcal{Z}(\boldsymbol{\phi}_1^0)\Big]_2+\frac{\partial\boldsymbol{\mathcal{S}}_{0,D}[\kappa h\boldsymbol{\phi}_0^0]}{\partial\nu_0}\Big|_+-\frac{\partial\boldsymbol{\mathcal{D}}^\sharp_{0,D}[h\boldsymbol{\phi}_0^0]}{\partial\nu_0}\Big|_+\\
=&h(div(\sigma_0\mathbf{D}(\mathbf{H}))\Big|_++h\kappa(\sigma_0\mathbf{D}(\mathbf{H}))\mathbf{n}\Big|_+-\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\sigma_0\mathbf{D}(\mathbf{H})\big)\boldsymbol{\tau}\Big)\\
&-\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\sigma_0\mathbf{D}(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0])\big)\boldsymbol{\tau}\Big)\Big|_+hdiv(\sigma_0\mathbf{D}(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0]))\Big|_+\\
&+h\kappa(\sigma_0\mathbf{D}(\boldsymbol{\mathcal{S}}_{0,D}[\boldsymbol{\phi}_0^0]))\mathbf{n}\Big|_+-hdiv(\sigma_1\mathbf{D}(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^0]))\Big|_-\\
&-h\kappa(\sigma_1\mathbf{D}(\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^0]))\mathbf{n}\Big|_-+\frac{\partial}{\partial\boldsymbol{\tau}}\Big(h\big(\mathbb{M}_{2,1}\widehat{\nabla}\boldsymbol{\mathcal{S}}_{1,D}[\boldsymbol{\phi}_1^0]\big)\boldsymbol{\tau}\Big)\Big|_-\\
=&\frac{\partial}{\partial\boldsymbol{\tau}}\big(h(\mathbb{M}_{2,1}\widehat{\nabla}\mathbf{u}^i)\boldsymbol{\tau}\big)-\frac{\partial}{\partial\boldsymbol{\tau}}\big(h(\sigma_0\mathbf{D}(\mathbf{u}^e))\boldsymbol{\tau}\big)\\
=&\frac{\partial}{\partial\boldsymbol{\tau}}\big(h([\mathbb{M}_{2,1}-\mathbb{M}_{0,1}]\mathbf{D}(\mathbf{u}^i))\boldsymbol{\tau}\big).
\end{aligned}$$

This completes the proof of the theorem 4.5.

The main theorem 4.2 immediately follows from the integral representation of $\mathbf{u}_1$ and the theorem 4.4.

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Let $(\mathbf{v}, q)$ be the solution of the following problem:

$$\begin{cases} -div\big(\sigma_0\mathbf{D}(\mathbf{v}) - qId\big) = 0 & \text{in } \Omega\backslash\overline{D}, \\ -div\big(\sigma_1\mathbf{D}(\mathbf{v}) - qId\big) = 0 & \text{in } D, \\ \mathbf{v}|_- = \mathbf{v}|_+ & \text{on } \partial D, \\ \dfrac{\partial \mathbf{v}}{\partial \nu_1}\Big|_- = \dfrac{\partial \mathbf{v}}{\partial \nu_0}\Big|_+ & \text{on } \partial D, \\ \nabla.v = 0 & \text{in } \Omega, \\ \mathbf{v}(x) = \mathbf{G}(x) & \text{on } \partial\Omega \end{cases} \tag{4.19}$$

As a consequence of the theorem 4.2, we obtain the following relationship between velocity measurements and the deformation h.

Theorem 4.6 *Let* $(\mathbf{u}_\delta, p_\delta)$*,* $(\mathbf{u}, p)$*, and* $(\mathbf{v}, q)$ *be the solutions to* (1.3)*,* (4.5)*, and* (4.19)*, respectively. Let* S *be a Lipschitz closed curve enclosing* D *away from* ∂D*. The following asymptotic expansion holds:*

$$\begin{aligned} &\int_S (\mathbf{u}_\delta - \mathbf{u}) \cdot \frac{\partial \mathbf{F}}{\partial \nu_0} d\sigma - \int_S (\frac{\partial \mathbf{u}_\delta}{\partial \nu_0} - \frac{\partial \mathbf{u}}{\partial \nu_0}) \cdot \mathbf{F} d\sigma \\ &= \delta \int_{\partial D} h\Big(([\mathbb{M}_{0,1} - \mathbb{M}_{2,1}]\boldsymbol{D}(\mathbf{u}^i)\boldsymbol{\tau} \cdot \boldsymbol{D}(\mathbf{v}^i)\boldsymbol{\tau}\Big) d\sigma + o(\delta), \end{aligned} \tag{4.20}$$

where the remainder $o(\delta)$ *depends only on* μ_s *for s=0,1,2, the* $\mathcal{C}^2$*-norm of* X*, the* $\mathcal{C}^1$*-norm of* h*, and* $dist(S, \partial D)$*. The dot denotes the scalar product in* $\mathbb{R}^2$*.*

4.2 Proof of the theorem 4.6

The following corollary can be proved as in exactly the same manner as Theorem 4.2.

Corollary 4.7 *Let* $(\mathbf{u}_\delta, p_\delta)$ *and* $(\mathbf{u}, p)$ *be the solutions to* (1.3) *and* (4.5)*, respectively and* $\mathbf{u}_1$ *is the unique solution of* (4.6)*.*
Let Ω *be a bounded region outside the inclusion* D *away from* ∂D*. For* $x \in \Omega$*, the following pointwise asymptotic expansion holds:*

$$\frac{\partial \mathbf{u}_\delta}{\partial \nu_0}(x) = \frac{\partial \mathbf{u}}{\partial \nu_0}(x) + \delta \frac{\partial \mathbf{u}_1}{\partial \nu_0}(x) + o(\delta), \tag{4.21}$$

where the remainder $o(\delta)$ *depends only on* μ_j *for j=0,1,2, the* $\mathcal{C}^2$*-norm of* X*, the* $\mathcal{C}^1$*-norm of* h*, and* $dist(\Omega, \partial D)$*.*

Let S be a Lipschitz closed curve enclosing D away from ∂D. Let $\mathbf{v}$ be the solution to (4.19). As is done in [18], and by integration by parts one may combine both relations (4.3) and (4.21) to get that,

$$\begin{aligned} \int_S (\mathbf{u}_\delta - \mathbf{u}) \cdot \frac{\partial \mathbf{F}}{\partial \nu_0} d\sigma - \int_S (\frac{\partial \mathbf{u}_\delta}{\partial \nu_0} - \frac{\partial \mathbf{u}}{\partial \nu_0}) \cdot \mathbf{F} d\sigma &= \delta \int_S \Big(\mathbf{u}_1 \cdot \frac{\partial \mathbf{v}}{\partial \nu_0} - \frac{\partial \mathbf{u}_1}{\partial \nu_0} \cdot \mathbf{v}\Big) d\sigma + o(\delta) \\ &= \int_S \Big(\frac{\partial \mathbf{v}^e}{\partial \nu_0} \cdot \mathbf{u}_1^e - \mathbf{v}^e \cdot \frac{\partial \mathbf{u}_1^e}{\partial \nu_0}\Big) d\sigma + o(\delta). \end{aligned}$$

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Taking into account relation (4.6), we immediately get

$$\int_S \Big(\frac{\partial \mathbf{v}^e}{\partial \nu_0}\cdot \mathbf{u}_1^e - \mathbf{v}^e \cdot \frac{\partial \mathbf{u}_1^e}{\partial \nu_0}\Big) d\sigma = \int_{\partial D} \Big(\frac{\partial \mathbf{v}^i}{\partial \nu_1}\cdot \mathbf{u}_1^i - \mathbf{v}^i \cdot \frac{\partial \mathbf{u}_1^i}{\partial \nu_1}\Big) d\sigma \\ + \int_{\partial D} \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{M}_{2,1} - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big)\cdot \mathbf{v}^i d\sigma. \tag{4.22}$$

It follows that

$$\int_{\partial D} \Big(\frac{\partial \mathbf{v}^i}{\partial \nu_1}\cdot \mathbf{u}_1^i - \mathbf{v}^i \cdot \frac{\partial \mathbf{u}_1^i}{\partial \nu_1}\Big) d\sigma = 0. \tag{4.23}$$

We have

$$\int_{\partial D} \frac{\partial}{\partial \boldsymbol{\tau}}\Big(h\big([\mathbb{M}_{2,1} - \mathbb{M}_{0,1}]\widehat{\nabla}\mathbf{u}^i\big)\boldsymbol{\tau}\Big)\cdot \mathbf{v}^i d\sigma = -\int_{\partial D} h\big([\mathbb{M}_{2,1} - \mathbb{M}_{0,1}]\mathbf{D}(\mathbf{u}^i)\big)\boldsymbol{\tau}\cdot \nabla \mathbf{v}^i \boldsymbol{\tau} d\sigma. \tag{4.24}$$

One can easily check that

$$\big([\mathbb{M}_{2,1} - \mathbb{M}_{0,1}]\mathbf{D}(\mathbf{u}^i)\big)\boldsymbol{\tau}\cdot \nabla \mathbf{v}^i \boldsymbol{\tau} = \big([\mathbb{M}_{2,1} - \mathbb{M}_{0,1}]\mathbf{D}(\mathbf{u}^i)\big)\boldsymbol{\tau}\cdot \mathbf{D}(\mathbf{v}^i)\boldsymbol{\tau}. \tag{4.25}$$

We finally obtain from (4.22)-(4.25) the relationship between stokes solution measurements and the shape deformation h (4.20), as desired.

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