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A STUDY OF SPACE TAUTOCHRONE CURVE

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ABSTRACT. We consider the tautochrone problem in the three-dimensional space. First, we derive the sliding time along a frictionless space curve from rest under the influence of gravity in terms of fractional calculus. Then we give the condition for an unrestricted space curve to satisfy tautochronism. Further we study the tautochrone curve restricted on a vertical cylindrical surface, in a tilting plane and on a general space surface, respectively. A tautochrone curve with the glide time T on a vertical cylindrical surface can stretch upwards and reach its maximum height $H = 2gT^2/\pi^2$ like an unrestricted tautochrone curve, where g is the gravitational acceleration, while a tautochrone curve restricted on a general space surface cannot reach such a height in general. Specific examples further show the proposed method. The tautochrone problems on the conical surface and the elliptic paraboloid lead to complicated nonlinear ordinary differential equations worthy of further study.

1. Introduction

In the 17th century, with the development of calculus, many mathematicians strived to explore practical problems posed in physics and mechanics, so that calculus in the scope of application continued to expand. Among many problems, one of the more famous is the tautochrone problem, alias isochrone problem [1]. The tautochrone problem is to determine a frictionless curve or wire, which lies in a vertical plane, such that the time taken by a bead sliding from rest along the curve under the influence of gravity to its lowest point is independent of its starting position.

The tautochrone problem was solved by Dutch scientist Huygens, who proved geometrically in his "Horologium Oscillatorium", published in 1673, that the curve was a cycloid [1,2]. Huygens also proved that the time of descent is equal to the time an object takes to fall vertically the same distance as diameter of the circle that generates the cycloid, multiplied by $\pi/2$. This means that the time of descent is $\pi\sqrt{r/g}$, where r is the radius of the circle which generates the cycloid, and g is the gravitational acceleration.

In the late 17th century, Huygens independently discovered a pendulum clock based on the principles of the tautochrone curve while seeking to improve the accuracy of timekeeping devices. Afterwards, this appliance was called as the Huygens pendulum. Unlike earlier pendulum clocks, which utilized simple harmonic motion, Huygens' design incorporated a pendulum whose period of oscillation remained constant, regardless of the amplitude of its swing. This crucial innovation was made possible by shaping the pendulum's suspension point and bob to follow a cycloidal path, which approximates the tautochrone curve.

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The Huygens pendulum clock revolutionized timekeeping, providing unprecedented accuracy and reliability compared to earlier mechanical clocks. Its widespread adoption heralded a new era of precision in science, navigation, and everyday life, laying the groundwork for modern timekeeping technologies. Beyond its practical applications, the Huygens pendulum exemplifies the intersection of theoretical inquiry and technological innovation. By harnessing and optimizing the path design, Huygens not only advanced the field of horology but also deepened our understanding of the fundamental dynamics of oscillatory systems.

Later, mathematicians Lagrange, Euler and Abel provided and developed different methods to solve the tautochrone problem [3]. In particular, the Abel integral equations came from the study for the tautochrone problem and fractional calculus was born in Abel's first paper on the generalization of the tautochrone problem, that was published in 1823 [4].

For the resolution of the tautochrone problem, the Laplace transforms and the fractional calculus were found out to be effective methods. In [5], the history of mathematics on the cycloid curve and proofs of its important properties were presented. In [6], systems of tautochrones in a general field of force were studied. In [7], a more general problem of finding tautochrone curves for a particle in a general potential was introduced. In [8], the Laplace transform formalism and the convolution theorem were used to solve tautochrone problem. In [9], a simplified method for the tautochrone problem was proposed based on experience with simple harmonic motion.

Generalization of the tautochrone problem was considered in [10–14]. In [11], potential energy functions that lead to periodic motions were investigated and it was found that there are an infinite number of tautochrone curves in addition to the cycloid solution. In [12], a tautochrone problem was considered by supposing that the xy-plane of the tautochrone curve was rotating about the y-axis with constant angular momentum. In [13], relativistic tautochrone was considered and it was shown that the methods of fractional calculus are more useful in the derivation of the exact relativistic tautochrone. In [14], using the fractional derivation method, the tautochrone curve for a rotating system was determined.

Research on the tautochrone problem promoted development of the Abel integral equation, including the analytical and numerical methods [15–19]. In addition, the tautochrone curve is related to the brachistochrone curve, which is also the cycloid. The brachistochrone problem is concerned with finding the shortest time trajectory of a particle sliding on a frictionless path under gravity. Johann Bernoulli posed the problem of the brachistochrone and published the solution in the Acta Eruditorum in 1697, and noted that the solution is the same curve as Huygens's tautochrone curve [20]. The cycloid is the solution to the tautochrone and brachistochrone problems, which is one of the most intriguing objects in the classical physics world [21]. The cycloid is widely applied in real life, e.g., in the design of the life-saving passage, the skiing venue, the roller coaster track and the building roof.

Research on the space tautochrone curves, especially the tautochrone curves restricted on a space surface, is meagre. In this work, we consider this problem. The Riemann-Liouville fractional integral and its basic properties will be used. Suppose $f(x)$ is a real function defined on the interval $(x_0, b]$ and $f(x) = (x-x_0)^{\lambda} g(x)$, where $\lambda > -1$ and $g(x)$ is a continuous function on the interval $[x_0, b]$, then the

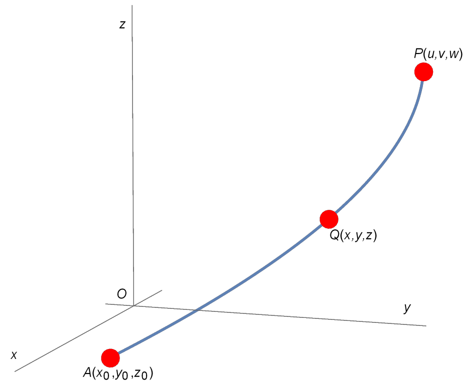

FIGURE 1. A sliding bead on a space wire.

Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $f(x)$ is defined as [22,23]

$$ {}_{x_0}I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \; x \in (x_0, b], \tag{1}$$

where Γ is the Euler gamma function defined by $\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt,\; z > 0$, and has the special valuation $\Gamma(1/2) = \sqrt{\pi}$. The fractional integral operator accepts the semigroup property

$$ {}_{x_0}I_x^{\alpha}\, {}_{x_0}I_x^{\beta} f(x) = {}_{x_0}I_x^{\alpha+\beta} f(x), \tag{2}$$

and has the formula

$$ {}_{x_0}I_x^{\alpha}\, c = \frac{c\,(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}, \tag{3}$$

for a constant c.

In this work, we consider the tautochrone problem in the three-dimensional space. In next section, we derive the sliding time along a space curve and calculate two numerical examples. In Section 3, conditions and properties satisfied by an unrestricted space tautochrone curve are investigated. In Section 4, we consider the tautochrone curve restricted on a vertical cylindrical surface. In Section 5, the tautochrone curve in a tilting plane is derived. In Section 6, the tautochrone curve restricted on a general surface is studied. Section 7 summarizes our conclusions.

2. Sliding time along a space curve

Suppose a bead is constrained to move on a smooth, frictionless and nondeformable wire, which lies in the three-dimensional space as shown in Figure 1. If the particle starts from rest at any point of the wire and falls under the influence of gravity, find the time of descent to the point A of the wire.

The wire is simulated by the space curve and suppose the parametric equation of the curve is

$$\begin{cases} x = \xi(p), \\ y = \eta(p), \\ z = \zeta(p), \end{cases} \tag{4}$$

where $\xi(p), \eta(p), \zeta(p)$ have continuous derivatives and $\zeta(p)$ is strictly monotonically increasing function of p. Thus the inverse function $p = \zeta^{-1}(z)$ exists and the curve has the form

$$\begin{cases} x = x(z) = \xi\left(\zeta^{-1}(z)\right), \\ y = y(z) = \eta\left(\zeta^{-1}(z)\right), \end{cases} \tag{5}$$

parameterized by the vertical coordinate z.

The particle has mass m and starts from rest at a point P with coordinates (u, v, w). The sliding time along the space curve to the terminal (lowest) point $A(x_0, y_0, z_0)$ is to be calculated. For this purpose, we take an intermediate point in the motion, $Q(x, y, z)$, and let σ be the length of the arc $\widehat{AQ}$. From the conservation of energy, we have

$$mgw = mgz + \frac{1}{2}m\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2, \tag{6}$$

where g is the gravitational acceleration and $-\frac{\mathrm{d}\sigma}{\mathrm{d}t}$ is the magnitude of the instantaneous speed of the particle at Q. Then it follows

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 = 2g(w - z).$$

Further, using the fact that σ decreases as time t increases yields,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\sqrt{2g(w - z)}. \tag{7}$$

On the other hand, the arc length differentiation has the form from Eq. (4),

$$\mathrm{d}\sigma = \sqrt{\xi'(p)^2 + \eta'(p)^2 + \zeta'(p)^2}\,\mathrm{d}p,$$

or from the parametric equation (5), the derivative of the arc length σ with respect to the vertical coordinate z is a function of z, denoted by $f(z)$,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = f(z) = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}. \tag{8}$$

From Eqs. (7) and (8), we obtain the form in variable separation,

$$\mathrm{d}t = -\frac{f(z)}{\sqrt{2g(w - z)}}\mathrm{d}z. \tag{9}$$

The total time $T(w)$ taken for the bead to go from point P to point A is given by integration as

$$T(w) = \int_0^{T(w)} \mathrm{d}t = \int_{z_0}^{w} \frac{f(z)}{\sqrt{2g(w - z)}}\mathrm{d}z,\ w \geq z_0. \tag{10}$$

In the Riemann-Liouville fractional integral, the sliding time is expressed as

$$T(w) = \sqrt{\frac{\pi}{2g}}\,{}_{z_0}I_w^{1/2} f(w). \tag{11}$$

Due to arbitrariness of w, we may replace w by z in Eq. (11) to obtain the sliding time from the point $Q(x,y,z)$ to the point $A(x_0,y_0,z_0)$,

$$T(z)=\sqrt{\frac{\pi}{2g}}\,{}_{z_0}I_z^{1/2}f(z). \tag{12}$$

Next, we calculate the sliding time of a bead for the two specific space curves: the conical helix and the Viviani curve.

Example 1. Consider the conical helix parameterized by $\theta \geq 0$,

$$\begin{cases} x=a\theta\cos\theta,\\ y=b\theta\sin\theta,\\ z=c\theta, \end{cases} \tag{13}$$

where $a,b,c>0$ are constants, and find the time for a bead to slide from any point $P(x,y,z)$, $z>0$, on the curve to the point $(0,0,0)$.

Taking z as the parameter, the equation of the conical helix has the form

$$\begin{cases} x=a\frac{z}{c}\cos\frac{z}{c},\\ y=b\frac{z}{c}\sin\frac{z}{c}. \end{cases} \tag{14}$$

Thus $f(z)$ in Eq. (8) is computed as

$$f(z)=\frac{1}{c}\sqrt{c^2+\left(a\cos\frac{z}{c}-a\frac{z}{c}\sin\frac{z}{c}\right)^2+\left(b\sin\frac{z}{c}+b\frac{z}{c}\cos\frac{z}{c}\right)^2}. \tag{15}$$

Substituting Eq. (15) into Eq. (12) yields the sliding time as

$$T(z)=\frac{1}{c\sqrt{2g}}\int_0^z\frac{1}{\sqrt{z-\tau}}\sqrt{c^2+\left(a\cos\frac{\tau}{c}-a\frac{\tau}{c}\sin\frac{\tau}{c}\right)^2+\left(b\sin\frac{\tau}{c}+b\frac{\tau}{c}\cos\frac{\tau}{c}\right)^2}\,\mathrm{d}\tau. \tag{16}$$

For the standard conical helix, i.e., the case of a=b, the integration in (16) has the expression in terms of the hypergeometric function,

$$\begin{aligned} T(z) &= \frac{1}{c\sqrt{2g}}\int_0^z\frac{1}{\sqrt{z-\tau}}\sqrt{c^2+a^2+\frac{a^2}{c^2}\tau^2}\,\mathrm{d}\tau \\ &= \sqrt{\frac{2z}{g}\left(1+\frac{a^2}{c^2}\right)}\,{}_3F_2\left(-\frac{1}{2},\frac{1}{2},1;\frac{3}{4},\frac{5}{4};-\frac{a^2z^2}{c^2\left(a^2+c^2\right)}\right), \end{aligned} \tag{17}$$

where the hypergeometric function is defined as

$${}_3F_2\left(a_1,a_2,a_3;b_1,b_2;w\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k(a_3)_k}{(b_1)_k(b_2)_k}\frac{w^k}{k!}, \tag{18}$$

where $(a)_k$ is the Pochhammer symbol

$$(a)_0=1,\quad (a)_k=a(a+1)\ldots(a+k-1).$$

In Figure 2, the curves of sliding time $T(z)$ versus z are shown for the conical helix ($a=1$, $b=2$, $c=2$) and the standard conical helix ($a=b=1$, $c=2$), respectively. Compared with the standard conical helix, an undulate rising of sliding time $T(z)$ is displayed for the conical helix.

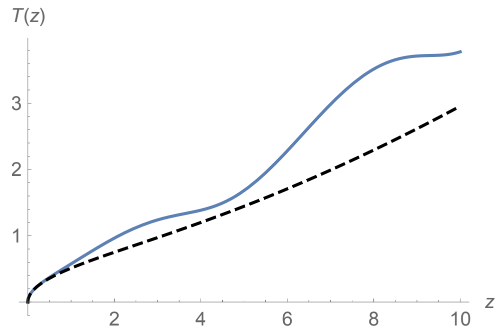

FIGURE 2. The sliding times $T(z)$ versus z along the conical helix for $a=1$, $b=2$ and $c=2$ (solid line) and the standard conical helix $a=b=1$ and $c=2$ (dash line) in Example 1.

Example 2. Consider a section of the Viviani curve

$$\begin{cases} x = a\cos^2\theta, \\ y = a\cos\theta\sin\theta, \\ z = a\sin\theta, \end{cases} \tag{19}$$

where $0 \le \theta < \frac{\pi}{2}$, $a > 0$, and find the time taken for a bead to slide from any point $P(x,y,z)$ to the point $(a,0,0)$ corresponding to $\theta = 0$.

We note that the Viviani curve is the intersection between the spherical surface $x^2+y^2+z^2=a^2$ and the circular cylindrical surface $x^2+y^2=ax$. The section we considered is in the first octant, as shown in Figure 3 for $a=2$. The curve equation has the form parameterized by z,

$$\begin{cases} x = a - \frac{z^2}{a}, \\ y = z\sqrt{1-\frac{z^2}{a^2}}. \end{cases} \tag{20}$$

So we calculate that

$$f(z) = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2} = \sqrt{\frac{2a^2-z^2}{a^2-z^2}},\ 0 \le z < a. \tag{21}$$

Substituting Eq. (21) into Eq. (12) yields the sliding time as

$$T(z) = \frac{1}{\sqrt{2g}}\int_0^z \frac{1}{\sqrt{z-\tau}}\sqrt{\frac{2a^2-\tau^2}{a^2-\tau^2}}\mathrm{d}\tau,\ 0 \le z < a. \tag{22}$$

Note that if $z=a$, which corresponds to $\theta = \frac{\pi}{2}$ in Eq. (19), the integral in Eq. (22) diverges. In fact, if $z=a$, then the sliding bead lies at the apex $(0,0,a)$, an equilibrium position, and so it will stay there the whole time. In Figure 4, the curve of the sliding time $T(z)$ versus z is shown for $a=2$. The curve of $T(z)$ extends up infinitely as $z \to 2$.

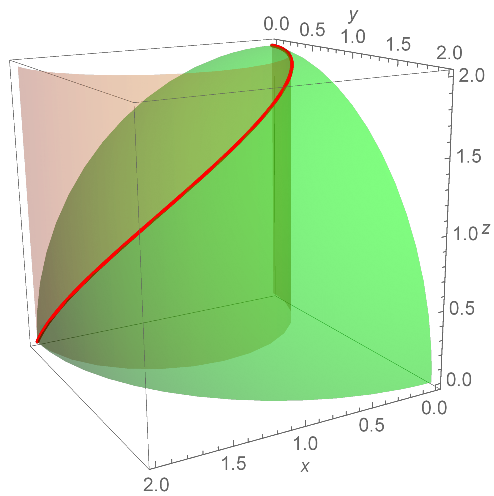

FIGURE 3. The Viviani curve in the first octant for $a = 2$ in Example 2.

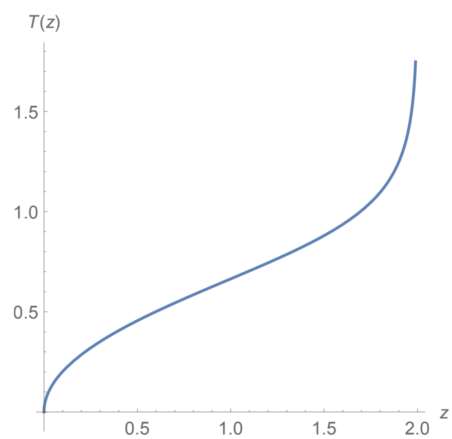

FIGURE 4. The sliding time $T(z)$ versus z for $a = 2$ in Example 2.

3. Space tautochrone curve

If in Eq. (12), the time taken to reach the point A is constant, i.e., is independent of the starting position, then we call this curve a space tautochrone curve or we say that the curve has tautochronism. We give the condition for tautochronism of a space curve (5) as follows.

Proposition 1. A space curve is tautochronic with the sliding time T, if and only if it satisfies the condition

$$(z - z_0)\left(1 + \left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\right) = H, \text{ where } H = \frac{2gT^2}{\pi^2}, \tag{23}$$

and z_0 is the vertical coordinate of the terminal point A of the slide.

Proof. If a space curve is tautochronic with the sliding time T, then from Eq. (12) we have

$$\sqrt{\frac{\pi}{2g}}\,{}_{z_0}I_z^{1/2} f(z) = T, \tag{24}$$

where

$$f(z) = \frac{\mathrm{d}\sigma}{\mathrm{d}z} = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}, \tag{25}$$

and σ denotes the arc length measured from the terminal A.

Operating the half-order integral on the both sides of Eq. (24) leads to

$$\sqrt{\frac{\pi}{2g}}\,{}_{z_0}I_z^{1} f(z) = {}_{z_0}I_z^{1/2} T = \frac{2T}{\sqrt{\pi}}\sqrt{z-z_0}. \tag{26}$$

Calculating the first order derivative on the both sides of Eq. (26) leads to

$$f(z) = \frac{\sqrt{2g}\,T}{\pi\sqrt{z-z_0}}. \tag{27}$$

From Eqs. (25) and (27), we have

$$\mathrm{d}\sigma = \frac{\sqrt{2g}\,T}{\pi\sqrt{z-z_0}}\mathrm{d}z, \tag{28}$$

or

$$\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2} = \frac{\sqrt{2g}\,T}{\pi\sqrt{z-z_0}}, \tag{29}$$

Introducing the notation $H = \frac{2gT^2}{\pi^2}$, we may rewrite Eq. (29) to

$$1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2 = \frac{H}{z-z_0}, \tag{30}$$

where the vertical coordinate of the space tautochrone curve is required to satisfy the restriction

$$z_0 \le z \le z_0 + H. \tag{31}$$

From Eq. (30), the condition (23) is derived.

Conversely, if the condition (23) holds, we can derive

$$f(z) = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2} = \frac{\sqrt{2g}\,T}{\pi\sqrt{z-z_0}}. \tag{32}$$

Applying the half-order integral on the both sides of Eq. (32) gives rise to

$${}_{z_0}I_z^{1/2} f(z) = \sqrt{\frac{2g}{\pi}}\,T. \tag{33}$$

From Eqs. (12) and (33), we obtain that the space curve is tautochronic with the sliding time T. The proof is completed. ■

Eq. (28) implies the characteristics of the tautochrone curve, i.e., the change rate of the arc length measured from $A(x_0, y_0, z_0)$ is inversely proportional to the square root of the lifting height. At the lowest point of a tautochrone curve, $z \to z_0$, from Eq. (30) we have

$$\frac{\mathrm{d}x}{\mathrm{d}z} \to \infty \quad \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}z} \to \infty.$$

This means that there is a horizontal tangent line at the lowest point of a tautochrone curve.

For an unrestricted space tautochrone curve, it occupies the entire range $z_0 \le z \le z_0 + H$, and at the highest point, $z = z_0 + H$, from Eq. (30) we have

$$\frac{\mathrm{d}x}{\mathrm{d}z} = 0 \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}z} = 0.$$

This means that there is a vertical tangent line at the highest point of an unrestricted space tautochrone curve.

H in Proposition 1 has definite geometric meaning, i.e., H is the vertical height of the unrestricted tautochrone curve. Further, the full length of an unrestricted space tautochrone curve can be calculated from Eq. (28) as

$$L = \int_0^L \mathrm{d}\sigma = \int_{z_0}^{z_0+H} \frac{\sqrt{2g}\,T}{\pi\sqrt{z-z_0}} \mathrm{d}z = \frac{4gT^2}{\pi^2} = 2H, \tag{34}$$

which is just twice the height of the tautochrone curve.

In the next three sections, we consider the space tautochrone curves restricted on space surfaces. In Sections 5 and 6, we will see such restricted space tautochrone curves where the height and length are strictly less than H and $2H$, respectively.

4. Tautochrone curve on a vertical cylindrical surface

Suppose that the equation of a vertical cylindrical surface is $F(x, y) = 0$, and its parametric equation is

$$\begin{cases} x = x(q), \\ y = y(q), \end{cases} \tag{35}$$

where there are two independent parameters q and z, $x(q)$ and $y(q)$ are not both constants and have continuous derivatives. We look for the relationship $q = q(z)$, such that the curve

$$\begin{cases} x = x(q(z)), \\ y = y(q(z)), \end{cases} \tag{36}$$

on the cylindrical surface (35) has tautochronism.

Proposition 2. The function $q = q(z)$ in the tautochrone curve (36) on a vertical cylindrical surface is determined by the equation

$$\int_{q_0}^{q} \sqrt{x'(q)^2 + y'(q)^2}\,\mathrm{d}q = \sqrt{(z_0 + H - z)(z - z_0)} + H \arcsin\sqrt{\frac{z - z_0}{H}}, \quad z_0 \le z \le z_0 + H, \tag{37}$$

where $H = \frac{2gT^2}{\pi^2}$ and T is the sliding time. The tautochrone curve has the vertical height H and the full length $L = 2H$.

Proof. From Proposition 1, we derive that

$$\sqrt{\mathrm{d}x^2+\mathrm{d}y^2}=\sqrt{\frac{H}{z-z_0}-1}\,\mathrm{d}z,$$

and further, using equations in (36), we obtain

$$\sqrt{x'(q)^2+y'(q)^2}\,\mathrm{d}q=\sqrt{\frac{H}{z-z_0}-1}\,\mathrm{d}z. \tag{38}$$

Let $q(z_0)=q_0$. Integrating both sides of Eq. (38) leads to

$$\int_{q_0}^{q}\sqrt{x'(q)^2+y'(q)^2}\,\mathrm{d}q-\sqrt{(z_0+H-z)(z-z_0)}-H\arcsin\sqrt{\frac{z-z_0}{H}}=0. \tag{39}$$

The partial derivative with respect to q of the left hand side of Eq. (39) is always greater than zero and $\frac{dq}{dz}>0$ for all $z_0<z\leq z_0+H$, according to Eq. (38). This means that Eq. (39) determines the function relationship $q=q(z)$ for $z_0\leq z\leq z_0+H$, and the tautochrone curve has the vertical height H and the full length

$$L=\int_{z_0}^{z_0+H}\sqrt{\frac{H}{z-z_0}}dz=2H.$$

The proof is completed. ■

The classical tautochrone curve is in the vertical plane, so it can be obtained as a special case. Now we consider the tautochrone curve in the xz coordinate plane. In Proposition 2, taking $y=0$, $q=z$, $z_0=0$ and $x(z_0)=0$, we have

$$x(z)=\sqrt{(H-z)z}+H\arcsin\sqrt{\frac{z}{H}},\ 0\leq z\leq H. \tag{40}$$

If $z=H\sin^2\frac{\phi}{2}$, $0\leq\phi\leq\pi$, we obtain the parametric equation of the classical tautochrone curve (cycloid),

$$\begin{cases} x=\frac{H}{2}(\phi+\sin\phi),\\ z=\frac{H}{2}(1-\cos\phi),\ 0\leq\phi\leq\pi. \end{cases} \tag{41}$$

Next, we consider two specific examples, one is the tautochrone curve on an elliptic cylindrical surface and the other is the tautochrone curve on a parabolic cylindrical surface.

Example 3. Consider the tautochrone curve on the elliptic cylindrical surface $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$, where $a,b>0$ are constants.

The elliptic cylindrical surface has the parametric equation

$$\begin{cases} x=a\cos\theta,\\ y=b\sin\theta. \end{cases} \tag{42}$$

In accordance with Proposition 2, we look for the tautochrone curve on the elliptic cylindrical surface in the form of

$$\begin{cases} x=a\cos\theta(z),\\ y=b\sin\theta(z),\ z_0\leq z\leq z_0+H. \end{cases} \tag{43}$$

Denoting $\theta(z_0) = \theta_0$ and substituting the derivatives $x'(\theta) = -a\sin\theta,\ y'(\theta) = b\cos\theta$ into the left hand side of Eq. (39), we have

$$\begin{aligned}\int_{\theta_0}^{\theta} \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\, \mathrm{d}\theta &= \int_{\theta_0}^{\theta} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, \mathrm{d}\theta \\ &= bE\left(\theta; 1 - \frac{a^2}{b^2}\right) - bE\left(\theta_0; 1 - \frac{a^2}{b^2}\right),\end{aligned} \tag{44}$$

where the result is expressed by using the elliptic integral of the second kind

$$E(\varphi; m) = \int_0^{\varphi} \sqrt{1 - m\sin^2\psi}\, \mathrm{d}\psi,\ m \le 1. \tag{45}$$

By Eq. (39) we obtain the relationship between θ and z as

$$bE\left(\theta; 1 - \frac{a^2}{b^2}\right) - bE\left(\theta_0; 1 - \frac{a^2}{b^2}\right) = \sqrt{(z_0 + H - z)(z - z_0)} + H\arcsin\sqrt{\frac{z - z_0}{H}}. \tag{46}$$

Eq. (46) determines an implicit function $\theta = \theta(z)$, $z_0 \le z \le z_0 + H$, so Eq. (43) denotes the tautochrone curve on the elliptic cylindrical surface.

Now we consider the case that the terminal point is $A(a, 0, 0)$, i.e., the case of $z_0 = 0$ and $\theta_0 = 0$. Thus Eq. (46) is simplified as

$$E\left(\theta; 1 - \frac{a^2}{b^2}\right) = \frac{1}{b}\left(\sqrt{(H - z)z} + H\arcsin\sqrt{\frac{z}{H}}\right),\ 0 < z \le H. \tag{47}$$

Further for a circular cylindrical surface, where $a = b$ holds, noting that $E(\varphi; 0) = \varphi$, Eq. (47) degenerates to an explicit function

$$\theta = \frac{1}{a}\sqrt{(H - z)z} + \frac{H}{a}\arcsin\sqrt{\frac{z}{H}},\ 0 \le z \le H. \tag{48}$$

By using the relation (47) or (48), the tautochrone curve can be determined from the parameter equation (43). In Figures 5 and 6, we plot the tautochrone curves for $H = 5$ on the elliptic cylindrical surface ($a = 1$ and $b = 2$) and the circular cylindrical surface ($a = b = 1$), respectively. The two tautochrone curves have the same heights and lengths, whereas their maximum angles of rotation, $\theta_{\max}$, are different: In Figure 5 the angle is 5.2 radians and in Figure 6 the angle is 2.5π radians.

Example 4. Consider the tautochrone curve on the parabolic cylindrical surface $y^2 = 2ax$, where $a > 0$ is a constant.

The parabolic cylindrical surface has the parametric equation

$$\begin{cases} x = 2aq^2, \\ y = 2aq. \end{cases} \tag{49}$$

Now we look for the tautochrone curve on the parabolic cylindrical surface in the form of

$$\begin{cases} x = 2aq^2(z), \\ y = 2aq(z),\ z_0 \le z \le z_0 + H, \end{cases} \tag{50}$$

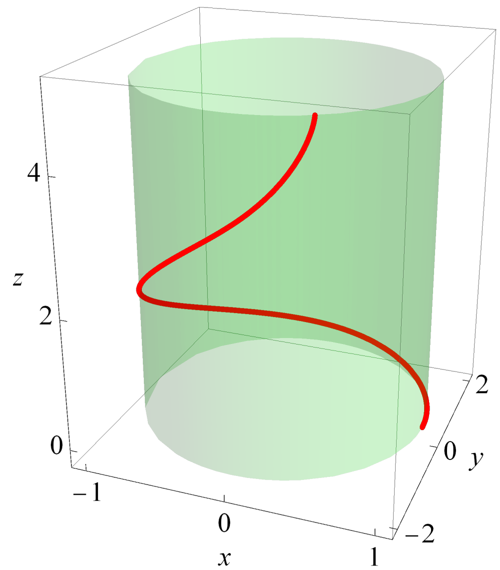

FIGURE 5. Tautochrone curve on the elliptic cylindrical surface for $a = 1$, $b = 2$, $H = 5$ in Example 3.

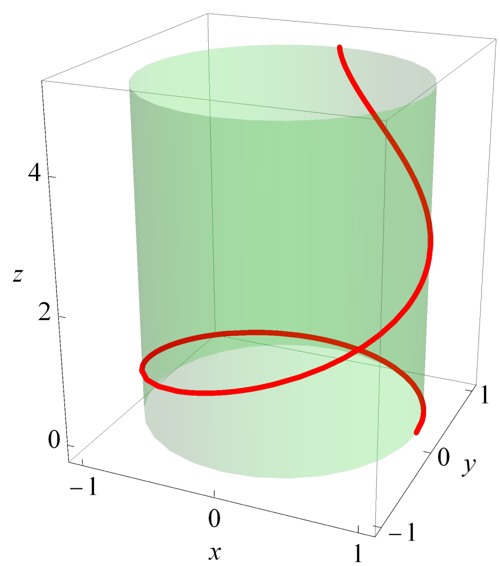

FIGURE 6. Tautochrone curve on the circular cylindrical surface for $a = b = 1$, $H = 5$ in Example 3.

with the lowest point at $q(z_0) = q_0$. Substituting the derivatives $x'(q) = 4aq$, $y'(q) = 2a$ into Eq. (39) and calculating the integral yield the relationship between q and z as

$$\begin{aligned} &\frac{a}{2}\left(2q\sqrt{4q^2+1}+\ln\left(2q+\sqrt{4q^2+1}\right)-2q_0\sqrt{4q_0^2+1}-\ln\left(2q_0+\sqrt{4q_0^2+1}\right)\right) \\ (51)\qquad &= \sqrt{(z_0+H-z)(z-z_0)}+H\arcsin\sqrt{\frac{z-z_0}{H}},\ z_0 \le z \le z_0+H. \end{aligned}$$

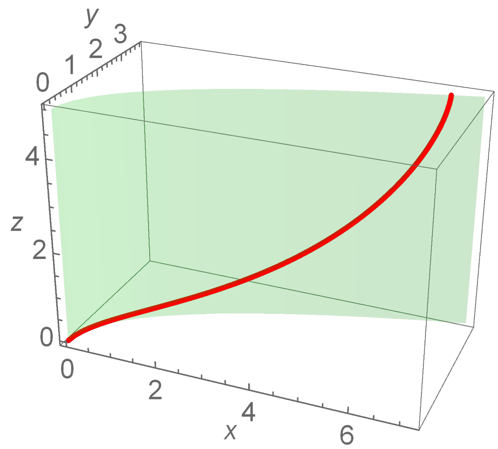

FIGURE 7. Tautochrone curve on the parabolic cylindrical surface in Example 4.

Here we take $z_0 = 0$ and $q_0 = 0$, i.e., the lowest point is the origin O, then Eq. (51) is simplified as

$$(52)\quad 2q\sqrt{4q^2+1}+\ln\left(2q+\sqrt{4q^2+1}\right)=\frac{2}{a}\left(\sqrt{(H-z)z}+H\arcsin\sqrt{\frac{z}{H}}\right),\ 0\le z\le H.$$

Eq. (52) determines an implicit function $q = q(z)$, $0 \le z \le H$. Thus Eq. (50) gives the tautochrone curve on the parabolic cylindrical surface. Taking $a = 1$ and $H = 5$, the tautochrone curve on the parabolic cylindrical surface is shown in Figure 7.

5. Tautochrone curve in a tilting plane

Consider the problem of tautochrone curve in the tilting plane $\Pi:\ y = z\tan\alpha$, $0 \le \alpha < \frac{\pi}{2}$. The plane Π goes through the x axis and intersects the z-axis with the deviation angle α. We seek a monotone increasing function $x = x(z)$, such that the tautochrone curve in the tilting plane has the form

$$(53)\quad \begin{cases} x = x(z), \\ y = z\tan\alpha, \end{cases}$$

with the lowest point at $(0,0,0)$. We note that the plane Π becomes the vertical xz coordinate plane as $\alpha = 0$ and it gets close to the horizontal xy coordinate plane as $\alpha \to \frac{\pi}{2}$.

Substituting Eq. (53) into Eq. (23), we derive that

$$\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 = \frac{H}{z} - 1 - \tan^2\alpha,$$

where the vertical coordinate is constrained as

$$(54)\quad 0 < z \le H\cos^2\alpha.$$

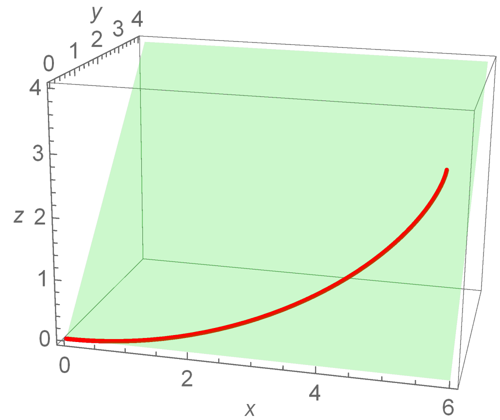

FIGURE 8. Tautochrone curve on tilting plane.

We note that the constraint in (54) is stronger than $0 < z \le H$ in (31) for the unrestricted case. Thus we obtain

$$\begin{aligned} x(z) &= \int_0^z \sqrt{\frac{H}{z} - \sec^2\alpha}\, \mathrm{d}z \\ &= \sqrt{(H - z\sec^2\alpha)z} + H\cos\alpha \arcsin\left(\sqrt{\frac{z}{H}}\sec\alpha\right). \end{aligned} \tag{55}$$

Inserting Eq. (55) into (53) determines the tautochrone curve in the tilting plane Π.

For the parameter equation of the tautochrone curve, we let $z = H\cos^2\alpha\sin^2\frac{\phi}{2}$, $0 \le \phi \le \pi$, and obtain

$$\begin{cases} x = \frac{H\cos\alpha}{2}(\phi + \sin\phi), \\ y = \frac{H\sin 2\alpha}{4}(1 - \cos\phi), \\ z = \frac{H\cos^2\alpha}{2}(1 - \cos\phi),\ 0 \le \phi \le \pi. \end{cases} \tag{56}$$

For the tautochrone curve in the plane Π, its vertical height is $H\cos^2\alpha$ and its length is

$$L = \int_0^\pi \sqrt{x'(\phi)^2 + y'(\phi)^2 + z'(\phi)^2}\, \mathrm{d}\phi = 2H\cos\alpha.$$

They are strictly less than H and $2H$ if $0 < \alpha < \frac{\pi}{2}$ and approach zero as $\alpha \to \frac{\pi}{2}$. In Figure 8, the tautochrone curve in the tilting plane for $H = 5$ and $\alpha = \pi/4$ is shown, where the vertical height of the curve is only 2.5.

If $\alpha = 0$, then Eq. (56) degenerates to the parameter equation of the tautochrone curve in the vertical xz plane,

$$\begin{cases} x = \frac{H}{2}(\phi + \sin\phi), \\ z = \frac{H}{2}(1 - \cos\phi),\ 0 \le \phi \le \pi. \end{cases}$$

6. Tautochrone curve restricted on a general surface

Suppose the parametric equation of the general surface is

$$\begin{cases} x = x(p,q), \\ y = y(p,q), \\ z = z(p,q), \end{cases} \tag{57}$$

where x, y, z have continuous partial derivatives with respect to p and q. We seek for the relationship $q = q(p)$, such that the equation parameterized by p,

$$\begin{cases} x = x(p,q(p)), \\ y = y(p,q(p)), \\ z = z(p,q(p)), \end{cases} \tag{58}$$

represents a tautochrone curve on the surface with the lowest point $A\,(x_0, y_0, z_0)$ corresponds to $p = p_0$. The function $q(p)$ is to be determined through the condition for the tautochronism in Section 3.

From Proposition 1 we have

$$\left(\frac{\mathrm{d}x}{\mathrm{d}p}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}p}\right)^2 = \left(\frac{H}{z(p,q(p)) - z(p_0,q(p_0))} - 1\right)\left(\frac{\mathrm{d}z}{\mathrm{d}p}\right)^2. \tag{59}$$

Substituting the differential relations

$$\mathrm{d}x = \left(\frac{\partial x}{\partial p} + \frac{\partial x}{\partial q}q'(p)\right)\mathrm{d}p,\ \mathrm{d}y = \left(\frac{\partial y}{\partial p} + \frac{\partial y}{\partial q}q'(p)\right)\mathrm{d}p,\ \mathrm{d}z = \left(\frac{\partial z}{\partial p} + \frac{\partial z}{\partial q}q'(p)\right)\mathrm{d}p,$$

we obtain the nonlinear differential equation on $q(p)$,

$$\begin{aligned} &\left(\frac{\partial x}{\partial p} + \frac{\partial x}{\partial q}q'(p)\right)^2 + \left(\frac{\partial y}{\partial p} + \frac{\partial y}{\partial q}q'(p)\right)^2 \\ = &\left(\frac{H}{z(p,q(p)) - z(p_0,q(p_0))} - 1\right)\left(\frac{\partial z}{\partial p} + \frac{\partial z}{\partial q}q'(p)\right)^2. \end{aligned} \tag{60}$$

Next, we consider two specific examples, one is the tautochrone curve on the upper conical surface and the other is the tautochrone curve on the elliptic paraboloid.

Example 5. Consider the tautochrone curve on the upper conical surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{z}{c}\right)^2$, $z \geq 0$, which has the parametric equation

$$\begin{cases} x = ap\cos\theta, \\ y = bp\sin\theta, \\ z = cp, \end{cases} \tag{61}$$

where $a, b, c > 0$ are constants and $p, \theta \geq 0$ are parameters.

We seek for the relationship $\theta = \theta(p)$, a monotonic increasing function, such that the equation

$$\begin{cases} x = ap\cos\theta(p), \\ y = bp\sin\theta(p), \\ z = cp, \end{cases} \tag{62}$$

forms a tautochrone curve on the upper conical surface, where $p \geq p_0$, $\theta(p_0) = \theta_0$. Calculating the derivatives

$$x'(p) = a\cos\theta - ap\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}p},\ y'(p) = b\sin\theta + bp\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p},\ z'(p) = c,$$

and substituting them into Eq. (59) yield

$$\left(a\cos\theta - ap\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 + \left(b\sin\theta + bp\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 = c^2\left(\frac{H}{cp - cp_0} - 1\right),$$

which leads to a complicated nonlinear differential equation

$$\begin{aligned}&\left(a^2\sin^2\theta + b^2\cos^2\theta\right)p^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 + 2\left(b^2 - a^2\right)p\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p} + a^2\cos^2\theta + b^2\sin^2\theta\\ &= \frac{cH}{p - p_0} - c^2.\end{aligned} \tag{63}$$

Here we consider the case of circular conical surface, i.e., the case of $a = b$. For this case, Eq. (63) and the constraint in Eq. (31) become

$$a^2 + a^2p^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 = \frac{cH}{p - p_0} - c^2, \tag{64}$$

and

$$p_0 \leq p \leq p_0 + \frac{H}{c}. \tag{65}$$

From Eq. (64), a stronger constraint $\frac{cH}{p-p_0} - c^2 \geq a^2$ than that in Eq. (65) is required and it leads to

$$p_0 \leq p \leq p_0 + \frac{cH}{c^2 + a^2}. \tag{66}$$

Eq. (64) has the form of separation of variables

$$\mathrm{d}\theta = \frac{1}{ap}\sqrt{\frac{cH}{p - p_0} - a^2 - c^2}\,\mathrm{d}p. \tag{67}$$

Taking $\theta_0 = 0$ and integrating on the both sides of Eq. (67) lead to

$$\theta = \frac{1}{a}\int_{p_0}^{p}\frac{1}{p}\sqrt{\frac{cH}{p - p_0} - a^2 - c^2}\,\mathrm{d}p, \tag{68}$$

where $p_0 > 0$ is imposed to guarantee convergence of the integral. Calculating the integral in Eq. (68), we obtain the relation between θ and p,

$$\begin{aligned}\theta &= \frac{2}{a}\left(\sqrt{a^2 + c^2}\arctan\sqrt{\frac{cH}{(a^2 + c^2)(p - p_0)} - 1}\right.\\ &\quad\left. - \sqrt{\frac{cH}{p_0} + a^2 + c^2}\arctan\sqrt{\frac{p_0(cH - (a^2 + c^2)(p - p_0))}{(p - p_0)(cH + (a^2 + c^2)p_0)}}\right)\\ &\quad + \frac{\pi}{a}\left(\sqrt{a^2 + c^2 + \frac{cH}{p_0}} - \sqrt{a^2 + c^2}\right),\end{aligned} \tag{69}$$

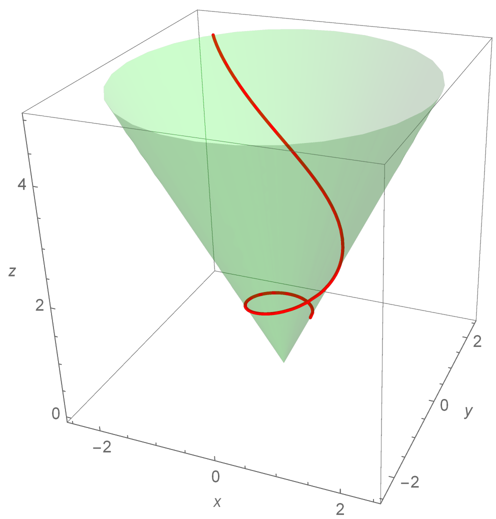

FIGURE 9. Tautochrone curve on the circular conical surface in Example 5.

where p is constrained in Eq. (66). By Eq. (69), the tautochrone curve on a circular conical surface is determined in Eq. (62).

From Eqs. (62) and (66), the height of the restricted tautochrone curve is $\frac{H}{1+(a/c)^2}$, which is strictly less than H. Further, the length of the restricted tautochrone curve can be exactly calculated as

$$L = \frac{2Hc}{\sqrt{a^2+c^2}},$$

which is strictly less than $2H$.

Taking $a = b = 1$, $c = 2$, $H = 5$ and $p_0 = 0.5$, the tautochrone curve is shown in Figure 9.

Example 6. Consider the tautochrone curve on the elliptic paraboloid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{z}{c}$, which has the parametric equation

$$\begin{cases} x = ap\cos\theta, \\ y = bp\sin\theta, \\ z = cp^2, \end{cases} \tag{70}$$

where $a, b, c > 0$ are constants, $p, \theta \geq 0$ are parameters.

We seek for $\theta = \theta(p)$, a monotonic increasing function, such that the tautochrone curve on the elliptic paraboloid has the parameter form

$$\begin{cases} x = ap\cos\theta(p), \\ y = bp\sin\theta(p), \\ z = cp^2, \end{cases} \tag{71}$$

where $p \geq p_0$, $\theta(p_0) = \theta_0$. Calculating the derivatives

$$x'(p) = a\cos\theta - ap\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}p},\ y'(p) = b\sin\theta + bp\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p},\ z'(p) = 2cp,$$

and substituting them into Eq. (59) lead to

$$\left(a\cos\theta - ap\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 + \left(b\sin\theta + bp\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 = 4c^2p^2\left(\frac{H}{cp^2 - cp_0^2} - 1\right),$$

i.e., a complicated nonlinear differential equation

$$\begin{aligned} &\left(a^2\sin^2\theta + b^2\cos^2\theta\right)p^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 + 2\left(b^2 - a^2\right)p\sin\theta\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}p} + a^2\cos^2\theta + b^2\sin^2\theta \\ (72) \quad &= \quad 4c^2p^2\left(\frac{H}{cp^2 - cp_0^2} - 1\right). \end{aligned}$$

Now we consider the case of the circular paraboloid, i.e., the case of $a = b$. So Eq. (72) becomes

$$a^2 + a^2p^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}p}\right)^2 = 4c^2p^2\left(\frac{H}{cp^2 - cp_0^2} - 1\right), \tag{73}$$

where a stronger constraint

$$s(p) = 4c^2p^2\left(\frac{H}{cp^2 - cp_0^2} - 1\right) \geq a^2, \tag{74}$$

than that in Eq. (31) is equipped with. Calculating the derivative we have

$$s'(p) = -8c^2p\left(\frac{cHp_0^2}{(cp^2 - cp_0^2)^2} + 1\right) < 0, \ p > p_0 \geq 0.$$

Thus by solving the equation $s(p) = a^2$ we obtain the upper bound of the parameter p as

$$p_1 = \frac{\sqrt{4Hc + 4c^2p_0^2 - a^2 + \sqrt{(4Hc + 4c^2p_0^2 - a^2)^2 + 16a^2c^2p_0^2}}}{2\sqrt{2}c}.$$

Rewrite Eq. (73) to a form of separation of variables,

$$\mathrm{d}\theta = \frac{1}{a}\sqrt{\frac{4cH}{p^2 - p_0^2} - 4c^2 - \frac{a^2}{p^2}}\,\mathrm{d}p.$$

Taking $\theta_0 = 0$, then θ has the expression

$$\theta = \frac{1}{a}\int_{p_0}^{p}\sqrt{\frac{4cH}{p^2 - p_0^2} - 4c^2 - \frac{a^2}{p^2}}\,\mathrm{d}p, \ p_0 < p \leq p_1, \tag{75}$$

where $p_0 > 0$ is imposed to ensure the convergence of the integral.

Thus by the relation (75) between θ and p, the tautochrone curve on the circular paraboloid is determined by Eq. (71). We note that the inequality in (74) is equivalent to

$$cp^2 - cp_0^2 \leq \frac{H}{1 + \frac{a^2}{4c^2p^2}}. \tag{76}$$

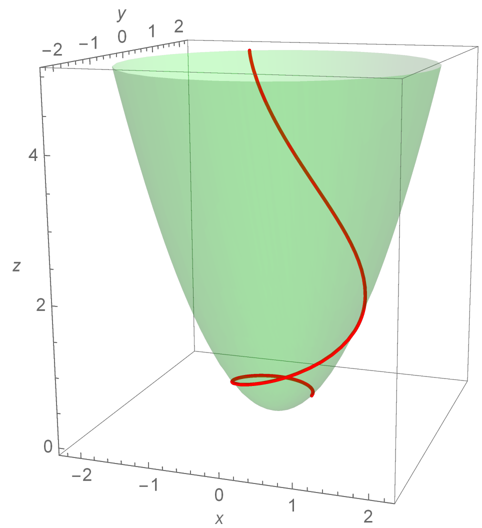

FIGURE 10. Tautochrone curve on the circular paraboloid in Example 6.

From Eq. (76) and the definition of p_1, we derive the height of the restricted tautochrone curve $H/\left(1+\frac{a^2}{4c^2p_1^2}\right)$, which is strictly less than H. Further, the length of the restricted tautochrone curve is calculated as

$$L = 2\sqrt{H(cp_1^2 - cp_0^2)} = \frac{2H}{\sqrt{1+\frac{a^2}{4c^2p_1^2}}},$$

which is strictly less than $2H$. Taking $a = b = c = 1$, $H = 5$ and $p_0 = 0.5$, the tautochrone curve is shown in Figure 10.

7. Conclusions

In this article, a generalization of the classical tautochrone problem in the three-dimensional space was presented, applying the methods of fractional calculus. First, a general formula for the sliding time along a wire of arbitrary shape from an arbitrary starting point $Q(x,y,z)$ to a fixed end point $A(x_0,y_0,z_0)$ was derived as

$$T(z) = \sqrt{\frac{\pi}{2g}}\, {}_{z_0}I_z^{1/2} f(z),$$

where g is the gravitational acceleration and

$$f(z) = \frac{\mathrm{d}\sigma}{\mathrm{d}z} = \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2},$$

and where σ is the arc length measured from the terminal A. Using this formula we calculated two numerical examples for the conical helix and the Viviani curve.

Next, the necessary and sufficient condition for an unrestricted space curve to satisfy tautochronism was given in Proposition 1, i.e.,

$$(z-z_0)\left(1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\right)=H,\ \text{where } H=\frac{2gT^2}{\pi^2},$$

and T is the sliding time. For an unrestricted tautochrone curve, its vertical height is H and its full length was shown to be exactly twice the height, $L=2H$.

Thereafter, we studied the restricted space tautochrone curves, i.e., tautochrone curves that lay on space surfaces of various type, including the vertical cylindrical surface, the tilting plane and a general surface expressed by parameter equations with two arguments. We showed that a tautochrone curve on a vertical cylindrical surface has the vertical height H and the full length $L=2H$, while a tautochrone curve restricted on a general space surface cannot reach such height and length in general. The solution of the classical tautochrone problem can be obtained as a special case of vertical cylindrical surfaces. The same solution can also be obtained by taking $\alpha=0$ from Eq. (56), which describes the tautochrone curve on a tilting plane.

We exemplified the tautochrone curves restricted on the elliptic cylindrical surface, the parabolic cylindrical surface, the tilting plane, the upper conical surface and the elliptic parabola surface, respectively. These space curves were drawn by using MATHEMATICA 11.3. The tautochrone problems on the conical surface and the elliptic paraboloid lead to complicated nonlinear ordinary differential equations worthy of further study.

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