A note on existence of unique solution for a class of weakly singular Fredholm-Hammerstein type integral equation

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June 23, 2024

Abstract

We propose the class of Fredholm-Hammerstein type integral equations by considering three different types of weakly singular kernels, such as the Hadamard type kernel, the mixed type kernel, and the algebraic kernel. As the Hadamard type kernel and mixed type kernel are the combination of $|x-s|^{-\alpha},\ 0<\alpha<\frac{1}{2}$ and $(\log|x-s|)^n,\ n\in\mathbb{N}$, therefore it is difficult to handle. Moreover, the algebraic kernel contains singularities at two points. Furthermore, due to the presence of singularities inside the domain, it isn't easy to analyze the solution. We have used fixed-point iteration to study the existence of solutions. Additionally, we have demonstrated that the solution is unique to the particular range of the parameters. The theoretical results are validated using numerical examples.

Keywords: Integral Equation; Weakly Singular Kernel; Existence; Uniqueness.
AMS Subject Classification: 45B05; 45G05.

1 Introduction

The study of integral equations has a long history. It has been investigated by many authors ([3, 8, 1]). The beginning of this theory was mainly made by many researcher like astrophysicists, mathematicians, etc. They found many applications and open questions in the theory of radioactive transfer, kinetic theory of gases, etc. However, over the last few years, the theory of integral equations with singular kernels has received a lot of attention ([3, 8, 5, 6, 1]).

In this work, we propose the following weakly singular Fredholm-Hammerstein type integral equation

$$\mu u(x) = f(x,u) + \lambda \int_0^T l(x,s)k(x,s)\psi(s,u(s))ds, \quad T>0, \tag{1}$$

where $k(x,s)$ is sufficiently smooth function, λ is the parameter, $f(x,u(x))$ and $\psi(s,u(s))$ are a nonlinear functions which satisfy the Lipschitz condition with respect to u, $f(x,u(x))$ and $\psi(s,u(s))$ are continuous on $[0,T]\times\mathbb{R}$, $l(x,s)$ is weakly singular kernel defined by

$$\text{Hadamard type kernel: } l(x,s) = |x-s|^{-\alpha}(\log|x-s|)^n + A_1; \quad 0<\alpha<\frac{1}{2}, \quad n\in\mathbb{N}, \ A_1 \text{ is constant}, \tag{2}$$

$$\text{Mixed type kernel: } l(x,s) = a_1|x-s|^{-\alpha} + a_2(\log|x-s|)^n + A_2; \tag{3}$$

where $a_1,\ a_2, A_2$ are constants and $0<\alpha<\frac{1}{2},\ \ n\in\mathbb{N}$ and

$$\text{Algebric type kernel: } l(x,s) = \int_0^T \frac{g_3(x,t)g_4(t,s)}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}; \text{ where } \alpha_1+\alpha_2\le 1, \tag{4}$$

and $g_3(x,t)\ge 0, g_4(t,s)\ge 0$ are sufficiently smooth and bounded functions on $[0,T]\times[0,T]$.

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22 Jun 2024 23:30:45 PDT
240322-Pandit Version 2 - Submitted to J. Integr. Eq. Appl.

Remark 1.1. *If we put $\psi(s,u(s)) = u(s)$ in equation (1), then we have*

$$\mu u(x) = f(x) + \lambda \int_0^T l(x,s)k(x,s)u(s)ds, \quad T > 0. \tag{5}$$

The equation (5) is known as Fredholm integral equation ([4]). The generalization of equation (5) is given by equation (1). Due to the presence of nonlinear function $\psi(s,u(s))$, the equation (1) is said to be Fredholm-Hammerstein type integral equation ([4, 12, 7, 34]).

These kinds of integral equations can be found in many mathematical and physical applications, including Dirichlet problems, radiative equilibrium problems in physics, transportation issues, potential issues, the description of hydrodynamic interactions between the elements of a polymer chain in solution ([11, 30, 17, 1, 10]), etc.

The solution of the equation (1) fully depends on the kernel $l(x,s)$. Also, the equations (2) and (3) indicates that $l(x,s)$ presents the singularity at the point $x = s$. Furthermore, the algebraic kernel in equation (3) contains singularities at two points $x = t$ and $t = s$. Since s and t are varying throughout the domain, therefore it is big challenge to analyze the equation theoretically as well as numerically. Here, we study equation (1) with respect to the kernels (2), (3) and (4). We show the existence of the unique solution in continuous space. We also derive the bounds of the parameters λ and μ. We place the numerical examples to verify the theoretical results.

Remark 1.2. *Many authors ([4, 14, 23, 29]) studied Hammerstein type integral equation by considering the kernel either $|x-s|^{-\alpha}$, $0 < \alpha < 1$ or $(\log|x-s|)^n$, $n \in \mathbb{N}$. Also, the behavior of the functions $|x-s|^{-\alpha}$ and $(\log|x-s|)^n$ are different. As the kernels (2) and (3) are the combination of $|x-s|^{-\alpha}$ and $(\log|x-s|)^n$, therefore it is difficult to handle. Moreover, the algebraic kernel (4) contains singularity at two points. Because of the above significance of the problem (1), we can see it isn't easy to analyze the solution. In this work, we provide an idea to derive the existence and uniqueness results of the solution for the governing equation (1) concerning the kernels (2), (3) and (4). To our knowledge, there is no work in the literature related to the proposed problem.*

Below, we have mentioned existing literature related to our methodology. In 19^{th} century a class of integrals with strong singularities encountered by Hadamard ([2]). In [4, 7] , authors studied weakly singular kernel of type $s^{1-\alpha}$, where $0 < \alpha < 1$ and $\log s$, and showed that the solutions satisfy certain regularity properties. Kabir et al. ([9]) applied the piece-wise quadratic polynomial technique to solve singular integral equations in the presence of logarithmic and Hadamard-type kernels. In [11], Kilbas et al. discussed the conditions for the existence of the solution of the Hammerstein integral equation concerning Hadamard-type kernel. After that, Yonis et al. ([24]) and Pandey et al. ([15]) found sufficient conditions for the existence of solutions. They developed an algorithm to solve a class of Hammerstein type integral equations. In [12, 29, 13], Chebyshev expansion is used to solve Volterra integral equations with logarithmic singularities in its kernel. The authors considered the system of Hammerstein integral equations to show the existence of at least one solution and applied a quadratic numerical method with respect to time for approximating the solution. Pathak et al. ([25]) used fixed point theory to show the existence result of the fractional order non-linear Hadamard type functional integral equations. In [20, 19, 19], some generalized Hermite-Hadamard type inequalities for fractional integrals that depends on a parameter have been given. Abdou et al. ([14, 18]) discussed the existence and uniqueness of solution for the nonlinear integral equation of Hammerstein type with discontinuous kernel. Li et al. ([21, 22]) showed the uniqueness of solutions to several Hadamard-type integral equations. Ahmat et al. ([17]) discussed the existence of solutions by using Banach's contraction principle to deal with a system of Hadamard-type integral boundary conditions. Abbas et al. ([26]) have shown the existence and uniqueness results for a generalized Hadamard fractional integral equation by using Picard and Picard-Krasnoselskii iteration methods and the Banach contraction principle. Pandit et al. ([32, 33]) converted the higher-order singular boundary value problems into equivalent Fredhlom-type integral equations and studied the existence of at least one solution. Recently, Agarwal et al. ([31]) investigated the existence and uniqueness of the solution of an integral equation involving convex and concave non-linearity. By using the Leray–Schauder alternative, Schauder fixed point theorem, and Banach's fixed point theorem, Paul et al. ([35]) have shown the existence and uniqueness of solution for the nonlinear Volterra–Fredholm integral equations. After that, Bhat et al. ([36]) extended the results of [35] by considering Volterra and Fredholm integral equations corresponding to the weakly singular kernel $k(x,s) = \frac{L(x,s)}{(x-s)^{1-\alpha}}$, where $L \in \mathcal{C}([0,b]\times[0,b])$ and $0 < \alpha < 1$. In [40], authors considered nonlinear integral equation of the form $u(x) = \mathcal{N}(x) + \frac{f(x)}{\Gamma(\lambda)}\int_0^x (x-\mu)^{\lambda-1}\xi(\mu)\mathcal{H}(u(\mu))d\mu$ to find the numerical approximation. To know more about the literature related to theoretical and numerical works based on weakly singular Fredholm-Hammerstein type integral equations, the reader can read the references ([23, 30, 27, 15, 16, 28, 29, 37, 38, 39]) and the reference therein.

We organize the remaining work in the following three sections. In Section 2, we have derived a few results which help us to show the existence and uniqueness of the solution. We have devoted Section 3 to examples to verify the theoretical results. Finally, we conclude the work in Section 4.

2 Existence of solution

Here, we show the existence and uniqueness of the solution. Below, we have derived some results which help us to prove the main theorem. Throughout the paper, we assume $\mathbb{X} = \mathcal{C}[0, T]$ as continuous space along with the norm $\|u\|_\infty = \max\limits_{x\in[0,T]} |u(x)|$. Therefore, it is easy to see that $(X, ||.||_\infty)$ is a Banach space.

Lemma 2.1. *Let $l(x, s)$ be the weakly singular kernel defined by (2) of the equation (1), then*

$$\sup_{x\in[0,T]} \int_0^T l(x, s)ds \le C_n + A_1 T, \tag{6}$$

where

$$C_n = 2\sum_{m=1}^{n} \frac{(-1)^n n!}{(1-\alpha)^{n+1-m} m!} \left(\frac{m}{1-\alpha}\right)^m e^{-m} + 2\frac{(-1)^n n!}{(1-\alpha)^{n+1}} T^{1-\alpha}. \tag{7}$$

Proof. For the kernel of type (2), we have

$$\begin{aligned} \int_0^T l(x, s)ds &= \int_0^x l(x, s)ds + \int_x^T l(x, s)ds \\ &= \int_0^x \frac{(\log(x-s))^n}{(x-s)^\alpha} ds + \int_x^T \frac{(\log(s-x))^n}{(s-x)^\alpha} ds + A_1 T. \end{aligned} \tag{8}$$

Now, we put $t = x - s$ and $v = -x + s$. Therefore we have

$$\int_0^T l(x, s)ds = \int_0^x \frac{(\log t)^n}{t^\alpha} dt + \int_0^{T-x} \frac{(\log v)^n}{v^\alpha} dv + A_1 T. \tag{9}$$

Let the first integral of equation (9) be $I_n = \int_0^x \frac{(\log t)^n}{t^\alpha} dt$. Therefore, by using integration by parts we have

$$I_n = \sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{(1-\alpha)^{n+1-m} m!} (\log x)^m x^{1-\alpha} + \frac{(-1)^n n!}{(1-\alpha)^{n+1}} x^{1-\alpha}.$$

To find the bound for I_n, we assume $g_1(x) = (\log x)^m x^{1-\alpha}$. Therefore, we have $g_1'(x) = \frac{(\log x)^{m-1}}{x^\alpha}[(1-\alpha)(\log x) + m]$. For extremum, $g_1'(x) = 0$. So, the roots will be either $x = 1$ or $x = e^{\frac{-m}{1-\alpha}}$ for every values of $m \in \mathbb{N}$. Now

$$g_1''(x) = \frac{(\log x)^{m-2}}{x^{1+\alpha}}[m(m-1) + m(1-2\alpha)\log x - \alpha(1-\alpha)(\log x)^2].$$

We get $g_1''(x) > 0$ when $m = (2n - 1)$ and $g_1''(x) < 0$ for $m = 2n$, $\forall n \in \mathbb{N}$. Therefore, the function $g_1(x)$ attains maximum at $x = e^{\frac{-m}{1-\alpha}}$ for $m = 2n$ and attains minimum at $x = e^{\frac{-m}{1-\alpha}}$ for $m = 2n-1$. Hence, we have

$$g_1(x) \le \left(\frac{-m}{1-\alpha}\right)^m e^{-m} \text{ for } m = 2n, \tag{10}$$

and

$$g_1(x) \ge \left(\frac{-m}{1-\alpha}\right)^m e^{-m} \text{ for } m = 2n-1,\ 0 < \alpha < \frac{1}{2}. \tag{11}$$

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240322-Pandit Version 2 - Submitted to J. Integr. Eq. Appl.

Similarly for the second integral of equation (9), we denote $I'_n = \int_0^{T-x} \frac{(\log z)^n}{z^\alpha} dz$. We consider $g_2(x) = [\log(T-x)]^m (T-x)^{1-\alpha}$. Therefore the roots of the equation $g'_2(x) = 0$ will be either $x = T - e^{\frac{-m}{1-\alpha}}$ or $x = T$. Hence, we have

$$g_2(x) \leq \left(\frac{-m}{1-\alpha}\right)^m e^{-m} \text{ for } m = 2n, \tag{12}$$

and

$$g_2(x) \geq \left(\frac{-m}{1-\alpha}\right)^m e^{-m} \text{ for } m = 2n-1,\ 0 < \alpha < \frac{1}{2}. \tag{13}$$

Now, by using equations (10), (11), (12) and (13), from equation (8) we have

$$\begin{aligned}
\int_0^T l(x,s)ds &= \sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{(1-\alpha)^{n+1-m} m!} (\log x)^m x^{1-\alpha} + \frac{(-1)^n n!}{(1-\alpha)^{n+1}} x^{1-\alpha} \\
&+ \sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{(1-\alpha)^{n+1-m} m!} [\log(T-x)]^m (T-x)^{1-\alpha} + \frac{(-1)^n n!}{(1-\alpha)^{n+1}} (T-x)^{1-\alpha} + A_1 T \\
&\leq 2\sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{(1-\alpha)^{n+1-m} m!} \left(\frac{-m}{1-\alpha}\right)^m e^{-m} + 2\frac{(-1)^n n!}{(1-\alpha)^{n+1}} T^{1-\alpha} + A_1 T.
\end{aligned}$$

Hence, we get the result. □

Lemma 2.2. *Let $l(x,s)$ be the weakly singular kernel defined by (2) of the equation (1), and if $l(x,s)$ satisfy the result of Lemma 2.1 i.e.,*

$$\sup_{x\in[0,T]} \int_0^T l(x,s)ds \leq C_n + |A_1|T, \tag{14}$$

then it also satisfies the following inequality

$$c_1 = \sup_{x\in[0,T]} \int_0^T [l(x,s)]^2 ds \leq B_{2n} + 2A_1 C_n + A_1^2 T, \tag{15}$$

where

$$B_{2n} = 2\left[\sum_{m=1}^{2n} \frac{2n!}{(1-2\alpha)^{2n+1-m} m!} \left(\frac{m}{1-2\alpha}\right)^m e^{-m} + \frac{2n!}{(1-2\alpha)^{2n+1}} T^{(1-2\alpha)}\right].$$

Proof. From equation (2), we get

$$\begin{aligned}
\int_0^T [l(x,s)]^2 ds &= \int_0^T \left[\frac{(\log|x-s|)^n}{|x-s|^\alpha} + A_1\right]^2 ds \\
&= \int_0^x \frac{(\log t)^{2n}}{t^{2\alpha}} dt + \int_0^{T-x} \frac{(\log z)^{2n}}{z^{2\alpha}} dz + 2A_1 \int_0^T \frac{(\log|x-s|)^n}{|x-s|^\alpha} ds + A_1^2 T.
\end{aligned}$$

Consider $I_{2n} = \int_0^x \frac{(\log t)^{2n}}{t^{2\alpha}} dt$ and $g_3(x) = (\log x)^m x^{1-2\alpha}$. Therefore the roots of the equation $g'_3(x) = 0$ are $x = 1$ and $x = e^{\frac{-m}{1-2\alpha}}$. So, we have

$$g_3(x) \leq \left(\frac{-m}{1-2\alpha}\right)^m e^{-m} \text{ for } m = 2n, \tag{16}$$

and

$$g_3(x) \geq \left(\frac{-m}{1-2\alpha}\right)^m e^{-m} \text{ for } m = 2n-1,\ 0 < \alpha < \frac{1}{2}. \tag{17}$$

By using equations (16) and (17), we have

$$I_{2n} \leq \sum_{m=1}^{2n} \frac{(-1)^{2n-m} 2n!}{(1-2\alpha)^{2n+1-m}} \left(\frac{-m}{1-2\alpha}\right)^m e^{-m} + \frac{2n!}{(1-2\alpha)^{2n+1}} x^{1-2\alpha};\ 0 \leq x \leq T.$$

By similar analysis, we get

$$\begin{aligned} I'_{2n} &= \int_0^{T-x} \frac{(\log z)^{2n}}{z^{2\alpha}} dt \\ &\leq \sum_{m=1}^{2n} \frac{(-1)^{2n-m} 2n!}{(1-2\alpha)^{2n+1-m}} \left(\frac{-m}{1-2\alpha}\right)^m e^{-m} + \frac{2n!}{(1-2\alpha)^{2n+1}} (T-x)^{1-2\alpha};\ 0 \leq x \leq T. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \int_0^T (l(x,s))^2 ds &\leq I_{2n} + I'_{2n} + 2A_1 C_n + A_1^2 T \\ &\leq \sum_{m=1}^{2n} \frac{2 \times 2n!}{(1-2\alpha)^{2n+1-m} m!} \left(\frac{m}{1-2\alpha}\right)^m e^{-m} + \frac{2 \times 2n!}{(1-2\alpha)^{2n+1}} T^{(1-2\alpha)} + 2A_1 C_n + A_1^2 T. \end{aligned}$$

Thus the proof is complete. □

Lemma 2.3. *Let $l(x,s)$ be the weakly singular kernel defined by (3) of the equation (1), then*

$$\sup_{x \in [0,T]} \int_0^T l(x,s) ds \leq \frac{2a_1}{1-\alpha} T^{(1-\alpha)} + a_2 A'_n + A_2 T, \tag{18}$$

where

$$A'_n = 2\left[\sum_{m=1}^{n} \frac{(-1)^n n!}{m!} m^m e^{-m} + (-1)^n n! T\right]. \tag{19}$$

Proof. From equation (3), we have

$$\begin{aligned} \int_0^T l(x,s) ds &= \int_0^T \left[a_1 |x-s|^{-\alpha} + a_2 (\log|x-s|)^n + A_2\right] ds \\ &= \int_0^x [a_1 (x-s)^{-\alpha} + a_2 (\log(x-s))^n] ds \\ &\quad + \int_x^T [a_1 (s-x)^{-\alpha} + a_2 [\log(s-x)]^n] ds + \int_0^T A_2 ds \\ &= \frac{a_1 x^{1-\alpha}}{1-\alpha} + \frac{a_1 (T-x)^{1-\alpha}}{1-\alpha} + a_2 \left[\int_0^x (\log t)^n dt + \int_0^{T-x} (\log z)^n dz\right] + A_2 T. \end{aligned} \tag{20}$$

We denote $M_n = \int_0^x (\log t)^n dt$. By similar calculation as in Lemma 2.2, we have

$$M_n = \sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{m!} x[\log x]^m + (-1)^n n! x,$$

and

$$x[\log x]^m \leq (-m)^m e^{-m} \text{ for } m = 2n, \tag{21}$$

and

$$x[\log x]^m \geq (-m)^m e^{-m} \text{ for } m = 2n-1. \tag{22}$$

Hence, we have $M_n \leq \sum_{m=1}^{n} \frac{(-1)^n n!}{m!} (m)^m e^{-m} + (-1)^n n! T$. Similarly,

$$\begin{aligned} M'_n &= \int_0^{T-x} (\log z)^n dz \\ &= \sum_{m=1}^{n} \frac{(-1)^{n-m} n!}{m!} (T-x)[\log(T-x)]^m + (-1)^n n! (T-x) \\ &\leq \sum_{m=1}^{n} \frac{(-1)^n n!}{m!} (m)^m e^{-m} + (-1)^n n! T. \end{aligned}$$

Now, from (20), we get

$$\sup_{x\in[0,T]}\int_0^T l(x,s)ds \leq \frac{2a_1}{1-\alpha}T^{(1-\alpha)} + a_2A'_n + A_2T,$$

where $A'_n = 2\sum_{m=1}^{n}\frac{(-1)^n n!}{m!}m^m e^{-m} + 2(-1)^n T\times n!$. □

Lemma 2.4. *Let $l(x,s)$ be the weakly singular kernel defined by (3) of the equation (1), and if $l(x,s)$ satisfy the result of Lemma 2.3, then it also satisfies the following inequality*

$$c_1 = \sup_{x\in[0,T]}\int_0^T [l(x,s)]^2 ds \leq \frac{2a_1^2}{1-2\alpha}T^{(1-2\alpha)} + a_2^2A'_{2n} + A_2^2T + 2a_1a_2C_n + 2a_2A_2A'_n + \frac{4a_1A_2}{1-\alpha}T^{(1-\alpha)}, \quad (23)$$

where

$$A'_{2n} = 2\bigg[\sum_{m=1}^{2n}\frac{2n!}{m!}e^{-m}.m^m + 2n!T\bigg].$$

Proof. From equation (3), we get

$$\begin{aligned}\int_0^T [l(x,s)]^2 ds &= \int_0^T \bigg[(a_1|x-s|^{-\alpha})^2 + [a_2(\log|x-s|)^n]^2 + A_2^2 \\ &\quad + 2a_1a_2\frac{(\log|x-s|)^n}{|x-s|^\alpha} + 2a_2A_2(\log|x-s|)^n + 2a_1A_2|x-s|^{-\alpha}\bigg]ds \\ &= \int_0^T \frac{a_1^2}{|x-s|^{2\alpha}}ds + a_2^2\int_0^T (\log|x-s|)^{2n}ds + 2a_1A_2\int_0^T |x-s|^{-\alpha}ds \\ &\quad + 2a_1a_2\int_0^T \frac{(\log|x-s|)^n}{|x-s|^\alpha} + 2a_2A_2\int_0^T (\log|x-s|)^n + A_2^2T. \end{aligned} \quad (24)$$

Now, the first integral of (24) can be written as

$$\begin{aligned}\int_0^T \frac{a_1^2}{|x-s|^{2\alpha}}ds &= a_1^2\bigg[\int_0^x \frac{1}{(x-s)^{2\alpha}}ds + \int_x^{T-x}\frac{1}{(s-x)^{2\alpha}}ds\bigg] \\ &= \frac{a_1^2}{(1-2\alpha)}x^{1-2\alpha} + \frac{a_1^2}{(1-2\alpha)}(T-x)^{1-2\alpha} \\ &\leq \frac{a_1^2}{(1-2\alpha)}T^{(1-2\alpha)} + \frac{a_1^2}{(1-2\alpha)}T^{(1-2\alpha)} \\ &\leq \frac{2a_1^2}{(1-2\alpha)}T^{(1-2\alpha)}.\end{aligned}$$

By similar analysis as in Lemma 2.3, we have

$$a_2^2\int_0^T (\log|x-s|)^{2n}ds \leq a_2^2\bigg[2\sum_{m=1}^{2n}\frac{2n!}{m!}e^{-m}.m^m + 2\times 2n!T\bigg] = a_2^2A'_{2n}.$$

Hence, by using Lemma 2.1 and Lemma 2.3, from equation (24) we conclude the result. □

Lemma 2.5. *Let $l(x,s)$ be the weakly singular kernel defined by (4) of the equation (1), then it satisfies the following inequality*

$$\sup_{x\in[0,T]}\int_0^T l(x,s)ds \leq \frac{3T^{(2-\alpha_1-\alpha_2)}}{(1-\alpha_1-\alpha_2)}m_1m_2, \quad (25)$$

where $m_1 = \max_{(x,t)\in[0,T]\times[0,T]}|g_3(x,t)|$, $m_2 = \max_{(x,t)\in[0,T]\times[0,T]}|g_4(x,t)|$ *and* $\alpha_1+\alpha_2<1$.

Proof. Equation (4) leads to the following

$$\begin{aligned}\int_0^T l(x,s)ds &= \int_0^T\bigg[\int_0^T \frac{g_3(x,t)g_4(t,s)}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt\bigg]ds \\ &\leq \max_{(x,t)\in[0,T]\times[0,T]}|g_3(x,t)|\max_{(t,s)\in[0,T]\times[0,T]}|g_4(t,s)|\int_0^T\bigg[\int_0^T\frac{1}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt\bigg]ds \\ &\leq m_1m_2\bigg[\int_0^T K ds\bigg].\end{aligned} \quad (26)$$

Now, we have to show $\int_0^T Kds$ is bounded. For $\alpha_1+\alpha_2<1$ and $0\le x\le s\le T$, we get

$$\begin{aligned}\int_0^T \frac{1}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt &= \int_0^x \frac{1}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt+\int_x^s \frac{1}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt+\int_s^T \frac{1}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt\\ &\le \int_0^x \frac{dt}{|x-t|^{\alpha_1+\alpha_2}}+\max\left\{\int_x^s \frac{dt}{|x-t|^{\alpha_1+\alpha_2}},\int_x^s \frac{dt}{|t-s|^{\alpha_1+\alpha_2}}\right\}\\ &\quad +\int_s^T \frac{dt}{|t-s|^{\alpha_1+\alpha_2}}.\end{aligned}$$

Since $t\le x$, therefore we have

$$\int_0^x \frac{dt}{|x-t|^{\alpha_1+\alpha_2}} = \frac{x^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}\le \frac{T^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}.$$

Since $x\le t\le s$, so we get

$$\int_x^s \frac{dt}{|t-s|^{\alpha_1+\alpha_2}} \le \frac{t^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}\le \frac{T^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}.$$

and

$$\int_x^s \frac{dt}{|x-t|^{\alpha_1+\alpha_2}}\le \int_x^s \frac{dt}{|t-s|^{\alpha_1+\alpha_2}}\le \frac{T^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}.$$

By using the inequality $x\le s\le t\le T$, we compute

$$\int_s^T \frac{dt}{|t-s|^{\alpha_1+\alpha_2}} = \frac{(T-s)^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}\le \frac{T^{1-\alpha_1-\alpha_2}}{1-\alpha_1-\alpha_2}.$$

Hence from inequality (26) we can conclude the result. □

Lemma 2.6. *Let $l(x,s)$ be the weakly singular kernel defined by (4) of the equation (1), and if $l(x,s)$ satisfy the result of Lemma 2.5, then it also satisfies the following inequality*

$$c_1=\sup_{x\in[0,T]}\int_0^T [l(x,s)]^2ds\le \frac{9T^{(5-2\alpha_1-2\alpha_2)}m_1^2m_2^2}{(1-\alpha_1-\alpha_2)^2}. \tag{27}$$

Proof. From equation (4), we have

$$\begin{aligned}\int_0^T [l(x,s)]^2ds &= \int_0^T\left[\int_0^T \frac{g_3(x,t)g_4(t,s)}{|x-t|^{\alpha_1}|t-s|^{\alpha_2}}dt\right]^2ds\\ &\le \frac{9T^{2(2-\alpha_1-\alpha_2)}m_1^2m_2^2}{(1-\alpha_1-\alpha_2)^2}\int_0^T ds,\ \text{since } g_3(x,t) \text{ and } g_4(t,s) \text{ are positive,}\\ &\le \frac{9T^{(5-2\alpha_1-2\alpha_2)}m_1^2m_2^2}{(1-\alpha_1-\alpha_2)^2}.\end{aligned}$$

This completes the proof. □

Let us define an operator $\mathcal{T}:\mathbb{X}\to\mathbb{X}$ such that

$$\mathcal{T}u(x)=\frac{f(x,u)}{\mu}+\frac{\lambda}{\mu}\int_0^T l(x,s)k(x,s)\psi(s,u(s))ds. \tag{28}$$

If we show $\mathcal{T}u(x)$ has a fixed point, then from equation (28) we have

$$u=\mathcal{T}u. \tag{29}$$

Now we consider a sequence $\{u_n\}$ such that

$$u_1=\mathcal{T}u_0, \tag{30}$$
$$u_2=\mathcal{T}u_0, \tag{31}$$
$$\vdots$$
$$u_{n+1}=\mathcal{T}u_n, \tag{32}$$
$$\vdots \tag{33}$$

Here we state the following theorem to show the existence and uniqueness of the solution u.

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Theorem 2.1. *Let $l(x,s)$ be the weakly singular kernel defined by the equation (2)(respectively, (3) and (4)) of the integral equation (1). Assume $f(x,u(x))$ and $\psi(s,u(s))$ are continuous in $[0,T]\times\mathbb{R}$ and Lipschitz with respect to u having l and c_2 as a Lipschitz constant given by $l=\sup_{(s,u(s))\in X\times\mathbb{R}}\left|\frac{\partial f}{\partial u}\right|$ and $c_2=\sup_{(s,u(s))\in X\times\mathbb{R}}\left|\frac{\partial \psi}{\partial u}\right|$, respectively. If $l(x,s)$ satisfy the inequality in Lemma 2.2 (respectively, Lemma 2.4 and Lemma 2.6), then for any $u_0\in X$ the sequence $\{u_n\}_{n=0}^{\infty}$ defined by equation (32) converge uniformly to the solution of the integral equation (29) in $\mathbb{X}$. Furthermore, the solution is unique.*

Proof. From equation (28), we have

$$\begin{aligned}||Tu-Tv||_\infty &\le \frac{1}{|\mu|}||f(x,u)-f(x,v)||_\infty+\left|\frac{\lambda}{\mu}\right|\left|\int_0^T l(x,s)k(x,s)[\psi(s,u(s))-\psi(s,v(s))]ds\right|\\
&\le \frac{l}{|\mu|}||u-v||_\infty+\frac{|\lambda|}{|\mu|}\|\psi(s,u(s))-\psi(s,v(s))\|_\infty\left(\int_0^T l(x,s)^2ds\right)^{\frac12}\left(\int_0^T (k(x,s))^2ds\right)^{\frac12}\\
&\le \left(\frac{l}{|\mu|}+\frac{|\lambda|}{|\mu|}c_2\sqrt{c_1c_3}\right)\|u(s)-v(s)\|_\infty,\ \text{Since } \psi(s,u(s)) \text{ is Lipschitz}\\
&\qquad\qquad \text{and } c_3=\sup_{x\in[0,T]}\int_0^T[k(x,s)]^2ds,\\
&\le M\|u(s)-v(s)\|_\infty,\ \text{where } M=\left(\frac{l}{|\mu|}+\frac{|\lambda|}{|\mu|}c_2\sqrt{c_1c_3}\right).\end{aligned}$$

Since, $(X,||.||_\infty)$ is a Banach Space, therefore it is complete. If $M<1$, by using Banach Fixed Point theorem we can conclude that the equation (28) has a fixed point, which is defined by (29). Now

$$\begin{aligned}u_2-u_1&=\mathcal{T}u_1-\mathcal{T}u_0\\
&=\frac{1}{\mu}\left(f(x,u_1(x)-f(x,u_0(x))\right)+\frac{\lambda}{\mu}\int_0^T l(x,s)k(x,s)[\psi(s,u_1(s))-\psi(s,u_0(s))]ds.\end{aligned}$$

Therefore, by similar analysis, we have

$$||u_2-u_1||_\infty\le M\|u_1(s)-u_0(s)\|_\infty,\ \text{where } M=\left(\frac{l}{|\mu|}+\frac{|\lambda|}{|\mu|}c_2\sqrt{c_1c_3}\right).$$

Similarly,

$$||u_3-u_2||_\infty=\mathcal{T}u_2-\mathcal{T}u_1\le M\|u_2(s)-u_1(s)\|_\infty\le M^2\|u_1(s)-u_0(s)\|_\infty.$$

In general,

$$\|u_{n+1}-u_n\|_\infty\le M^n\|u_1(s)-u_0(s)\|_\infty.$$

If $m>n>0$,

$$\begin{aligned}\|u_m-u_n\|_\infty &\le \|u_m-u_{m-1}+u_{m-1}-u_{m-2}+u_{m-2}-\cdots+u_{n+1}-u_n\|_\infty\\
&\le \|u_m-u_{m-1}\|_\infty+\|u_{m-1}-u_{m-2}\|_\infty+\cdots+\|u_{n+1}-u_n\|_\infty\\
&\le M^{m-1}\|u_1(s)-u_0(s)\|_\infty+\cdots+M^n\|u_1(s)-u_0(s)\|_\infty\\
&\le [M^{m-1}+M^{m-2}+\cdots+M^n]\|u_1(s)-u_0(s)\|_\infty.\end{aligned}$$

Taking limit on both side we get

$$\begin{aligned}\|u_m-u_n\|_\infty &\le \|u_1(s)-u_0(s)\|_\infty\lim_{m\to 0}M^n(1+M+M^2+\cdots+M^{m-1-n})\\
&\le M^n\|u_1(s)-u_0(s)\|_\infty(1+M+M^2+\cdots)\\
&\le \frac{M^n}{1-M}\|u_1(s)-u_0(s)\|_\infty.\end{aligned}$$

Hence, $\{u_n\}$ is a Cauchy sequence and it converge to a point u^* in X. If $m\to\infty$ we get

$$\|u^*-u_n\|_\infty\le\frac{M^n}{1-M}\|u_1(s)-u_0(s)\|_\infty.$$

Therefore;

$$\mathcal{T}(u^*) = \lim_{n\to\infty} \mathcal{T}(u_n) = \lim_{n\to\infty} u_n = u^*,$$

where u^* be the solution of equation (1). To show uniqueness of the solution, we assume equation (1) has two solutions u_1^* and u_2^*. Therefore we have $u_1^* = \mathcal{T}u_1^*$ and $u_2^* = \mathcal{T}u_2^*$. Now

$$\|u_2^* - u_1^*\|_\infty \leq M\|u_2^* - u_1^*\|_\infty$$
$$\implies \|u_2^* - u_1^*\|_\infty(1 - M) \leq 0.$$

Since $M < 1$, hence we can conclude $\|u_2^* - u_1^*\|_\infty = 0$ which implies $u_1^* = u_2^*$. This completes the proof. □

Corollary 2.1. *Let $l(x, s)$ be the weakly singular kernel defined by equation (2) (respectively, equations (3) and (4)) of the integral equation (1). If $l(x, s)$ satisfy the result of Lemma 2.2 (respectively, Lemma 2.4 and Lemma 2.6), then the solution will exists and unique for*

$$|\lambda| < \frac{|\mu| - l}{\sqrt{c_1 c_2 \sqrt{c_3}}}, \tag{34}$$

where c_1, c_2 and c_3 are defined by Theorem 2.1.

Proof. For existence and uniqueness of the solution, from Theorem 2.1 we have $M < 1$. Therefore, we have $\left(\frac{l}{|\mu|} + \frac{|\lambda|}{|\mu|} c_2 \sqrt{c_1 c_3}\right) < 1$. Hence, the proof is complete. □

Remark 2.1. *From Theorem 2.1 and Corrolary 2.1, we can see that the condition (34) is sufficient but not necessary. This indicates for $|\lambda| > \frac{|\mu|-l}{\sqrt{c_1 c_2 \sqrt{c_3}}}$ the solution of equation (1) may or may not be exists.*

Remark 2.2. *In view of Theorem 2.1 and Corrolary 2.1, we have the classical solution ([41]) $u(x)$ of equation (1) exist, unique and it belongs to the class $C[0, T]$ where $T > 0$ subject to the condition (34). Moreover, we could not find the higher regularity of the classical solution $u(x)$.*

3 Examples

By using Theorem 2.1, we have placed three numerical constructions of the following examples:

$$\mu u(x) = \cos u + \lambda \int_0^1 \left(|x - s|^{-\frac{1}{3}} \log|x - s| + 1\right) x \sin u(s) ds. \tag{35}$$

Corollary 3.1. *Let the condition $|\lambda| < \frac{|\mu|-1}{12.87}$ be true. Then for any $u_0 \in X$ the sequence $\{u_n\}_{n=0}^\infty$ defined by*

$$\mu u_{n+1}(x) = \cos u_n + \lambda \int_0^1 \left(|x - s|^{-\frac{1}{3}} \log|x - s| + 1\right) x \sin u_n(s) ds, \;\; n = 0, \cdots \tag{36}$$

converge uniformly to the solution u^* of equation (35) in X. Moreover the solution is unique.

Proof. By using Theorem 2.1 and equation (32), we have $u_{n+1} = \frac{\mathcal{T}u_n}{\mu}$. Now,

$$u_2 - u_1 = \frac{1}{\mu}\left[\cos u_1 - \cos u_0 + \lambda \int_0^1 \left(|x - s|^{-\frac{1}{3}} \log|x - s| + 1\right) x \left(\sin u_1(s) - \sin u_0(s)\right) ds\right].$$

So,

$$\begin{aligned} \|u_2 - u_1\|_\infty &\leq \frac{1}{|\mu|}\left[\|\cos u_1 - \cos u_0\|_\infty\right] \\ &+ \left[\frac{|\lambda|}{|\mu|}\|\sin u_1 - \sin u_0\|_\infty \left(\int_0^1 \left(|x - s|^{-\frac{1}{3}} \log|x - s| + 1\right)^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 x^2 ds\right)^{\frac{1}{2}}\right]. \\ \|u_2 - u_1\|_\infty &\leq \frac{1}{|\mu|}\left[l\|u_1 - u_0\|_\infty + |\lambda| c_2 \|u_1 - u_0\|_\infty \sqrt{c_1 c_3}\right], \end{aligned}$$

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where $l = \sup_{x\in[0,1]} \left|\frac{\partial \cos u}{\partial u}\right| = \sup_{x\in[0,1]} |\sin u| = 1$ and $c_2 = \sup_{s\in[0,1]} \left|\frac{\partial \sin u}{\partial u}\right| = \sup_{x\in[0,1]} |\cos u| = 1$. Hence, we get

$$\|u_2 - u_1\|_\infty \quad \leq \quad \|u_1 - u_0\|_\infty \left[\frac{1}{|\mu|} + \frac{|\lambda|\sqrt{c_1 c_3}}{|\mu|}\right].$$

In general

$$\|u_{n+1} - u_n\|_\infty \leq \|u_1 - u_0\|_\infty \left[\frac{1}{|\mu|} + \frac{|\lambda|\sqrt{c_1 c_3}}{|\mu|}\right]^n. \tag{37}$$

From Lemma 2.2, we have, $c_1 = B_{2n} + 2A_1 C_n + A_1^2 T = 165.65 \implies \sqrt{c_1} = 12.87$. Since, $k(x,s) = x$, so we have $c_3 = \sup_{x\in[0,1]} \int_0^1 x^2 ds = 1$. Now,

$$\|u_1 - u_0\|_\infty \qquad \leq \frac{1}{|\mu|}\left|\cos u_0 + \lambda \int_0^1 \left(|x-s|^{-\frac{1}{3}} \log|x-s| + 1\right) x \sin u_0(s) ds - |\mu| u_0\right| \tag{38}$$

$$\leq \frac{1}{|\mu|} + \frac{|\lambda|}{|\mu|}\sqrt{c_1 c_3} + |u_0|. \tag{39}$$

Since u_0 is in X, therefore we have $\|u_1 - u_0\|_\infty < +\infty$. As $|\lambda| < \frac{|\mu|-1}{12.87}$, hence by using Corollary 2.1 and Theorem 2.1, we can conclude the sequence $\{u_n\}$ converges uniformly to the solution u^* of equation (35) in X. Hence, the proof is complete. □

Now, we consider weakly singular Fredholm-Hammerstein type integral equation as follows:

$$\mu u(x) = |u(x)| + \lambda \int_0^2 \left(|x-s|^{\frac{-1}{3}} + \log|x-s| + 1\right) e^s \sin(u(s)) ds. \tag{40}$$

Corollary 3.2. *Assume $|\lambda| < \frac{|\mu|-1}{10.65}$ is true. Then for any $u_0 \in X$ the sequence $\{u_n\}_{n=0}^\infty$ defined by*

$$\mu u_{n+1}(x) = |u_n(x)| + \lambda \int_0^2 \left(|x-s|^{-\frac{1}{3}} + \log|x-s| + 1\right) e^s \sin(u_n(s)) ds \tag{41}$$

converge uniformly to the solution u^* of equation (40) in X. Moreover the solution is unique.

Proof. By using similar calculation as in Corollary 3.1, we have

$$|u_2 - u_1| \quad \leq \quad \frac{1}{|\mu|}\left[||u_1| - |u_0||\right] + \left[\frac{|\lambda|}{|\mu|}||\sin u_1 - \sin u_0||_\infty \left(\int_0^1 \left(|x-s|^{-\frac{1}{3}} + \log|x-s| + 1\right)^2 ds\right)^{\frac{1}{2}} \times \left(\int_0^1 e^{2s} ds\right)^{\frac{1}{2}}\right].$$

Since, $||u_1| - |u_0|| \leq |u_1 - u_0|$, we have

$$\|u_2 - u_1\|_\infty \leq \|u_1 - u_0\|_\infty \left[\frac{1}{|\mu|} + \frac{|\lambda|\sqrt{c_1 c_3}}{|\mu|}\right].$$

Now, from Lemma 2.4, we have $\sqrt{c_1} = 2.13$. Again, $k(x,s) = e^s$ implies $c_3 = \sup_{x\in[0,2]} \int_0^2 e^{2s} ds = 26.80$. So, $\sqrt{c_3} = 5.18$. Hence, from Corollary 2.1 we determine the range of λ as $|\lambda| < \frac{|\mu|-1}{10.65}$. By using Theorem 2.1, we can conclude the sequence $\{u_n\}_{n=0}^\infty$ converges uniformly to the solution u^* of equation (40) in X. This completes the proof. □

In the following, we consider Fredholm-Hammerstein type integral equation corresponding to weakly singular kernel (4) as follows:

$$\mu u(x) = x + \sin u(x) + \lambda \int_0^1 \left(\int_0^1 \frac{x e^s}{|x-t|^{\frac{1}{2}} |t-s|^{\frac{1}{3}}} dt\right) x \cos(u(s)) ds. \tag{42}$$

Corollary 3.3. *Assume* $|\lambda| < \frac{|\mu|-1}{48.90}$ *is true. Then for any* $u_0 \in X$ *the sequence* $\{u_n\}_{n=0}^{\infty}$ *defined by*

$$\mu u_{n+1}(x) = x + \sin u_n(x) + \lambda \int_0^1 \left(\int_0^1 \frac{xe^s}{|x-t|^{\frac{1}{2}}|t-s|^{\frac{1}{3}}} dt \right) x \cos(u_n(s)) ds \tag{43}$$

converge uniformly to the solution u^* *of equation (42) in* X*. Moreover the solution is unique.*

Proof. Here, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}$, $g_3(x,t) = x$, $g_4(t,s) = e^s$ and $k(x,s) = x$. By similar analysis as in Corollary 3.1, we have

$$\begin{aligned} \|u_2 - u_1\|_\infty &\leq \frac{\|u_1 - u_0\|_\infty}{|\mu|} \left[l + |\lambda| c_2 \int_0^1 \left(\int_0^1 \frac{xe^s}{|x-t|^{\frac{1}{2}}|t-s|^{\frac{1}{3}}} dt \right) x ds \right] \\ &\leq \|u_1 - u_0\|_\infty \left[\frac{l + |\lambda| c_2 \sqrt{c_1 c_3}}{|\mu|} \right], \end{aligned}$$

where $l = \sup_{x\in[0,1]} \left|\frac{\partial \sin u}{\partial u}\right| = 1$ and $c_2 = \sup_{s\in[0,1]} \left|\frac{\partial \cos u}{\partial u}\right| = 1$. By using Lemma 2.6, we compute $c_1 = 2394.05 \implies \sqrt{c_1} = 48.93$ and $\sqrt{c_3} = 1$. Hence by using Theorem 2.1, we can conclude that for any $u_0 \in X$ the sequence $\{u_n\}_{n=0}^{\infty}$ converges uniformly to the solution u^* of equation (42) in X subject to the condition $|\lambda| < \dfrac{|\mu|-1}{48.90}$. Hence the proof is complete. □

Remark 3.1. *Exact solution of problems (35), (40) and (42) are not known. By using Corollary 3.1, Corollary 3.2 and Corollary 3.3, we can guarantee the existence and uniqueness of the solution of problems (35), (40) and (42), respectively.*

In below, we provide another three numerical examples with exact solutions. Corresponding to kernel (2), we consider Fredholm-Hammerstein type integral equation given by

$$\mu u(x) = f(x) + \lambda \int_0^1 \left(|x-s|^{-\frac{1}{3}} \log|x-s| + 1 \right) u^2(s) ds, \tag{44}$$

where

$$f(x) = \mu(x + x^2) - \lambda \int_0^1 \left(|x-s|^{-\frac{1}{3}} \log|x-s| + 1 \right) (s + s^2)^2 ds. \tag{45}$$

The unique solution of the problem (44) is $u(x) = x + x^2$.

Corollary 3.4. *Let the condition* $|\lambda| < \frac{|\mu|}{51.48}$ *be true and* $f(x)$ *is given by equation (45). Then for any* $u_0 \in X$ *the sequence* $\{u_n\}_{n=0}^{\infty}$ *defined by*

$$\mu u_{n+1}(x) = f(x) + \lambda \int_0^1 \left(|x-s|^{-\frac{1}{3}} \log|x-s| + 1 \right) u_n^2(s) ds, \ n = 0, 1, \cdots \tag{46}$$

converge uniformly to the solution u^* *of equation (44) in* X*. Moreover the solution is unique.*

Proof. The proof follows from Corollary 3.1. □

Now, we consider weakly singular Fredholm-Hammerstein type integral equation with respect to kernel (3) as follows:

$$\mu u(x) = g(x) + \lambda \int_0^2 \left(|x-s|^{\frac{-1}{3}} + \log|x-s| + 1 \right) u^3(s) ds, \tag{47}$$

where

$$g(x) = \mu e^x - \lambda \int_0^2 \left(|x-s|^{\frac{-1}{3}} + \log|x-s| + 1 \right) e^{3s} ds. \tag{48}$$

The exact solution is $u(x) = e^x$.

Corollary 3.5. *Assume* $|\lambda| < \frac{|\mu|}{493.38}$ *is true and* $g(x)$ *is given by equation (48). Then for any* $u_0 \in X$ *the sequence* $\{u_n\}_{n=0}^{\infty}$ *defined by*

$$\mu u_{n+1}(x) = g(x) + \lambda \int_0^2 \left(|x-s|^{\frac{-1}{3}} + \log|x-s| + 1 \right) u_n^3(s) ds \tag{49}$$

converge uniformly to the solution u^* *of equation (47) in* X*. Moreover the solution is unique.*

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Proof. The proof is similar to the Corollary 3.2. □

Again, corresponding to weakly singular kernel (4), we consider Fredholm-Hammerstein type integral equation in the following form:

$$\mu u(x) = h(x) + \lambda \int_0^1 \left(\int_0^1 \frac{xe^s}{|x-t|^{\frac{1}{2}} |t-s|^{\frac{1}{3}}} dt \right) \cos(u(s)) ds, \tag{50}$$

where

$$h(x) = 2\mu x - \lambda \int_0^1 \left(\int_0^1 \frac{xe^s}{|x-t|^{\frac{1}{2}} |t-s|^{\frac{1}{3}}} dt \right) \cos(2s) ds. \tag{51}$$

The unique solution of the given problem (50) is $u(x) = 2x$.

Corollary 3.6. *Assume* $|\lambda| < \frac{|\mu|}{48.90}$ *is true and* $h(x)$ *is given by equation (51). Then for any* $u_0 \in X$ *the sequence* $\{u_n\}_{n=0}^{\infty}$ *defined by*

$$\mu u_{n+1}(x) = h(x) + \lambda \int_0^1 \left(\int_0^1 \frac{xe^s}{|x-t|^{\frac{1}{2}} |t-s|^{\frac{1}{3}}} dt \right) \cos(u_n(s)) ds \tag{52}$$

converge uniformly to the solution u^* *of equation (50) in* X*. Moreover the solution is unique.*

Proof. The proof follows from Corollary 3.3. □

Remark 3.2. *We have observed that the range of the* λ *depends on* T*. If we increase the value of* T *then we are getting a smaller range of the parameter for a fixed value of the parameter* μ*.*

4 Conclusions

We successfully proposed the class of Fredholm-Hammerstein type integral equation corresponding to different types of weakly singular kernels. We derived results to show the existence of the solution. In view of Theorem 2.1 we conclude that the solution to the proposed problem exists and is unique in the continuous space. We have constructed numerical iterations corresponding to different types of problems to validate the theoretical results. Finally, we conclude that our proposed problem is reliable and valid, which helps researchers to explore the literature.

Future Remark 4.1. *Since our proposed problems are novel, therefore lots of theoretical and numerical works can be done. Researcher may try to apply any numerical technique on (1) to compute the numerical approximation.*

Future Remark 4.2. *Here, we have computed smaller range of* λ *corresponding to a fixed value of* μ *because it is depending on the domain. Researcher may try to improve the bounds of* λ *which is independent of the domain with respect to any norm.*

Acknowledgments

We are thankful to all the reviewers and editors for their valuable suggestions to improve the quality of the work.

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