New numerical method for solving a new generalized American options under $\psi-$Caputo time-fractional derivative Heston model

Hassen Arfaoui *

Abstract

In this paper, a new generalized American Options under $\psi-$Caputo Fractional-Order Derivative Heston (AOψCFDH) model was investigated. Moreover, a new Numerical Implicit Scheme Method NISM has been developed for solving the AOψCFDH model. Also, we have analyzed the stability and convergence of the NISM. Finally, two numerical examples are proposed in order to show the robustness and the efficiency of both the model AOψCFDH and the NISM.

Keywords. The $\psi-$Riemann-Liouville fractional derivative, The $\psi-$Caputo fractional derivative, American Options, Heston model, Numerical method, Stability, Convergence.

AMS Subject Classification. 65M15, 65N12, 65M22, 65M06, 26A33.

1 Introduction

The recent period has seen a major revolution in the fields of scientific research concerned with the study and analysis of partial differential equations. Particularly, the interest focused, not so long ago, on the study of Partial Differential Equations with Fractional Derivative (PDEFD) [12, 13, 19, 22, 24, 28, 30, 32, 33, 35, 38, 39]. The PDEFD has been studied and applied in various scientific fields such as medicine [11, 17, 21, 23, 27], engineering, pure and applied mathematics, physics [2, 3, 5, 6, 7, 8, 9, 10, 21, 25, 29, 36], Stochastic [4, 26] and other fields. In the literature there are several types of fractional derivatives. We list the most frequently used, for example, Riemann-Liouville derivative, Caputo derivative, Caputo-Hadamard derivative [1, 15, 20, 37], Atangana-Baleanu fractional derivative [3, 9, 34] and Caputo-Fabrizio fractional derivative [10].

The application of PDEFD in the mathematical modeling of many natural and realistic phenomena in the economic and financial fields is still very weak compared to other fields [2, 14, 25, 36, 40]. In [36, 2015], the authors investigated the time-fractional Black-Scholes equations. The Black-Scholes option pricing equations have been studied with Caputo generalized fractional derivative in [2, 2019]. In [25, 2022], by using a modified right Riemann-Liouville fractional derivative and Caputo fractional derivative with $\psi(t) = t$, we studied the Mittag-Leffler stability and the numerical resolution of a pricing European options under time-fractional Vasicek model. For this reason, our interest was focused when we realized the importance of this proposition in the development of fractal modeling in the economic and financial fields.

*Institut Supérieur de Sciences Appliquées et de Technologie de Mateur, Université de Carthage, Tunisia.

In [31, 2020], the authors studied the American options under integer-order derivative Heston Model. In this work, we investigated a new generalized American options under $\psi-$Caputo Fractional-Order Derivative Heston Model AOψCFDH described by an evolution advection-diffusion-reaction PDE posed in a fixed two dimensional domain. Moreover, we proposed a new numerical implicit scheme method NISM for solving the AOψCFDH model. First, we analyzed the stability and convergence of the NISM. Then, some examples are proposed with numerical simulations in order to show the robustness and the efficiency of both the model AOψCFDH and the NISM.

Let's summarize the novelties of this work. In fact, we have developed several goals which are as follows:

- Propose and validate a new generalized American options under $\psi-$Caputo Fractional-Order Derivative Heston model, (Section 2).
- Adaptation of the splitting method to avoid the undesirable numerical effects of the cross-derivative term in the PDE, (Section 3).
- Propose a new Numerical Implicit Scheme Method NISM for solving the generalized AOψCFDH model, (Section 3).
- Analyze the stability and convergence of the NISM, (Sections 4, 5).
- Numerical implementation of the NISM, (Section 6).

The paper is structured as follows. In Section 2, we presented the new generalized American options under $\psi-$Caputo Fractional-Order Derivative Heston model. The new numerical implicit scheme method with the splitting method was given in Section 3. In Section 4, we studied the stability of the NISM and the convergence of the NISM has been investigated in Section 5. Finally, the numerical implementation of the NISM and the interpretation of the numerical results have been given in Section 6.

2 The $\psi-$Caputo fractional derivative model

Consider the new generalized American options under modified right $\psi-$Riemann-Liouville fractional-order derivative Heston model described by an evolution advection-diffusion-reaction (PDE) posed in a fixed two dimensional domain:

$$\begin{aligned} {}^{mr}D^{\varrho,\psi}\Upsilon(\mathbf{x},\xi) \quad + \quad & 0.5zS^2\frac{\partial^2\Upsilon(\mathbf{x},\xi)}{\partial S^2} + \rho\eta zS\frac{\partial^2\Upsilon(\mathbf{x},\xi)}{\partial S\partial z} + 0.5\eta^2 z\frac{\partial^2\Upsilon(\mathbf{x},\xi)}{\partial z^2} + \\ & \mathbf{r}S\frac{\partial\Upsilon(\mathbf{x},\xi)}{\partial S} + a(b-z)\frac{\partial\Upsilon(\mathbf{x},\xi)}{\partial z} - \mathbf{r}\Upsilon(\mathbf{x},\xi) + f(K,S,\Upsilon) = 0, \end{aligned} \tag{2.1}$$

where the variable $\mathbf{x} = (S, z)$ and $(\mathbf{x}, \xi) \in (0, \infty) \times (0, \infty) \times B_T$, $B_T = (0, T)$. The parameters η is the volatility, $a = \kappa + \varsigma$ and $b = \frac{\kappa}{\kappa+\varsigma}\theta$, κ is the mean reversion rate, θ is the long-run variance, ς is a positive constant [18]. In fact, the term $a(b - z)$ is the risk-neutral drift rate. The correlation coefficient $\rho \in (-1, 1)$, $\mathbf{r}$ is the interest rate, z is the volatility of the interest rate, S is the asset price, K is the strike price and $f(K,S,\Upsilon) = \Sigma \max\{K - S - \Upsilon, 0\}$ is a penalty function, where Σ is a positive penalty parameter, Σ tends to ∞, (see [16]).

The initial condition at T is given by:

$$\Upsilon(\mathbf{x}, T) = \max(0, K - S). \tag{2.2}$$

The boundary conditions are defined by:

$$\lim_{S \longmapsto 0} \Upsilon(S, z, \xi) = K, \quad \lim_{S \longmapsto \infty} \Upsilon(S, z, \xi) = 0, \tag{2.3}$$
$$\lim_{z \longmapsto 0} \Upsilon(S, z, \xi) = \lim_{z \longmapsto \infty} \Upsilon(S, z, \xi) = \max(0, K - S). \tag{2.4}$$

The term ${}^{mr}D^{\varrho,\psi}\Upsilon(\mathbf{x}, \xi)$ is the Modified Right $\psi-$Riemann-Liouville Fractional Derivative (MRψRLFD) defined as follows [30]:

$${}^{mr}D^{\varrho,\psi}\Upsilon(\mathbf{x}, \xi) = \frac{1}{\Gamma(1-\varrho)} \Big(\frac{1}{\psi'(\xi)} \frac{d}{d\xi} \Big) \int_{\xi}^{T} \psi'(\tau)[\psi(\tau) - \psi(\xi)]^{-\varrho} [\Upsilon(\mathbf{x}, \tau) - \Upsilon(\mathbf{x}, T)]\, d\tau, \tag{2.5}$$

where $\psi \in \mathcal{C}^1(\overline{B}_T)$ is a positive and strictly increasing function on the bounded interval B_T. Moreover, ψ satisfies $\psi'(h) \neq 0$ for all $h \in B_T$.

Let $\zeta = T - \xi$. Then, (2.5) can be rewritten as:

$$\begin{aligned}&{}^{mr}D^{\varrho,\psi}\Upsilon(\mathbf{x}, T-\zeta) = \\ &\frac{1}{\Gamma(1-\varrho)} \Big(\frac{1}{\psi'(T-\zeta)} \frac{d}{d\zeta} \Big) \int_{T-\zeta}^{T} \psi'(\tau)[\psi(\tau) - \psi(T-\zeta)]^{-\varrho} [\Upsilon(\mathbf{x}, \tau) - \Upsilon(\mathbf{x}, T)]\, d\tau.\end{aligned} \tag{2.6}$$

Let $q = T - \tau$. Then, the equation (2.6) becomes:

$$\begin{aligned}{}^{mr}D^{\varrho,\psi}\Upsilon(\mathbf{x}, T-\zeta) &= \frac{1}{\Gamma(1-\varrho)} \Big(\frac{-1}{\psi'(T-\zeta)} \frac{d}{d\zeta} \Big) \int_{0}^{\zeta} \psi'(T-q)[\psi(T-q) - \psi(T-\zeta)]^{-\varrho} \times \\ &\quad [\Upsilon(\mathbf{x}, T-q) - \Upsilon(\mathbf{x}, T)]\, dq.\end{aligned} \tag{2.7}$$

Let $\upsilon(\mathbf{x}, \zeta) = \Upsilon(\mathbf{x}, T-\zeta)$. Then, the equation (2.7) can be rewritten as follows:

$$\begin{aligned}{}^{mr}D^{\varrho,\psi}\upsilon(\mathbf{x}, \zeta) &= \frac{1}{\Gamma(1-\varrho)} \Big(\frac{-1}{\psi'(T-\zeta)} \frac{d}{d\zeta} \Big) \int_{0}^{\zeta} \psi'(T-q)[\psi(T-q) - \psi(T-\zeta)]^{-\varrho} \times \\ &\quad [\upsilon(\mathbf{x}, q) - \upsilon(\mathbf{x}, 0)]\, dq.\end{aligned} \tag{2.8}$$

Lemma 2.1. *Suppose that $\upsilon(\cdot, \zeta)$ is an absolutely continuous differentiable function with respect to ζ on $[0, T]$. Then, the MRψRLFD given in relation (2.8) satisfies:*

$${}^{mr}D^{\varrho,\psi}\upsilon(\mathbf{x}, \zeta) = -{}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}, \zeta),$$

where ${}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}, \zeta)$ is the ψCFD of order $\varrho \in (0,1)$ defined by:

$${}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}, \zeta) = \frac{1}{\Gamma(1-\varrho)} \int_{0}^{\zeta} [\psi(T-q) - \psi(T-\zeta)]^{-\varrho} \frac{\partial \upsilon(\mathbf{x}, q)}{\partial q}\, dq.$$

Proof. Let us assume that $\upsilon(\cdot,\zeta)$ is an absolutely continuous differentiable function with respect to ζ on the interval $\overline{B}_T$. Then, from the MRψRLFD defined in equation (2.8), we get:

$$\begin{aligned}
{}^{mr}D^{\varrho,\psi}\upsilon(\mathbf{x},\zeta) &= \frac{1}{\Gamma(1-\varrho)}\Big(\frac{-1}{\psi'(T-\zeta)}\frac{d}{d\zeta}\Big)\int_0^\zeta \psi'(T-q)[\psi(T-q)-\psi(T-\zeta)]^{-\varrho}\upsilon(\mathbf{x},q)\,dq - \\
&\quad \frac{\upsilon(\mathbf{x},0)}{\Gamma(1-\varrho)}\Big(\frac{-1}{\psi'(T-\zeta)}\frac{d}{d\zeta}\Big)\int_0^\zeta \psi'(T-q)[\psi(T-q)-\psi(T-\zeta)]^{-\varrho}\,dq, \\
&= \frac{1}{\Gamma(1-\varrho)}\Big(\frac{-1}{\psi'(T-\zeta)}\frac{d}{d\zeta}\Big)\int_0^\zeta \psi'(T-q)[\psi(T-q)-\psi(T-\zeta)]^{-\varrho}\upsilon(\mathbf{x},q)\,dq - \\
&\quad \frac{\upsilon(\mathbf{x},0)}{\Gamma(1-\varrho)}\Big(\frac{-1}{\psi'(T-\zeta)}\frac{d}{d\zeta}\Big)\Big[\frac{1}{1-\varrho}[\psi(T)-\psi(T-\zeta)]^{-\varrho+1}\Big], \\
&= \frac{1}{\Gamma(1-\varrho)}\Big(\frac{-1}{\psi'(T-\zeta)}\frac{d}{d\zeta}\Big)\int_0^\zeta \psi'(T-q)[\psi(T-q)-\psi(T-\zeta)]^{-\varrho}\upsilon(\mathbf{x},q)\,dq + \\
&\quad \frac{\upsilon(\mathbf{x},0)}{\Gamma(1-\varrho)}[\psi(T)-\psi(T-\zeta)]^{-\varrho}, \\
&= -\frac{1}{\Gamma(1-\varrho)}\int_0^\zeta [\psi(T-q)-\psi(T-\zeta)]^{-\varrho}\frac{\partial\upsilon(\mathbf{x},q)}{\partial q}\,dq, \\
&= -{}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x},\zeta),
\end{aligned}$$

where the operator ${}^{C}D^{\varrho,\psi}\upsilon(\cdot,\cdot)$ is the $\psi-$Caputo Fractional Derivative (ψCFD) of order $\varrho \in (0,1)$. $\square$

Thus, from system (2.1)-(2.4) and Lemma 2.1, we get the new generalized AOψCFDH model described by the following system:

$$\begin{aligned}
{}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x},\zeta) &= 0.5zS^2\frac{\partial^2\upsilon(\mathbf{x},\zeta)}{\partial S^2} + \rho\eta zS\frac{\partial^2\upsilon(\mathbf{x},\zeta)}{\partial S\partial z} + 0.5\eta^2 z\frac{\partial^2\upsilon(\mathbf{x},\zeta)}{\partial z^2} + \\
&\quad \mathbf{r}S\frac{\partial\upsilon(\mathbf{x},\zeta)}{\partial S} + a(b-z)\frac{\partial\upsilon(\mathbf{x},\zeta)}{\partial z} - \mathbf{r}\upsilon(\mathbf{x},\zeta) + f(K,S,\upsilon).
\end{aligned} \tag{2.9}$$

The initial condition at $t=0$ is given by:

$$\upsilon(\mathbf{x},0) = \max(0, K-S). \tag{2.10}$$

The boundary conditions are defined by:

$$\lim_{S\longmapsto 0}\upsilon(S,z,\zeta) = K, \quad \lim_{S\longmapsto\infty}\upsilon(S,z,\zeta) = 0, \tag{2.11}$$

$$\lim_{z\longmapsto 0}\upsilon(S,z,\zeta) = \lim_{z\longmapsto\infty}\upsilon(S,z,\zeta) = \max(0,K-S). \tag{2.12}$$

As mentioned above, the new ψCFD model (2.9)-(2.12) has never been studied before. It is for this reason that we have proposed a new numerical scheme for solving numerically this problem. Moreover, we adapted the well-known splitting technique as a relaxation method in solving this problem.

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3 Three-steps splitting method and discretization

Remark that the system (2.9)-(2.12) is posed on an infinite domain $(0, \infty) \times (0, \infty) \times B_T$. So, for the numerical study we need to consider a truncated bounded domain $\mathcal{D}$ as follows:

$$(\mathbf{x}, t) = (S, z, t) \in \mathcal{D} = (\ell_S, \Bbbk_S) \times (\ell_z, \Bbbk_z) \times B_T, \quad \text{where} \quad \ell_S, \ell_z, \Bbbk_S, \Bbbk_z > 0,$$

where the values ℓ_S and ℓ_z are chosen very close to 0, and those of $\Bbbk_S$ and $\Bbbk_z$ are very large.

Thus, the considered system on the bounded domain $\mathcal{D}$ is given by:

$$\begin{aligned}
{}^C D^{\varrho,\psi} \upsilon(\mathbf{x}, t) \quad = \quad & 0.5 z S^2 \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial S^2} + \gamma z S \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial S \partial z} + \varepsilon z \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial z^2} + \\
& \mathbf{r} S \frac{\partial \upsilon(\mathbf{x}, t)}{\partial S} + a(b - z) \frac{\partial \upsilon(\mathbf{x}, t)}{\partial z} - \mathbf{r} \upsilon(\mathbf{x}, t) + f(\mathbf{x}, t), && (3.13) \\
& \upsilon(\mathbf{x}, 0) = \hbar(\mathbf{x}), && (3.14) \\
& \upsilon(\ell_S, z, t) = \chi_0(z, t), \quad \upsilon(\Bbbk_S, z, t) = \chi_1(z, t), && (3.15) \\
& \upsilon(S, \ell_z, t) = \digamma_0(S, t), \quad \upsilon(S, \Bbbk_z, t) = \digamma_1(S, t), && (3.16)
\end{aligned}$$

for all $t \in [0, T]$, where the parameters γ and ε are given by:

$$\gamma = \rho \eta, \quad \varepsilon = 0.5 \eta^2.$$

Recall that the ψCFD is defined by:

$${}^C D^{\varrho,\psi} \upsilon(\mathbf{x}, t) = \frac{1}{\Gamma(1 - \varrho)} \int_0^t [\psi(T - q) - \psi(T - t)]^{-\varrho} \upsilon'(\mathbf{x}, q) \, dq, \quad \forall t \in [0, T], \tag{3.17}$$

where $\upsilon'(\mathbf{x}, q) = \frac{\partial \upsilon(\mathbf{x}, q)}{\partial q}$.

Let us divide the time domain $\overline{B}_T$ into $\mathcal{N}$ equal subintervals as follows:

$$[0, T] = \bigcup_{p=0}^{\mathcal{N}-1} [t^p, t^{p+1}],$$

where the $\mathcal{N} + 1$ equidistant points t^p are given by:

$$t^p = p\nu, \quad p = 0, \cdots, \mathcal{N},$$

where the time step ν is defined by:

$$\nu = T \mathcal{N}^{-1} = t^{p+1} - t^p, \quad \forall p = 0, \cdots, \mathcal{N}. \tag{3.18}$$

3.1 Three-steps splitting method

Remark that the equation (3.13) contains two different types of differential operators; an elliptic operator and a hyperbolic operator. Moreover, the equation contains a mixed differential term which is difficult to manipulate numerically. Thus, we proposed to apply the well-known splitting method in the following way:

- **Step:1–** On the time interval $[t^p, t^{p+1/3}]$, we solve the hyperbolic system:

$$ {}^C D^{\varrho,\psi} \upsilon(\mathbf{x}, t) = \gamma z S \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial S \partial z}. \tag{3.19} $$

- **Step:2–** On the time interval $[t^{p+1/3}, t^{p+2/3}]$, we solve the elliptic system:

$$ {}^C D^{\varrho,\psi} \upsilon(\mathbf{x}, t) = 0.5 z S^2 \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial S^2} + \mathbf{r} S \frac{\partial \upsilon(\mathbf{x}, t)}{\partial S} - \mathbf{r} \upsilon(\mathbf{x}, t). \tag{3.20} $$

- **Step:3–** On the time interval $[t^{p+2/3}, t^{p+1}]$, we solve the elliptic system:

$$ {}^C D^{\varrho,\psi} \upsilon(\mathbf{x}, t) = \varepsilon z \frac{\partial^2 \upsilon(\mathbf{x}, t)}{\partial z^2} + a(b - z) \frac{\partial \upsilon(\mathbf{x}, t)}{\partial z} + f(\mathbf{x}, t). \tag{3.21} $$

We solve the problems (3.19), (3.20) and (3.21) simultaneously on each interval $[t^p, t^{p+1}]$: the solution $\upsilon(\mathbf{x}, t^{p+1/3})$ to (3.19) is used as the initial condition to (3.20), the solution $\upsilon(\mathbf{x}, t^{p+2/3})$ to (3.20) is used as the initial condition to (3.21) and on the interval $[t^{p+1}, t^{p+4/3}]$ we solve again the problem (3.19) using the solution $\upsilon(\mathbf{x}, t^{p+1})$ to the problem (3.21) as an initial condition and so on until the final stage T. In this way each operator deals with the appropriate numerical scheme. The bibliographic research shows that this technique is very effective for this type of problem. Here we started in the first step of the splitting method by solving the hyperbolic problem defined by the mixed derivative $\frac{\partial^2 \upsilon(\mathbf{x},t)}{\partial S \partial z}$ because this term has known numerical drawbacks [31]. On the other hand, the elliptic operator in (3.20) and (3.21) is numerically more stable. Therefore, the fact of starting with the hyperbolic then the elliptic is a way of relaxation of the disturbances preventing from the hyperbolic part.

3.2 Discretization

For the spacial discretization, we divide every interval $[\ell_S, \Bbbk_S]$ and $[\ell_z, \Bbbk_z]$ into $\mathcal{M} \geq 3$ and $\mathcal{K} \geq 3$ equal subintervals, respectively. So, we get:

$$ [\ell_S, \Bbbk_S] = \bigcup_{m=0}^{\mathcal{M}-1} [S_m, S_{m+1}], \qquad [\ell_z, \Bbbk_z] = \bigcup_{n=0}^{\mathcal{K}-1} [z_n, z_{n+1}], $$

where the equidistant points S_m and z_n are defined by:

$$ \begin{aligned} S_m &= \ell_S + m\delta, \quad \delta = (\Bbbk_S - \ell_S)\mathcal{M}^{-1}, \quad m = 0, \cdots, \mathcal{M}, \\ z_n &= \ell_z + n\lambda, \quad \lambda = (\Bbbk_z - \ell_z)\mathcal{K}^{-1}, \quad n = 0, \cdots, \mathcal{K}. \end{aligned} $$

First, we begin by the approximation of the $\psi-$Caputo operator. At the point $(\mathbf{x}_{mn}, t^p) = (S_m, z_n, t^p)$, we have:

$$ \begin{aligned} {}^C D^{\varrho,\psi} \upsilon(\mathbf{x}_{mn}, t^p) &= \frac{1}{\Gamma(1-\varrho)} \int_0^{t^p} [\psi(T - q) - \psi(T - t^p)]^{-\varrho} \upsilon'(\mathbf{x}_{mn}, q)\, dq, \\ &= \frac{1}{\Gamma(1-\varrho)} \sum_{j=0}^{p-1} \int_{t^j}^{t^{j+1}} [\psi(T - q) - \psi(T - t^p)]^{-\varrho} \upsilon'(\mathbf{x}_{mn}, q)\, dq. \end{aligned} \tag{3.22} $$

Recall that the left rectangular rule for approximating the integral of a function $Q \in \mathcal{C}^1$ on the interval $[\xi, \zeta]$ is given by:

$$\int_{\xi}^{\zeta} Q(r)\,dr = (\zeta-\xi)Q(\xi) + E_2(Q), \quad \text{where} \quad E_2(Q) = (\zeta-\xi)^2 \sup_{s\in[\xi,\zeta]} (\frac{1}{2}|Q'(s)|).$$

So, on the interval $[t^j, t^{j+1}]$, we have:

$$\int_{t^j}^{t^{j+1}} Q(r)\,dr = \nu Q(t^j) + \mathcal{O}(\nu^2), \tag{3.23}$$

where ν is defined by (3.18). Consequently, by using the rule (3.23) in the equation (3.22), we get:

$$\begin{aligned} {}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}_{mn}, t^p) &= \frac{1}{\Gamma(1-\varrho)} \sum_{j=0}^{p-1} \nu[\psi(T-t^j) - \psi(T-t^p)]^{-\varrho}\upsilon'(\mathbf{x}_{mn}, t^j) + \mathcal{O}(\nu^2), \\ &= \frac{1}{\Gamma(1-\varrho)} \sum_{j=0}^{p-1} \nu[\psi(t^{\mathcal{N}-j}) - \psi(t^{\mathcal{N}-p})]^{-\varrho}\upsilon'(\mathbf{x}_{mn}, t^j) + \mathcal{O}(\nu^2). \end{aligned}$$

Let us consider the following approximation:

$$\upsilon'(\mathbf{x}_{mn}, t^j) = \frac{1}{\nu}\Big(\upsilon(\mathbf{x}_{mn}, t^{j+1}) - \upsilon(\mathbf{x}_{mn}, t^j)\Big) + \mathcal{O}(\nu).$$

Then, we get:

$${}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}_{mn}, t^p) = \frac{1}{\Gamma(1-\varrho)} \sum_{j=0}^{p-1} [\psi(t^{\mathcal{N}-j}) - \psi(t^{\mathcal{N}-p})]^{-\varrho}\Big(\upsilon(\mathbf{x}_{mn}, t^{j+1}) - \upsilon(\mathbf{x}_{mn}, t^j)\Big) + \mathcal{O}(\nu).$$

By considering the following notations:

$$\begin{aligned} \psi^j &= \psi(t^j), \quad \forall\, j = 0, \cdots, \mathcal{N}, \\ \upsilon^j_{m,n} &= \upsilon(\mathbf{x}_{mn}, t^j) = \upsilon(S_m, z_n, t^j), \quad \forall\, j = 0, \cdots, \mathcal{N}, \quad m = 0, \cdots, \mathcal{M}, \quad n = 0, \cdots, \mathcal{K}, \end{aligned}$$

we obtain:

$${}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x}_{mn}, t^p) = \frac{1}{\Gamma(1-\varrho)} \sum_{j=0}^{p-1} \Big(\upsilon^{j+1}_{m,n} - \upsilon^j_{m,n}\Big) c_{p,j} + \mathcal{O}(\nu), \tag{3.24}$$

where:

$$c_{p,j} = [\psi^{\mathcal{N}-j} - \psi^{\mathcal{N}-p}]^{-\varrho}, \quad \forall\, j = 0, \cdots, p-1.$$

Lemma 3.1. *The coefficients* $c_{p,j} = [\psi^{\mathcal{N}-j} - \psi^{\mathcal{N}-p}]^{-\varrho}$, $\forall j = 0, \cdots, p-1$, $p = 1, \cdots, \mathcal{N}$ *satisfy the following identities:*

1. *There exists two constants* $\mathbf{a}$, $\mathbf{b}$, $(\mathbf{a}, \mathbf{b} > 0)$, *such that:* $\mathbf{a}^{-\varrho} \leq c_{p,j} \leq \mathbf{b}^{-\varrho}$, *for all* $j = 0, \cdots, p-1$, $p = 1, \cdots, \mathcal{N}$.

2. $0 < c_{p,j} < c_{p,j+1}$, *for all* $j = 0, \cdots, p-2$.

3. $\sum_{j=1}^{p-1}(c_{p,j} - c_{p,j-1}) + c_{p,0} = c_{p,p-1}$, *for all* $p \geq 2$.

4. There exists a positive constant $M_0 \geq 0$ *such that:*

$$\sum_{j=1}^{p-1}(c_{p,j} - c_{p,j-1})c_{j,j-1}^{-1} \leq M_0. \tag{3.25}$$

Proof. 1. Recall that $c_{p,j}$, for every p, j, is defined by the function ψ that is strictly increasing and belongs to $\mathcal{C}^1$ on the bounded interval B_T. Since ψ is bounded on B_T, then $c_{p,j}$ is also bounded on B_T, for every p, j. Moreover, since ψ is strictly increasing on B_T, then we have $\psi^{\mathcal{N}-p} < \psi^{\mathcal{N}-j}$ for all $j < p$, and consequently $c_{p,j} > 0$. Thus, there exists two constants $\mathbf{a}$, $\mathbf{b} > 0$, such that: $\mathbf{a}^{-\varrho} \leq c_{p,j} \leq \mathbf{b}^{-\varrho}$, for all $j = 0, \cdots, p-1$, $p = 1, \cdots, \mathcal{N}$.

2. The sequence $(c_{p,j})_j$ is increasing. Indeed:

$$c_{p,j+1}^{\frac{1}{\varrho}} - c_{p,j}^{\frac{1}{\varrho}} = \frac{\psi^{\mathcal{N}-j} - \psi^{\mathcal{N}-(j+1)}}{(\psi^{\mathcal{N}-(j+1)} - \psi^{\mathcal{N}-p})(\psi^{\mathcal{N}-j} - \psi^{\mathcal{N}-p})} > 0,$$

for all $j = 0, \cdots, p-2$. Then, we get: $c_{p,j} < c_{p,j+1}$.

3. With simple calculation, we can prove the result.

4. Since we have a finite sum and the function $\psi \in \mathcal{C}^1(\overline{B}_T)$, then there exists a positive constant M_0 such that the identity (3.25) holds.

□

Now, we give the approximation of each spacial differential term in the different steps in the three-steps splitting method. At the point $(\mathbf{x}_{mn}, t^p) = (S_m, z_n, t^p)$, we have:

$$0.5 z_n S_m^2 \frac{\partial^2 \upsilon(\mathbf{x}_{mn}, t^p)}{\partial S^2} = \frac{0.5 z_n S_m^2}{\delta^2}(\upsilon_{m+1,n}^p - 2\upsilon_{m,n}^p + \upsilon_{m-1,n}^p) + \mathcal{O}(\delta^2), \tag{3.26}$$

$$\gamma z_n S_m \frac{\partial^2 \upsilon(\mathbf{x}_{mn}, t^p)}{\partial S \partial z} = \frac{\gamma z_n S_m}{4\delta\lambda}\Big(\upsilon_{m+1,n+1}^p - \upsilon_{m+1,n-1}^p - \upsilon_{m-1,n+1}^p + \upsilon_{m-1,n-1}^p\Big) + \mathcal{O}(\delta^2\lambda^2),$$

$$\varepsilon z_n \frac{\partial^2 \upsilon(\mathbf{x}_{mn}, t^p)}{\partial z^2} = \frac{\varepsilon z_n}{\lambda^2}(\upsilon_{m,n+1}^p - 2\upsilon_{m,n}^p + \upsilon_{m,n-1}^p) + \mathcal{O}(\lambda^2), \tag{3.27}$$

$$\mathbf{r} S_m \frac{\partial \upsilon(\mathbf{x}_{mn}, t^p)}{\partial S} = \frac{\mathbf{r} S_m}{\delta}(\upsilon_{m+1,n}^p - \upsilon_{m,n}^p) + \mathcal{O}(\delta), \tag{3.28}$$

$$a(b - z_n)\frac{\partial \upsilon(\mathbf{x}_{mn}, t^p)}{\partial z} = \frac{a(b - z_n)}{\lambda}(\upsilon_{m,n+1}^p - \upsilon_{m,n}^p) + \mathcal{O}(\lambda), \tag{3.29}$$

$$f(K, S_m, \upsilon(\mathbf{x}_{mn}, t^p)) = f_{m,n}^p, \tag{3.30}$$

$$\mathbf{r}\upsilon(\mathbf{x}_{mn}, t^p) = \mathbf{r}\upsilon_{m,n}^p. \tag{3.31}$$

- **Step:1**– The total discretization of the equation (3.19) is obtained as follows:

$$\begin{aligned} \frac{1}{\Gamma(1-\varrho)}\sum_{j=0}^{p-1}\Big(\upsilon_{m,n}^{j+1} - \upsilon_{m,n}^{j}\Big)c_{p,j} &= \frac{\gamma z_n S_m}{4\delta\lambda}\Big(\upsilon_{m+1,n+1}^p - \upsilon_{m+1,n-1}^p \\ &- \upsilon_{m-1,n+1}^p + \upsilon_{m-1,n-1}^p\Big) + \mathcal{O}(\delta^2\lambda^2 + \nu). \end{aligned} \tag{3.32}$$

The equation (3.32) can also be written in the form:

$$\sum_{j=0}^{p-1}\Big(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\Big)c_{p,j} = \omega_{mn}\upsilon_{m+1,n+1}^{p}-\omega_{mn}\upsilon_{m+1,n-1}^{p} - \omega_{mn}\upsilon_{m-1,n+1}^{p}+\omega_{mn}\upsilon_{m-1,n-1}^{p}+\mathcal{O}(\delta^2\lambda^2+\nu), \quad (3.33)$$

where the coefficient ω_{mn} in the equation (3.33) is given by:

$$d=\Gamma(1-\varrho), \quad \omega_{mn}=\frac{d\gamma z_n S_m}{4\delta\lambda}.$$

- For $p=1$. Then, from equation (3.33), we get:

$$\Big(\upsilon_{m,n}^{1}-\upsilon_{m,n}^{0}\Big)c_{1,0} = \omega_{mn}\upsilon_{m+1,n+1}^{1}-\omega_{mn}\upsilon_{m+1,n-1}^{1} - \omega_{mn}\upsilon_{m-1,n+1}^{1}+\omega_{mn}\upsilon_{m-1,n-1}^{1}+\mathcal{O}(\delta^2\lambda^2+\nu),$$

  or equivalently:

$$-c_{1,0}\upsilon_{m,n}^{0} = -c_{1,0}\upsilon_{m,n}^{1}+\omega_{mn}\upsilon_{m+1,n+1}^{1}-\omega_{mn}\upsilon_{m+1,n-1}^{1} - \omega_{mn}\upsilon_{m-1,n+1}^{1}+\omega_{mn}\upsilon_{m-1,n-1}^{1}+\mathcal{O}(\delta^2\lambda^2+\nu). \quad (3.34)$$

- For $p\geq 2$. Then, from equation (3.33), we get:

$$\sum_{j=1}^{p-1}\Big(c_{p,j-1}-c_{p,j}\Big)\upsilon_{m,n}^{j}-c_{p,0}\upsilon_{m,n}^{0} = -c_{p,p-1}\upsilon_{m,n}^{p}+\omega_{mn}\upsilon_{m+1,n+1}^{p}-\omega_{mn}\upsilon_{m+1,n-1}^{p} - \omega_{mn}\upsilon_{m-1,n+1}^{p}+\omega_{mn}\upsilon_{m-1,n-1}^{p}+\mathcal{O}(\delta^2\lambda^2+\nu),$$

with boundary conditions:

$$\begin{aligned}\upsilon_{0,n}^{p} &= \chi_0^{n,p}, \quad \upsilon_{\mathcal{M},n}^{p}=\chi_1^{n,p},\\ \upsilon_{m,0}^{p} &= F_0^{m,p}, \quad \upsilon_{m,\mathcal{K}}^{p}=F_1^{m,p}.\end{aligned}$$

Let $\widetilde{\upsilon}_{m,n}^{p}$ be an approximation to $\upsilon_{m,n}^{p}$. Then, we get the following system:

$$-c_{1,0}\widetilde{\upsilon}_{m,n}^{0} = -c_{1,0}\widetilde{\upsilon}_{m,n}^{1}+\omega_{mn}\widetilde{\upsilon}_{m+1,n+1}^{1} - \omega_{mn}\widetilde{\upsilon}_{m+1,n-1}^{1} - \omega_{mn}\widetilde{\upsilon}_{m-1,n+1}^{1}+\omega_{mn}\widetilde{\upsilon}_{m-1,n-1}^{1}, \quad (3.35)$$

$$\sum_{j=1}^{p-1}\Big(c_{p,j-1}-c_{p,j}\Big)\widetilde{\upsilon}_{m,n}^{j}-c_{p,0}\widetilde{\upsilon}_{m,n}^{0} = -c_{p,p-1}\widetilde{\upsilon}_{m,n}^{p}+\omega_{mn}\widetilde{\upsilon}_{m+1,n+1}^{p}-\omega_{mn}\widetilde{\upsilon}_{m+1,n-1}^{p} -\omega_{mn}\widetilde{\upsilon}_{m-1,n+1}^{p} + \omega_{mn}\widetilde{\upsilon}_{m-1,n-1}^{p}, \quad \forall p\geq 2, \quad (3.36)$$

with boundary conditions:

$$\widetilde{\upsilon}_{0,n}^{p} = \chi_0^{n,p}, \quad \widetilde{\upsilon}_{\mathcal{M},n}^{p}=\chi_1^{n,p}, \quad (3.37)$$

$$\widetilde{\upsilon}_{m,0}^{p} = F_0^{m,p}, \quad \widetilde{\upsilon}_{m,\mathcal{K}}^{p}=F_1^{m,p}. \quad (3.38)$$

- **Step:2–** The total discretization of the equation (3.20) is obtained as follows:

$$\begin{aligned}\frac{1}{\Gamma(1-\varrho)}\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\right)c_{p,j} &= \frac{0.5z_nS_m^2}{\delta^2}(\upsilon_{m+1,n}^{p}-2\upsilon_{m,n}^{p}+\upsilon_{m-1,n}^{p})\\ &+ \frac{\mathbf{r}S_m}{\delta}(\upsilon_{m+1,n}^{p}-\upsilon_{m,n}^{p})-\mathbf{r}\upsilon_{m,n}^{p}+\mathcal{O}(\delta+\nu).\end{aligned}$$

This equation can also be written in the form:

$$\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\right)c_{p,j} = \alpha_{mn}\upsilon_{m+1,n}^{p}-\beta_{mn}\upsilon_{m,n}^{p}+\gamma_{mn}\upsilon_{m-1,n}^{p}+\mathcal{O}(\delta+\nu), \tag{3.39}$$

where the coefficients in the equation (3.39) are given by:

$$\alpha_{mn}=\frac{0.5z_ndS_m^2}{\delta^2}+\frac{d\mathbf{r}S_m}{\delta},\quad \gamma_{mn}=\frac{0.5z_ndS_m^2}{\delta^2},\quad \beta_{mn}=\frac{z_ndS_m^2}{\delta^2}+d\mathbf{r}+\frac{d\mathbf{r}S_m}{\delta}.$$

- For $p=1$. Then, from equation (3.39), we get:

$$\left(\upsilon_{m,n}^{1}-\upsilon_{m,n}^{0}\right)c_{1,0}=\alpha_{mn}\upsilon_{m+1,n}^{1}-\beta_{mn}\upsilon_{m,n}^{1}+\gamma_{mn}\upsilon_{m-1,n}^{1}+\mathcal{O}(\delta+\nu),$$

  or equivalently:

$$-c_{1,0}\upsilon_{m,n}^{0}=\alpha_{mn}\upsilon_{m+1,n}^{1}-(\beta_{mn}+c_{1,0})\upsilon_{m,n}^{1}+\gamma_{mn}\upsilon_{m-1,n}^{1}+\mathcal{O}(\delta+\nu). \tag{3.40}$$

- For $p\geq 2$. Then, from equation (3.39), we get:

$$\begin{aligned}\sum_{j=1}^{p-1}\left(c_{p,j-1}-c_{p,j}\right)\upsilon_{m,n}^{j}-c_{p,0}\upsilon_{m,n}^{0} &= \alpha_{mn}\upsilon_{m+1,n}^{p}-(\beta_{mn}+c_{p,p-1})\upsilon_{m,n}^{p}\\ &+ \gamma_{mn}\upsilon_{m-1,n}^{p}+\mathcal{O}(\delta+\nu).\end{aligned}$$

Let $\widetilde{\upsilon}_{m,n}^{p}$ be an approximation to $\upsilon_{m,n}^{p}$. Then, we get the following system:

$$-c_{1,0}\widetilde{\upsilon}_{m,n}^{0}=\alpha_{mn}\widetilde{\upsilon}_{m+1,n}^{1} - (\beta_{mn}+c_{1,0})\widetilde{\upsilon}_{m,n}^{1}+\gamma_{mn}\widetilde{\upsilon}_{m-1,n}^{1}, \tag{3.41}$$

$$\begin{aligned}\sum_{j=1}^{p-1}\left(c_{p,j-1}-c_{p,j}\right)\widetilde{\upsilon}_{m,n}^{j}-c_{p,0}\widetilde{\upsilon}_{m,n}^{0} &= \alpha_{mn}\widetilde{\upsilon}_{m+1,n}^{p}-(\beta_{mn}+c_{p,p-1})\widetilde{\upsilon}_{m,n}^{p}\\ &+ \gamma_{mn}\widetilde{\upsilon}_{m-1,n}^{p},\quad \forall p\geq 2.\end{aligned} \tag{3.42}$$

- **Step:3–** The total discretization of the equation (3.21) is obtained as follows:

$$\begin{aligned}\frac{1}{\Gamma(1-\varrho)}\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1} - \upsilon_{m,n}^{j}\right)c_{p,j} &= \frac{\varepsilon z_n}{\lambda^2}(\upsilon_{m,n+1}^{p}-2\upsilon_{m,n}^{p}+\upsilon_{m,n-1}^{p})\\ &+ \frac{da(b-z_n)}{\lambda}(\upsilon_{m,n+1}^{p}-\upsilon_{m,n}^{p})+df_{m,n}^{p}+\mathcal{O}(\lambda+\nu).\end{aligned} \tag{3.43}$$

The equation (3.43) can also be rewritten in the form:

$$\begin{aligned}\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\right)c_{p,j} &= r_n\upsilon_{m,n+1}^{p}-q_n\upsilon_{m,n}^{p}+a_n\upsilon_{m,n-1}^{p}+\\ &\quad df_{m,n}^{p}+\mathcal{O}(\lambda+\nu),\end{aligned}\tag{3.44}$$

where the coefficients in the equation (3.44) are given by:

$$q_n=\frac{2d\varepsilon z_n}{\lambda^2}+\frac{da}{\lambda}(b-z_n),\quad r_n=\frac{d\varepsilon z_n}{\lambda^2}+\frac{da}{\lambda}(b-z_n),\quad a_n=\frac{d\varepsilon z_n}{\lambda^2}.$$

- For $p=1$. Then, from equation (3.44), we get:

  $$\left(\upsilon_{m,n}^{1}-\upsilon_{m,n}^{0}\right)c_{1,0}=r_n\upsilon_{m,n+1}^{1}-q_n\upsilon_{m,n}^{1}+a_n\upsilon_{m,n-1}^{1}+df_{m,n}^{1}+\mathcal{O}(\lambda+\nu),$$

  or equivalently:

  $$-c_{1,0}\upsilon_{m,n}^{0}=r_n\upsilon_{m,n+1}^{1}-(q_n+c_{1,0})\upsilon_{m,n}^{1}+a_n\upsilon_{m,n-1}^{1}+df_{m,n}^{1}+\mathcal{O}(\lambda+\nu).\tag{3.45}$$

- For $p\geq 2$. Then, from equation (3.44), we get:

  $$\begin{aligned}\sum_{j=1}^{p-1}\left(c_{p,j-1}-c_{p,j}\right)\upsilon_{m,n}^{j}-c_{p,0}\upsilon_{m,n}^{0} &= r_n\upsilon_{m,n+1}^{p}-(q_n+c_{p,p-1})\upsilon_{m,n}^{p}\\ &+ a_n\upsilon_{m,n-1}^{p}+df_{m,n}^{p}+\mathcal{O}(\lambda+\nu).\end{aligned}$$

Let $\widetilde{\upsilon}_{m,n}^{p}$ be an approximation to $\upsilon_{m,n}^{p}$. Then, we get the following system:

$$-c_{1,0}\widetilde{\upsilon}_{m,n}^{0}=r_n\widetilde{\upsilon}_{m,n+1}^{1}-(q_n+c_{1,0})\widetilde{\upsilon}_{m,n}^{1}+a_n\widetilde{\upsilon}_{m,n-1}^{1},\tag{3.46}$$

$$\begin{aligned}\sum_{j=1}^{p-1}\left(c_{p,j-1}-c_{p,j}\right)\widetilde{\upsilon}_{m,n}^{j}-c_{p,0}\widetilde{\upsilon}_{m,n}^{0} &= r_n\widetilde{\upsilon}_{m,n+1}^{p}-(q_n+c_{p,p-1})\widetilde{\upsilon}_{m,n}^{p}\\ &+ a_n\widetilde{\upsilon}_{m,n-1}^{p}+df_{m,n}^{p},\quad \forall p\geq 2.\end{aligned}\tag{3.47}$$

The stability of the NISM and the convergence of the NISM are studied in Section 4 and Section 5, respectively.

4 Stability analysis of the NISM

Let us denote by $\widehat{\upsilon}_{m,n}^{p}$ the numerical approximation to the solution $\widetilde{\upsilon}_{m,n}^{p}$. It's obvious that the states $\widehat{\upsilon}_{m,n}^{p}$ and $\widetilde{\upsilon}_{m,n}^{p}$ have the same boundary conditions, since they are the solutions of the same system.

Let $Æ_{m,n}^{p}$ be the state defined by:

$$Æ_{m,n}^{p}=\widetilde{\upsilon}_{m,n}^{p}-\widehat{\upsilon}_{m,n}^{p},\quad m=0,\cdots,\mathcal{M},\quad n=0,\cdots,\mathcal{K},\quad p=0,\cdots,\mathcal{N}.$$

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Consequently, we can deduce that the state $\AE^p_{m,n}$ has zero boundary conditions, since the states $\widehat{\upsilon}^p_{m,n}$ and $\widetilde{\upsilon}^p_{m,n}$ have the same boundary conditions. Therefore, we have:

$$\begin{aligned} \AE^p_{0,n} &= \AE^p_{\mathcal{M},n} = 0, && (4.48)\\ \AE^p_{m,0} &= \AE^p_{m,\mathcal{K}} = 0. && (4.49)\end{aligned}$$

Consider the grid function $\AE^p$, for $p = 0, \cdots, \mathcal{N}$, defined by:

$$\AE^p(S,z) = \\ \begin{cases} \AE^p_{m,n}, \quad (S,z) \in (S_{m-\frac{1}{2}}, S_{m+\frac{1}{2}}] \times (z_{n-\frac{1}{2}}, z_{n+\frac{1}{2}}], \quad m = 1, \cdots, \mathcal{M}, \quad n = 1, \cdots, \mathcal{K}, \\ \quad 0, \quad (S,z) \in \Big([\ell_S, \Bbbk_S] \times [\ell_z, \Bbbk_z]\Big) \setminus \Big([\ell_S + \frac{\delta}{2}, \Bbbk_S - \frac{\delta}{2}] \times [\ell_z + \frac{\lambda}{2}, \Bbbk_z - \frac{\lambda}{2}]\Big). \end{cases} \tag{4.50}$$

Taking into account conditions (4.48)-(4.49) and (4.50), we can make a periodic extension for $\AE^p_{m,n}$ on $[0, \mathcal{L}_S] \times [0, \mathcal{L}_z]$, where $\mathcal{L}_S = \Bbbk_S - \ell_S$ and $\mathcal{L}_z = \Bbbk_z - \ell_z$. Then, the function $\AE^p$ can be expanded into double Fourier series as follows:

$$\AE^p(S,z) = \sum_{m,n\in\mathbb{Z}} v^p_{m,n} e^{i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}, \quad (i^2 = -1).$$

where:

$$v^p_{m,n} = \frac{1}{\mathcal{L}_S\mathcal{L}_z} \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} \AE^p(S,z) e^{-i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}\, dzdS.$$

From Parseval identity, we deduce that:

$$\|\AE^p\|^2_{L^2} = \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} |\AE^p(S,z)|^2\, dzdS = \mathcal{L}_S\mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |v^p_{m,n}|^2.$$

Define the following norm:

$$\|\AE^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\AE^p_{m,n}|^2.$$

We conclude that:

$$\|\AE^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\AE^p_{m,n}|^2 = \mathcal{L}_S\mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |v^p_{m,n}|^2, \quad \forall p = 0, \cdots, \mathcal{N}. \tag{4.51}$$

Thus, let us assume that the solution $\AE^p_{m,n}$ can be written as follows:

$$\AE^p_{m,n} = \varphi^p e^{i(\Lambda_S S_m + \Lambda_z z_n)}, \qquad \Lambda_S = \frac{2\pi}{\mathcal{L}_S}, \quad \Lambda_z = \frac{2\pi}{\mathcal{L}_z}. \tag{4.52}$$

With the above new expression of $\AE^p_{m,n}$ given by (4.52), we get:

- **Step:1**– By substituting (4.52) in the system (3.35)-(3.36), we obtain:

$$\begin{aligned} -c_{1,0}\varphi^0 = \Big(- c_{1,0} + \omega_{mn} e^{i(\Lambda_S\delta + \Lambda_z\lambda)} &- \omega_{mn} e^{i(\Lambda_S\delta - \Lambda_z\lambda)} \\ &- \omega_{mn} e^{-i(\Lambda_S\delta - \Lambda_z\lambda)} + \omega_{mn} e^{-i(\Lambda_S\delta + \Lambda_z\lambda)}\Big)\varphi^1, \\ \sum_{j=1}^{p-1} \Big(c_{p,j-1} - c_{p,j}\Big)\varphi^j - c_{p,0}\varphi^0 &= \Big(- c_{p,p-1} + \omega_{mn} e^{i(\Lambda_S\delta + \Lambda_z\lambda)} - \omega_{mn} e^{i(\Lambda_S\delta - \Lambda_z\lambda)} \\ -\omega_{mn} e^{-i(\Lambda_S\delta - \Lambda_z\lambda)} &+ \omega_{mn} e^{-i(\Lambda_S\delta + \Lambda_z\lambda)}\Big)\varphi^p, \quad \forall p \geq 2. \end{aligned}$$

Which is equivalent to:

$$\Big[c_{1,0} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})\Big]\varphi^1 = c_{1,0}\varphi^0, \tag{4.53}$$

$$\Big[c_{p,p-1} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})\Big]\varphi^p = \sum_{j=1}^{p-1}\Big(c_{p,j} - c_{p,j-1}\Big)\varphi^j + c_{p,0}\varphi^0,\ \forall p \geq 2. \tag{4.54}$$

Let us notice that we have:

$$\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K}) > 0, \quad \forall\mathcal{M} \geq 3, \quad \forall\mathcal{K} \geq 3.$$

Lemma 4.1. *We have $|\varphi^p| < |\varphi^0|$, for all $p \geq 1$.*

Proof. For $p = 1$. From equation (4.53), we have:

$$|c_{1,0} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})| > c_{1,0},$$

then:

$$|\varphi^1| = \frac{c_{1,0}}{|c_{1,0} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})|}|\varphi^0| < |\varphi^0|.$$

Suppose that we have $|\varphi^p| < |\varphi^0|$, for $2 \leq p \leq k$. From equation (4.54) and Lemma 3.1, we have:

$$\begin{aligned}
\Big|c_{p+1,p} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})\Big||\varphi^{p+1}| &\leq \sum_{j=1}^{p}\Big(c_{p+1,j} - c_{p+1,j-1}\Big)|\varphi^j| + c_{p+1,0}|\varphi^0|,\\
&\leq \Big[\sum_{j=1}^{p}\Big(c_{p+1,j} - c_{p+1,j-1}\Big) + c_{p+1,0}\Big]|\varphi^0|,\\
&\leq c_{p+1,p}|\varphi^0|.
\end{aligned}$$

We deduce that:

$$|\varphi^{p+1}| < \frac{c_{p+1,p}|\varphi^0|}{\Big|c_{p+1,p} + 4\omega_{mn}\sin(2\pi/\mathcal{M})\sin(2\pi/\mathcal{K})\Big|} < |\varphi^0|.$$

Thus: $|\varphi^{p+1}| < |\varphi^0|$. □

Theorem 4.1. *The numerical implicit scheme (3.35)-(3.36) is unconditional stable.*

Proof. By using the relation (4.51) and the Lemma 4.1 we deduce that:

$$\|Æ^p\| \leq \|Æ^0\|, \quad p = 1, \cdots, \mathcal{N}.$$

Thus, the implicit scheme (3.35)-(3.36) is unconditional stable. □

- **Step:2–** By substituting (4.52) in the system (3.41)-(3.42), we obtain:

$$\begin{aligned} -c_{1,0}\varphi^0 &= \Big(\alpha_{mn}e^{i\Lambda_S\delta} - (\beta_{mn}+c_{1,0}) + \gamma_{mn}e^{-i\Lambda_S\delta}\Big)\varphi^1, \\ \sum_{j=1}^{p-1}\Big(c_{p,j-1}-c_{p,j}\Big)\varphi^j &- c_{p,0}\varphi^0 = \\ & \Big(\alpha_{mn}e^{i\Lambda_S\delta} - (\beta_{mn}+c_{p,p-1}) + \gamma_{mn}e^{-i\Lambda_S\delta}\Big)\varphi^p, \quad \forall p \geq 2. \end{aligned}$$

Which is equivalent to:

$$\begin{aligned} c_{1,0}\varphi^0 &= \Big[c_{1,0} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) \\ &- i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big]\varphi^1, \end{aligned} \tag{4.55}$$

$$\begin{aligned} \sum_{j=1}^{p-1}\Big(c_{p,j}-c_{p,j-1}\Big)\varphi^j &+ c_{p,0}\varphi^0 = \Big[c_{p,p-1} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) \\ &- i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big]\varphi^p, \quad \forall p \geq 2, \end{aligned} \tag{4.56}$$

where $D_m = \frac{d\mathbf{r}S_m}{\delta}$.

Lemma 4.2. *We have $|\varphi^p| < |\varphi^0|$, for all $p \geq 1$.*

Proof. For $p = 1$. From equation (4.55), we have:

$$\Big|c_{1,0} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) - i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big| > c_{1,0},$$

then:

$$|\varphi^1| = \frac{c_{1,0}|\varphi^0|}{\Big|c_{1,0} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) - i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big|} < |\varphi^0|.$$

Suppose that we have $|\varphi^p| < |\varphi^0|$, for $2 \leq p \leq k$. From equation (4.56) and Lemma 3.1, we have:

$$\begin{aligned} \Big|c_{p+1,p} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) &- i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big||\varphi^{p+1}| \\ &\leq \sum_{j=1}^{p}\Big(c_{p+1,j}-c_{p+1,j-1}\Big)|\varphi^j| + c_{p+1,0}|\varphi^0|, \\ &\leq \Big[\sum_{j=1}^{p}\Big(c_{p+1,j}-c_{p+1,j-1}\Big) + c_{p+1,0}\Big]|\varphi^0|, \\ &\leq c_{p+1,p}|\varphi^0|. \end{aligned}$$

Knowing that:

$$\Big|c_{p+1,p} + d\mathbf{r} + 2(2\gamma_{mn}+D_m)\sin^2(\Lambda_S\delta/2) - i2D_m\cos(\Lambda_S\delta/2)\sin(\Lambda_S\delta/2)\Big| > c_{p+1,p},$$

we deduce that: $|\varphi^{p+1}| < |\varphi^0|$. □

Theorem 4.2. *The numerical implicit scheme* (3.41)-(3.42) *is unconditional stable.*

Proof. By using the relation (4.51) and the Lemma 4.2 we deduce that:

$$\|Æ^p\| \leq \|Æ^0\|, \quad p = 1, \cdots, \mathcal{N}.$$

Thus, the implicit scheme (3.41)-(3.42) is unconditional stable. $\square$

- **Step:3–** By substituting (4.52) in the system (3.46)-(3.47), we obtain:

$$\begin{aligned} -c_{1,0}\varphi^0 &= \Big(r_n e^{i\Lambda_z\lambda} \quad - \quad (q_n + c_{1,0}) + a_n e^{-i\Lambda_z\lambda}\Big)\varphi^1, \\ \sum_{j=1}^{p-1}\Big(c_{p,j-1} - c_{p,j}\Big)\varphi^j - c_{p,0}\varphi^0 &= \Big(r_n e^{i\Lambda_z\lambda} - (q_n + c_{p,p-1}) + a_n e^{-i\Lambda_z\lambda}\Big)\varphi^p, \quad \forall p \geq 2. \end{aligned}$$

Which is equivalent to:

$$c_{1,0}\varphi^0 = \Big(c_{1,0} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) - i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big)\varphi^1, \tag{4.57}$$

$$\begin{aligned} &\sum_{j=1}^{p-1}\Big(c_{p,j} - c_{p,j-1}\Big)\varphi^j + c_{p,0}\varphi^0 = \Big(c_{p,p-1} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) \\ &-i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big)\varphi^p, \quad \forall p \geq 2, \end{aligned} \tag{4.58}$$

where $V_n = \frac{da}{\lambda}(b - z_n)$.

Lemma 4.3. *If the following identity holds:*

$$\lambda \leq \frac{\eta^2}{\kappa + \varsigma}, \tag{4.59}$$

then $|\varphi^p| < |\varphi^0|$, *for all* $p \geq 1$.

Proof. We have:

$$\begin{aligned} 2a_n + V_n &= 2\left(\frac{d\varepsilon z_n}{\lambda^2}\right) + \frac{da}{\lambda}(b - z_n), \\ &= \frac{da}{\lambda}b + \frac{d}{\lambda}z_n\left(\frac{2\varepsilon}{\lambda} - a\right), \\ &= \frac{da}{\lambda}b + \frac{d}{\lambda}z_n\left(\frac{\eta^2}{\lambda} - (\kappa + \varsigma)\right). \end{aligned}$$

Consequently, under the condition (4.59), we get:

$$2a_n + V_n \geq 0, \quad \forall n = 0, \cdots, \mathcal{K}.$$

For $p = 1$. From equation (4.57) and Lemma 3.1, we have:

$$\Big|c_{1,0} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) - i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big| > c_{1,0},$$

then:

$$|\varphi^1| = \frac{c_{1,0}|\varphi^0|}{\Big|c_{1,0} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) - i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big|} < |\varphi^0|.$$

Suppose that we have $|\varphi^p| < |\varphi^0|$, for $2 \leq p \leq k$. From equation (4.58) and Lemma 3.1, we have:

$$\begin{aligned}
\Big|c_{p+1,p} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) &- i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big||\varphi^{p+1}| \\
&\leq \sum_{j=1}^{p}\Big(c_{p+1,j} - c_{p+1,j-1}\Big)|\varphi^j| + c_{p+1,0}|\varphi^0|, \\
&\leq \Big[\sum_{j=1}^{p}\Big(c_{p+1,j} - c_{p+1,j-1}\Big) + c_{p+1,0}\Big]|\varphi^0|, \\
&\leq c_{p+1,p}|\varphi^0|.
\end{aligned}$$

Knowing that:

$$\Big|c_{p+1,p} + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) - i2V_n\cos(\Lambda_z\lambda/2)\sin(\Lambda_z\lambda/2)\Big| > c_{p+1,p},$$

we deduce that: $|\varphi^{p+1}| < |\varphi^0|$. □

Theorem 4.3. *If the identity* (4.59) *holds, then the numerical implicit scheme* (3.46)-(3.47) *is stable.*

Proof. By using the relation (4.51) and the Lemma 4.3 we deduce that:

$$\|Æ^p\| \leq \|Æ^0\|, \quad p = 1, \cdots, \mathcal{N}.$$

Thus, the implicit scheme (3.46)-(3.47) is stable. □

Theorem 4.4. *If the identity* (4.59) *holds, then the global numerical implicit scheme used for the problem* (3.13)-(3.16) *is stable.*

Proof. By using Theorem 4.1, Theorem 4.2 and Theorem 4.3, we deduce that the global numerical implicit scheme associated to problem (3.13)-(3.16) is conditionally stable about the initial condition under the condition (4.59). □

5 Global convergence of the NISM

In this section, we consider the global scheme for the problem (3.13)-(3.16). Using the approximations given in (3.24) and in the system (3.26)-(3.31), we get:

$$\begin{aligned}
\frac{1}{\Gamma(1-\varrho)}\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\right)c_{p,j} &= \frac{0.5z_nS_m^2}{\delta^2}(\upsilon_{m+1,n}^{p}-2\upsilon_{m,n}^{p}+\upsilon_{m-1,n}^{p})\\
&+ \frac{\gamma z_nS_m}{4\delta\lambda}\left(\upsilon_{m+1,n+1}^{p}-\upsilon_{m+1,n-1}^{p}-\upsilon_{m-1,n+1}^{p}+\upsilon_{m-1,n-1}^{p}\right)\\
&+ \frac{\varepsilon z_n}{\lambda^2}(\upsilon_{m,n+1}^{p}-2\upsilon_{m,n}^{p}+\upsilon_{m,n-1}^{p})+\frac{\mathbf{r}S_m}{\delta}(\upsilon_{m+1,n}^{p}-\upsilon_{m,n}^{p})\\
&+ \frac{a(b-z_n)}{\lambda}(\upsilon_{m,n+1}^{p}-\upsilon_{m,n}^{p})-\mathbf{r}\upsilon_{m,n}^{p}+f_{m,n}^{p}+\mathcal{O}(\delta+\lambda+\nu).
\end{aligned}$$

$$\begin{aligned}
\sum_{j=0}^{p-1}\left(\upsilon_{m,n}^{j+1}-\upsilon_{m,n}^{j}\right)c_{p,j} &= \alpha_{mn}\upsilon_{m+1,n}^{p}-(\beta_{mn}+q_n)\upsilon_{m,n}^{p}+\gamma_{mn}\upsilon_{m-1,n}^{p}+r_n\upsilon_{m,n+1}^{p}\\
&+ a_n\upsilon_{m,n-1}^{p}+\omega_{mn}\upsilon_{m+1,n+1}^{p}-\omega_{mn}\upsilon_{m+1,n-1}^{p}-\omega_{mn}\upsilon_{m-1,n+1}^{p}\\
&+ \omega_{mn}\upsilon_{m-1,n-1}^{p}+df_{m,n}^{p}+\mathcal{O}(\delta+\lambda+\nu).
\end{aligned}$$

- For $p=1$. We have:

$$\begin{aligned}
-\upsilon_{m,n}^{0}c_{1,0} &= \alpha_{mn}\upsilon_{m+1,n}^{1}-(c_{1,0}+\beta_{mn}+q_n)\upsilon_{m,n}^{1}+\gamma_{mn}\upsilon_{m-1,n}^{1}+r_n\upsilon_{m,n+1}^{1}\\
&+ a_n\upsilon_{m,n-1}^{1}+\omega_{mn}\upsilon_{m+1,n+1}^{1}-\omega_{mn}\upsilon_{m+1,n-1}^{1}-\omega_{mn}\upsilon_{m-1,n+1}^{1}\\
&+ \omega_{mn}\upsilon_{m-1,n-1}^{1}+df_{m,n}^{1}+\mathcal{O}(\delta+\lambda+\nu). \qquad (5.60)
\end{aligned}$$

- For $p\geq 2$. We have:

$$\begin{aligned}
\sum_{j=1}^{p-1}\left(c_{p,j-1}-c_{p,j}\right)\upsilon_{m,n}^{j} &- c_{p,0}\upsilon_{m,n}^{0}=\alpha_{mn}\upsilon_{m+1,n}^{p}-(c_{p,p-1}+\beta_{mn}+q_n)\upsilon_{m,n}^{p}\\
&+ \gamma_{mn}\upsilon_{m-1,n}^{p}+r_n\upsilon_{m,n+1}^{p}+a_n\upsilon_{m,n-1}^{p}\\
&+ \omega_{mn}\upsilon_{m+1,n+1}^{p}-\omega_{mn}\upsilon_{m+1,n-1}^{p}-\omega_{mn}\upsilon_{m-1,n+1}^{p}\\
&+ \omega_{mn}\upsilon_{m-1,n-1}^{p}+df_{m,n}^{p}+\mathcal{O}(\delta+\lambda+\nu). \qquad (5.61)
\end{aligned}$$

The boundary conditions:

$$\begin{aligned}
\upsilon_{m,n}^{0} &= \hbar_{m,n},\\
\upsilon_{0,n}^{p} &= \chi_0^{n,p}, \quad \upsilon_{\mathcal{M},n}^{p}=\chi_1^{n,p}, \qquad (5.62)\\
\upsilon_{m,0}^{p} &= F_0^{m,p}, \quad \upsilon_{m,\mathcal{K}}^{p}=F_1^{m,p}.
\end{aligned}$$

Let $\widetilde{\upsilon}^p_{m,n}$ be the approximate solution to the solution $\upsilon^p_{m,n}$ of the system (5.60)-(5.61)-(5.62). Then, we have:

$$\begin{aligned}
-\widetilde{\upsilon}^0_{m,n}c_{1,0} &= \alpha_{mn}\widetilde{\upsilon}^1_{m+1,n} - (c_{1,0}+\beta_{mn}+q_n)\widetilde{\upsilon}^1_{m,n} + \gamma_{mn}\widetilde{\upsilon}^1_{m-1,n} + r_n\widetilde{\upsilon}^1_{m,n+1} \\
&+ a_n\widetilde{\upsilon}^1_{m,n-1} + \omega_{mn}\widetilde{\upsilon}^1_{m+1,n+1} - \omega_{mn}\widetilde{\upsilon}^1_{m+1,n-1} - \omega_{mn}\widetilde{\upsilon}^1_{m-1,n+1} \\
&+ \omega_{mn}\widetilde{\upsilon}^1_{m-1,n-1} + df^1_{m,n}, \qquad (5.63)\\
\sum_{j=1}^{p-1}\Big(c_{p,j-1}-c_{p,j}\Big)\widetilde{\upsilon}^j_{m,n} &- c_{p,0}\widetilde{\upsilon}^0_{m,n} = \alpha_{mn}\widetilde{\upsilon}^p_{m+1,n} - (c_{p,p-1}+\beta_{mn}+q_n)\widetilde{\upsilon}^p_{m,n} \\
&+ \gamma_{mn}\widetilde{\upsilon}^p_{m-1,n} + r_n\widetilde{\upsilon}^p_{m,n+1} + a_n\widetilde{\upsilon}^p_{m,n-1} \\
&+ \omega_{mn}\widetilde{\upsilon}^p_{m+1,n+1} - \omega_{mn}\widetilde{\upsilon}^p_{m+1,n-1} - \omega_{mn}\widetilde{\upsilon}^p_{m-1,n+1} \\
&+ \omega_{mn}\widetilde{\upsilon}^p_{m-1,n-1} + df^p_{m,n}. \qquad (5.64)
\end{aligned}$$

The boundary conditions:

$$\begin{aligned}
\widetilde{\upsilon}^0_{m,n} &= \hbar_{m,n}, \\
\widetilde{\upsilon}^p_{0,n} &= \chi_0^{n,p}, \quad \widetilde{\upsilon}^p_{\mathcal{M},n} = \chi_1^{n,p}, \qquad (5.65)\\
\widetilde{\upsilon}^p_{m,0} &= F_0^{m,p}, \quad \widetilde{\upsilon}^p_{m,\mathcal{K}} = F_1^{m,p}.
\end{aligned}$$

Let us denote by $\upsilon^p = (\upsilon^p_{m,n})_{m,n}$ and $\widetilde{\upsilon}^p = (\widetilde{\upsilon}^p_{m,n})_{m,n}$. We introduce the error state $\mathbb{E}^p = (\mathbb{E}^p_{m,n})_{m,n}$ defined by:

$$\mathbb{E}^p = \upsilon^p - \widetilde{\upsilon}^p, \quad \forall p = 0,\cdots,\mathcal{N}.$$

or equivalently:

$$\mathbb{E}^p_{m,n} = \upsilon^p_{m,n} - \widetilde{\upsilon}^p_{m,n}, \quad \forall m = 0,\cdots,\mathcal{M}, \quad \forall n = 0,\cdots,\mathcal{K}, \quad \forall p = 0,\cdots,\mathcal{N}.$$

Then, we deduce that $\mathbb{E}^p_{m,n}$ is a solution to the following system:

$$\begin{aligned}
-\mathbb{E}^0_{m,n}c_{1,0} &= \alpha_{mn}\mathbb{E}^1_{m+1,n} - (c_{1,0}+\beta_{mn}+q_n)\mathbb{E}^1_{m,n} + \gamma_{mn}\mathbb{E}^1_{m-1,n} + r_n\mathbb{E}^1_{m,n+1} \\
&+ a_n\mathbb{E}^1_{m,n-1} + \omega_{mn}\mathbb{E}^1_{m+1,n+1} - \omega_{mn}\mathbb{E}^1_{m+1,n-1} - \omega_{mn}\mathbb{E}^1_{m-1,n+1} \\
&+ \omega_{mn}\mathbb{E}^1_{m-1,n-1} + \mathcal{R}^1_{m,n}. \qquad (5.66)\\
\sum_{j=1}^{p-1}\Big(c_{p,j-1}-c_{p,j}\Big)\mathbb{E}^j_{m,n} &- c_{p,0}\mathbb{E}^0_{m,n} = \alpha_{mn}\mathbb{E}^p_{m+1,n} - (c_{p,p-1}+\beta_{mn}+q_n)\mathbb{E}^p_{m,n} \\
&+ \gamma_{mn}\mathbb{E}^p_{m-1,n} + r_n\mathbb{E}^p_{m,n+1} + a_n\mathbb{E}^p_{m,n-1} \\
&+ \omega_{mn}\mathbb{E}^p_{m+1,n+1} - \omega_{mn}\mathbb{E}^p_{m+1,n-1} - \omega_{mn}\mathbb{E}^p_{m-1,n+1} \\
&+ \omega_{mn}\mathbb{E}^p_{m-1,n-1} + \mathcal{R}^p_{m,n}. \qquad (5.67)
\end{aligned}$$

where:

$$\mathcal{R}^p_{m,n} = \mathcal{O}(\delta+\lambda+\nu), \quad \forall m = 0,\cdots,\mathcal{M}, \quad \forall n = 0,\cdots,\mathcal{K}, \quad \forall p = 0,\cdots,\mathcal{N}.$$

The initial boundary are given by:

$$\begin{array}{rcl} \mathbb{E}^0_{m,n} &=& 0, \\ \mathbb{E}^p_{0,n} &=& 0, \quad \mathbb{E}^p_{\mathcal{M},n} = 0, \\ \mathbb{E}^p_{m,0} &=& \mathbb{E}^p_{m,\mathcal{K}} = 0, \end{array} \tag{5.68}$$

Let $\mathbb{E}^p$ and $\mathcal{R}^p$ be two grid functions defined as follows:

$$\mathbb{E}^p(S,z) = \begin{cases} \mathbb{E}^p_{m,n}, & (S,z) \in (S_{m-\frac{1}{2}}, S_{m+\frac{1}{2}}] \times (z_{n-\frac{1}{2}}, z_{n+\frac{1}{2}}], \quad m = 1, \cdots, \mathcal{M}, \quad n = 1, \cdots, \mathcal{K}, \\ 0, & (S,z) \in \Big([\ell_S, \Bbbk_S] \times [\ell_z, \Bbbk_z]\Big) \setminus \Big([\ell_S + \frac{\delta}{2}, \Bbbk_S - \frac{\delta}{2}] \times [\ell_z + \frac{\lambda}{2}, \Bbbk_z - \frac{\lambda}{2}]\Big). \end{cases}$$

$$\mathcal{R}^p(S,z) = \begin{cases} \mathcal{R}^p_{m,n}, & (S,z) \in (S_{m-\frac{1}{2}}, S_{m+\frac{1}{2}}] \times (z_{n-\frac{1}{2}}, z_{n+\frac{1}{2}}], \quad m = 1, \cdots, \mathcal{M}, \quad n = 1, \cdots, \mathcal{K}, \\ 0, & (S,z) \in \Big([\ell_S, \Bbbk_S] \times [\ell_z, \Bbbk_z]\Big) \setminus \Big([\ell_S + \frac{\delta}{2}, \Bbbk_S - \frac{\delta}{2}] \times [\ell_z + \frac{\lambda}{2}, \Bbbk_z - \frac{\lambda}{2}]\Big). \end{cases}$$

The functions $\mathbb{E}^p(S,z)$ and $\mathcal{R}^p(S,z)$ have the following double Fourier series:

$$\begin{array}{rcl} \mathbb{E}^p(S,z) &=& \displaystyle\sum_{m,n\in\mathbb{Z}} \mathbb{E}^p_{m,n} e^{i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}, \\ \mathcal{R}^p(S,z) &=& \displaystyle\sum_{m,n\in\mathbb{Z}} \mathcal{R}^p_{m,n} e^{i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}, \end{array}$$

where ($i^2 = -1$) and $\mathbb{E}^p_{m,n}$, $\mathcal{R}^p_{m,n}$ are defined as follows:

$$\begin{array}{rcl} \mathbb{E}^p_{m,n} &=& \displaystyle\frac{1}{\mathcal{L}_S \mathcal{L}_z} \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} \mathbb{E}^p(S,z) e^{-i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}\, dz dS, \\ \mathcal{R}^p_{m,n} &=& \displaystyle\frac{1}{\mathcal{L}_S \mathcal{L}_z} \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} \mathcal{R}^p(S,z) e^{-i2\pi(\frac{mS}{\mathcal{L}_S} + \frac{nz}{\mathcal{L}_z})}\, dz dS. \end{array}$$

From Parseval identity, we deduce that:

$$\|\mathbb{E}^p\|^2_{L^2} = \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} |\mathbb{E}^p(S,z)|^2 \, dz dS = \mathcal{L}_S \mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |\mathbb{E}^p_{m,n}|^2,$$

$$\|\mathcal{R}^p\|^2_{L^2} = \int_0^{\mathcal{L}_S} \int_0^{\mathcal{L}_z} |\mathcal{R}^p(S,z)|^2 \, dz dS = \mathcal{L}_S \mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |\mathcal{R}^p_{m,n}|^2.$$

We define the following two norms by:

$$\|\mathbb{E}^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\mathbb{E}^p_{m,n}|^2, \qquad \|\mathcal{R}^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\mathcal{R}^p_{m,n}|^2. \tag{5.69}$$

We conclude, as in the previous section, that: $\forall p = 0, \cdots, \mathcal{N}$

$$\|\mathbb{E}^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\mathbb{E}^p_{m,n}|^2 = \mathcal{L}_S \mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |\mathbb{E}^p_{m,n}|^2, \tag{5.70}$$

$$\|\mathcal{R}^p\|^2 = \sum_{m=1}^{\mathcal{M}-1} \sum_{n=1}^{\mathcal{K}-1} \delta\lambda |\mathcal{R}^p_{m,n}|^2 = \mathcal{L}_S \mathcal{L}_z \sum_{m,n\in\mathbb{Z}} |\mathcal{R}^p_{m,n}|^2. \tag{5.71}$$

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Thus, let us assume that $\mathbb{E}^p_{m,n}$ and $\mathcal{R}^p_{m,n}$ can be written as follows:

$$\mathbb{E}^p_{m,n} = \theta^p e^{i(\Lambda_S S_m + \Lambda_z z_n)}, \tag{5.72}$$
$$\mathcal{R}^p_{m,n} = \varphi^p e^{i(\Lambda_S S_m + \Lambda_z z_n)}, \tag{5.73}$$

where $\Lambda_S = \frac{2\pi}{\mathcal{L}_S}$, $\Lambda_z = \frac{2\pi}{\mathcal{L}_z}$. By substituting (5.72) and (5.73) in (5.66)-(5.67) and under the conditions (5.68), we get:

$$\begin{aligned}
-\varphi^1 &= \Big[\alpha_{mn} e^{i\Lambda_S\delta} - (c_{1,0} + \beta_{mn} + q_n) + \gamma_{mn} e^{-i\Lambda_S\delta} + r_n e^{i\Lambda_z\lambda} \\
&+ a_n e^{-i\Lambda_z\lambda} + \omega_{mn} e^{i(\Lambda_S\delta + \Lambda_z\lambda)} - \omega_{mn} e^{i(\Lambda_S\delta - \Lambda_z\lambda)} - \omega_{mn} e^{-i(\Lambda_S\delta - \Lambda_z\lambda)} \\
&+ \omega_{mn} e^{-i(\Lambda_S\delta + \Lambda_z\lambda)}\Big]\theta^1, \qquad (5.74)\\
\sum_{j=1}^{p-1}\Big(c_{p,j-1} - c_{p,j}\Big)\theta^j &= \Big[\alpha_{mn} e^{i\Lambda_S\delta} - (c_{p,p-1} + \beta_{mn} + q_n) \\
&+ \gamma_{mn} e^{-i\Lambda_S\delta} + r_n e^{i\Lambda_z\lambda} + a_n e^{-i\Lambda_z\lambda} \\
&+ \omega_{mn} e^{i(\Lambda_S\delta + \Lambda_z\lambda)} - \omega_{mn} e^{i(\Lambda_S\delta - \Lambda_z\lambda)} - \omega_{mn} e^{-i(\Lambda_S\delta - \Lambda_z\lambda)} \\
&+ \omega_{mn} e^{-i(\Lambda_S\delta + \Lambda_z\lambda)}\Big]\theta^p + \varphi^p. \qquad (5.75)
\end{aligned}$$

Using the fact that:

$$e^{ic} - 2 + e^{-ic} = -4\sin^2(c/2),$$

the system (5.74)-(5.75) can be rewritten as follows:

$$\begin{aligned}
\Big[c_{1,0} + W_{m,n} - iP_{m,n}\Big]\theta^1 &= \varphi^1, \qquad (5.76)\\
\Big[c_{p,p-1} + W_{m,n} - iP_{m,n}\Big]\theta^p &= \sum_{j=1}^{p-1}\Big(c_{p,j} - c_{p,j-1}\Big)\theta^j + \varphi^p, \qquad (5.77)
\end{aligned}$$

where:

$$\begin{aligned}
W_{m,n} &= 2(2\gamma_{mn} + D_m)\sin^2(\Lambda_S\delta/2) + 2(2a_n + V_n)\sin^2(\Lambda_z\lambda/2) + d\mathbf{r}.\\
P_{m,n} &= 2V_n\cos^2(\Lambda_z\lambda/2)\sin^2(\Lambda_z\lambda/2) + 2D_m\cos^2(\Lambda_S\delta/2)\sin^2(\Lambda_S\delta/2).
\end{aligned}$$

Notice that if the identity (4.59) holds, then $W_{m,n} \geq 0$.

Proposition 5.1. *Assume that θ^p verifies the system* (5.74)-(5.75)*. Then, under the condition* (4.59)*, there exists a positive constant k_0:*

$$|\theta^p| \leq k_0 c_{p,p-1}^{-1}|\varphi^1|, \quad \forall p = 1, \cdots, \mathcal{N}.$$

Proof. Knowing that:

$$\mathcal{R}^p_{m,n} = \mathcal{O}(\delta + \lambda + \nu), \quad \forall m = 0, \cdots, \mathcal{M}, \quad \forall n = 0, \cdots, \mathcal{K}, \quad \forall p = 0, \cdots, \mathcal{N},$$

then there exists a positive constant k_1:

$$|\mathcal{R}_{m,n}^p| \leq k_1(\delta+\lambda+\nu), \quad \forall m=0,\cdots,\mathcal{M}, \quad \forall n=0,\cdots,\mathcal{K}, \quad \forall p=0,\cdots,\mathcal{N}.$$

By using the relation (5.71), we deduce that:

$$\|\mathcal{R}^p\| \leq k_1\sqrt{\mathcal{L}_S\mathcal{L}_z}(\delta+\lambda+\nu). \tag{5.78}$$

Also, from relation (5.71), the series in the right converges. Thus, the term $|\mathcal{R}_{m,n}^p|$ tends to zero when m,n tens to ∞. Consequently: there exists a positive constant k_3 such that:

$$|\varphi^p| = |\mathcal{R}_{m,n}^p| \leq k_3|\mathcal{R}_{m,n}^1| = k_3|\varphi^1|, \quad \forall p=2,\cdots,\mathcal{N}.$$

From equation (5.76), we have:

$$|\theta^1| = \frac{|\varphi^1|}{\left|c_{1,0}+W_{m,n}-iP_{m,n}\right|} \leq \frac{|\varphi^1|}{c_{1,0}} = c_{1,0}^{-1}|\varphi^1|.$$

Assume that $|\theta^p| \leq k_4 c_{p,p-1}^{-1}|\varphi^1|$. Then, from equation (5.77), we have:

$$\begin{aligned}
|\theta^{p+1}| &\leq \frac{\sum_{j=1}^{p}\Big(c_{p+1,j}-c_{p+1,j-1}\Big)|\theta^j|}{\left|c_{p+1,p}+W_{m,n}-iP_{m,n}\right|} + \frac{|\varphi^p|}{\left|c_{p+1,p}+W_{m,n}-iP_{m,n}\right|}, \\
&\leq \frac{\sum_{j=1}^{p}\Big(c_{p+1,j}-c_{p+1,j-1}\Big)k_4c_{j,j-1}^{-1}|\varphi^1|}{c_{p+1,p}} + \frac{k_3|\varphi^1|}{c_{p+1,p}}, \\
&\leq \Big(k_5c_{p+1,p}^{-1}|\varphi^1|\Big)\Big(\sum_{j=1}^{p}\Big(c_{p+1,j}-c_{p+1,j-1}\Big)c_{j,j-1}^{-1}+1\Big).
\end{aligned}$$

Using Lemma 3.1, we obtain:

$$|\theta^{p+1}| \leq (k_5c_{p+1,p}^{-1}|\varphi^1|)(M_0+1) = k_0c_{p+1,p}^{-1}|\varphi^1|.$$

$\square$

Theorem 5.1. *Under the condition* (4.59)*, the implicit scheme* (5.63)-(5.65) *converges. Moreover, we have the following identity:*

$$\|\upsilon^p - \widetilde{\upsilon}^p\| \leq q\mathbf{a}^{\varrho}(\delta+\lambda+\nu), \quad \forall p=1,\cdots,\mathcal{N},$$

where q is a positive constant and $\mathbf{a}$ is given Lemma 3.1.

Proof. From Proposition 5.1, we have:

$$|\theta^p| \leq k_0c_{p,p-1}^{-1}|\varphi^1|, \quad \forall p=1,\cdots,\mathcal{N}.$$

Using Lemma 3.1, we get:

$$|\theta^p| \leq k_0\mathbf{a}^{\varrho}|\varphi^1|, \quad \forall p=1,\cdots,\mathcal{N}.$$

Now, by using the relations (5.70)-(5.71), (5.72)-(5.73) and (5.78), we obtain:

$$\|\mathbb{E}^p\| \leq k_0\mathbf{a}^{\varrho}\|\mathcal{R}^p\| \leq (k_0k_1\sqrt{\mathcal{L}_S\mathcal{L}_z})\mathbf{a}^{\varrho}(\delta+\lambda+\nu) = q\mathbf{a}^{\varrho}(\delta+\lambda+\nu),$$

where $q = k_0k_1\sqrt{\mathcal{L}_S\mathcal{L}_z}$. $\square$

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6 Numerical simulation

In Example 6.1, we validate both our AOψCFDH model and NISM proposed in this work. We consider an exact solution and thanks to a very detailed study of the relative and absolute errors, we show the efficiency of our proposed numerical scheme.
In Example 6.2, we apply our numerical scheme for an American Options problem under $\psi-$Caputo Fractional-Order Derivative Heston Model. By comparison with the results obtained in [31], we note that our results obtained in this work are very satisfactory and in excellent agreement with those of [31].

Example 6.1. *The considered system is given by:*

$$\begin{aligned}
{}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x},t) &= 0.5zS^2\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial S^2}+\gamma zS\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial S\partial z}+\varepsilon z\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial z^2}+\\
&\quad \mathbf{r}S\frac{\partial\upsilon(\mathbf{x},t)}{\partial S}+a(b-z)\frac{\partial\upsilon(\mathbf{x},t)}{\partial z}-\mathbf{r}\upsilon(\mathbf{x},t)+f(\mathbf{x},t), && (6.79)\\
&\quad \upsilon(\mathbf{x},0)=(S-S^2)(z-z^2)+0.8, && (6.80)\\
&\quad \upsilon(\ell_S,z,t)=0.8(1+t)^2,\quad \upsilon(\Bbbk_S,z,t)=0.8(1+t)^2, && (6.81)\\
&\quad \upsilon(S,\ell_z,t)=0.8(1+t)^2,\quad \upsilon(S,\Bbbk_z,t)=0.8(1+t)^2, && (6.82)
\end{aligned}$$

where $\mathbf{x}=(S,z)$ *and the parameters* $\gamma=\rho\eta$, $\varepsilon=0.5\eta^2$, $a=\kappa+\varsigma$ *and* $b=\frac{\kappa\theta}{a}$. *In this experience the exact solution of the problem* (6.79)-(6.82) *is given by* $\upsilon(\mathbf{x},t)=[(S-S^2)(z-z^2)+0.8](1+t)^2$ *and the function* $\psi(t)=t$. *Consequently, the source term is as the following form:*

$$\begin{aligned}
f(\mathbf{x},t) &= \frac{2[(S-S^2)(z-z^2)+0.8]}{\Gamma(2-\varrho)}\Big[t^{1-\varrho}+\frac{t^{2-\varrho}}{2-\varrho}\Big]+(1+t)^2\Big[z(z-z^2)S^2-\\
&\quad \gamma zS(1-2z)(1-2S)+2\varepsilon z(S-S^2)-rS(1-2S)(z-z^2)-\\
&\quad a(b-z)(1-2z)(S-S^2)+r((S-S^2)(z-z^2)+0.8)\Big].
\end{aligned}$$

The data of the simulation are as follows:

$$\begin{aligned}
&T=1,\ \mathcal{M}=\mathcal{K}=20,\ [\ell_S,\Bbbk_S]=[\ell_z,\Bbbk_z]=[0,1],\ \varrho=0.9,\\
&\rho=0.01,\ \eta=11,\ r=0.1,\ \kappa=5.1,\ \theta=0.1,\ \varsigma=0.3.
\end{aligned}$$

The initial condition, (the solution at $t=0$*), was plotted in Figure 1. In Figure 2, we have plotted the exact solution and the numerical solution. In Table 1, the relative error, the absolute error and the order of convergence are calculated. Recall that the relative error* E_r *and the absolute error* E_a *are given by:*

$$E_r=\max\Big(\frac{|\upsilon_e-\upsilon_a|}{|\upsilon_e|}\Big),\qquad E_a=\max\Big(|\upsilon_e-\upsilon_a|\Big),$$

where υ_e *is the exact solution and* υ_a *is an approximation of* υ_e. *In general, the order of convergence is determined by:*

$$Order=\log_{\frac{\Delta_1}{\Delta_2}}\Big(\frac{Error(\Delta_1)}{Error(\Delta_2)}\Big),$$

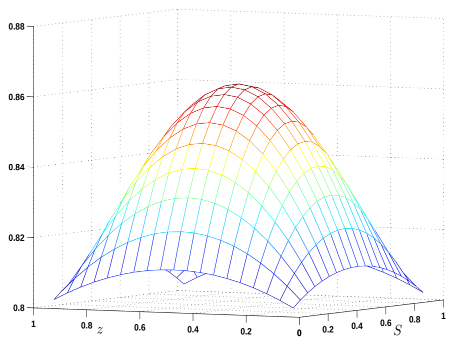

Figure 1: The initial condition, (the solution at $t = 0$).

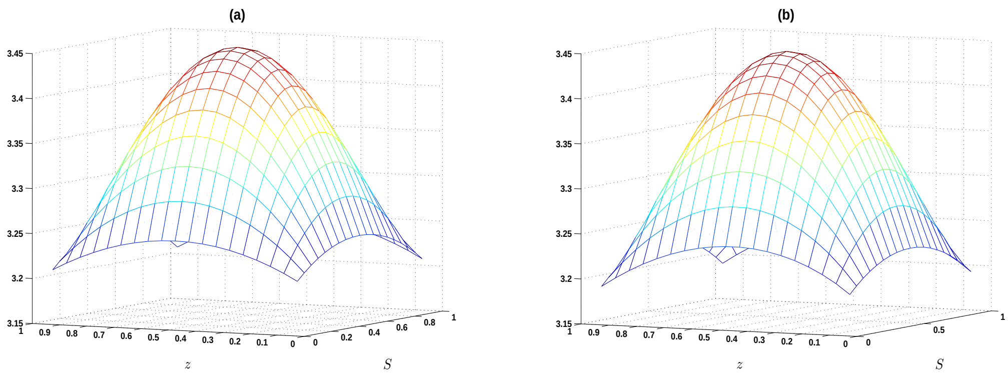

Figure 2: (a) Exact solution, (b) numerical solution.

where Δ_i*,* $i = 1, 2$*, are the temporal steps size. Table 1 clearly shows that the order of convergence in time of the scheme is equal* 1*. In fact, this result was predicted from Theorem 5.1. From Figure 2, Figure 3 and Table 1, it is clear that the numerical solution obtained by the numerical implicit scheme is in excellent consistency with the analytical solution.*

Example 6.2. *The considered system is an American Options problem under* $\psi-$*Caputo Fractional-*

Δt	E_r	Order	E_a	Order
$1/80$	11.80×10^{-3}		3.77×10^{-2}	
$1/100$	9.30×10^{-3}	1.066	2.98×10^{-2}	1.053
$1/130$	7.00×10^{-3}	1.082	2.25×10^{-2}	1.071
$1/140$	6.47×10^{-3}	1.062	2.08×10^{-2}	1.060
$1/160$	5.60×10^{-3}	1.116	1.79×10^{-2}	1.124

Table 1: Relative error, absolute error and order of convergence in time.

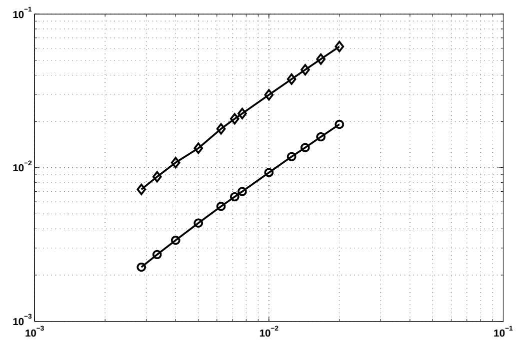

Figure 3: The errors plotted against Δt on a log-log scale.

Order Derivative Heston Model given by:

$$\begin{aligned} {}^{C}D^{\varrho,\psi}\upsilon(\mathbf{x},t) &= 0.5zS^2\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial S^2} + \gamma zS\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial S\partial z} + \varepsilon z\frac{\partial^2\upsilon(\mathbf{x},t)}{\partial z^2} + \\ & \mathbf{r}S\frac{\partial\upsilon(\mathbf{x},t)}{\partial S} + a(b-z)\frac{\partial\upsilon(\mathbf{x},t)}{\partial z} - \mathbf{r}\upsilon(\mathbf{x},t) + \Sigma f(K,S,\upsilon), \quad (6.83) \\ & \upsilon(\mathbf{x},0) = \max(0, K-S), \quad (6.84) \\ & \upsilon(\ell_S, z, t) = K, \quad \upsilon(\Bbbk_S, z, t) = 0, \quad (6.85) \\ & \upsilon(S,\ell_z,t) = \upsilon(S,\Bbbk_z,t) = \max(0, K-S), \quad (6.86) \end{aligned}$$

where $\mathbf{x} = (S, z)$ *and the parameters* $\gamma = \rho\eta$, $\varepsilon = 0.5\eta^2$, $a = \kappa+\varsigma$ *and* $b = \frac{\kappa\theta}{a}$. *In this experience, the source term is as the following form:*

$$f(K,S,\upsilon) = \max(K-S-\upsilon, 0).$$

The data of the simulation are as follows:

$$T = 0.25,\ \mathcal{N} = 150,\ \mathcal{M} = \mathcal{K} = 24,\ [\ell_S, \Bbbk_S] = [0.25, 40],\ [\ell_z, \Bbbk_z] = [0.002, 1.2],\ \varrho = 0.9,$$
$$\rho = 0.9,\ \eta = 0.9,\ r = 0.9,\ \kappa = 5,\ \theta = 3.5,\ \varsigma = 0.3,\ \Sigma = 100.$$

Let us notice that in Figure 4 the numerical solution is unstable because the stability condition (4.59) *is violated. Indeed, for* $\eta = 0.1$ *and* $\kappa = 0.2$, *we have* $\lambda = 0.0499 \not\leq \frac{\eta^2}{\kappa+\varsigma} = 0.02$.
In all Figures 5-10, the stability condition (4.59) *is satisfied. Indeed, we have* $\lambda = 0.0499 \leq \frac{\eta^2}{\kappa+\varsigma} = 0.1528$, *(for* $\eta = 0.9$ *and* $\kappa = 5$*).*
We can study and interpret the numerical solution from the first order partial derivatives given by the terms $\Delta = \frac{\partial\upsilon}{\partial S}$ *and* $\nabla = \frac{\partial\upsilon}{\partial z}$ *whose curves are plotted in Figures 6, 8 and 10. For* $S << K$, Δ *is very close to* (-1). *When* $S \geq K$, Δ *grows rapidly towards* 0 *and stays there. Moreover,* ∇ *tends to zero for* $S \geq K$. *It appears clearly that when the price of the asset is high, the price of the put option turns to zero, in fact, this behavior was expected.*

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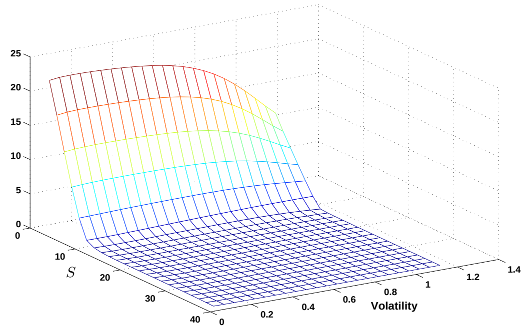

Figure 4: Unstable numerical solution for $\psi(t) = t$, $\eta = 0.1$ and $\kappa = 0.2$. The stability condition (4.59) is not satisfied: $\lambda = 0.0499 \nleq \frac{\eta^2}{\kappa+\varsigma} = 0.02$.

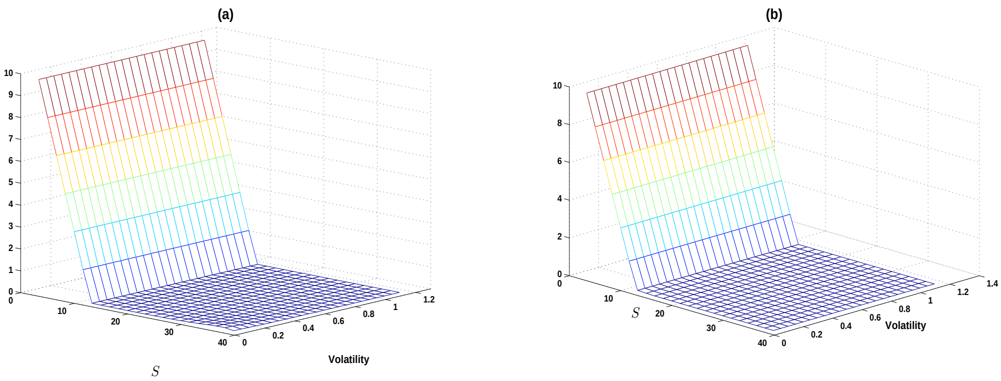

Figure 5: Numerical solution for $\psi(t) = t$ and correlation: (a) $\rho = 0.9$, (b) $\rho = 0.1$.

7 Conclusion

The American options under $\psi-$Caputo time-fractional derivative Heston model is a generalization of the classical American options under Heston model. The study of AOψCFDH model represents major difficulties compared to the classical model. Indeed, the study of the stability, the convergence and the numerical implementation of the associated numerical method NISM is more difficult than the integer-order model. In this paper, first we transformed the modified right $\psi-$Riemann-Liouville time-fractional derivative to the $\psi-$Caputo time-fractional derivative, (see Lemma 2.1). Then, a new numerical implicit scheme method has been developed for solving the AOψCFDH model. Also, we have analyzed the stability (Theorem 4.4) and convergence (Theorem 5.1) of the NISM. Thanks to the proposed numerical methods, we were able to reach a correlation coefficient $\rho = 0.9$, knowing that in the literature the widely considered values of this coefficient are $\rho = 0.1, 0.7$, (see [31]). Finally, two numerical examples are proposed in order to show the robustness and the efficiency of both the model AOψCFDH and the NISM. In a future work, we will extend the application of the new numerical methods proposed in this paper to solve models with generalized time-fractional derivatives such as Vasiceik model and

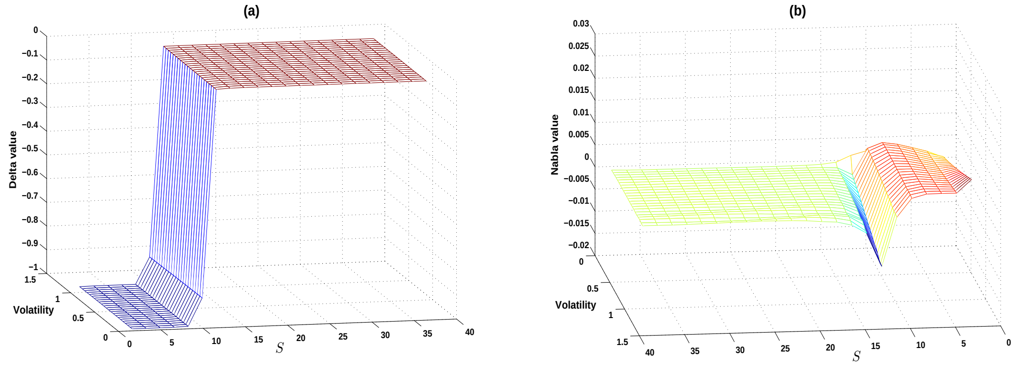

Figure 6: (a) Delta $\Delta = \frac{\partial v}{\partial S}$ of the option and (b) Nabla $\nabla = \frac{\partial v}{\partial z}$ of the option for $\psi(t) = t$ and correlation $\rho = 0.9$.

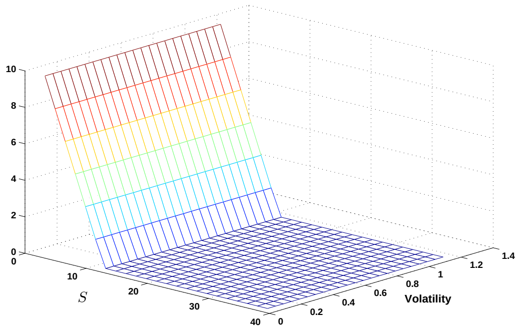

Figure 7: Numerical solution for $\psi(t) = \log(t + 1)$ and correlation $\rho = 0.9$.

Black-scholes model.

Ethical approval and Informed Consent: We affirm that this manuscript is original, has not been published before and is not currently being considered for publication elsewhere.

Conflict of interest: This work does not have any conflicts of interest.

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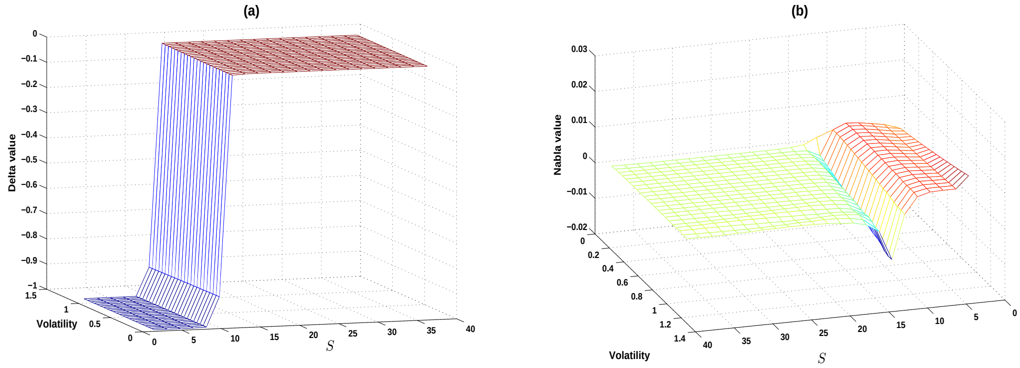

Figure 8: (a) Delta $\Delta = \frac{\partial v}{\partial S}$ of the option and (b) Nabla $\nabla = \frac{\partial v}{\partial z}$ of the option for $\psi(t) = \log(t+1)$ and correlation $\rho = 0.9$.

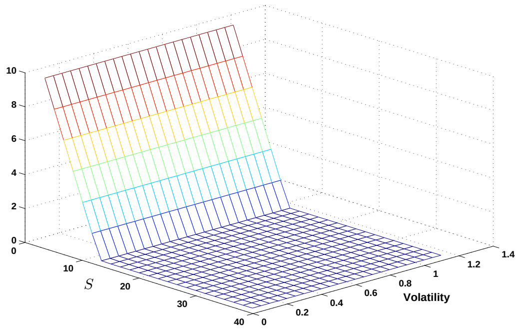

Figure 9: Numerical solution for $\psi(t) = \sqrt{t+1}$ and correlation $\rho = 0.9$.

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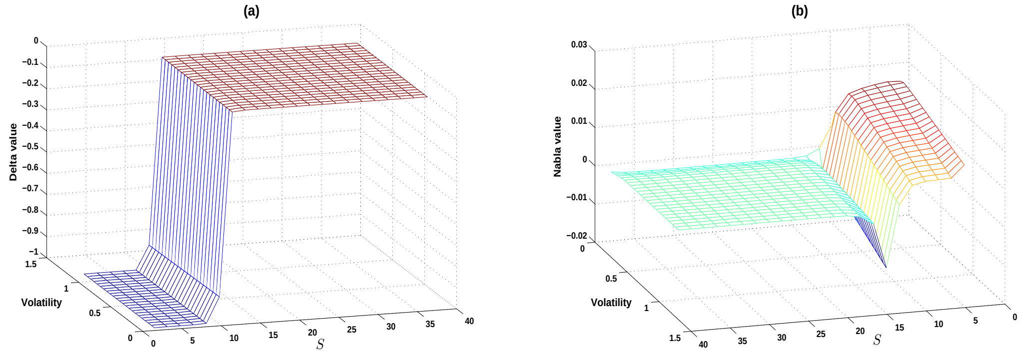

Figure 10: (a) Delta $\Delta = \frac{\partial v}{\partial S}$ of the option and (b) Nabla $\nabla = \frac{\partial v}{\partial z}$ of the option for $\psi(t) = \sqrt{t+1}$ and correlation $\rho = 0.9$.

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