WELL-POSEDNESS OF
TRICOMI-GELLERSTEDT-KELDYSH-TYPE FRACTIONAL
ELLiptic PROBLEMS

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Abstract. In this paper Tricomi-Gellerstedt-Keldysh-type fractional elliptic equations are studied. The results on the well-posedness of fractional elliptic boundary value problems are obtained for general positive operators with discrete spectrum and for Fourier multipliers with positive symbols. As examples, we discuss results in half-cylinder, star-shaped graph, half-space and other domains.

1. Introduction

1.1. Statement of the problem and historical background. The main purpose of this paper is to study the following fractional elliptic equation

\[ D^{2\alpha} u(x, y) - x^{2\beta} L u(x, y) = 0, \quad (x, y) \in \mathbb{R}^+_+ \times \Omega, \]

where \( 1/2 < \alpha \leq 1, \beta > -\alpha, \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary or \( \Omega = \mathbb{R}^N \), and \( D^{2\alpha} \) means \( D^{2\alpha} = \partial_0^\alpha, x \partial_0^\alpha, x \). Here \( \partial_0^\alpha, x \) is a Caputo fractional derivatives of order \( \alpha \):

\[ \partial_0^\alpha, x u(x, y) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-s)^{-\alpha} \partial_s u(s, y) ds, \]

and \( L \) satisfies one of the following properties (A):

- a linear self-adjoint positive operator with a discrete spectrum \( \{\lambda_k \geq 0 : k \in \mathbb{N}\} \) on the Hilbert space \( L^2(\Omega) \). According to \( \lambda_k \), the operator \( L \) has the system of orthonormal eigenfunctions \( \{e_k : k \in \mathbb{N}\} \) on \( L^2(\Omega) \).

As an example of \( L \), we can consider all self-adjoint positive operators that were given in [22, 23]. For example:

- Dirichlet-Laplacian, Neumann-Laplacian or fractional Dirichlet-Laplacian in a bounded domain;
- Sturm-Liouville operator or its involution perturbations in a finite interval;
- integro-differential operators with fractional derivatives.

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(B): Fourier multiplier \( a(D) \) with symbol \( a(\xi) \geq 0, \xi \in \mathbb{R}^N \), i.e. \( a(D) = \mathcal{F}^{-1}(a(\xi)\mathcal{F}), \xi \in \mathbb{R}^N \), where \( \mathcal{F} \) is the Fourier transform and \( \mathcal{F}^{-1} \) is the inverse Fourier transform.

As an example of \( \mathcal{L} \), we can consider all operators with nonnegative symbol (see [21]). For example:
- Laplace operator \(-\Delta\) with symbol \(|\xi|^2\) or fractional Laplacian \((-\Delta)^s, s \in (0,1)\), with symbol \(|\xi|^{2s}\);
- Linear partial differential operator \(\sum_{|\beta| \leq m} a_{\beta}D^\beta\), \(a_{\beta} \geq 0\), with nonnegative symbol \(\sum_{|\beta| \leq m} a_{\beta}|\xi|^\beta \geq 0\), with \(D^\beta = \left(\frac{1}{i} \partial_{x_1} \right)^{\beta_1} \cdots \left(\frac{1}{i} \partial_{x_N} \right)^{\beta_N}\).

The need to study the boundary value problems for the fractional elliptic equations to describe the production processes in mathematical modeling of socio-economic systems was shown in [19]. In [19] the attention was drawn to the fact that the problem of finding a generalized two-factor Cobb-Douglas function is reduced to the Dirichlet problem for the fractional elliptic equation.

The equation (1.1) is a generalization of the following well-known equations:
- If \(\alpha = 1, \beta = 0\) and \(\mathcal{L} = -\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\), then the equation (1.1) coincides with the classical Laplace equation
  \[ u_{xx}(x,y) + \Delta_y u(x,y) = 0, \quad x > 0, \quad y \in \mathbb{R}^N; \]
- If \(N = 1, \alpha = 1, \beta = \frac{1}{2}\) and \(\mathcal{L} = -\frac{\partial^2}{\partial y^2}\), then the equation (1.1) coincides with the classical Tricomi equation ([26])
  \[ u_{xx}(x,y) + xu_{yy}(x,y) = 0, \quad x > 0, \quad y \in \mathbb{R}; \]
- If \(N = 1, \alpha = 1, \beta = m > 0\) and \(\mathcal{L} = -\frac{\partial^2}{\partial y^2}\), then the equation (1.1) coincides with the classical Gellerstedt equation ([10])
  \[ u_{xx}(x,y) + x^m u_{yy}(x,y) = 0, \quad x > 0, \quad y \in \mathbb{R}; \]
- If \(N = 1, \alpha = 1, \beta = -k \in (-2,0)\) and \(\mathcal{L} = -\frac{\partial^2}{\partial y^2}\), then the equation (1.1) coincides with the classical Keldysh equation ([11])
  \[ u_{xx}(x,y) + x^{-k} u_{yy}(x,y) = 0, \quad x > 0, \quad y \in \mathbb{R}. \]

The above equations are used in transonic gas dynamics [7], and in mathematical models of cold plasma [20].

Note that the study of Tricomi, Gellerstedt and Keldysh equations was done in many papers [1, 4, 5, 6, 18, 29]. The boundary value problems for the fractional elliptic equations are studied in [2, 9, 16, 15].

1.2. **Three-parameter Mittag-Leffler (Kilbas-Saigo) function.** First, we recall the definition of the Kilbas-Saigo function (three-parameter Mittag-Leffler function) and some of its particular cases.
• **Classical Mittag-Leffler function.** The classical Mittag-Leffler function $E_{\alpha,1}(z)$ defined by ([17])

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \: z \in \mathbb{C},$$

is a natural extension of the exponential function $E_{1,1}(z) = \exp(z)$, and also of the hyperbolic cosine function $E_{2,1}(z) = \cosh \sqrt{z}$.

The most interesting properties of Mittag-Leffler function are associated with its upper-lower estimates for $0 < \alpha < 1$ as follows ([24]):

$$\frac{1}{1 + \Gamma(1-\alpha)z} \leq E_{\alpha,1}(-z) \leq \frac{1}{1 + \frac{1}{\Gamma(1+\alpha)} z}, \quad z \geq 0.$$  \hspace{1cm} (1.2)

• **Two-parameter Mittag-Leffler function.** The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \: \beta > 0, \: z \in \mathbb{C}.$$  

This function, sometimes called a Mittag-Leffler-type function, first appeared in [28]. When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the classical Mittag-Leffler function $E_{\alpha,1}(z)$.

• **Three-parameter (Kilbas-Saigo) Mittag-Leffler function.** Another generalization of the Mittag-Leffler function was introduced by Kilbas and Saigo [13] in terms of a special function of the form

$$E_{\alpha,m,n}(z) = 1 + \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} \frac{\Gamma(\alpha jm + n + 1)}{\Gamma(\alpha jm + n + 1 + 1)} z^k,$$  \hspace{1cm} (1.3)

where $\alpha, m$ are real numbers and $n \in \mathbb{C}$ such that

$$\alpha > 0, \: m > 0, \: \alpha jm + n + 1 \neq -1, -2, -3, \ldots (j \in \mathbb{N}_0).$$  \hspace{1cm} (1.4)

In particular, if $m = 1$, the function $E_{\alpha,m,n}(z)$ is reduced to the two-parameter Mittag-Leffler function:

$$E_{\alpha,1,n}(z) = \Gamma(\alpha n + 1) E_{\alpha,\alpha n+1}(z),$$

and if $m = 1, n = 0$, then it coincides with the classical Mittag-Leffler function:

$$E_{\alpha,1,0}(z) = E_{\alpha,1}(z).$$

Recently Simon et al. [8] obtained the following interesting estimates of the Kilbas-Saigo functions:

$$\frac{1}{1 + \Gamma(1-\alpha)z} \leq E_{\alpha,m,m-1}(-z) \leq \frac{1}{1 + \frac{1}{\Gamma(1+(m-1)\alpha)} z}, \quad z \geq 0,$$  \hspace{1cm} (1.5)

where $m > 0$ and $0 < \alpha < 1$. 
1.3. **Ill-posedness of the non-sequential problem.** As generally
\[ \partial_x^\alpha \partial_x^\alpha \neq \partial_x^{2\alpha}, \]
the equation (1.1) is different from the following non-sequential equation
\[ \partial_x^{2\alpha} u(x, y) - x^{2\beta} \mathcal{L} u(x, y) = 0, (x, y) \in \mathbb{R}_+ \times \Omega. \tag{1.6} \]
However, we cannot consider the problem of bounded solutions of equation (1.6) in \( x \in \mathbb{R}_+ \), since for such class of functions, nontrivial solutions of equation (1.6) may not exist. We demonstrate this with the following example:

Let \( 1 < 2\alpha < 2, \beta = 0, \) and \( \mathcal{L} = -\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} \) in (1.6). Then using the Fourier transform to (1.6) with respect to \( y \) we have
\[ \partial_x^{2\alpha} \hat{u}(x, \xi) - |\xi|^2 \hat{u}(x, \xi) = 0, x > 0, \xi \in \mathbb{R}^N. \tag{1.7} \]

The general solution to the equation (1.7) has the form [14, Example 4.10]
\[ \hat{u}(x, \xi) = C_1(\xi) E_{\alpha,1}(|\xi|^2 x^\alpha) + C_2(\xi) t E_{\alpha,2}(|\xi|^2 x^\alpha), \]
where \( C_1(\xi), C_2(\xi) \) are arbitrary constants and \( E_{\alpha,\beta}(z) \) is the Mittag-Leffler function. From the asymptotic estimate of the Mittag-Leffler function
\[ E_{\alpha,\beta}(z) \sim z^{1-\alpha} e^{z^{1-\beta}}, \ z \to \infty, \]
it follows that
\[ \lim_{x \to \infty} E_{\alpha,1}(|\xi|^2 x^\alpha) \to \infty \quad \text{and} \quad \lim_{z \to \infty} E_{\alpha,2}(|\xi|^2 x^\alpha) \to \infty. \]
Therefore, the equation (1.7) does not have a bounded solution in \( x \in \mathbb{R}_+ \).

1.4. **One dimensional fractional differential equation.** Let \( 0 < \alpha \leq 1, \mu \) is a positive real number. For further exposition we need to give some information about the exact solutions of differential equations of the form:
\[ D^{2\alpha} h(x) - \mu^2 x^{2\beta} h(x) = 0, x > 0. \tag{1.8} \]

Using the method of constructing the solution of the fractional-order differential equations developed in [3, 27], one can show that the functions
\[ \left\{ E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(\mu x^{\alpha+\beta}), E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(-\mu x^{\alpha+\beta}) \right\}, \tag{1.9} \]
are solutions of the equation (1.8).

It is easy to show that the functions (1.9) are linearly independent. Hence, the system of functions (1.9) is a fundamental system for the equation (1.8), and therefore the general solution of this equation has the form:
\[ h(x) = C_1 E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(\mu x^{\alpha+\beta}) + C_2 E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(-\mu x^{\alpha+\beta}), \tag{1.10} \]
where \( C_1 \) and \( C_2 \) are arbitrary constants.

It is easy to see that, if \( x \to +\infty \), then
\[ E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(\mu x^{\alpha+\beta}) \to +\infty, \]
since
\[ E_{\alpha,1+\beta, \frac{\alpha-\beta}{\alpha}}(\mu x^{\alpha+\beta}) \geq \frac{\mu \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} x^{\alpha+\beta}, x > 0. \tag{1.11} \]
And for the function $E_{\alpha,1+\frac{\beta}{2},\frac{\beta}{2}} (-\mu x^{\alpha+\beta})$, the following estimate holds ([8]):

$$E_{\alpha,1+\frac{\beta}{2},\frac{\beta}{2}} (-\mu x^{\alpha+\beta}) \leq \frac{1}{1 + \Gamma(\beta+1)\Gamma(\alpha+\beta+1)\mu x^{\alpha+\beta}}, \quad x > 0. \quad (1.12)$$

2. Well-posedness in a bounded domain

Let $\mathcal{L}$ be a self-adjoint, positive operator with the discrete spectrum $\{\lambda_k \geq 0 : k \in \mathbb{N}\}$ on $L^2(\Omega)$. The main assumption in this section is that the system of eigenfunctions $\{e_k \in L^2(\Omega) : k \in \mathbb{N}\}$ of the operator $\mathcal{L}$ is an orthonormal basis in $L^2(\Omega)$.

The Hilbert space $H^\mathcal{L}(\Omega)$ is defined by

$$H^\mathcal{L}(\Omega) = \{u \in L^2(\Omega) : \sum_{k=0}^{\infty} \lambda_k^2 |(u, e_k)|^2 < \infty\},$$

with the norm

$$\|u\|_{H^\mathcal{L}(\Omega)} = \sum_{k=0}^{\infty} \lambda_k^2 |(u, e_k)|^2.$$

**Definition 2.1.** The generalised solution of equation (1.1) in $\Omega \subset \mathbb{R}^N$ is a bounded function $u \in C(\mathbb{R}^+; L^2(\Omega))$, such that $x^{-2\beta}D_x^{2\alpha}u, \mathcal{L}u \in C(\mathbb{R}^+; L^2(\Omega))$.

**Theorem 2.2.** Let $\phi \in H^\mathcal{L}(\Omega)$. Then the generalised solution of equation (1.1) satisfying conditions

$$u(0, y) = \phi(y), \quad y \in \Omega, \quad (2.1)$$

and

$$\lim_{x \to +\infty} u(x, y) \text{ is bounded for almost every } y \in \Omega, \quad (2.2)$$

exists, it is unique and can be represented as

$$u(x, y) = \sum_{k=0}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{2},\frac{\beta}{2}} (-\sqrt{\lambda_k} x^{\alpha+\beta}) e_k(y), \quad (x, y) \in [0, \infty) \times \Omega, \quad (2.3)$$

where $\phi_k = \int_{\Omega} \phi(y) e_k(y) dy$, $k \in \mathbb{Z}_+ = 0, 1, 2, \ldots$, and $E_{\alpha,m,l}(z)$ is a Kilbas-Saigo function.

In addition, the solution $u$ satisfies the following estimates:

$$\|u\|_{C(\mathbb{R}^+; L^2(\Omega))} \leq \|\phi\|_{L^2(\Omega)},$$

$$\sup_{x \in (0, \infty)} \|x^{-2\beta}D_x^{2\alpha}u(x, \cdot)\|_{L^2(\Omega)} \leq \|\phi\|_{H^\mathcal{L}(\Omega)},$$

and

$$\sup_{x \in (0, \infty)} \|\mathcal{L}u(x, \cdot)\|_{L^2(\Omega)} \leq \|\phi\|_{H^\mathcal{L}(\Omega)}.$$
bounded solution to Problem (1.1), (2.1) has the form (2.3). However, if we take into account condition (2.4), then, for the existence of a solution to problem (1.1), (2.1), (2.4), it is necessary and sufficient to have the condition

\[ \int_{\Omega} \phi(y) dy = 0. \]

2.1. Particular cases. We now specify Theorem 2.2 to several concrete cases.

2.1.1. Laplace equation in the half-strip and in the star-shaped graphs. Our first example will focus on the Laplace equation.

- Let \( \Omega = (0, 1) \), \( \alpha = 1 \), \( \beta = 0 \) and \( L = -\frac{\partial^2}{\partial y^2}, D(L) := \{ u \in W^1_2([0, 1]), u(0) = u(1) = 0 \} \).

Then the equation (1.1) coincides with the classical Laplace equation on the half-strip

\[ u_{xx}(x, y) + u_{yy}(x, y) = 0, \quad (x, y) \in \mathbb{R}_+ \times (0, 1). \]

(2.5)

It is known that the unique solution to problem (2.5), (2.1), (2.2) is represented in the form

\[ u(x, y) = \sum_{k=1}^{\infty} \phi_k e^{-k\pi x} \sin k\pi y. \]

- Let \( \Omega \) be a star-shaped metric graph consisting of \( d \) segments of equal length, \( \alpha = 1 \), \( \beta = 0 \), and let \( L \) be a differential operator \( L = -\frac{\partial^2}{\partial y_j^2}, j = 1, \ldots, d \), with boundary conditions

\[ v_j(0) = 0, \quad j = 1, \ldots, d, \]
\[ v_1(\pi) = v_2(\pi) = \cdots = v_d(\pi), \]
\[ v_1'(\pi) + v_2'(\pi) + \cdots + v_d'(\pi) = 0. \]

It is known ([30]) that the above operator is self-adjoint in \( L^2([0, \pi]) = \bigotimes_{i=1}^{d} L^2([0, \pi]) \) and has discrete spectrum \( \lambda_k^d = (k - \frac{1}{2})^2, k \in \mathbb{N} \). Then the equation (1.1) coincides with the Laplace equation on the star-shaped graphs

\[ \Delta u(x, y) \equiv \Delta \begin{pmatrix} u_1(x, y) \\ u_2(x, y) \\ \vdots \\ u_d(x, y) \end{pmatrix} = 0. \]

(2.6)

Then the unique solution to problem (2.6), (2.1), (2.2) is represented in the form

\[ u(x, y) \equiv \begin{pmatrix} u_1(x, y) \\ u_2(x, y) \\ \vdots \\ u_d(x, y) \end{pmatrix} = \sum_{k=1}^{\infty} \phi_k e^{- (k - \frac{1}{2}) x} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \sin (k - \frac{1}{2}) y. \]
2.1.2. Fractional analogue of the Laplace equation with involution. Let $\Omega = (-\pi, \pi)$, $\beta = 0$, and
\[
\mathcal{L} u(x) = -\frac{\partial^2}{\partial y^2} u(x) + \varepsilon \frac{\partial^2}{\partial y^2} u(-x), \ |\varepsilon| < 1,
\]
\[
D(\mathcal{L}) := \{ u \in W^1_2([-\pi, \pi]), \ u(-\pi) = u(\pi) = 0 \}.
\]
Then the equation (1.1) coincides with the fractional analogue of the Laplace equation with involution
\[
D_x^{2\alpha} u(x, y) + u_{yy}(x, y) - \varepsilon u_{yy}(x, -y) = 0, \ (x, y) \in \mathbb{R}^+ \times (-\pi, \pi).
\]
(2.7)
It is known ([15]) that there exist a unique solution to problem (2.7), (2.1), (2.2) and it can be represented in the form
\[
u (x, y) = \sum_{k=1}^{\infty} \phi_k E_{\alpha,1} \left( - (1 + (-1)^k \varepsilon) k \pi x^\alpha \right) \sin k \pi y.
\]

2.1.3. Elliptic Tricomi and Gellerstedt equation. Let $\alpha = 1$, $\beta > -2$, and $\mathcal{L} = -\frac{\partial^2}{\partial y^2}$, $D(\mathcal{L}) := \{ u \in W^1_2([0, 1]), \ u(0) = u(1) = 0 \}$.

• If $\beta = 1$ then the equation (1.1) coincides with the classical Tricomi equation
\[
u_{xx}(x, y) + xu_{yy}(x, y) = 0, \ x > 0, \ y \in (0, 1),
\]
(2.8)
and the unique solution to problem (2.8), (2.1), (2.2) can be written as
\[
u (x, y) = \sum_{k=1}^{\infty} \phi_k \text{Ai}(-k \pi x) \sin k \pi y,
\]
where $\text{Ai}(z)$ is the Airy function.

• If $\beta > -2$ then the equation (1.1) coincides with the classical Gellerstedt equation
\[
u_{xx}(x, y) + x^\beta u_{yy}(x, y) = 0, \ x > 0, \ y \in (0, 1),
\]
(2.9)
and the unique solution to problem (2.9), (2.1), (2.2) can be written as (see [18])
\[
u (x, y) = \sum_{k=1}^{\infty} \phi_k \sqrt{x} K_{\nu} \left( \frac{2\pi k x^{2+\beta}}{\beta + 2} \right) \sin k \pi y,
\]
where $K_{\nu}(z)$ is the Macdonald function.

2.1.4. Fractional elliptic equation with variable coefficients. If $\beta = 0$ and
\[
\mathcal{L} = (1 - y)^\mu (1 + y)^\mu D_{1-y}^\mu \partial_{1+y}^\mu,
\]
\[
u(-1) = J_{1-y}^{1-\mu} \partial_{1+y}^{\mu} u(1) = 0,
\]
then the equation (1.1) coincides with the equation
\[
u_{xx}(x, y) + (1 - y)^\mu (1 + y)^\mu D_{1-y}^\mu \partial_{1+y}^\mu u(x, y) = 0, \ x > 0, \ y \in (-1, 1),
\]
(2.10)
where $\mu \in (0, 1)$, $D_{1-y}^\mu$ is a right-side Riemann-Liouville fractional derivative of order $\mu \in (0, 1)$
\[
D_{1-y}^\mu u(x, s) = \frac{1}{\Gamma(1-\mu)} \frac{\partial}{\partial y} \int_y^1 (s - y)^{-\mu} u(x, s) ds,
\]
\( \partial_{-1+y}^{\mu} \) is a left-side Caputo fractional derivative of order \( \mu \in (0,1) \)

\[
\partial_{-1+y}^{\mu} u(x, y) = \frac{1}{\Gamma(1-\mu)} \int_{-1}^{y} (y-s)^{-\mu} u_s(x,s) ds,
\]

\( I_{1-y}^{1-\mu} \) is a right-side Riemann-Liouville fractional integral of order \( \mu \in (0,1) \)

\[
I_{1-y}^{1-\mu} u(x, y) = \frac{1}{\Gamma(1-\mu)} \int_{y}^{1} (s-y)^{-\mu} u(x, s) ds.
\]

The unique solution of problem (2.10), (2.1), (2.2) can be written as

\[
u(x, y) = \sum_{k=1}^{\infty} \phi_k \exp \left( -\frac{\Gamma(k+\mu)}{\Gamma(k-\mu)} x \right) (1+y)^{\mu} P_{k-1-\mu}^{\alpha, \beta}(y),
\]

where \( P_{k-1-\mu}^{\alpha, \beta}(y) \) is the Jacobi polynomial ([31])

\[
P_{k-1-\mu}^{\alpha, \beta}(y) = \sum_{n=0}^{k-1} \left( \frac{k-1-\mu}{k-1-n} \right) \left( \frac{k-1+\mu}{n} \right) \left( \frac{y-1}{2} \right)^{n} \left( \frac{y+1}{2} \right)^{k-1-n}.
\]

2.2. Proof of Theorem 2.2.

2.2.1. Existence of solution. As \( \mathcal{L} \) is self-adjoint in \( L^2(\Omega) \), any solution of problem (1.1), (2.1)–(2.2) can be represented as:

\[
u(x, y) = \sum_{k=0}^{\infty} u_k(x) e_k(y), (x, y) \in \mathbb{R}_+ \times \Omega.
\]

(2.11)

It is clear that if \( \phi \in \mathcal{H}^\mathcal{L}(\Omega) \), then it can be represented in the form

\[
\phi(y) = \sum_{k=0}^{\infty} \phi_k e_k(y), y \in \Omega,
\]

(2.12)

where \( \phi_k = \int_{\Omega} \phi(y) \overline{e_k(y)} dy \).

Substituting function (2.11) into equation (1.1), we obtain the following problem for \( u_k(x) \),

\[
D^{2\alpha} u_k(x) - \lambda_k x^{2\beta} u_k(x) = 0, x > 0,
\]

(2.13)

\[
u_k(0) = \phi_k, u_k(\infty) \leq C, C = \text{const},
\]

(2.14)

where \( \lambda_k > 0 \) are eigenvalues of \( \mathcal{L} \).

According to formula (1.10), the general solution to equation (2.13) has the form:

\[
u_k(x) = C_1 E_{\alpha,1+\frac{\alpha}{\alpha}} \left( \sqrt{\lambda_k} x^{\alpha+\beta} \right) + C_2 E_{\alpha,1+\frac{\alpha}{\alpha}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right),
\]

where \( C_1 \) and \( C_2 \) are arbitrary constants.

Since

\[
E_{\alpha,1+\frac{\alpha}{\alpha}} \left( \sqrt{\lambda_k} x^{\alpha+\beta} \right) \to +\infty, \text{ as } x \to +\infty,
\]

we have \( C_1 = 0 \).
Since
\[ E_{\alpha+1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) \to 0, \quad \text{as} \quad x \to +\infty, \]
then by (2.14) we have
\[ u_k (x) = \phi_k E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right), \quad (2.15) \]
hence
\[ u (x, y) = \sum_{k=0}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) e_k (y), \quad (x, y) \in \mathbb{R}_+ \times \Omega. \]

2.2.2. Convergence of solution. The estimate (1.12) gives
\[ |u_k (x)| \leq \frac{|\phi_k|}{1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \sqrt{\lambda_k} x^{\alpha+\beta}}, \]
which implies
\[
\sup_{x \geq 0} \| u (x, \cdot) \|_{L^2(\Omega)}^2 \leq \sup_{x \geq 0} \sum_{k=0}^{\infty} |\phi_k|^2 \| E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) \|^2 \| e_k \|_{L^2(\Omega)}^2 \nleq \sup_{x \geq 0} \sum_{k=0}^{\infty} |\phi_k|^2 \left( 1 + \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \sqrt{\lambda_k} x^{\alpha+\beta} \right)^2 \nleq \sum_{k=0}^{\infty} |\phi_k|^2 = \| \phi \|_{L^2(\Omega)}^2 < \infty,
\]
thanks to Parseval’s identity. Let us calculate \( D_x^{2\alpha} u \) and \( L u \). We have
\[
D_x^{2\alpha} u (x, y) = \sum_{k=0}^{\infty} \phi_k D_x^{2\alpha} E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) e_k (y) \n= x^{2\beta} \sum_{k=0}^{\infty} \lambda_k \phi_k E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) e_k (y), \quad (x, y) \in \mathbb{R}_+ \times \Omega,
\]
and
\[
L u (x, y) = \sum_{k=0}^{\infty} \phi_k E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) \mathcal{L} e_k (y) \n= \sum_{k=0}^{\infty} \lambda_k \phi_k E_{\alpha,1+\frac{\beta}{n}, \frac{\beta}{n}} \left( -\sqrt{\lambda_k} x^{\alpha+\beta} \right) e_k (y), \quad (x, y) \in \mathbb{R}_+ \times \Omega.
\]
Applying the above calculations and Parseval’s identity we have
\[
\sup_{x \in (0, \infty)} \| x^{-2\beta} D_x^{2\alpha} u (x, \cdot) \|_{L^2(\Omega)}^2 \leq \sum_{k=0}^{\infty} \lambda_k^2 |\phi_k|^2 = \| \phi \|_{H^2(\Omega)}^2 < \infty,
\]
and
\[
\sup_{x \in (0, \infty)} \| L u (x, \cdot) \|_{L^2(\Omega)}^2 \leq \sum_{k=0}^{\infty} \lambda_k^2 |\phi_k|^2 = \| \phi \|_{\mathcal{H}^2(\Omega)}^2 < \infty.
\]
2.2.3. Uniqueness of solution. Suppose that there are two solutions \( u_1(x, y) \) and \( u_2(x, y) \) of problem (1.1), (2.1)–(2.2). Let \( u(x, y) = u_1(x, y) - u_2(x, y) \). Then \( u(x, y) \) satisfies the equation (1.1) and homogeneous conditions (2.1)–(2.2).

Let us consider the function

\[
  u_k(x) = \int_{\Omega} u(x, y) e_k(y) dy, \quad k \in \mathbb{Z}_+, \quad x \geq 0. \tag{2.16}
\]

Applying \( D^{2\alpha} \) to the function (2.16) by (1.1) we have

\[
  D^{2\alpha} u_k(x) = \int_{\Omega} D_x^{2\alpha} u(x, y) e_k(y) dy = x^{2\beta} \int_{\Omega} \mathcal{L} u(x, y) e_k(y) dy \\
  = x^{2\beta} \int_{\Omega} u(x, y) \mathcal{L} e_k(y) dy = x^{2\beta} \lambda_k \int_{\Omega} u(x, y) e_k(y) dy \\
  = x^{2\beta} \lambda_k u_k(x), \quad k \in \mathbb{Z}_+, \quad x \geq 0.
\]

Also from (2.1) and (2.2) we have \( u_k(0) = 0 \), \( u_k(\infty) \) is bounded. Then from (2.15) we conclude that \( u_k(x) = 0, \quad x \geq 0 \). This implies \( \int_{\Omega} u(x, y) e_k(y) dy = 0 \), and the completeness of the system \( e_k(x), k \in \mathbb{Z}_+ \), gives \( u(x, y) \equiv 0, \quad (x, y) \in [0, \infty) \times \Omega \).

3. Well-posedness in \( \mathbb{R}^N \)

The Sobolev space \( \mathcal{H}^L(\mathbb{R}^N) \) is defined by

\[
  \mathcal{H}^L(\mathbb{R}^N) = \{ f \in L^2(\mathbb{R}^N) : a(\xi) \hat{f} \in L^2(\mathbb{R}^N) \},
\]

where \( \hat{f}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot y} f(y) dy \), \( \xi \in \mathbb{R}^N \).

The space \( \mathcal{H}^L(\mathbb{R}^N) \) is a Hilbert space; it is equipped with the norm

\[
  \| f \|_{\mathcal{H}^L(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} |a(\xi) \hat{f}(\xi)|^2 d\xi.
\]

**Definition 3.1.** The generalised solution of equation (1.1) in \( \mathbb{R}^N \) is a function \( u \in C \left( (0, \infty); L^2(\mathbb{R}^N) \right) \), such that \( x^{-2\beta} D_x^{2\alpha} u, \mathcal{L} u \in C \left( (0, \infty); L^2(\mathbb{R}^N) \right) \).

**Theorem 3.2.** Let \( \phi \in \mathcal{H}^L(\mathbb{R}^N) \). Then the generalised solution of equation (1.1) satisfying conditions

\[
  u(0, y) = \phi(y), \quad y \in \mathbb{R}^N, \tag{3.1}
\]

and

\[
  \lim_{x \to +\infty} u(x, y) \quad \text{is bounded for almost every} \quad y \in \mathbb{R}^N, \tag{3.2}
\]

exists, it is unique and can be represented as

\[
  u(x, y) = \int_{\mathbb{R}^N} e^{-iy \cdot \xi} \hat{\phi}(\xi) E_{\alpha, 1+\beta, \frac{2\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha+\beta} \right) d\xi, \quad (x, y) \in \mathbb{R}_+ \times \mathbb{R}^N, \tag{3.3}
\]

where \( \hat{\phi}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-i\xi \cdot s} \phi(s) ds \).
In addition, the solution $u$ satisfies the following estimates:

$$\|u\|_{C([0,\infty) \times L^2(\mathbb{R}^N))} \leq \|\phi\|_{L^2(\mathbb{R}^N)},$$

$$\sup_{x \in (0, \infty)} \|x^{-2\beta} D_x^{2\alpha} u (x, \cdot)\|_{L^2(\mathbb{R}^N)} \leq \|\phi\|_{H^L(\mathbb{R}^N)},$$

and

$$\sup_{x \in (0, \infty)} \|L u (x, \cdot)\|_{L^2(\mathbb{R}^N)} \leq \|\phi\|_{H^L(\mathbb{R}^N)}.$$

### 3.1. Particular cases

We now specify Theorem 3.2 to several concrete cases.

#### 3.1.1. Laplace equation in the half-space

Our first example will focus on the Laplace equation.

Let $\alpha = 1$, $\beta = 0$ and $L = -\Delta = \sum_{j=1}^{N} \partial^2_{y_j^2}$. Then the equation (1.1) coincides with the classical Laplace equation on the half-space

$$u_{xx}(x, y) + \Delta_y u(x, y) = 0, \; (x, y) \in \mathbb{R}_+ \times \mathbb{R}^N. \tag{3.4}$$

It is known that the unique solution to problem (3.4), (3.1), (3.2) is represented by the Poisson integral ([25])

$$u(x, y) = \frac{\Gamma((N + 1)/2)}{\pi^{(N+1)/2}} \int_{\mathbb{R}^N} \frac{x \phi(s)}{|y - s|^2 + x^2(N+1)/2} ds.$$

#### 3.1.2. Multidimensional degenerate elliptic equations

Let $\alpha = 1$, $\beta > -2$ and $L = -\Delta_y$.

- If $\beta = 1$, then the equation (1.1) coincides with the multidimensional Tricomi equation

$$u_{xx}(x, y) + x \Delta_y u(x, y) = 0, \; x > 0, \; y \in \mathbb{R}^N, \tag{3.5}$$

and the solution to problem (3.5), (3.1), (3.2) can be written as ([1])

$$u(x, y) = \frac{3^{n+1/2} \Gamma(2/3) \Gamma(N/2 + 1/3)}{21/3 \pi^{N/2+1}} \int_{\mathbb{R}^N} \frac{x \phi(s)}{(9|y - s|^2 + 4x^3)^{N/2+1/3}} ds.$$

- If $\beta = m > -2$ then the equation (1.1) coincides with the multidimensional Gellerstedt equation

$$u_{xx}(x, y) + x^m \Delta_y u(x, y) = 0, \; x > 0, \; y \in \mathbb{R}^N, \tag{3.6}$$

and the unique solution to problem (3.6), (3.1), (3.2) can be written as ([1])

$$u(x, y) = \frac{(m + 2)^{n+1/2} \Gamma \left( \frac{N}{2} + \frac{1}{m+2} \right)}{2^{N} \pi^{N/2} \Gamma \left( \frac{1}{m+2} \right)} \int_{\mathbb{R}^N} \frac{x \phi(s)}{\left( x^{m+2} + \left( \frac{m+2}{2} \right)^2 |y - s|^2 \right)^{N/2+1/2}} ds.$$
3.1.3. **Fractional Laplace equation.** Let $\beta = 0$ and

$$L v = (-\Delta)^s v = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{(v(y) - v(s))}{|x-y|^{N+2s}} \, dy,$$

where $s \in (0,1)$ and $C_{N,s}$ is a normalizing constant (whose value is not important here). Then the equation (1.1) coincides with the equation

$$D_x^{2a} u(x,y) + (-\Delta)^s_y u(x,y) = 0, \ x > 0, \ y \in \mathbb{R}^N. \tag{3.7}$$

From Theorem 3.2 we have the unique solution of the problem (3.7), (3.1), (3.2) in the form

$$u(x,y) = \int e^{-iy \xi} \phi(\xi) E_{\alpha,1} (-|\xi|^a x^\alpha) \, d\xi, \ (x,y) \in \mathbb{R}_+ \times \mathbb{R}^N.$$  

Rearranging the order of integration in the last representation, according to Fubini’s Theorem, we have

$$u(x,y) = \pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} \phi(s) \int_{\mathbb{R}^N} e^{-iy(y-s)} E_{\alpha,1} (-|\xi|^a x^\alpha) \, d\xi \, ds, \ (x,y) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

Using the calculation of the Fourier transform of Mittag-Leffler functions from [12], we have

$$u(x,y) = \pi^{-\frac{N}{2}} \int_{\mathbb{R}^N} \phi(s) H_{12}^{32} \left( \frac{2^s x^\alpha}{|y-s|^s}, (0,1), (0,s/2) \right) \, ds.$$

Here $H^{m,n}_{pq}(\cdot)$ is the Fox H-function defined via a Mellin-Barnes type integral as

$$H_{pq}^{m,n}(z) \left( \frac{(a_1^1, a_1^2, \ldots, a_1^p)}{(b_1^1, b_1^2, \ldots, b_1^q)} \right) = \frac{1}{2\pi i} \int_I H_{pq}^{m,n}(\tau) z^{-\tau} d\tau,$$

where $(a_1^1, a_1^2, \ldots, a_1^p) = ((a_1^1, a_1^2), (a_2^1, a_2^2, \ldots), (a_p^1, a_p^2))$ and

$$H_{pq}^{m,n}(\tau) = \prod_{j=1}^p \Gamma(b_j^1 + b_j^2 \tau) \prod_{i=1}^n \Gamma(1 - a_i^1 - a_i^2 \tau) \prod_{i=n+1}^m \Gamma(1 - b_i^1 - b_i^2 \tau) \prod_{j=m+1}^{n+1} \Gamma(1 - b_j^1 - b_j^2 \tau).$$

3.2. **Proof of Theorem 3.2.**

3.2.1. **Existence of solution.** Applying the Fourier transform $\mathcal{F}$ to problem (1.1), (3.1)–(3.2) with respect to space variable $y$ yields

$$D_x^{2a} \hat{u}(x,\xi) - a(\xi) x^{2\beta} \hat{u}(x,\xi) = 0, \ x > 0, \ \xi \in \mathbb{R}^N, \tag{3.8}$$

$$\hat{u}(0,\xi) = \hat{\phi}(\xi), \ \hat{u}(\infty,\xi) \text{ is bounded for } \xi \in \mathbb{R}^N, \tag{3.9}$$

thank to $\mathcal{F}\{L u(x,y)\} = a(\xi) \hat{u}(x,\xi)$. Then the solution of problem (3.8)-(3.9) can be represented as

$$\hat{u}(x,\xi) = \hat{\phi}(\xi) E_{\alpha,1+\frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha+\beta} \right). \tag{3.10}$$
By applying the inverse Fourier transform $\mathcal{F}^{-1}$ we have (3.3), i.e.

$$u(x, y) = \int_{\mathbb{R}^N} e^{iy\xi} \hat{\phi}(\xi) E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) d\xi, (x, y) \in \mathbb{R}_+ \times \mathbb{R}^N.$$  

### 3.2.2. Convergence of solution

Now we prove the convergence of the obtained solution. Applying estimate (1.12) and Plancherel theorem we have

$$\sup_{x \in [0, \infty)} \int_{\mathbb{R}^N} |u(x, y)|^2 dy = \sup_{x \in [0, \infty)} \int_{\mathbb{R}^N} |\hat{u}(x, \xi)|^2 d\xi \leq \sup_{x \in [0, \infty)} \int_{\mathbb{R}^N} |\hat{\phi}(\xi)|^2 \left| E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) \right|^2 d\xi \leq \int_{\mathbb{R}^N} |\hat{\phi}(\xi)|^2 d\xi = ||\hat{\phi}||_{L^2(\mathbb{R}^N)}^2 < \infty.$$

Let us calculate $D_{x}^{2\alpha} u$:

$$D_{x}^{2\alpha} u(x, y) = \int_{\mathbb{R}^N} e^{iy\xi} \hat{\phi}(\xi) D_{x}^{2\alpha} E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) d\xi = x^{2\beta} \int_{\mathbb{R}^N} e^{iy\xi} \hat{\phi}(\xi) a(\xi) E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) d\xi, (x, y) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

Hence

$$\sup_{x \in (0, \infty)} \left\| x^{-2\beta} D_{x}^{2\alpha} u(x, \cdot) \right\|_{L^2(\mathbb{R}^N)}^2 \leq \sup_{x \in (0, \infty)} \int_{\mathbb{R}^N} a^2(\xi) |\hat{\phi}(\xi)|^2 \left| E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) \right|^2 d\xi \leq \int_{\mathbb{R}^N} |a(\xi) \hat{\phi}(\xi)|^2 d\xi = ||\phi||_{H^\infty(\mathbb{R}^N)}^2 < \infty.$$

Similarly, for $Lu$ we have

$$\sup_{x \in (0, \infty)} \left\| Lu(x, \cdot) \right\|_{L^2(\mathbb{R}^N)}^2 \leq \sup_{x \in (0, \infty)} \int_{\mathbb{R}^N} a^2(\xi) |\hat{\phi}(\xi)|^2 \left| E_{\alpha, 1 + \frac{\beta}{\alpha}, \frac{\beta}{\alpha}} \left( -\sqrt{a(\xi)} x^{\alpha + \beta} \right) \right|^2 d\xi \leq ||\phi||_{H^\infty(\mathbb{R}^N)}^2 < \infty.$$

### 3.2.3. Uniqueness of solution

Suppose that there are two solutions $u_1(x, y)$ and $u_2(x, y)$ of problem (1.1), (3.1)–(3.2). Let $u(x, y) = u_1(x, y) - u_2(x, y)$. Then $u(x, y)$ satisfies the equation (1.1) and homogeneous conditions (3.1)–(3.2).

Let us consider the function

$$\hat{u}(x, \xi) = \int_{\mathbb{R}^N} e^{-iy\xi} u(x, y) dy, x \geq 0, \xi \in \mathbb{R}^N.$$  

(3.11)
As $u$ is bounded continuous in $x$ function, applying $\mathcal{D}_x^{2\alpha}$ to the function (3.11) by (1.1) we have

$$
\mathcal{D}_x^{2\alpha} \hat{u}(x, \xi) = \int_{\mathbb{R}^N} e^{-iy\xi} \mathcal{D}_x^{2\alpha} u(x, y) dy
$$

$$
= x^{2\beta} \int_{\mathbb{R}^N} e^{-iy\xi} \mathcal{L}u(x, y) dy
$$

$$
= x^{2\beta} \mathcal{F} \left[ \mathcal{F}^{-1}(a(\xi) \hat{u}(x, y)) \right]
$$

$$
= x^{2\beta} a(\xi) \hat{u}(x, \xi), \ x \geq 0, \xi \in \mathbb{R}^N.
$$

Also from (3.1) and (3.2) we have $\hat{u}(0, \xi) = 0, \hat{u}(\infty, \xi)$ is bounded. Then from (3.10) we conclude that $\hat{u}(x, \xi) = 0, x \geq 0, \xi \in \mathbb{R}^N$. Applying the inverse Fourier transform we have $u(x, y) \equiv 0, (x, y) \in [0, \infty) \times \mathbb{R}^N$. The proof is complete.

**References**


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