INTEGRODIFFERENTIAL EQUATIONS OF VOLterra TYPE WITH NON LOCAL AND IMPULSIVE CONDITIONS

AMADOU DIOP, MOUSTAPHA DIEYE, MAMADOU ABDOU DIOP, AND KHALIL EZZINBI

Abstract. This work is devoted to the study of a class of nonlocal impulsive integrodifferential equations of Volterra type. We investigate the situation when the resolvent operator corresponding to the linear part of (1) is norm continuous. Our results are obtained by using noncompactness Hausdorff measure and fixed point theorems. An example is provided to illustrate the basic theory of this work.

1. Introduction

Consider on a Banach space \((E, \| \cdot \|)\) the following Volterra integral equation with nonlocal initial value:

\[
\begin{aligned}
&\frac{dx}{dt}(t) = A(t)x(t) + \int_0^t \Gamma(t,s)x(s)ds + f(t,x(t)), \quad t \in I = [0,T], \quad t \neq t_i, \quad i = 1,2,3,\ldots,m, \\
&x(t_i^+) = x(t_i) + J_i(x(t_i)), \\
&x(0) = x_0 + g(x),
\end{aligned}
\]

where \(0 = t_0 < t_1 < t_2 < \ldots < t_{m+1} = T < \infty\), \(A(t) : D(A) \subset E \to E\) is a closed linear operator with domain \(D(A)\) which is independent of \(t\), for \(0 \leq s \leq t, \Gamma(t,s) : D(\Gamma) \subset E \to E\) is a closed linear operator with \(D(\Gamma) \supset D(A)\), and \(J_i : E \to E, f : J \times E \to E, g : PC([0,T],E) \to E\) are given \(E\)-valued functions.

In evolution equation theory, integrodifferential equations are of great importance. They have attracted much attention because of their applications in many areas: physics, finance, biology, ecology, sociology, population dynamics, electrical engineering and other areas of science and engineering. Qualitative and quantitative properties such as existence, uniqueness, controllability and stability for various integrodifferential equations have been extensively studied. We refer the reader to [1, 3, 8, 9, 10, 14] and the references therein.

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To model phenomena such as the dynamics of populations subject to abrupt changes (harvesting, diseases, etc), the nonlocal condition has advantages over traditional initial value problems. The study of abstract nonlocal Cauchy problem was motivated by the paper of Byszewski [6, 7]. Since it is demonstrated that the non local problems have better effects in applications than the classical ones $x(0) = x_0$, differential equations with nonlocal problems have been studied extensively in the literature, see [36, 19, 32, 30, 31] and the references therein. In this work we discuss the existence of solution on bounded intervals of (1). We introduce a relation between the resolvent operator which is generated by the linear part of (1) and the evolution family generated by the families operator $A(t)$. This enable us to find conditions that guarantee the existence of mild solutions of equation (1) without any assumptions on the compactness of $f$; $J_i : i = 1, \ldots, m$; and $R(t, s)$ the eventual resolvent operator see below. The results are established with the use of the theory of resolvent operator in the sense of Grimmer. We use the fact that the operator-norm continuity of the resolvent operator yields from to the evolution family. This property allows us to drop the supposition that the evolution family is compact and to show that the operator solution meets the requirements of the Mönch conditions.

The proofs of results obtained in this work are based on a calculation method which employs the technique of measures of noncompactness in the Banach space $PC([0, T], E)$ and fixed point theory. We will assume the function $f$ satisfies Caratheodory conditions and Lipschitzian with respect to a measure of non compactness. Also the functions $g$ is completely continuous and $J_i$ are assumed to be only Lipschitz with respect to a measure of non compactness (MNC). Therefore we recall some needed facts on the MNC. We use the approach developed in [21] to show the existence of a (mild) solution. The resolvent operator gives the expression of solutions which existence follows by Mönch fixed theorem. Several authors have contributed to the development of this theory by solving differential equations across various fixed point theorems. Moreover, to the best of our knowledge, up to now no work has been reported yet regarding the non-autonomous partial integrodifferential equations with nonlocal initial conditions and noncompact resolvent operator of the form (1), which motivates this present study. The main contributions of this work are summarized as follows:

1. The study of existence integrodifferential equations via measure of noncompactness in the form (1) is an untreated topic in the literature and this is an additional motivation for writing this paper.
2. We establish an outcome that guarantee the operator-norm continuity of the resolvent operator.
3. We assume the nonlinear term satisfies a weakly compactness conditions and does not require the compactness of the resolvent operator. This extends the results in [13].
4. We establish some sufficient conditions for the nonlocal existence when the solution operators are only equicontinuous, by means of Mönch fixed-point Theorem via the noncompactness measure.
5. Our results guarantee the effectiveness of nonlocal existence results under some weakly compactness condition.
The rest of the work is organized as follows. In Section 2 we given notations and auxiliary facts needed further on. In Section 3 we formulate and prove the existence of local mild solutions of Equation (1). In Section 4, an example is presented to show the application of our main results.

2. Preliminaries

This section presents the notations and auxiliary results that will be used. Let $X, Y$ be two linear topological spaces, denote by $X, \text{Conv}(X), \lambda X$ and $X + Y$ stand for the closure, convex closure of $X$ and algebraic operations on sets, respectively.

Let $C(I, E)$ denote the space of all $E$-valued continuous functions defined on the interval $I \subseteq \mathbb{R}$. It is well known that if $I$ is a compact interval of $\mathbb{R}_+$, the space $C(I, E)$ endowed with the standard norm $\| x \|_{\infty} = \sup\{ \| x(t) \| : t \in I \}$, $x \in C(I, E)$ is a Banach space.

We shall consider a subdivision of $[0, T]$, that is, a finite set $\{0, t_1, \ldots, t_{m+1}\}$ such that $0 = t_0 < t_1 < t_2 < \ldots < t_{m+1} = T$. We set $I_0 = [0, t_1]$ and $I_i = (t_i, t_{i+1}]$, for $i = 1, \ldots, m$. For a given function $x : [0, T] \to E$, the notations $x|_{I_i}$ and $x(t^+)$ stand for the restriction of the function $x$ on $I_i$ and the right limit of $x$ at $t$, respectively.

Let $\mathcal{PC}([0, T], E) \equiv \mathcal{PC}$ be the set of all piecewise continuous functions $x : [0, T] \to E$ such that $x|_{I_i} \in C(I_i, E)$ and $x(t^+_i) \in E$ for $i = 1, \ldots, m$. Clearly the space $\mathcal{PC}$ furnished with the supremum norm becomes a Banach space.

2.1. Measure of noncompactness.

Definition 2.1. Let $\mathcal{E}^+$ be a positive cone of an ordered Banach space $(\mathcal{E}, \leq)$. A function $\beta$ defined on the set of all bounded subsets of the Banach space $\mathcal{E}$ with values in $\mathcal{E}^+$ is called a measure of noncompactness $\text{MNC}$ on $\mathcal{E}$ if $\beta(\text{Conv}(\Omega)) = \beta(\Omega)$ for all bounded subsets $\Omega \subseteq \mathcal{E}$.

Now we recall some basic definitions and properties of Hausdorff measure of noncompactness, which will be used in proof of the main results.

Definition 2.2. Let $\Omega$ be a bounded subset of $\mathcal{E}$. The Hausdorff measure of noncompactness is defined by:

$$\beta(\Omega) = \inf \{ \epsilon > 0 : \text{has a finite cover by balls of radius less than } \epsilon \}.$$

From now on, we denote by $\beta$ and $\beta_{\mathcal{PC}}$ the Hausdorff measure of noncompactness in the spaces $\mathcal{E}$ and $\mathcal{PC}$, respectively.

Theorem 2.1. [2] Let $\Omega, \Omega_1$ and $\Omega_2$ be bounded subsets of a Banach space $\mathcal{E}$. The Hausdorff MNC has the following properties:

1. If $\Omega_1 \subset \Omega_2$, then $\beta(\Omega_1) \leq \beta(\Omega_2)$ (monotonicity).
2. $\beta(\Omega) = \beta(\overline{\Omega})$ (invariance under closure)
3. $\beta(\overline{\Omega}) = 0$ if only if $\Omega$ is relatively compact (regularity),
4. $\beta(\{a\} \cup \Omega) = \beta(\Omega)$, for every $a \in \mathcal{E}$ (invariance under translations),
5. $\beta(\Omega_1 \cup \Omega_2) = \max \{ \beta(\Omega_1), \beta(\Omega_2) \},$
6. $\beta(\Omega_1 \cap \Omega_2) \leq \min \{ \beta(\Omega_1), \beta(\Omega_2) \}.$
(7) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$, where $\Omega_1 + \Omega_2 = \{x + y : x \in \Omega_1, y \in \Omega_2\}$,
(8) $\beta(\lambda \Omega) \leq |\lambda| \beta(\Omega)$,
(9) if the map $G : D(G) \to E$ is Lipschitz continuous with constant $k$, then $\beta(G(\Omega)) \leq k \beta(\Omega)$,
(10) if $(E_n)_{n=1}^\infty$ is a sequence of nonempty, bounded and closed subsets of $E$ such that $E_{n+1} \subset E_n$ and $\lim_{n \to \infty} \beta(E_n) = 0$, then the intersection $E_\infty = \cap_{n=0}^{\infty} E_n$ is nonempty and compact.

A collection $K \subset PC$ of $E$-valued functions on $[0,T]$ is called piecewise equicontinuous if it is equicontinuous with respect to the topology on $PC$.

**Lemma 2.2.** [26] If $K \subset PC$ is bounded and piecewise equicontinuous, then $\beta(K(t))$ is piecewise continuous for $t \in [0,T]$ and

$$\beta_{PC}(K) = \sup \{\beta(K(t)) : 0 \leq t \leq T\},$$

where $K(t) = \{x(t) : x \in K\}$.

**Lemma 2.3.** [26] If $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $L^1([0,T],E)$ satisfying

$$\sup_{n \geq 1} \|f_n(t)\| \leq \mu(t), \quad a.e \ t \in [0,T],$$

where $\mu \in L^1([0,T],\mathbb{R})$. Then, the function

$$\psi(t) = \beta(\{f_n(t) : n \geq 1\})$$

is integrable and satisfies the following estimation:

$$\beta\left(\left\{\int_0^T f_n(s)\,ds : n \geq 1\right\}\right) \leq 2 \int_0^T \psi(s)\,ds.$$

### 2.2. Integrodifferential equations.

The resolvent operator is a valuable tool in the study of the existence of solutions and give a variation of constants formula for nonlinear systems. In this subsection, we recall some tools and facts that play an important role for the existence of the resolvent operators. For more details, we refer to [21]. Let $Z_1$ and $Z_2$ be Banach spaces. Then $\mathcal{L}(Z_1)$ and $\mathcal{L}(Z_1, Z_2)$ denote respectively the space of bounded linear operators on $Z_1$ and the space of bounded linear operators from $Z_1$ to $Z_2$.

#### 2.2.1. Resolvent operators.

Consider the following linear homogeneous equation:

\begin{align*}
(2) \quad \begin{cases} 
  x'(t) = A(t)x(t) + \int_0^t \Gamma(t,s)x(s)\,ds & \text{for } t \in [0,T], \\
  x(0) = x_0 \in E,
\end{cases}
\end{align*}

In what follows $Y$ is the Banach space $D(A)$ provided with the following norm

$$\|y\|_Y := \|A(0)y\| + \|y\| \text{ for } y \in Y.$$

**Definition 2.3.** [21].

A family $\{R(t,s), 0 \leq s \leq t\}$ of bounded linear operator on $E$ is called resolvent operator for Eq. (2) if the following properties are verified:
(i) $R(t,s)$ is strongly continuous in $s$ and $t$, $R(s,s) = \text{id}_E$ (the identity map of $E$), $0 \leq s \leq t$ and $\|R(t,s)\| \leq Me^{\gamma(t-s)}$ for some constants $M$ and $\gamma$,
(ii) $R(t,s)Y \subset Y$ and $R(t,s)$ is strongly continuous in $s$ and $t$ on $Y$,
(iii) For every $x \in Y$, $R(t,s)x$ is strongly continuously differentiable in $s$ and $t$ and nonempty

\[
\begin{align*}
\frac{\partial R}{\partial t}(t,s)x &= A(t)R(t,s)x + \int_s^t \Gamma(t,r)R(r,s)xdr, \\
\frac{\partial R}{\partial s}(t,s)x &= -R(t,s)A(s)x - \int_s^t R(t,r)\Gamma(r,s)xdr.
\end{align*}
\]

**Definition 2.4.** Let $(A(t))_{t \in [0,T]}$ be a family of infinitesimal generators of $C_0$-semigroups. $(A(t))_{t \in [0,T]}$ is called stable if there exist constants $M_0 \geq 1$ and $\alpha_0$ such that

\[
\left\| (A(s_k) - \lambda \text{id}_E)^{-1}(A(s_{k-1}) - \lambda \text{id}_E)^{-1} \cdots (A(s_1) - \lambda \text{id}_E)^{-1} \right\| \leq \frac{M_0}{(\lambda - \alpha_0)^k}
\]

for all $\lambda > \alpha_0, 0 \leq s_1 \leq s_2 \leq \ldots \leq s_k < T$ for $k = 1, 2, 3, \ldots$

To obtain the existence of the resolvent operator of Eq.(2), we assume the following hypotheses due to [21].

Let $\mathcal{B}(\cdot)$ be defined by

\[
(\mathcal{B}(x))(s) = \Gamma(t+s,t)x \text{ for } x \in Y \text{ and } t, s \geq 0.
\]

Let $BUC(\mathbb{R}^+, E)$ be the space of bounded uniformly continuous functions on $\mathbb{R}^+$ into $E$ provided with the sup norm.

(1) The family $(A(t))_{t \in [0,T]}$ is a stable family of infinitesimal generators of $C_0$-semigroups such that $A(t)x$ is strongly continuously differentiable on $[0,T]$ for $x \in Y$. In addition, $\mathcal{B}(t)x$ is strongly continuously differentiable on $[0,T]$ for $x \in Y$.

(2) $\mathcal{B}(t)$ is continuous on $[0, +\infty]$ into $\mathcal{L}(Y, \mathfrak{F})$, where $\mathfrak{F} \subset BUC(\mathbb{R}^+, E)$ is a Banach space with a norm stronger than the sup norm on $BUC(\mathbb{R}^+, E)$.

(3) $\mathcal{B}(t) : Y \to D(D_s)$ for all $t \geq 0$, where $D_s$ is infinitesimal generator of the $C_0$-semigroup $(T(t))_{t \geq 0}$ on $\mathfrak{F}$ defined by $T(t)\ell(s) = \ell(t+s)$, for $t, s \geq 0$.

(4) $D_s \mathcal{B}(t)$ is continuous on $\mathbb{R}^+$ into $\mathcal{L}(Y, \mathfrak{F})$.

If all the above four conditions are satisfied, we say that condition (R) is verified.

**Lemma 2.4.** [21] Suppose that condition (R) is verified, then Eq. (2) has a unique resolvent operator.

**Lemma 2.5.** [21] Suppose that condition (R) is verified. If $x_0 \in D(A)$ and $h \in C^1([0,T], E)$ then (2) has a classical solution given by

\[
x(t) = R(t,0)x_0 + \int_0^t R(t,s)h(s)ds.
\]

It is from this latter fact that we shall define mild solution (see below) of (1) using the eventual resolvent operator.

**Lemma 2.6.** [21] If $A(t) \equiv A$ and $\Gamma(t,s) \equiv \Gamma(t-s)$ and there exists a resolvent operator $R(t,s)$, then $R(t,s) = R(t-s,0)$. 
Du to the previous Lemma, the resolvent operator for autonomous system with convolution perturbation is a one parameter operator valued function and the operator \( B(\cdot) \) is constant.

Then conditions 1 and 2 are satisfied if \( A \) generates a semigroup. Another important fact is

\[
1 \leq M_T := \sup_{0 \leq s \leq t \leq T} ||R(t,s)|| < \infty \quad \text{by definition of} \quad R \quad \text{and the uniform boundedness principle.}
\]

### 2.3. Norm-continuity of the resolvent operators

In the sequel, we establish a sufficient condition to obtain the norm continuity of the resolvent operator for the system (2).

Now, consider the following Cauchy problem:

\[
\begin{align}
\begin{cases}
x'(t) &= A(t)x(t), & 0 \leq s \leq t \leq T, \\
x(s) &= x \in E.
\end{cases}
\end{align}
\]

**Definition 2.5.** A family \( \{S(t,s) : 0 \leq s \leq t \leq T\} \) of linear, bounded operators on a Banach space \( X \) is called an evolution family for (4) if

1. \( S(t,t) = I \) and \( S(t,s) = S(t,r)S(r,s) \) for every \( 0 \leq s \leq t \leq T \),
2. The mapping \( \{(t,s) \in \mathbb{R}^2 : s \leq t\} \ni (t,s) \mapsto S(t,s) \) is strongly continuous,
3. \( ||S(t,s)|| \leq N \exp(\omega(t-s)) \) for some \( N \geq 1, \omega \in \mathbb{R} \) and all \( t \geq s \in \mathbb{R} \),
4. For each \( z \in Y, S(t,s) \) is strongly continuously differentiable in \( t \) and \( s \) with

\[
(\partial/\partial t)S(t,s)z = A(t)S(t,s)z,
\]

\[
(\partial/\partial s)S(t,s)z = -S(t,s)A(t)z.
\]

**Lemma 2.7.** [28] Let \( \{A(t)\}_{t \in [0,T]} \) a family of linear operators on a Banach space \( E \) and consider the corresponding non-autonomous Cauchy problem (4). The following assertions are equivalent.

1. Eq. (4) is well posed (with exponentially bounded solutions).
2. There exists a unique strongly continuous (exponentially bounded) evolution family \( \{S(t,s) : 0 \leq s \leq t \leq T\} \) on \( E \) solving Eq. (4).

\( (A(t))_{t \in [0,T]} \) is a stable family of generators and \( A(t)z \) is strongly continuously differentiable on \([0,T], \) so Eq. (4) has a fundamental solution [24, p.120].

In the following, we consider the following perturbation of equation (4):

\[
\begin{align}
\begin{cases}
x'(t) &= A(t)x(t) + \int_s^t \Gamma(t,u)x(u)du, & 0 \leq s \leq t \leq T \\
x(s) &= x \in E
\end{cases}
\end{align}
\]

The variation of constants formula related to (4) combined with the resolvent operator of (5), give a concise relation between the resolvent operator and the evolution family.

**Lemma 2.8.** Let \( \{A(t)\}_{t \in [0,T]} \) be the generator of an evolution family \( \{S(t,s) : 0 \leq s \leq t \leq T\} \) and the condition \( (R) \) holds. If \( \{R(t,s) : 0 \leq s \leq t \leq T\} \) is the resolvent operator for Eq. (5) then we have the following representation

\[
R(t,s)x = S(t,s)x + \int_s^t S(t,r)Q(r)xdr
\]
with
\[ Q(r)x = \Gamma(r,r) \int_s^r R(\tau,s)x \, d\tau - \int_s^r \frac{\partial \Gamma(r,u)}{\partial u} \int_s^u R(\tau,s)x \, d\tau \, du \]
such that \( \{Q(\cdot)x : t \in [0,T]\} \) are uniformly bounded and for each \( x \in E \), \( Q(\cdot)x \in C([0,T],E) \).

**Remark 2.1.** The above Lemma is essential for the continuation of the proof of the main result. Let us point out that by the uniform boundedness principle \( \sup_{t \in [0,T]} \|Q(t)\|_{L(E)} < \infty \).

**Proof of Lemma 2.8.** For all \( x \in Y \), the map \( P(\cdot)x : [0,T] \to (Y,||\cdot||_Y) \) defined by
\[ P(t)x = \int_t^s R(u,s)x \, du \]
is a continuous function and for all \( x \in Y \)
\[ \sup_{0 \leq t \leq T} |P(t)x|_Y < \infty. \]

By the variation of constants formula, a mild solution of Eq. (5) is given by
\[ x(t) = S(t,s)x + \int_s^t S(t,r)L(r) \, dr, \]
where \( L(r) = \int_s^r \Gamma(r,u)x(u) \, du \). From the definition of the resolvent operator of (5), we have \( x(t) = R(t,s)x \). Then, we have
\[ R(t,s) = S(t,s)x + \int_s^t S(t,r)L(r) \, dr \]
\[ = S(t,s)x + \int_s^t S(t,r) \int_s^r \Gamma(r,u)x(u) \, du \, dr \]
\[ = S(t,s)x + \int_s^t S(t,r) \int_s^r \Gamma(r,u)R(u,s) \, du \, dr \]
\[ = S(t,s)x + \int_s^t S(t,r)Q(r)x \, dr \]
where \( Q(r)x = \int_s^r \Gamma(r,u)R(u,s)x \, du \).

The derivative with respect to \( t \) of \( P(t)x \) is given by \( R(t,s)x \). Indeed, the function \( t \mapsto R(t,s)x \) is continuous. It should be noted that
\[ Q(r)x = \int_s^r \Gamma(r,u)R(u,s)x \, du \]
\[ = \int_s^r \Gamma(r,u)Q'(r) \, du \]
\[ = \Gamma(r,r)P(r)x - \int_s^r \frac{\partial \Gamma(r,u)}{\partial u} P(u)x \, du \]
\[ = \Gamma(r,r)P(r)x - \int_s^r \frac{\partial \Gamma(r,u)}{\partial u} P(u)x \, du \]
\[ = \Gamma(r,r) \int_s^r R(\tau,s)x \, d\tau - \int_s^r \frac{\partial \Gamma(r,u)}{\partial u} \int_s^u R(\tau,s)x \, d\tau \, du \]
where we have applied integration by part formula to get (9). (See [12, Proposition 1] for more details). Using (7) and the uniform boundedness principle, the family \( \{ P(t) : t \in [0, T] \} \) of continuous linear operators are uniformly bounded.

Combining the above and the condition on \( \Gamma(r, s) \) from condition (R), we have that \( \Gamma(r, r)P(r)x - \int_s^t \frac{\partial \Gamma(r, u)}{\partial u} P(u)x \, du \) is uniformly bounded in \([0, T]\). Therefore \( \text{Q}(.)x \in C([0, T], Y) \).

By continuous extension, the claim holds for all \( x \in E \).

\( \square \)

**Definition 2.6.** The resolvent operator \( \{ R(t, s) : 0 \leq s \leq t \leq T \} \) is said to be norm-continuous if the function \( (t, s) \mapsto R(t, s) \) is continuous by operator norm for \( 0 \leq s < t \leq T \).

**Theorem 2.9.** Let \( A(t) \) be a family of linear operators generating an evolution family \( \{ S(t, s) : 0 \leq s \leq t \leq T \} \) and conditions (R) holds. Then the resolvent operator \( \{ R(t, s) : 0 \leq s \leq t \leq T \} \) for equation (5) is norm-continuous for \( 0 \leq s < t \leq T \) if the evolution family \( \{ S(t, s) : 0 \leq s \leq t \leq T \} \) is norm-continuous for \( 0 \leq s < t \leq T \).

**Proof.** From Lemma 2.8, we have that

\[
R(t, s)x = S(t, s)x + \int_0^{t-s} S(t, r+s)Q(r+s)x \, dr.
\]

Define \( \tilde{q} = \sup_{0 \leq t \leq T} \| \text{Q}(t) \|_{E} \) and let \( x \in E \) be an arbitrary and such that \( \| x \|_E \leq 1 \). Moreover, there exists \( N > 0, \omega \in \mathbb{R} \) such that \( \| \text{S}(t, s) \|_{E} \leq N \exp(\omega(t-s)) \) for \( 0 \leq s \leq t \leq T \). Assume that the evolution family \( \text{S}(t, s) \) is norm-continuous for \( t > s \). The rest of the proof is divided in two cases.

**Case 1:** For \( 0 \leq s < t \leq T, 0 \leq h_2 \leq h_1 \) and \( h_1 - h_2 \in (0, T-(t-s)) \), we have

\[
\| R(t+h_1, s+h_2)-R(t, s)x \|_E \leq \| S(t+h_1, s+h_2)x + \int_0^{t-h_1}(s+h_2) S(t+h_1, r+(s+h_2))Q(r+s+h_2)xdr - S(t, s)x - \int_0^{t-s} S(t, r+s)Q(r+s)xdr \|_E
\]

\[
= \| S(t+h_1, s+h_2)x + \int_0^{t-h_1}(s+h_2) S(t+h_1, r+(s+h_2))Q(r+s+h_2)xdr - S(t, s)x - \int_0^{t-s} S(t, r+s)Q(r+s)xdr \|
\]

\[
\leq \| S(t+h_1, s+h_2)x - S(t, s)x \|_E + \int_0^{t-h_1}(s+h_2) S(t+h_1, r+s+h_2)Q(r+s+h_2)xdr - \int_0^{t-s} S(t, r+s)Q(r+s)xdr 
\]

\[
\leq \| S(t+h_1, s+h_2)x - S(t, s)x \|_E + \int_0^{t-s}(s+h_2) S(t+h_1, r+s+h_2)Q(r+s+h_2)xdr
\]

\[
- \int_0^{t-s} S(t, r+s)Q(r+s)xdr \|
\]
Then,

\[
\|R(t+h_1,s+h_2)x - R(t,s)x\|_E \\
\leq \|S(t+h_1,s+h_2)x - S(t,s)x\| \\
\quad + \left\| \int_0^{t-s} S(t+h_1,r+s+h_2)Q(r+s+h_2)x \, dr \right\| \\
\quad + \left\| \int_0^{(t-s)+(h_1-h_2)} S(t+h_1,r+s+h_2)Q(r+s+h_2)x \, dr - \int_0^{t-s} S(t,r+s)Q(r+s)x \, dr \right\| \\
\quad + \left\| \int_0^{t-s} S(t+r+s+h_2)Q(r+s+h_2)x \, dr - \int_0^{t-s} S(t+r+s)Q(r+s)x \, dr \right\| \\
\leq \|S(t+h_1,s+h_2)x - S(t,s)x\| + \int_{t-s}^{(t-s)+(h_1-h_2)} \|S(t+h_1,r+s+h_2)Q(r+s+h_2)x\| \, dr \\
\quad + \int_0^{t-s} \|\{S(t+h_1,r+s+h_2) - S(t+r+s)\} Q(r+s+h_2)x\| \, dr \\
\quad + \int_0^{t-s} \|S(t+r+s)[Q(r+s+h_2)x - Q(r+s)x]\| \, dr.
\]

Hence,

\[
\|R(t+h_1,s+h_2) - R(t,s)\|_{L(E)} \\
\leq \|S(t+h_1,s+h_2) - S(t,s)\|_{L(E)} + \tilde{q}N \int_{t-s}^{(t-s)+(h_1-h_2)} \exp(\gamma(t+h_1-r-s-h_2)) \, dr \\
\quad + \tilde{q} \int_0^{t-s} \|S(t+h_1,r+s+h_2) - S(t+r+s)\|_{L(E)} \, dr \\
\quad + N \int_0^{t-s} \exp(\gamma(t-r-s)) \|Q(r+s+h_2) - Q(r+s)\|_{L(E)} \, dr.
\]

Using the continuity of \((t,s) \mapsto S(t,s)\) in the operator norm for \(0 \leq s < t \leq T\), the continuity of \(t \mapsto Q(t)\) in the operator norm for \(t \in [0,T]\) and Lebesgue dominated convergence Theorem, we get that

\[
\|R(t+h_1,s+h_2) - R(t,s)\|_{L(E)} \to 0 \text{ as } (h_1,h_2) \to (0^+,0^+).
\]
**Case 2:** For $0 \leq s < t < T$, $0 \leq h_1 \leq h_2$ and $h_1 - h_2 \in (-t, s)$, we have that
\[
\|R(t+h_1, s+h_2) x - R(t, s) x\|_E
\]
\[
\leq \|S(t+h_1, s+h_2) x - S(t, s) x\|_E + \int_{(t-s)+(h_1-h_2)}^{(t-s)} \|S(t+h_1, r+s+h_2) Q(r+s+h_2) x\| dr
\]
\[
+ \int_0^{t-s} \|S(t+h_1, r+s+h_2) - S(t, r+s)\| Q(r+s+h_2) x\| dr
\]
\[
+ \int_0^{t-s} \|S(t, r+s)\| Q(r+s+h_2) x - Q(r+s) x\| dr.
\]
It follows that
\[
\|R(t+h_1, s+h_2) - R(t, s)\|_{\mathcal{L}(E)}
\]
\[
\leq \|S(t+h_1, s+h_2) - S(t, s)\|_{\mathcal{L}(E)} + \tilde{q}N \int_{(t-s)+(h_1-h_2)}^{(t-s)} \exp(\gamma(t+h_1-r-s-h_2)) dr
\]
\[
+ \tilde{q} \int_0^{t-s} \|S(t+h_1, r+s+h_2) - S(t, r+s)\|_{\mathcal{L}(E)} dr
\]
\[
+ N \int_0^{t-s} \exp(\gamma(t-r-s)) \|Q(r+s+h_2) - Q(r+s)\|_{\mathcal{L}(E)} dr.
\]
The continuity of $t \mapsto S(t, s)$ in the operator norm for $0 \leq s < t \leq T$, combined with the continuity of $t \mapsto Q(t)$ in the operator norm for $t \in [0, T]$ and Lebesgue dominated convergence Theorem yields that
\[
\|R(t+h_1, s+h_2) - R(t, s)\|_{\mathcal{L}(E)} \to 0 \text{ as } (h_1, h_2) \to (0^-, 0^-).
\]
Accordingly, if the evolution family $\{S(t, s) : 0 \leq s \leq t \leq T\}$ is norm-continuous for $0 \leq s < t \leq T$ then the resolvent operator $\{R(t, s) : 0 \leq s \leq t \leq T\}$ is norm-continuous for $0 \leq s < t \leq T$. \[\square\]

**2.3.1. Mönch’s fixed point Theorem.** Here we recall the Mönch’s fixed point Theorem that is the key tool for proving solvability of the integrodifferential equation (1).

**Theorem 2.10.** [27] Let $K$ be an open neighborhood for the origin in Banach space $E$. Suppose that $\mathcal{F}: K \to E$ is a continuous map which satisfies the following conditions:

(a) If $D$ is countable set such that $D \subseteq K$ and $D \subseteq \text{Conv}(\{0\} \cup \mathcal{F}(D))$ then $D$ is compact.

(b) $\mathcal{F}(x) \neq \lambda x$ for all $x \in \partial K$ and $\lambda > 1$.

Then, $\mathcal{F}$ has a fixed point.

The points (a) and (b) in above Theorem are called Mönch’s condition and Leray-Schauder condition, respectively. We now state the Mönch’s fixed point Theorem which we shall use for selfmappings in the sequel.
Theorem 2.11. Let $K$ be a closed, convex and bounded subset of a Banach space $E$ and $0 \in K$. Suppose that $F : K \to K$ is a continuous map which satisfies Mönch condition then $F$ has a fixed point.

We end this section by recalling the following useful Lemma.

Lemma 2.12 (Lemma 12, [19]). Let $E$ be a Banach space and $(U_n)_{n \geq 1} \subset L(E)$. Assume that $U_n x \to U x$ for all $x \in E$ for some $U \in L(E)$. Then, for any compact set $K$ in $E$, $U_n$ converges to $U$ uniformly in $K$, namely,

$$\sup_{x \in K} \|U_n(x) - U(x)\| \to 0, \quad \text{as } n \to +\infty.$$ 

3. Existence of mild solutions for equation (1)

In this section, using definitions and lemmas given in Section 2, we will study the existence of mild solutions for nonlocal impulsive integrodifferential equations (1) by means of Mönch fixed point theorem. In order to obtain the required result, we impose the conditions on $R$, $f$, $g$ as follows:

Definition 3.1. A function $x \in PC$ is said to be a mild solution of the nonlocal initial value problem (1) if for every $t \in [0, T]$,

$$x(t) = R(t, 0)[x_0 + g(x)] + \int_0^t R(t, s)f(s, x(s))ds + \sum_{0 < t_k < t} R(t, t_k)J_k(x(t_k)).$$

Remark 3.1. Theorem 2.9 gives a sufficient condition that guaranteed assumption (H0). Assumption (H4) is hold when we choose the involving Lipschitz coefficients with respect to Hausdorff MNC small enough.

(H0) The resolvent operator $(t, s) \mapsto R(t, s)$ of the linear part of (1) is norm-continuous in $E$ for $0 \leq s < t \leq T$.

(H1) The function $f : [0, T] \times E \to E$ satisfies the following:

(f1) For $t \in [0, T]$, $f(t, \cdot) : E \to E$ is continuous and for all $x \in E$, $f(\cdot, x) : [0, T] \to E$ is strongly mesurable.

(f2) There exist an integrable function $\eta : [0, T] \to [0, +\infty]$ such that $\beta[f(t, D)] \leq \eta(t)\beta(D)$ for all $t \in [0, T]$ and $D \subset E$.

(f3) there exist $L_f \in L^1([0, T], \mathbb{R}^+)$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ a nondecreasing continuous function such that

$$\|f(t, x)\| \leq L_f(t)\phi(\|x\|)$$

for $t \in [0, T]$, $x \in E$ and

$$\lim_{r \to \infty} \inf_{r} \frac{\phi(r)}{r} = 0,$$

(H2) $J_i : E \to E$, $i = 1, 2, \ldots, m$, are continuous such that
(I1) there are nondecreasing functions $L_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that
\[
\|J_i(x(t))\| \leq L_i(\|x(t)\|),
\]
for all $t \in [0, T], x \in \mathcal{PC}, i = 1, 2, \ldots, m$ and
\[
\liminf_{\rho \to \infty} \frac{L_i(\rho)}{\rho} = 0, \quad i = 1, 2, \ldots, m.
\]

(I2) $J_i$ is Lipschitzian with respect to the Hausdorff measure of noncompactness i.e there exist $K_i \geq 0$ such that
\[
\beta(J_i(S)) \leq K_i \beta(S)
\]
for any nonempty and bounded $S \subset E, i = 1, 2, \ldots, m$.

(H3) The function $g : \mathcal{PC} \to E$ is continuous, compact and satisfies the following condition :
\[
\lim_{r \to +\infty} \inf \frac{g_r}{r} = 0
\]
where $g_r = \sup \left\{ \|g(x)\| : \|x\| \leq r \right\}$ with $r > 0$.

(H4) The following inequality is true :
\[
\kappa := M_T \left( 2\|\eta\|_L + \sum_{i=0}^{m} K_i \right) < 1,
\]
where $M_T := \sup_{0 \leq s \leq t \leq T} \|R(t, s)\|$.

For each $r > 0$, we define $B_r := \{ x \in \mathcal{PC} : \| x \|_{\mathcal{PC}} \leq r \}$. The existence result for the problem (1) can be stated as following :

**Theorem 3.1.** Suppose that (H0)-(H4) are satisfied. Then, the impulsive nonlocal system (1) has at least one mild solution on $[0, T]$.

**Proof.** Consider the operator $\mathcal{F} : \mathcal{PC} \to \mathcal{PC}$ defined by
\[
\mathcal{F}(x)(t) = R(t, 0) \left[ x_0 + g(x) \right] + \int_0^t R(t, s)f(s, x(s))ds + \sum_{0 < t_i < t} R(t, t_i)J_i(x(t_i)).
\]
It is clear that any fixed point of the map $\mathcal{F}$ is a mild solution of (1). We split the proof in three steps.

**Step 1 : Claim** There exists $r > 0$ such that $\mathcal{F}(B_r) \subset B_r$.

If this condition fails, then for every $r > 0$, there exists $x_r \in B_r$ such that $\mathcal{F}(x_r) \notin B_r$ i.e. $\|\mathcal{F}(x)(\tau)\| > r$ for some $\tau \in [0, T]$ where $\tau$ depends on $r$. Now
$r^{-1} \|F(x_r)(\tau)\| > 1$ implies that $\lim_{r \to +\infty} \inf \frac{\|F(x_r)(\tau)\|}{r} \geq 1$. We have

$$\|F(x_r)(\tau)\| = \|F_1(x_r)(\tau) + F_2(x_r)(\tau)\|$$

$$\leq \|R(\tau,0)[x_0 + g(x_r)]\| + \int_0^\tau |f(s, x_r(s))| ds + \sum_{0 < t_i < \tau} R(\tau, t_i) J_i(x_r(t_i))$$

$$\leq M_T \|x_0 + g(x_r)\| + M_T \int_0^\tau |f(s, x_r(s))| ds + M_T \sum_{0 < t_i < \tau} J_i(x_r(t_i))$$

$$\leq M_T \|x_0\| + M_T \sup_{x \in B_r} \|g(x)\| + M_T \int_0^T L_f(s) \phi(\|x_r(s)\|) ds + M_T \sum_{0 < t_i < \tau} L_i(\|x_r(t_i)\|)$$

$$\leq M_T \|x_0\| + M_T g_r + M_T \int_0^T L_f(s) \phi(\|x_r(s)\|) ds + M_T \sum_{0 < t_i < \tau} L_i(\|x_r(t_i)\|)$$

$$\leq M_T \|x_0\| + M_T g_r + M_T \|L_f\| L_1 \phi(r) + M_T \sum_{i=1}^m L_i(r),$$

where we have used the non decreasing property of $\phi$ and $L_i, i = 1, \ldots, m$.

Dividing both sides by $r$ and let $r \to +\infty$, we get

$$\lim_{r \to +\infty} \inf \frac{\|F(x_r)(\tau)\|}{r} \leq \lim_{r \to +\infty} \inf \frac{\|x_0\|}{r} + \frac{g_r}{r} + \frac{M_T \|L_f\| L_1 \phi(r)}{r} ds$$

$$+ M_T \sum_{i=1}^m \lim_{r \to +\infty} \frac{L_i(r)}{r}$$

$$\leq 0.$$
Similarly, if \( t \in I_i, 1 \leq i \leq m \), then by continuity of \( J_i \), we get

\[
\lim_{n \to \infty} \| (\mathcal{F}_2 x_n)(t) - (\mathcal{F}_2 x)(t) \| \leq \lim_{n \to \infty} M_T \| g(x_n) - g(x) \|
\]

\[
+ M_T \lim_{n \to \infty} \sum_{k=1}^{k=i} \| J_i(x_n(t_k)) - J_i(x(t_k)) \|
\]

\[
+ \lim_{n \to \infty} M_T \int_0^t \| f(s, x_n(s)) - f(s, x(s)) \| ds
\]

\[
= 0.
\]

Hence \( \mathcal{F} \) is continuous on \( B_r \).

**Step 3: Claim** The Mönch's condition holds.

For this event, let us assume that \( D \) is a countable subset of \( B_r \) and \( D \subset \text{Conv}\{\{0\} \cup \mathcal{F}(D)\} \). Then, we demonstrate that \( \beta(D) = 0 \), where \( \beta \) is the Hausdorff measure of noncompactness.

For this purpose, without loss of generality, we may consider that \( D = \{x^n\}_{n=1}^\infty \). If we are able to show that \( \{(\mathcal{F}x^n)\}_{n=1}^\infty \) is equicontinuous on \( [0, T] \), then \( D \subset \text{Conv}\{\{0\} \cup \mathcal{F}(D)\} \) if also equicontinuous on \( [0, T] \).

Firstly, we show that \( \mathcal{F}(D) \) is piecewise equicontinuous on \( I = [0, T] \) i.e. \( \mathcal{F}(D) \) is equicontinuous on \( I_0 = [0, t_1], I_i = (t_i, t_{i+1}] \) and also equicontinuous at \( t = t_i^+ \). \( i = 1, 2, \ldots, m \). Indeed, we only need to prove that \( \mathcal{F}(D) \) is equicontinuous in \( I_i, i = 0, 1, \ldots, m \). Consider the following cases:

**Case 1.** If \( i = 0 \), we have the following subcases:

1. Let \( s_1 = 0 \) and \( s_2 \in (0, t_1] \). Observe that

\[
\| (\mathcal{F}x^n)(s_2) - (\mathcal{F}x^n)(0) \| \leq \| [R(s_2, 0) - R(0, 0)](x_0 + g(x^n)) \|
\]

\[
+ \left\| \int_0^{s_2} R(s_2, s) f(s, x_n(s)) ds \right\|
\]

\[
\leq \| [R(s_2, 0) - R(0, 0)] x_0 \| + \| [R(s_2, 0) - R(0, 0)] g(x^n) \|
\]

\[
+ \int_0^{s_2} \| R(s_2, s) \|_{L(E)} L_f(s) \phi(\| x^n(s) \|) ds
\]

\[
\leq \| [R(s_2, 0) - R(0, 0)] x_0 \| + \| [R(s_2, 0) - R(0, 0)] g(x^n) \|
\]

\[
+ \phi(r) M_T \int_{s_1}^{s_2} L_f(s) ds.
\]

We have that

\[
\| R(h, 0) g(x^n) - g(x^n) \| \leq \sup_{y \in g(D)} \| R(h, 0) y - y \| \to 0 \text{ as } h \to 0^+
\]
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by Lemma 2.12 since \(g(D)\) is compact. Therefore, independently on \(x^n\) we have,

\[
\lim_{s_2 \to s_1} \| (\mathcal{F} x^n)(s_2) - (\mathcal{F} x^n)(s_1) \| = 0.
\]

2. Let \(0 < s_1 < s_2 \in (0, t_1)\), then

\[
\| (\mathcal{F} x^n)(s_2) - (\mathcal{F} x^n)(s_1) \| \leq \left\| R(s_2, 0) - R(s_1, 0) \right\| \left( x_0 + g(x^n) \right)
\]

\[
+ \left\| \int_0^{s_2} R(s_2, s) f(s, x_n(s)) ds - \int_0^{s_1} R(s_1, s) f(s, x_n(s)) ds \right\|
\]

\[
\leq \left\| R(s_2, 0)x_0 - R(s_1, 0)x_0 \right\|
\]

\[
+ \left\| R(s_2, 0)g(x^n) - R(s_1, 0)g(x^n) \right\|
\]

\[
+ \phi(r) \int_0^{s_1} \left\| R(s_2, s) - R(s_1, s) \right\|_{\mathcal{L}(E)} L_f(s) ds
\]

\[
+ \phi(r) M_T \int_{s_1}^{s_2} L_f(s) ds
\]

\[
\leq P_1 + P_2 + P_3 + P_4,
\]

where

\[
P_1 = \left\| R(s_2, 0)x_0 - R(s_1, 0)x_0 \right\|
\]

\[
P_2 = \left\| R(s_2, 0)g(x^n) - R(s_1, 0)g(x^n) \right\|
\]

\[
P_3 = \phi(r) \int_0^{s_1} \left\| R(s_2, s) - R(s_1, s) \right\|_{\mathcal{L}(E)} L_f(s) ds
\]

\[
P_4 = \phi(r) M_T \int_{s_1}^{s_2} L_f(s) ds
\]

\[
P_5 = \sum_{i=0}^m \left\| \left( R(s_2, t_i) - R(s_1, t_i) \right) \right\|_{\mathcal{L}(E)} L_i(r).
\]

We will show that \(P_k \to 0\) for \(k = 1, 2, 3, 4, 5\) as \(s_2 - s_1 \to 0\). Definition 2.3 point (ii), give

\[
\lim_{s_2 \to s_1} P_1 = \lim_{s_2 \to s_1} \left\| R(s_2, 0)x_0 - R(s_1, 0)x_0 \right\|
\]

\[
\leq \left\| x_0 \right\| \left\| R(s_2, 0) - R(s_1, 0) \right\|
\]

\[
= 0,
\]

independent of \(x^n\). For the term \(P_2\) we have that

\[
\left\| R(s_2, 0)g(x^n) - R(s_1, 0)g(x^n) \right\| \leq \left\| R(s_2, 0) - R(s_1, 0) \right\|_{\mathcal{L}(E)} g_r.
\]
Therefore by the continuity of \( t \to R(t,0) \) for \( t > 0 \) in the operator-norm topology, \( P_2 \to 0 \) as \( |s_2 - s_1| \to 0 \). For \( P_3 \), by the continuity of the resolvent operator \( R(t,s) \) for \( t > 0 \) in the operator norm topology and the fact that \( L_f(.) \) is Lebesgue integrable on \([0,T]\), we obtain \( P_3 \to 0 \) as \( s_2 \to s_1 \). For \( P_4 \), we use the fact that the function \( L_f(s) \) is Lebesgue integrable on \([0,T]\) to get \( P_4 \to 0 \) as \( s_2 - s_1 \to 0 \).

Therefore, independently on \( x^n \) we have
\[
\lim_{s_2 \to s_1} (\mathcal{F} x^n)(s_2) = (\mathcal{F} x^n)(s_1) = 0.
\]

3. When \( t = t_1 \). Let \( s_1 > 0, s_2 > 0 \) such that \( t_1 < s_1 < s_2 < t_2 \). The definition of \( \mathcal{F} \) implies that
\[
\|(\mathcal{F} x^n)(s_2) - (\mathcal{F} x^n)(s_1)\|
\leq \left\| R(s_2,0) - R(s_1,0) \right\| (x_0 + g(x^n)) + \sum_{i=0}^{m} \left\| R(s_2, t_i) - R(s_1, t_i) \right\| \|J_i(x(t_i))\|
+ \left\| \int_{0}^{s_2} R(s_2, s) f(x_n(s))ds - \int_{0}^{s_1} R(s_1, s) f(x_n(s))ds \right\|.
\]

With similar argument as in the previous way, we have
\[
\lim_{s_2 \to s_1} (\mathcal{F} x^n)(s_2) = (\mathcal{F} x^n)(s_1) = 0.
\]

Case 2. In case of \( 1 \leq i \leq m \), use the same way as Case 1.

\[
\lim_{s_2 \to s_1} (\mathcal{F} x^n)(s_2) = (\mathcal{F} x^n)(s_1) = 0.
\]

From (11)-(14), \( \mathcal{F}(D) \) is equicontinuous on \( I_i \) for every \( 0 \leq i \leq m \).

We now show that \( \mathcal{F}(D) = \{ \mathcal{F}(x^n); x^n \in D \} \) is relatively compact in \( \mathcal{PC} \). we achieve this by using the measure of noncompactness. We have for \( t = 0 \), the set
\[
\{(\mathcal{F} x)(0); \ x \in D \} = \{x_0 + g(x); \ x \in D \} = x_0 + g(D)
\]
is relatively compact in \( \mathcal{PC} \). Since \( g \) compact then \( g(D) \) is compact also.

For \( 0 < t \leq T \), we have the following inequality
\[
\beta \left( (\mathcal{F} D)(t) \right) \leq \beta \left( (\mathcal{F}_1 D)(t) \right) + \beta \left( (\mathcal{F}_2 D)(t) \right) + \beta \left( (\mathcal{F}_2^* D)(t) \right)
\]
where
\[
(\mathcal{F}_1 x^n)(t) = R(t,0) [x_0 + g(x^n)]
\]
\[
(\mathcal{F}_2 x^n)(t) = \int_{0}^{t} R(t,s) f(s, x^n(s))ds
\]
\[
(\mathcal{F}_2^* x^n)(t) = \sum_{0 \leq t_i < t} R(t,t_i) x^n(t_i).
\]
We have that
\[ (15) \quad \beta\{\{\mathcal{F}_1 x^n(t)\}\}_{n=1}^\infty \leq \beta\{\{R(t,0)(x_0 + g(x^n))\}_{n=1}^\infty\} = 0. \]

Moreover using Lemma 2.3 and considering assumption \((H1-f2)\) we see that
\[ \beta((\mathcal{F}_2 D)(t)) = \beta\left(\left\{ \int_0^t R(t,s)f(s,x_n(s))ds \right\}_{n=1}^\infty \right) \leq 2MT \int_0^t \eta(s)\beta\{x_n(s)\}_{n=1}^\infty ds \]
\[ \leq 2MT \int_0^t \eta(s) \sup_{0 \leq s \leq t} \beta\{x_n(s)\}_{n=1}^\infty ds. \]

Then
\[ (16) \quad \sup_{0 \leq t \leq T} \beta((\mathcal{F}_2 D)(t)) \leq 2MT \sup_{0 \leq t \leq T} \beta\{x_n(t)\}_{n=1}^\infty \int_0^T \eta(s)ds. \]

From the regularity of the functions \(L_i\) with respect to the Hausdorff measure of noncompactness, that is assumption \((H2)-(I2)\), we have
\[ \beta((\mathcal{F}_2^{**} D)(t)) = \beta\left\{ \sum_{0 < t_i < t} R(t,t_i)J_i(x_n(t_i)) \right\} \]
\[ \leq MT \sum_{0 < t_i < t} \beta\left\{ J_i(x_n(t_i)) \right\}_{n=1}^\infty \]
\[ \leq MT \sum_{0 < t_i < t} K_i \beta\left\{ x_n(t_i) \right\}_{n=1}^\infty \]
\[ \leq MT \sum_{0 < t_i < t} \beta\left\{ x_n(t_i) \right\}_{n=1}^\infty \sum_{0 < t_i < t} K_i \]
\[ (17) \quad \sup_{0 \leq t \leq T} \beta((\mathcal{F}_2^{**} D)(t)) \leq MT \sup_{0 \leq t_1 \leq T} \beta\left\{ x_n(t_i) \right\}_{n=1}^\infty \sum_{i=0}^p K_i. \]

Combining (15), (16) and (17), we obtain the following
\[ \sup_{0 \leq t \leq T} \beta\left(\{\mathcal{F} D(t)\}\right) \leq 2MT \sup_{0 \leq s \leq T} \beta\{x_n(s)\}_{n=1}^\infty \int_0^T \eta(s)ds \]
\[ + MT \sup_{0 \leq t_i \leq b} \beta\left\{ x_n(t_i) \right\}_{n=1}^\infty \sum_{i=0}^p K_i \]
\[ \leq MT \left(2 \int_0^T \eta(s)ds + \sum_{i=0}^p K_i \right) \sup_{0 \leq s \leq T} \beta\{x_n(s)\}_{n=1}^\infty \]
\[ \leq MT \left(2\|\eta\|_{L_1} + \sum_{i=0}^p K_i \right) \sup_{0 \leq s \leq T} \beta\{x_n(s)\}_{n=1}^\infty \].
Since $D$ and $\mathcal{F}(D)$ are bounded and piecewise equicontinuous. Using Lemma 2.2 we see that

$$\beta(\mathcal{F}(D)) \leq M_T \left( 2\|\eta\|_{L^1} + \sum_{i=0}^{p} K_i \right) \beta(D) = \kappa \beta(D),$$

(19)

where $\kappa$ is the constant given in assumption (H4). From the Mönch’s conditions, we obtain that

$$\beta(D) \leq \beta \left( \text{Conv} \left( \{0\} \cup \mathcal{F}(D) \right) \right) = \beta(\mathcal{F}(D)) \leq \kappa \beta(D).$$

Consequently $\beta(D) = 0$ then $D$ is relatively compact in $\mathcal{P}C$.

By Mönch fixed point Theorem, Theorem 2.11, implies that $F$ has a fixed point $x \in B_r$ which is a mild solution of (1). □

4. Example

We study the impulsive effects on the following problem of heat conduction in materials with memory:

$$\frac{\partial}{\partial t} \vartheta(t,\xi) = \left[ \frac{\partial^2}{\partial \xi^2} \vartheta(t,\xi) + a_1 \frac{\partial}{\partial \xi} \vartheta(t,\xi) + a_2 \vartheta(t,\xi) \right]$$

$$+ \int_0^t \alpha e^{-\mu(t-s)} \left[ \frac{\partial^2}{\partial \xi^2} \vartheta(s,\xi) + a_1 \frac{\partial}{\partial \xi} \vartheta(s,\xi) + a_2 \vartheta(s,\xi) \right] ds + p_0 \sin(\vartheta(t,\xi)),

\begin{align*}
\vartheta(t,0) &= \vartheta(t,\pi) = 0, & t &\in [0,T], \\
\vartheta(t_i^+,\xi) - \vartheta(t_i^-,\xi) &= \pi \alpha_i(r) \cos(r \vartheta(t_i,\xi)) dr, & i &= 1, \ldots, m, \\
\vartheta(0,\xi) &= \vartheta_0(\xi) + \int_0^\pi \alpha_1(r) \cos(r \vartheta(t,\xi)) dr, & \xi &\in [0,\pi],
\end{align*}

where $0 < t_1 < \ldots < t_m < T$, $a_1, a_2 \in \mathbb{R}$, $\alpha > 0$, $\mu \in [0,\pi]$, $\alpha_i \in L^1([0,\pi])$ and $\Xi$ is a real valued uniformly continuous function on $[0, T] \times [0, \pi]$.

Let $E = L^2([0, \pi])$, $A(t) \equiv A : E \to E$ be defined by

$$\left( A \vartheta \right)(t,\xi) = \left[ \frac{\partial^2}{\partial \xi^2} \vartheta(t,\xi) + a_1 \frac{\partial}{\partial \xi} \vartheta(t,\xi) + a_2 \vartheta(t,\xi) \right]$$

with domain

$$\mathcal{D}(A) = H^2((0,\pi)) \cap H^1_0((0,\pi))$$

and

$$\Gamma(t,s) \equiv \Gamma(t - s) = \alpha e^{-\mu(t-s)} A = : p(t - s) A.$$
In this case, the condition (R) is reduced to the following assumptions (for more details, see [21])

(a) The operator A is the infinitesimal generator of a strongly continuous semigroup \( (S(t))_{t \geq 0} \) on \( E \).

(b) \( (\Gamma(t))_{t \geq 0} \) is a family of linear operators on \( E \), such that \( \Gamma(t) \) is continuous from \( Y \) to \( E \) for almost all \( t \geq 0 \). Moreover, there is a locally integrable function \( p : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), such that \( \Gamma(t)x \) is measurable and \( \|\Gamma(t)x\|_E \leq p(t)\|x\|_Y \) for all \( x \in Y \) and \( t \geq 0 \).

(c) For any \( x \in Y \), the map \( t \rightarrow \Gamma(t)x \) belongs to \( W^{1,1}_{loc}(\mathbb{R}^+, E) \) and:

\[
\left\| \frac{d\Gamma(t)x}{dt} \right\|_E \leq p(t)\|x\|_Y \quad \text{for} \quad x \in Y \quad \text{and a.e.} \quad t \geq 0.
\]

From [18, p. 173], we know that \( A \) is the infinitesimal generator of an analytic \( C_0 \) semigroup \( (S(t))_{t \geq 0} \) on \( E \). Moreover, for any \( t \geq 0 \) and any \( x \in Y \), we have:

\[
\|\Gamma(t)x\|_E \leq |\alpha e^{-\mu t}|\|Ax\|_E \leq p(t)\|x\|_Y,
\]

\[
\left\| \frac{d\Gamma(t)x}{dt} \right\|_E \leq \mu|\alpha e^{-\mu t}|\|Ax\|_E \leq p(t)\|x\|_Y.
\]

Referring to Lemma 2.4, we see that Eq. (20) admits a resolvent operator. Since the semigroup generated by \( A \) is analytic, then it is norm continuous for \( t > 0 \). Thus by Lemma 2.6 and 2.9, the corresponding resolvent operator \( R(t,s) \equiv R(t-s) \) is operator-norm continuous for \( t > s > 0 \).

We define the operators \( f : [0,T] \times E \rightarrow E \), \( g : \mathcal{P}C([0,T], E) \rightarrow E \) and \( J_i : E \rightarrow E \) by

\[
f(t,x(t))(\xi) = p_0 \sin(x(t)(\xi)), \quad \text{for} \quad \xi \in [0,\pi] \text{and} \quad x \in E,
\]

\[
g(x)(\xi) = \int_0^\pi \int_0^T \Xi(s,\xi)\log(1 + |x(s)(r)|^{1/2})dsdr,
\]

\[
J_i(x)(\xi) = \int_0^\pi \alpha_i(r)\cos(rx(\xi))dr.
\]

If we put \( x(t)(\xi) = \vartheta(t,\xi) \), then problem (20) takes the abstract form (1).

From the above facts, we obtain that

\[\text{(L1):} \quad f \text{ is Lipschitzian with respect to the second variable } x, \text{ with Lipschitz constant } k_x = |p_0|. \]

Therefore, by Theorem 2.1, (H1)-(f2) is satisfied with \( \eta(t) = |p_0| \) and

\[
\|f(t,x)\|_E \leq |p_0|\sqrt{\pi}
\]

with \( L_f(t) = |p_0|\sqrt{\pi} \) and \( \phi(t) = 1 \), that implies \( f \) satisfies condition (H1)-(f3). Therefore \( f \) satisfies the hypothesis (H1).

\[\text{(L2):} \quad J_i : E \rightarrow E \text{ is a continuous function for each } i = 1,2,\ldots,m \text{ and we have}
\]

\[
\|J_i(x(t))\|_E \leq \sqrt{\pi}\|\alpha_i\|_{L^2(0,\pi)}.
\]
It is clear that for \( L_i(\rho) = \sqrt{\pi} \| \alpha_i \|_{L^2(0,\pi)} \), we get \( \lim_{\rho \to \infty} \frac{L_i(\rho)}{\rho} = 0 \), then \( (H2) - (I1) \) is satisfied.

Thus \( J_i(\cdot) \) is Lipschitz with constant \( K_i = \pi^{3/2} \| \alpha_i \|_{L^2(0,\pi)} > 0 \). Therefore, by Theorem 2.1, we get

\[
\beta(J_i(B)) \leq K_i \beta(B) \quad \text{for every} \quad B \subset E, i = 1, \ldots, m,
\]

which satisfies \( (H2) - (I2) \). From the above facts, we conclude that condition \( (H2) \) holds.

(L3):

Lemma 4.1. The map \( g : PC([0,T],E) \to E \) defined by

\[
g(x)(\xi) = \int_0^\pi \int_0^T \Xi(t,\xi) \log(1 + |x(t)(r)|^{1/2}) dt dr
\]

is compact.

Proof. Let \( W \subset PC([0,T],E) \) be bounded. Then

\[
\|g(x)\|_E \leq (\pi T) \max_{t \in [0,T], \xi \in [0,\pi]} \| \Xi(t,\xi) \| \left( |x|_{PC} \right)^{1/2}, \quad \text{for all} \quad x \in W.
\]

So \( g(B) \) is bounded. Now since \( \Xi \) is uniformly continuous on \([0,T] \times [0,\pi]\), it follows that \( g(B) \) is equicontinuous on \([0,\pi]\). Therefore, by Ascoli-Alzela’s theorem, \( g(B) \) is relatively compact. Hence \( g \) is compact. \( \square \)

It is clear that for \( g_r = \sup \{ \|g(x)\| : |x|_{PC} \leq r \} \), we obtain \( \lim_{r \to +\infty} \frac{g_r}{r} = 0 \).

Lemma 4.2 ( [20], Lemma 5.4 ). Assume that \( 2\alpha \left( \frac{\mu T^2}{2} + 1 \right) \leq 1 \). Then

\[
M_T \leq \frac{1}{1 - 2\alpha \left( \frac{\mu T^2}{2} + 1 \right)}
\]

where \( M_T = \sup_{0 \leq s \leq T} \|R(s)\| \).

Theorem 4.3. If

\[
2|p_0| + \sum_{i=0}^{m} K_i \left( 1 - 2\alpha \left( \frac{\mu T^2}{2} + 1 \right) \right) < 1.
\]

Then Eq. (20) has a mild solution in \([0,T]\).

Proof. The proof follows from Theorem 3.1, (L1), (L2) and (L3). \( \square \)
INTEGRODIFFERENTIAL EQUATIONS OF VOLTERRA TYPE WITH NON LOCAL AND IMPULSIVE CONDITIONS

References


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