An existence result for functional integral equations via Petryshyn’s fixed point theorem

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Abstract

In this article, using Petryshyn’s fixed point theorem associated with the measure of non-compactness, we discuss the existence result for functional integral equations in Banach algebra, which covers many existence results for functional integral equations as a particular case under some weaker conditions. Further, we provide some examples of functional integral equations to illustrate our analytical findings.

**Keywords:** Functional integral equation, Fixed point theorem, Banach algebra, Measure of non-compactness.

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1 Introduction

The theory of fixed point plays a significant role in nonlinear analysis and is known to be very helpful in proving the existence and uniqueness results for nonlinear integral and differential equations. Integral equations as one of the branches of mathematical analysis play a significant role in several fields. It has importance not only for the expert in the field but additionally for those whose interests lie in different branches of mathematics such as mathematical physics, neutron transport, control theory, engineering, and population

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dynamics (see [12, 19, 21]). Many authors have studied functional integral equations in connection with some applications in the field of the kinetic theory of gases and radiative transfer, for instance, see [11, 24, 27]. Numerous papers have reported for the problem of the existence result of functional integral equations, one can see [1, 2, 13, 18, 22, 23, 42].

Motivated by the above literature, in this work we consider the equation:

\[ z(s) = \left( f(s, z(\varphi_1(s)), \ldots, z(\varphi_k(s))) \\
+ p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), \ldots, z(\beta_n(t))) dt, z(\gamma_1(s)), \ldots, z(\gamma_m(s))) \right) \\
\times \left( h(s, z(\theta_1(s)), \ldots, z(\theta_r(s))) \\
+ q(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), \ldots, z(\phi_m(t))) dt, z(\tau_1(s)), \ldots, z(\tau_n(s))) \right), s \in I = [0, a]. \] (1)

Caballero et al. [7] and Özdemir et al. [36] studied the existence result for the following functional integral equations

\[ z(s) = p(s, \int_0^s g(s, t, z(\beta(t))) dt, z(\gamma(s))) \times q(s, \int_0^a z(s) u(s, t, z(\phi(t))) dt, z(\tau(s))) \] (2)

and

\[ z(s) = p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta(t))) dt, z(\gamma(s))) \\
\times q(s, z(\tau_2(s)) \int_0^1 u(s, t, z(\phi(t))) dt, z(\tau_1(s))), s \in I = [0, a]. \] (3)

Deepmala and Pathak [18] studied the existence of solutions for functional integral equation

\[ z(s) = \left( f(s, z(s)) + p(s, \int_0^s g(s, t, z(\beta(t))) dt, z(\gamma(s))) \right) \\
\times q(s, \int_0^a u(s, t, z(\phi(t))) dt, z(\tau(s))), s \in I = [0, a] \] (4)

Banaś and Sadarangani [3] as well as Maleknejad et al. [29] studied the existence result for functional integral equation

\[ z(s) = p(s, \int_0^s g(s, t, z(\beta(t))) dt, z(\gamma(s))) \times q(s, \int_0^a u(s, t, z(\phi(t))) dt, z(\tau(s))), s \in I = [0, a]. \] (5)
Kazemi and Ezzati [26] studied the existence of solutions for functional integral equation

\[ z(s) = p\left(s, z(\varphi(s)), z(\gamma(s)), \int_0^{\alpha(s)} g(s, t, z(\beta(t)))dt\right), s \in I = [0, a]. \tag{6} \]

Maleknejad et al. [30, 31] examined the existence result for functional integral equations

\[ z(s) = f(s, z(s)) + p\left(s, \int_0^s g(s, t, z(t))dt, z(\beta(s))\right), \tag{7} \]

and

\[ z(s) = f(s, z(\beta(s))) \int_0^s g(s, t, z(t))dt, s \in I = [0, a]. \tag{8} \]

Banaś and Rzepka [4, 5] discussed the existence result for the non-linear functional integral equation and non-linear quadratic integral equations

\[ z(s) = h(s, z(s)) \int_0^s u(s, t, z(t))dt, \tag{9} \]

\[ z(s) = \hat{b}(s) + h(s, z(s)) \int_0^s u(s, t, z(t))dt, s \in I = [0, a]. \tag{10} \]

Dhage [17] studied the following non-linear integral equation

\[ z(s) = \hat{b}(s) \int_0^a g(s, t, z(t))dt + \int_0^s u(s, t, z(t))dt \times \int_0^a g(s, t, z(t))dt, s \in I = [0, a]. \tag{11} \]

Çakan and Özdemir [8], Özdemir et al. [35], Özdemir and Çakan [37] studied the equations

\[ z(s) = f(s, z(\varphi(s))) + f_1(s, z(\gamma(s))) \int_0^{\alpha_1(s)} g(s, t, z(\beta(t)))dt \tag{12} \]

and

\[ z(s) = h(s, z(\theta(s))) + q\left(s, \int_0^{\alpha(s)} u(s, t, z(\phi(t)))dt, z(\tau(s))\right), s \in I = [0, a]. \tag{13} \]

Banaś and Sadarangani [6] studied the existence of solutions for functional integral equations

\[ z(s) = f(s, z(\varphi_1(s)), \ldots, z(\varphi_k(s))) + \int_0^{\alpha(s)} g(s, t, z(\beta_1(t)), \ldots, z(\beta_n(t)))dt, s \in [0, \infty). \tag{14} \]

Çakan and Özdemir [9], Özdemir and Çakan [38] studied the equations

\[ z(s) = f_1(s, z(\tau_1(s)), \ldots, z(\tau_n(s))) + h_1(s, z(\gamma_1(t)), \ldots, z(\gamma_m(t)))dt \times \int_0^{\alpha(s)} g(s, t, z(\beta_1(t)), \ldots, z(\beta_n(t)))dt \tag{15} \]
Further, some famous integral equations of Volterra integral equation, Urysohn integral equation and Chandrasekhar type [11] have the form

\[
z(s) = \hat{b}(s) + \int_0^s g(s, t, z(t)) dt
\]  

(17)

\[
z(s) = \hat{b}(s) + \int_0^a u(s, t, z(t)) dt
\]  

(18)

\[
z(s) = 1 + z(s) \int_0^a \frac{s}{s + t} \phi(t) z(t) dt.
\]  

(19)

It can be seen that the equation (1) is more general and involves the equations (2)-(19) as particular cases. Such equations are very useful in real-world problems of physics, engineering, modeling, etc (we refer to [12, 19, 21, 24, 27]). Different authors have studied similar types of the above mentioned equations and used the concept of the measure of non-compactness to derive their results, for more details we refer to [9, 10, 14, 16, 18, 41].

In [25], Kazemi and Ezzati used Petryshyn’s fixed point theorem to investigate the solution of nonlinear functional integral equations. Furthermore, recently many researchers used Petryshyn’s fixed point theorem to investigate the existence of solution of nonlinear functional integral equations in Banach spaces as well as Banach algebra (for instance see [15, 26, 32, 43, 44, 45, 46] and references therein). In our problem, we use Petryshyn’s fixed point theorem (instead of Darbo’s fixed point theorem) to investigate the solvability of the equation (1). The following statements explain the main causes why we consider the equation (1). The first reason is that the equation (1) unifies many relevant work in this area. Secondly, the bounded condition shows that the “sublinear condition” that have been discussed in several literature (see the references [9, 10, 14, 18, 29, 30, 31]) will not play a significant role to obtain the existence of solution of functional integral equations.

The paper is organized into five sections including the introduction. In Section 2, we present some preliminaries and define the concept of measure of non-compactness. In Section 3, we prove an existence result for equations including condensing operators using
Petryshyn’s fixed point theorem. In Section 4, we give examples that test the utilization of this kind of functional integral equation. Finally in Section 5, we add few open problems for the interested researchers.

2 Preliminaries

In this study, we have some notations:

- \( E \): Real Banach space;
- \( B_{\rho}(z) \): Open ball with center \( z \) and radius \( \rho \);
- \( \text{co}Q \): Convex hull of a set \( Q \);
- \( \text{co} \bar{Q} \): Closed convex hull of a set \( Q \);
- \( M_E \): Set of all bounded subsets of \( E \);
- \( N_E \): Set of all relatively compact subsets of \( E \).

**Definition 2.1.** [28] The Kuratowski measure of non-compactness of \( Q \) is defined as

\[
\hat{\gamma}(Q) = \inf \left\{ \delta > 0 : Q = \bigcup_{j=1}^{n} Q_j \text{ with } \text{diam}(Q_j) \leq \delta, \; j = 1, 2, \ldots, m \right\}.
\]

**Definition 2.2.** [20] The Hausdorff measure of non-compactness of \( Q \) is defined by

\[
\mu(Q) = \inf \{ \delta > 0 : \text{there exists a finite } \delta\text{-net for } Q \text{ in } E \}, \tag{20}
\]

where a finite \( \delta\)-net for \( Q \) in \( E \), we mean, as usual, a set \( \{z_1, z_2, \ldots, z_m\} \subset E \) such that the ball \( B_{\delta}(E; z_1), B_{\delta}(E; z_2), \ldots, B_{\delta}(E; z_m) \) over \( Q \). These measures of non-compactness are respectively similar in the sense that

\[
\mu(Q) \leq \hat{\gamma}(Q) \leq 2\mu(Q),
\]

for any bounded set \( Q \subset E \).

**Theorem 2.1.** Let \( Q, P \in M_E \) and \( \lambda \in \mathbb{R} \). Then

(i) \( \mu(Q) = 0 \) if and only if \( Q \in N_E \);

(ii) \( Q \subseteq P \implies \mu(Q) \leq \mu(P) \);
(iii) $\mu(Q) = \mu(coQ) = \mu(Q)$;

(iv) $\mu(Q \cup P) = \max\{\mu(Q), \mu(P)\}$;

(v) $\mu(\lambda Q) = |\lambda|\mu(Q)$, where $\lambda Q = \{\lambda z : z \in Q\}$;

(vi) $\mu(Q + P) \leq \mu(Q) + \mu(P)$, where $Q + P = \{z + \hat{z} : z \in Q, \hat{z} \in P\}$.

Here, $C[0, a]$ is the space of all real valued and continuous functions defined on $[0, a]$. The space, $C[0, a]$ is a Banach space with standard norm

$$||z|| = \max\{|z(s)| : s \in [0, a]\}.$$  

Obviously, the space $C[0, a]$ is also the structure of Banach algebra. In our studies, we will use regular measure of non-compactness defined in [2]. Fix a set $Q \in M_{C[0, a]}$, $z \in Q$ and for a given $\delta > 0$, denote by $\omega(z, \delta)$ the modulus of continuity $z$ is defined by

$$\omega(z, \delta) = \sup\{|z(s) - z(\hat{s})| : s, \hat{s} \in [0, a], |s - \hat{s}| \leq \delta\}.$$  

Further,

$$\omega(Q, \delta) = \sup\{\omega(z, \delta) : z \in Q\}, \quad \omega_0(Q) = \lim_{\delta \to 0} \omega(Q, \delta).$$

**Theorem 2.2.** [26] The Hausdorff measure of non-compactness is equivalent to

$$\mu(Q) = \lim_{\delta \to 0} \sup_{\omega \in Q} \omega(z, \delta)$$

for all bounded sets $Q \subset C[0, a]$.

**Definition 2.3.** [33] Let $F : E \to E$ be a continuous mapping of $E$. $F$ is called a $k$-set contraction if for all $B \subset E$ with $B$ bounded, $F(B)$ is bounded and $\hat{\gamma}(FB) \leq \hat{k}\hat{\gamma}(B), \hat{k} \in (0, 1)$. If

$$\hat{\gamma}(FB) < \hat{\gamma}(B), \text{ for all } \hat{\gamma}(B) > 0,$$

then $F$ is called a condensing or densifying mapping.

**Remark 2.3.** [40] It is clear that every $k$-set-contraction with $k < 1$ is a condensing mapping.
Theorem 2.4. [39, Petryshyn’s] Let $B_{\rho}$ be an open ball about the origin in a Banach space $E$. Suppose that $F : \overline{B}_{\rho} \to E$ is a condensing mapping, which fulfills the boundary condition

$$F(z) = \hat{k}z, \text{ for } z \in \partial \overline{B}_{\rho} \text{ then } \hat{k} \leq 1.$$ 

Then, the set of fixed points of $F$ in $\overline{B}_{\rho}$ is nonempty.

The following lemmas are useful to discuss our main result in Section 3.

Lemma 2.5. [34, Proposition 6] Let $(X_i, d_i), i = 1, 2, 3$ be metric spaces. Assume that $T_1 : X_1 \to X_2$ is a $k_1$-set-contraction and $T_2 : X_2 \to X_3$ is a $k_2$-set-contraction. Then $T_2 T_1$ is a $k_1 k_2$-set-contraction, where $k_1, k_2 \in (0, 1)$.

Lemma 2.6. [40, Proposition 2(a)] Let $X$ be a Banach space and $G \subset X$. If $T_i : G \to X$ is $k_i$-set-contractive, $i = 1, 2$, and $T_3 : T_1(G) \to X$ is $k_3$-set-contractive, then $(T_1 + T_2) : G \to X$ is $(k_1 + k_2)$-set-contractive, and $T_3 T_1 : G \to X$ is $k_3 k_1$-set-contractive.

Lemma 2.7. [33, 39] Suppose that $E$ is a Banach space. If the operators $T$ and $H$ satisfies the Petryshyn’s condition on a bounded set $Q$ of $E$ with constant $k_1$ and $k_2$, respectively, then the operator $F = T H : Q \to E$ satisfies the Petryshyn’s condition (condensing map) on $Q$ with the constant $\hat{k} = k_1 k_2 < 1$, where $k_1 < 1, k_2 < 1$.

3 Main results

Here, we study the existence of solution of the equation (1) under the following assumptions:

(i) Functions $\varphi_i : I \to I$, for $1 \leq i \leq k$, $\beta_j : [0, N_1] \to I$, for $1 \leq j \leq n$, $\gamma_i : I \to I$, for $1 \leq i \leq m$, $\theta_i : I \to I$, for $1 \leq i \leq r$, $\phi_j : [0, N_2] \to I$, for $1 \leq j \leq \xi$, $\tau_i : I \to I$, for $1 \leq i \leq \eta$, $g : I \times [0, N_1] \times \mathbb{R}^n \to \mathbb{R}$, $u : I \times [0, N_2] \times \mathbb{R}^\xi \to \mathbb{R}$, and $\alpha_i : I \to \mathbb{R}^+$ are continuous for $1 \leq i \leq 2, \alpha_1(s) \leq N_1, \alpha_2(s) \leq N_2$ for each $s \in I$.

(ii) Functions $f : I \times \mathbb{R}^k \to \mathbb{R}$, $h : I \times \mathbb{R}^r \to \mathbb{R}$, $p : I \times \mathbb{R}^{m+1} \to \mathbb{R}$ and $q : I \times \mathbb{R}^{\eta+1} \to \mathbb{R}$ are continuous and there exist non-negative constants $b_i, c_j, d_i, l_j, k \sum_{i=1}^{k} b_i + \sum_{i=1}^{m} d_i < 1$. 

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\[ \sum_{j=1}^{r} c_j + \sum_{j=1}^{\xi} l_{j+1} < 1 \text{ for } 1 \leq i \leq k, 1 \leq j \leq r, 1 \leq i \leq m+1, 1 \leq j \leq \eta+1, \] respectively such that

\[ |f(s, z_1, z_2, ..., z_k) - f(s, x_1, x_2, ..., x_k)| \leq \sum_{i=1}^{k} b_i |z_i - x_i|, \]
\[ |h(s, z_1, z_2, ..., z_r) - h(s, x_1, x_2, ..., x_r)| \leq \sum_{j=1}^{r} c_j |z_j - x_j|, \]
\[ |p(s, z_1, z_2, ..., z_{m+1}) - p(s, x_1, x_2, ..., x_{m+1})| \leq \sum_{i=1}^{m+1} d_i |z_i - x_i|, \]
\[ |q(s, z_1, z_2, ..., z_{\eta+1}) - q(s, x_1, x_2, ..., x_{\eta+1})| \leq \sum_{j=1}^{\eta+1} l_j |z_j - x_j|. \]

(iii) There exists a \( \rho > 0 \) such that the following bounded condition is satisfied

\[ \sup \left\{ |(f + M_1) \times (h + M_2)| \right\} \leq \rho, \]

where

\[ \sup f = \sup \{|f(s, z_1, z_2, ..., z_k)| : \text{for all } s \in I, z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq k\}, \]
\[ \sup h = \sup \{|h(s, z_1, z_2, ..., z_r)| : \text{for all } s \in I, z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq r\}, \]
\[ \sup M_1 = \sup \{|p(s, z_1, z_2, ..., z_{m+1})| : \text{for all } s \in I, z_1 \in [-L_1 N_1, L_1 N_1] \text{ and } z_{i+1} \in [-\rho, \rho], 1 \leq i \leq m+1\}, \]
\[ \sup M_2 = \sup \{|q(s, z_1, z_2, ..., z_{\eta+1})| : \text{for all } s \in I, z_1 \in [-L_2 N_2, L_2 N_2] \text{ and } z_{i+1} \in [-\rho, \rho], 1 \leq i \leq \eta+1\}, \]
\[ L_1 = \sup \{|g(s, t_1, z_1, z_2, ..., z_n)| : \text{for all } s \in I, t_1 \in [0, N_1] \text{ and } z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq n\}, \]
\[ L_2 = \sup \{|u(s, t_2, z_1, z_2, ..., z_\xi)| : \text{for all } s \in I, t_2 \in [0, N_2] \text{ and } z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq \xi\}. \]

**Theorem 3.1.** Under the assumptions (i) – (iii) the equation (1) has at least one solution in \( E = C(I) \).

**Proof.** We will apply Theorem 2.4 as our main tool. Define the operators \( T, H : \overline{B}_\rho \to E \) as

\[ (Tz)(s) = \left( f(s, z(\varphi_1(s)), ..., z(\varphi_k(s)) + p \left( s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), ..., z(\beta_n(t))) dt, z(\gamma_1(s)), ..., z(\gamma_m(s)) \right) \right), \]

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(Hz)(s) = \left( h(s, z(\theta_1(s)), ..., z(\theta_r(s))) + q\left(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), ..., z(\phi_\xi(t)))dt, z(\tau_1(s)), ..., z(\tau_\eta(s)) \right) \right)

for s \in [0, a].

Further, we define the operator \( F : \overline{B}_\rho \to E \) by setting

\[ Fz = (Tz)(Hz). \]

Obviously, \( F \) is well-defined. Now, we show that \( T \) is continuous on \( \overline{B}_\rho \). For this, take \( \delta > 0 \) and any \( z, x \in \overline{B}_\rho \) such that \( ||z - x|| < \delta \). Then, we have

\[
|(Tz)(s) - (Tx)(s)| \\
= \left| f(s, z(\varphi_1(s)), ..., z(\varphi_k(s))) + p\left(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, \\
(\gamma_1(s), ..., z(\gamma_m(s))) - f(s, x(\varphi_1(s)), ..., x(\varphi_k(s))) \\
- p\left(s, \int_0^{\alpha_1(s)} g(s, t, x(\beta_1(t)), ..., x(\beta_n(t)))dt, x(\gamma_1(s)), ..., x(\gamma_m(s)) \right) \right) \right| \\
\leq \sum_{i=1}^k b_i |z(\varphi_i(s)) - x(\varphi_i(s))| + \sum_{i=1}^m d_i |z(\gamma_i(s)) - x(\gamma_i(s))| \\
+ d_1 \int_0^{\alpha_1(s)} |g(s, t, z(\beta_1(t)), ..., z(\beta_n(t))) - g(s, t, x(\beta_1(t)), ..., x(\beta_n(t)))|dt \\
\leq \sum_{i=1}^k b_i ||z - x|| + \sum_{i=1}^m d_i ||z - x|| + d_1 \int_0^{\alpha_1(s)} \omega(g, \delta)dt \\
\leq \left( \sum_{i=1}^k b_i + \sum_{i=1}^m d_i \right) \delta + d_1 N_1 \omega(g, \delta)
\]

and similarly, we have

\[
|(Hz)(s) - (Hx)(s)| \\
= \left| h(s, z(\theta_1(s)), ..., z(\theta_r(s))) + q\left(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), ..., z(\phi_\xi(t)))dt, \\
(\tau_1(s), ..., z(\tau_\eta(s))) - h(s, x(\theta_1(s)), ..., x(\theta_r(s))) \\
- q\left(s, \int_0^{\alpha_2(s)} u(s, t, x(\phi_1(t)), ..., x(\phi_\xi(t)))dt, x(\tau_1(s)), ..., x(\tau_\eta(s)) \right) \right) \right|
\]
\[ \leq \sum_{j=1}^{r} c_j |z(\theta_j(s)) - x(\theta_j(s))| + \sum_{j=1}^{\eta} l_{j+1} |z(\tau_j(s)) - x(\tau_j(s))| \\
+ l_1 \int_{0}^{\alpha_2(s)} |u(s, t, z(\phi_1(t)), ..., z(\phi_\xi(t))) - u(s, t, z(x(\phi_1(t)), ..., z(\phi_\xi(t))))| dt \\
\leq \sum_{j=1}^{r} c_j |z - x| + \sum_{j=1}^{\eta} l_{j+1} |z - x| + l_1 \int_{0}^{\alpha_2(s)} \omega(u, \delta) dt \\
\leq \left( \sum_{j=1}^{r} c_j + \sum_{j=1}^{\eta} l_{j+1} \right) \delta + l_1 N_2 \omega(u, \delta), \]

where

\[ \omega(g, \delta) = \sup\{ |g(s, t, z_1, ..., z_n) - g(s, t, x_1, ..., x_n)| : s \in I, t \in [0, N_1], z_i, x_i \in [-\rho, \rho], \]

\[ 1 \leq i \leq n, ||z_i - x_i|| \leq \delta, \]

\[ \omega(u, \delta) = \sup\{ |u(s, t, z_1, ..., z_\xi) - u(s, t, x_1, ..., x_\xi)| : s \in I, t \in [0, N_2], z_j, x_j \in [-\rho, \rho], \]

\[ ||z_j - x_j|| \leq \delta, \]

Since we know that \( g = g(s, t, z_1, ..., z_n) \) and \( u = u(s, t, z_1, ..., z_\xi) \) are uniformly continuous on \( I \times [0, N_1] \times [-\rho, \rho]^n \) and \( I \times [0, N_2] \times [-\rho, \rho]^\xi \), respectively, we infer that \( \omega(g, \delta) \to 0 \) and \( \omega(u, \delta) \to 0 \) as \( \delta \to 0 \). Thus the above fact shows that operators \( T \) and \( H \) are continuous on \( \overline{B}_\rho \). Hence, \( F \) is also continuous on \( \overline{B}_\rho \). Further, we show that the operators \( T \) and \( H \) satisfies the condensing condition with respect to the measure \( \mu \). For this, take a fixed arbitrary \( \delta > 0 \) and \( z \in Q \), where \( Q \) is bounded subset of \( E, s_1, s_2 \in I \) with \( \alpha(s_1) \leq \alpha(s_2) \) and \( s_2 - s_1 \leq \delta \), we get

\[ |(Tz)(s_2) - (Tz)(s_1)| \\
= \left| f(s_2, z(\varphi_1(s_2)), ..., z(\varphi_k(s_2))) + p(s_2, \int_{0}^{\alpha_1(s_2)} g(s_2, t, z(\beta_1(t)), ..., z(\beta_n(t))) dt, \right. \\
\left. z(\gamma_1(s_2)), ..., z(\gamma_m(s_2))) - f(s_1, z(\varphi_1(s_1)), ..., z(\varphi_k(s_1)) \\
- p(s_1, \int_{0}^{\alpha_1(s_1)} g(s_1, t, z(\beta_1(t)), ..., z(\beta_n(t))) dt, z(\gamma_1(s_1)), ..., z(\gamma_m(s_1))) \right| \]
\[
\begin{align*}
&\leq |f(s_2, z(\varphi_1(s_2)), ..., z(\varphi_k(s_2))) - f(s_2, z(\varphi_1(s_1)), ..., z(\varphi_k(s_1)))| \\
&+ |f(s_2, z(\varphi_1(s_1)), ..., z(\varphi_k(s_1))) - f(s_1, z(\varphi_1(s_1)), ..., z(\varphi_k(s_1)))| \\
&+ \left| p\left(s_2, \int_0^{\alpha_1(s_2)} g(s_2, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, z(\gamma_1(s_2)), ..., z(\gamma_m(s_2))\right) - \right| \\
&+ \left| p\left(s_2, \int_0^{\alpha_1(s_1)} g(s_1, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, z(\gamma_1(s_1)), ..., z(\gamma_m(s_1))\right) - \right| \\
&+ \left| p\left(s_1, \int_0^{\alpha_1(s_1)} g(s_1, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, z(\gamma_1(s_1)), ..., z(\gamma_m(s_1))\right) - \right| \\
&\leq \sum_{i=1}^{k} b_i |z(\varphi_i(s_2)) - x(\varphi_i(s_1))| + \omega_f(I, \delta) \\
&+ d_1 \left| \int_0^{\alpha_1(s_2)} g(s_2, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt - \int_0^{\alpha_1(s_1)} g(s_1, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt \right| \\
&+ \sum_{i=1}^{m} \sum_{i=1}^{n} d_{i+1} |z(\gamma_i(s_2)) - x(\gamma_i(s_1))| + \omega_p(I, \delta) \\
&\leq \sum_{i=1}^{k} b_i |z(\varphi_i(s_2)) - x(\varphi_i(s_1))| + \omega_f(I, \delta) + \sum_{i=1}^{m} \sum_{i=1}^{n} d_{i+1} |z(\gamma_i(s_2)) - x(\gamma_i(s_1))| + \omega_p(I, \delta) \\
&+ d_1 \int_0^{\alpha_1(s_1)} |g(s_2, t, z(\beta_1(t)), ..., z(\beta_n(t))) - g(s_1, t, z(\beta_1(t)), ..., z(\beta_n(t)))|dt \\
&+ d_1 \int_0^{\alpha_1(s_2)} |g(s_2, t, z(\beta_1(t)), ..., z(\beta_n(t)))|dt \\
&\leq \sum_{i=1}^{k} b_i |z(\varphi_i(s_2)) - x(\varphi_i(s_1))| + \omega_f(I, \delta) + \sum_{i=1}^{m} \sum_{i=1}^{n} d_{i+1} |z(\gamma_i(s_2)) - x(\gamma_i(s_1))| + \omega_p(I, \delta) \\
&+ d_1 \int_0^{\alpha_1(s_1)} \omega_g(I, \delta)dt + d_1 L_1 \omega(\alpha_1, \delta) \\
&\leq \sum_{i=1}^{k} b_i \omega(z, \omega(\varphi_i, \delta)) + \omega_f(I, \delta) + \sum_{i=1}^{m} \sum_{i=1}^{n} d_{i+1} \omega(z, \omega(\gamma_i, \delta)) + \omega_p(I, \delta) \\
&+ d_1 (N_1 \omega_g(I, \delta) + L_1 \omega(\alpha_1, \delta)).
\end{align*}
\]
Again

\[
\begin{aligned}
&\left| (Hz)(s_2) - (Hz)(s_1) \right| \\
= &\left| h(s_2, z(\theta_1(s_2)), \ldots, z(\theta_r(s_2))) + q(s_2, \int_0^{\alpha_2(s_2)} u(s_2, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, \\
&z(\tau_1(s_2)), \ldots, z(\tau_\eta(s_2)) \right) - h(s_1, z(\theta_1(s_1)), \ldots, z(\theta_r(s_1))) \\
&- q(s_1, \int_0^{\alpha_2(s_1)} u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, z(\tau_1(s_1)), \ldots, z(\tau_\eta(s_1))) \right| \\
\leq &\left| h(s_2, z(\theta_1(s_2)), \ldots, z(\theta_r(s_2))) - h(s_2, z(\theta_1(s_1)), \ldots, z(\theta_r(s_1))) \right| \\
&+ \left| h(s_2, z(\theta_1(s_1)), \ldots, z(\theta_r(s_1))) - h(s_1, z(\theta_1(s_1)), \ldots, z(\theta_r(s_1))) \right| \\
&+ \left| q(s_2, \int_0^{\alpha_2(s_2)} u(s_2, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, z(\tau_1(s_2)), \ldots, z(\tau_\eta(s_2))) \right| \\
&- q(s_2, \int_0^{\alpha_2(s_1)} u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, z(\tau_1(s_1)), \ldots, z(\tau_\eta(s_1))) \right| \\
&+ \left| q(s_2, \int_0^{\alpha_2(s_1)} u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, z(\tau_1(s_1)), \ldots, z(\tau_\eta(s_1))) \right| \\
&- q(s_1, \int_0^{\alpha_2(s_1)} u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt, z(\tau_1(s_1)), \ldots, z(\tau_\eta(s_1))) \right| \\
\leq &\sum_{j=1}^{r} c_j |z(\theta_j(s_2)) - x(\theta_j(s_1))| + \omega_h(I, \delta) \\
&+ l_1 \left| \int_0^{\alpha_2(s_2)} u(s_2, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt - \int_0^{\alpha_2(s_1)} u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))dt \right| \\
&+ \sum_{j=1}^{\xi} l_{\xi+1} |z(\phi_j(s_2)) - x(\phi_j(s_1))| + \omega_q(I, \delta) \\
\leq &\sum_{j=1}^{r} c_j |z(\theta_j(s_2)) - x(\theta_j(s_1))| + \omega_h(I, \delta) + \sum_{j=1}^{\xi} l_{\xi+1} |z(\phi_j(s_2)) - x(\phi_j(s_1))| + \omega_q(I, \delta) \\
&+ l_1 \int_0^{\alpha_2(s_1)} |u(s_2, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t))) - u(s_1, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))|dt \\
&+ l_1 \int_{\alpha_2(s_1)}^{\alpha_2(s_2)} |u(s_2, t, z(\phi_1(t)), \ldots, z(\phi_\xi(t)))|dt
\end{aligned}
\]
From above relation, we get

\[
\sum_{j=1}^{r} c_j |z(\phi_j(s_2)) - x(\phi_i(s_1))| + \omega_h(I, \delta) + \sum_{j=1}^{\xi} l_{j+1} |z(\phi_j(s_2)) - x(\phi_i(s_1))| + \omega_q(I, \delta)
\]

\[
+ l_1 \int_0^{\alpha_2(s_1)} \omega_u(I, \delta) dt + l_1 L_2 \omega(\alpha_2, \delta)
\]

\[
\leq \sum_{j=1}^{r} c_j \omega(z, \omega(\phi_j, \delta)) + \omega_h(I, \delta) + \sum_{j=1}^{\xi} l_{j+1} \omega(z, \omega(\phi_j, \delta)) + \omega_q(I, \delta)
\]

\[
+ l_1 (N_2 \omega_u(I, \delta) + L_2 \omega(\alpha_2, \delta)),
\]

where

\[
\omega_f(I, \delta) = \sup \{|f(s, z_1, ..., z_k) - f(\hat{s}, z_1, ..., z_k)| : s, \hat{s} \in I, z_i \in [-\rho, \rho], 1 \leq i \leq k, |s - \hat{s}| \leq \delta\},
\]

\[
\omega_h(I, \delta) = \sup \{|h(s, z_1, ..., z_r) - h(\hat{s}, z_1, ..., z_r)| : s, \hat{s} \in I, z_i \in [-\rho, \rho], 1 \leq i \leq r, |s - \hat{s}| \leq \delta\},
\]

\[
\omega_p(I, \delta) = \sup \{|p(s, z_1, ..., z_{m+1}) - p(\hat{s}, z_1, ..., z_{m+1})| : s, \hat{s} \in I, z_i \in [-N_1 L_1, N_1 L_1], z_i \in [-\rho, \rho]
\]

\[
\text{for } 2 \leq i \leq m + 1, |s - \hat{s}| \leq \delta\},
\]

\[
\omega_q(I, \delta) = \sup \{|q(s, z_1, ..., z_{\eta+1}) - q(\hat{s}, z_1, ..., z_{\eta+1})| : s, \hat{s} \in I, z_i \in [-N_2 L_2, N_2 L_2], z_i \in [-\rho, \rho]
\]

\[
\text{for } 2 \leq i \leq \eta + 1, |s - \hat{s}| \leq \delta\},
\]

\[
\omega_g(I, \delta) = \sup \{|g(s, t, z_1, ..., z_n) - g(\hat{s}, t, z_1, ..., z_n)| : s, \hat{s} \in I, t \in [0, N_1], z_i \in [-\rho, \rho],
\]

\[
\text{for } 1 \leq i \leq n, |s - \hat{s}| \leq \delta\},
\]

\[
\omega_u(I, \delta) = \sup \{|u(s, t, z_1, ..., z_\xi) - u(\hat{s}, t, z_1, ..., z_\xi)| : s, \hat{s} \in I, t \in [0, N_2], z_i \in [-\rho, \rho],
\]

\[
\text{for } 1 \leq i \leq \xi, |s - \hat{s}| \leq \delta\}.
\]

Also,

\[
\omega(\gamma_i, \delta) = \sup \{\gamma_i(s) - \gamma_i(\hat{s})| : s, \hat{s} \in I, 1 \leq i \leq m, |s - \hat{s}| \leq \delta\},
\]

\[
\omega(\tau_i, \delta) = \sup \{\tau_i(s) - \tau_i(\hat{s})| : s, \hat{s} \in I, 1 \leq i \leq \eta, |s - \hat{s}| \leq \delta\},
\]

\[
\omega(\alpha_i, \delta) = \sup \{\alpha_i(s) - \alpha_i(\hat{s})| : s, \hat{s} \in I, 1 \leq i \leq 2, |s - \hat{s}| \leq \delta\}.
\]

From above relation, we get

\[
|(Tz)(s_2) - (Tz)(s_1)| \leq \sum_{i=1}^{k} b_i \omega(Q, \omega(\varphi_i, \delta)) + \omega_f(I, \delta) + \sum_{i=1}^{m} d_{i+1} \omega(Q, \omega(\gamma_i, \delta)) + \omega_p(I, \delta)
\]

\[
+ d_1 (N_1 \omega_g(I, \delta) + L_1 \omega(\alpha_1, \delta))
\]
and

\[ |(Hz)(s_2) - (Hz)(s_1)| \leq \sum_{j=1}^r c_j \omega(Q, \omega(\phi_j, \delta)) + \omega_h(I, \delta) + \sum_{j=1}^\xi l_{j+1} \omega(Q, \omega(\phi_j, \delta)) + \omega_q(I, \delta) + l_1(N_2 \omega_u(I, \delta) + L_2 \omega(\alpha_2, \delta)). \]

Taking limit as \( \delta \to 0 \), we obtain

\[ \mu(TQ) \leq \left( \sum_{i=1}^k b_i + \sum_{i=1}^m d_{i+1} \right) \mu(Q) \tag{22} \]

and

\[ \mu(HQ) \leq \left( \sum_{j=1}^r c_j + \sum_{j=1}^\xi l_{j+1} \right) \mu(Q), \tag{23} \]

here \( \left( \sum_{i=1}^k b_i + \sum_{i=1}^m d_{i+1} \right) = k_1 < 1 \) and \( \left( \sum_{j=1}^r c_j + \sum_{j=1}^\xi l_{j+1} \right) = k_2 < 1 \).

From (22) and (23), we get \( F \) is a condensing mapping. Now, assume \( z \in \partial B_\rho \) and if \( Fz = \hat{k}z \), then we get \( ||Fz|| = \hat{k}||z|| = \hat{k}\rho \) and by assumption \((iii)\), we have

\[ ||Fz(s)|| = \left| f(s, z(\varphi_1(s)), ..., z(\varphi_k(s))) + p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, \right. \]

\[ \left. z(\gamma_1(s)), ..., z(\gamma_m(s)) \right) \times \left( h(s, z(\theta_1(s)), ..., z(\theta_r(s)) \right. \]

\[ + g\left(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), ..., z(\phi_\xi(t)))dt, z(\tau_1(s)), ..., z(\tau_\eta(s)) \right) \right| \leq \rho, \]

for all \( s \in I \), hence \( ||Fz|| \leq \rho \), i.e., \( \hat{k} \leq 1 \).

\[ \square \]

**Corollary 3.2.** Assume that

(i) \( \varphi_i : I \to I, \) for \( 1 \leq i \leq k, \) \( \beta_j : [0, N_1] \to I, \) for \( 1 \leq j \leq n, \) \( \gamma_i : I \to I, \) for \( 1 \leq i \leq m, \)

\( \phi_j : [0, N_2] \to I, \) for \( 1 \leq j \leq \xi, \) \( \tau_i : I \to I, \) for \( 1 \leq i \leq \eta, \) \( g : I \times [0, N_1] \times \mathbb{R}^n \to \mathbb{R}, \)

\( u : I \times [0, N_2] \times \mathbb{R}^\xi \to \mathbb{R}, \) and \( \alpha_i : I \to \mathbb{R}^+ \) are continuous for \( 1 \leq i \leq 2, \alpha_1(s) \leq N_1, \alpha_2(s) \leq N_2 \) for each \( s \in I. \)
(ii) \( f : I \times \mathbb{R}^k \to \mathbb{R}, p : I \times \mathbb{R}^{m+1} \to \mathbb{R} \) and \( q : I \times \mathbb{R}^{n+1} \to \mathbb{R} \) are continuous and there exist non-negative constants \( b_i, d_i, l_j, \sum_{i=1}^{k} b_i + \sum_{i=1}^{m} d_i + l_j < 1 \) for \( 1 \leq i \leq k, 1 \leq i \leq m, 1 \leq j \leq 1 \) such that

\[
|f(s, z_1, z_2, ..., z_k) - f(s, x_1, x_2, ..., x_k)| \leq \sum_{i=1}^{k} b_i |z_i - x_i|,
\]

\[
|p(s, z_1, z_2, ..., z_{m+1}) - p(s, x_1, x_2, ..., x_{m+1})| \leq \sum_{i=1}^{m+1} d_i |z_i - x_i|,
\]

\[
|q(s, z_1, z_2, ..., z_{n+1}) - q(s, x_1, x_2, ..., x_{n+1})| \leq \sum_{j=1}^{n+1} l_j |z_j - x_j|.
\]

(iii) There exists a \( \rho > 0 \) such that the following bounded condition is satisfied

\[
\sup \left\{ |(f + M_1) \times (M_2)| \right\} \leq \rho,
\]

where

\[
\sup f = \sup \{|f(s, z_1, z_2, ..., z_k)| : \text{for all } s \in I, \ z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq k\},
\]

\[
\sup M_1 = \sup \{|p(s, z_1, z_2, ..., z_{m+1})| : \text{for all } s \in I, \ z_1 \in [-L_1N_1, L_1N_1] \text{ and } z_{i+1} \in [-\rho, \rho] \text{ for } 1 \leq i \leq m+1\},
\]

\[
\sup M_2 = \sup \{|q(s, z_1, z_2, ..., z_{n+1})| : \text{for all } s \in I, \ z_1 \in [-L_2N_2, L_2N_2] \text{ and } z_{i+1} \in [-\rho, \rho] \text{ for } 1 \leq i \leq n+1\},
\]

\[
L_1 = \sup \{|g(s, t_1, z_1, z_2, ..., z_n)| : \text{for all } s \in I, t_1 \in [0, N_1] \text{ and } z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq n\},
\]

\[
L_2 = \sup \{|u(s, t_2, z_1, z_2, ..., z_\xi)| : \text{for all } s \in I, t_2 \in [0, N_2] \text{ and } z_i \in [-\rho, \rho] \text{ for } 1 \leq i \leq \xi\}.
\]

Then

\[
z(s) = \left( f(s, z(\varphi_1(s)), ..., z(\varphi_k(s))) + p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), ..., z(\beta_n(t)))dt, \right.
\]

\[
z(\gamma_1(s)), ..., z(\gamma_m(s))) \times \left( q(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), ..., z(\phi_\xi(t)))dt, \right.
\]

\[
z(\tau_1(s)), ..., z(\tau_\eta(s))) \right)
\]

has at least one solution in \( C(I), I = [0, a] \).
Corollary 3.3. [10] Suppose that

(i) $\beta_j : [0, N_1] \to I$, for $1 \leq j \leq n$, $\gamma_i : I \to I$, for $1 \leq i \leq m$, $\phi_j : [0, N_2] \to I$, for $1 \leq j \leq \xi$, $\tau_i : I \to I$, for $1 \leq i \leq \eta$, $g : I \times [0, N_1] \times \mathbb{R}^n \to \mathbb{R}$, $u : I \times [0, N_2] \times \mathbb{R}^\xi \to \mathbb{R}$, and $\alpha_i : I \to \mathbb{R}^+$ are continuous for $1 \leq i \leq 2$, $\alpha_1(s) \leq N_1, \alpha_2(s) \leq N_2$ for each $s \in I$.

(ii) $p : I \times \mathbb{R}^{m+1} \to \mathbb{R}$ and $q : I \times \mathbb{R}^{\eta+1} \to \mathbb{R}$ are continuous and there exist non-negative constants $d_i, l_j, \sum_{i=1}^{m} d_{i+1} < 1, \sum_{j=1}^{\xi} l_{j+1} < 1$ for $1 \leq i \leq k$, $1 \leq i \leq m+1$, $1 \leq j \leq \eta+1$, respectively, such that

$$|p(s, z_1, z_2, ..., z_{m+1}) - p(s, x_1, x_2, ..., x_{m+1})| \leq \sum_{i=1}^{m+1} d_i |z_i - x_i|,$$

$$|q(s, z_1, z_2, ..., z_{\eta+1}) - q(s, x_1, x_2, ..., x_{\eta+1})| \leq \sum_{j=1}^{\eta+1} l_j |z_j - x_j|.$$

(iii) There exists a $\rho > 0$ such that the following bounded condition is satisfied

$$\sup \left\{ \left| (M_1) \times (M_2) \right| \right\} \leq \rho,$$

where

$$\sup M_1 = \sup \{ |p(s, z_1, z_2, ..., z_{m+1})| : \text{for all } s \in I, z_1 \in [-L_1 N_1, L_1 N_1] \}
and z_{i+1} \in [-\rho, \rho], 1 \leq i \leq m + 1\},$$

$$\sup M_2 = \sup \{ |q(s, z_1, z_2, ..., z_{\eta+1})| : \text{for all } s \in I, z_1 \in [-L_2 N_2, L_2 N_2] \}
and z_{i+1} \in [-\rho, \rho], 1 \leq i \leq \eta + 1\},$$

$$L_1 = \sup \{ |q(s, t_1, z_1, z_2, ..., z_n)| : \text{for all } s \in I, t_1 \in [0, N_1] \}
and z_i \in [-\rho, \rho], \text{for } 1 \leq i \leq n\},$$

$$L_2 = \sup \{ |u(s, t_2, z_1, z_2, ..., z_\xi)| : \text{for all } s \in I, t_2 \in [0, N_2] \}
and z_i \in [-\rho, \rho], \text{for } 1 \leq i \leq \xi\}. $$
Then we
\[ z(s) = \left( p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta_1(t)), ..., z(\beta_n(t))) dt, z(\gamma_1(s)), ..., z(\gamma_m(s))) \right) \times \left( q(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi_1(t)), ..., z(\phi_n(t))) dt, z(\tau_1(s)), ..., z(\tau_n(s))) \right) \]
has at least one solution in \( C(I), I = [0, a] \).

Proof. Proof is similar to Theorem 3.1 and hence we omit the proof. \( \square \)

4 Examples

In this part, we give some examples of functional integral equations to explain the advantage of our results.

Example 4.1. Consider the following non-linear integral equation
\[
\begin{align*}
z(s) &= \left( \frac{1}{2}(se^{-2s} + z(\sqrt{s})) + \frac{1}{4} \int_0^{s^2} e^{-2t}(s^2 \sin(\sqrt{t}) + e^{\sqrt{t}} + \frac{1}{2}z(\sqrt{t})) dt \right) \\
&\times \left( \frac{1}{3} \cos \left( \frac{\sqrt{1 + s^2}}{2 + s^3} \right) + \frac{1}{5} \int_0^{\sqrt{t}} \sin \left( \frac{z(\sqrt{t}) dt}{1 + z(\sqrt{t})} \right) \right),
\end{align*}
\]
\( s \in [0, 1] \). (24)

Here \( k = n = m = r = \xi = \eta = 1, f(s, z_1) = \frac{1}{2}(se^{-2s} + z(\sqrt{s})), \ p(s, z_1, z_2) = \frac{1}{2}z_1, \)
\[
\begin{align*}
h(s, z_1) &= \frac{1}{3} \cos \left( \frac{\sqrt{1 + s^2}}{2 + s^3} \right), \ q(s, z_1, z_2) = \frac{1}{5}z_1, \\
g(s, t, z_1) &= e^{-2t}(s^2 \sin(\sqrt{t}) + e^{\sqrt{t}} + \frac{1}{2}z(\sqrt{t})) \), \ u(s, t, z_1) = \sin \left( \frac{z(\sqrt{t})(1 + z(\sqrt{t}))}{1 + z(\sqrt{t})} \right).
\end{align*}
\]
Moreover, all \( s \in [0, 1] \)
\[
\begin{align*}
|f(s, z_1) - f(s, x_1)| &\leq \frac{1}{2}|z_1 - x_1|, \\
|h(s, z_1) - h(s, x_1)| &\leq 0|z_1 - x_1|, \\
|p(s, z_1, z_2) - p(s, x_1, x_2)| &\leq \frac{1}{4}|z_1 - x_1|, \\
|q(s, z_1, z_2) - q(s, x_1, x_2)| &\leq \frac{1}{5}|z_1 - x_1|.
\end{align*}
\]
Now, we can see that these functions satisfy the conditions (i) and (ii). We check that (iii) also holds. Choose $\rho = \frac{3 + e}{5}$, then, we obtain $b_1 = \frac{1}{2}, c_1 = 0, d_1 = \frac{1}{4}, l_1 = \frac{1}{5}, N_1 = N_2 = 1, L_1 = \frac{11e}{10} + \frac{13}{10}, L_2 = 1$ and
\[
\sup \left| \left( f(s, z(\varphi(s))) + p(s, \int_0^{\alpha_1(s)} g(s, t, z(\beta(t)))dt, z(\gamma(s))) \right) \right| \\
\times \left| \left( h(s, z(\theta(s))) + q(s, \int_0^{\alpha_2(s)} u(s, t, z(\phi(t)))dt, z(\tau(s))) \right) \right| \\
\leq 3 + e.
\]

Hence, from Theorem 3.1 the functional integral equation (24) has at least one solution in $C[0, 1]$.

**Example 4.2.** Consider the following functional integral equation
\[
z(s) = \frac{1}{2}(s^2 e^{-s^3} + \sqrt{s} z(s)) + \frac{1}{4(e^{2s} + \cos(|z(s^2)|))} \int_0^{\sqrt{s}} (e^{t^2} + \sin(\sqrt{t}) + \frac{1}{2} z(t))dt, \ s \in [0, 1].
\]

The equation (25) is a special case of equation (1) as $f(s, z_1, ... z_\eta) = h(s, z_1, ... z_\eta) = 0, q(s, z_1, ... z_{\eta+1}) = 1, n = 1, m = 2, \gamma_1(s) = \gamma(s), \gamma_2(s) = \varphi(s), \alpha_1(s) = \alpha(s)$ and $p(s, z_1, z_2, z_3) = p(s, z_3, z_2, z_1)$. Here $p : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \varphi, \beta, \gamma : [0, 1] \to [0, 1]$ and $g : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$. From (25), we obtain
\[
\varphi(s) = \beta(s) = s, \ \gamma(s) = s^2, \ \alpha(s) = \sqrt{s}, \text{ for all } s \in [0, 1],
\]
\[
p(s, z_1, z_2, z_3) = p_1(s, z_1) + p_2(s, z_2, z_3),
\]
where
\[
p_1(s, z_1) = \frac{1}{2}(s^2 e^{-s^3} + \sqrt{s} z(s)), \ p_2(s, z_2, z_3) = \frac{z_3}{4(e^{2s} + \cos(|z(s^2)|))},
\]
\[
z_3 = \int_0^{\sqrt{s}} g(s, t, z(\beta(t)))dt, g(s, t, z(\beta(t))) = (e^{t^2} + \sin(\sqrt{t}) + \frac{1}{2} z(t)).
\]

Now, we examine the solution in $C[0, 1]$. It is clear that all functions satisfy the conditions
(i) and (ii). We prove that (iii) also satisfies. Choose \( \rho = \frac{2}{3}(3 + e) \) then \( L_1 \leq 2 + \frac{4e}{3} \) and
\[
\sup\{ |p(s, z_1, z_2, z_3)| : s \in [0,1], z_1, z_2 \in [-\rho, \rho], z_3 \in [-\frac{2 + 4e}{3}, \frac{2 + 4e}{3}] \}
\]
\[
\leq \sup\{ \frac{1}{2} (s^2e^{-s^3} + \sqrt{s}z(s)) + \frac{1}{4} z_3 ; s \in [0,1], -(2 + \frac{4e}{3}) \leq z_3 \leq (2 + \frac{4e}{3}) \}
\]
\[
\leq \frac{2}{3}(3 + e).
\]
Hence, by Theorem 3.1, the equation (25) has at least one solution in \( C[0,1] \).

**Example 4.3.** Consider the following non-linear functional integral equation
\[
z(s) = \left( \frac{\sin^2(\sqrt{s+1})}{3} + \frac{1}{2} \int_0^s (\cos \sqrt{t^3} + \frac{1}{2} z(t))dt \right) \times \left( s^2e^{-s^3} + \frac{1}{2} \int_0^1 \sin z(t^3)dt \right), \quad s \in [0,1].
\]  
(26)

The equation (26) is a particular case of the equation (1). It is easy to prove the conditions (i) and (ii). Now, we check that (iii) also holds. Choose \( \|z\| \leq \hat{\rho}, \hat{\rho} > 0 \), then we get
\[
\|z(s)\| = \left| \left( \frac{\sin^2(\sqrt{s+1})}{3} + \frac{1}{2} \int_0^s (\cos \sqrt{t^3} + \frac{1}{2} z(t))dt \right) \times \left( s^2e^{-s^3} + \frac{1}{2} \int_0^1 \sin z(t^3)dt \right) \right|
\]
\[
\leq \left( \frac{5}{6} + \frac{1}{4} \hat{\rho} \right) \frac{3}{2}
\]
for all \( s \in [0,1] \). Hence, (iii) holds if \( \left( \frac{5}{6} + \frac{1}{4} \hat{\rho} \right) \frac{3}{2} \leq \hat{\rho} \). This shows that \( \rho = 2 \). So, from Theorem 3.1 the functional integral equation (26) has at least one solution in \( C[0,1] \).

**Example 4.4.** Consider the following non-linear functional integral equation:
\[
z(s) = s^4 + \frac{1}{3} z(s^3) \int_0^{s^2} \left( \frac{\sqrt{s}}{2} + \sin \left( \frac{z^2(t)}{1 + z^4(t)} \right) \right) dt, \quad s \in [0,1].
\]  
(27)

The equation (27) is a particular case of the equation (1). It is easy to prove the conditions (i) and (ii). Now, we check that (iii) also holds. Choose \( \|z\| \leq \hat{\rho}, \hat{\rho} > 0 \), then we have
\[
\|z(s)\| = \left| s^4 + \frac{1}{3} z(s^3) \int_0^{s^2} \left( \frac{\sqrt{s}}{2} + \sin \left( \frac{z^2(t)}{1 + z^4(t)} \right) \right) dt \right| \leq 1 + \frac{\hat{\rho}}{2}
\]
for all \( s \in [0,1] \). Hence, (iii) holds if \( 1 + \frac{\hat{\rho}}{2} \leq \hat{\rho} \). This shows we can choose \( \rho \geq 2 \). So, from Theorem 3.1 the functional integral equation (27) has at least one solution in \( C[0,1] \).
5 Open problems

The interested researchers may obtain the existence of solution of the equation (1) in different Banach function spaces, e.g., Orlicz space, Sobolev space, Hölder space, etc. Also the researchers can think about the solvability of infinite systems of the equation (1) in Banach sequence spaces, tempered sequences, Lorentz sequence space, weighted sequence spaces, etc.

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References


