CONSTRUCTION OF COMPLEX POTENTIALS FOR MULTIPLY CONNECTED DOMAINS WITH GIVEN BOUNDARY CIRCULATIONS AND FLUXES

ABSTRACT. We present the approximate analytical method of construction for the complex potential of the locally sourceless, locally irrotational plane flow in a multiply connected infinite domain. The velocity at the infinity, the circulations around the impermeable boundary components and the fluxes across the permeable boundary components are given. The method is based on reduction of Fredholm integral equation to the linear system. This method is easily computable.

Introduction

Methods of complex analysis play an important role in solving many problems of mechanics and mathematics, especially in the case of flat potential fields and solving the Laplace equation [1]. It is known that the complex potential of a flat locally sourceless locally irrotational steady flow in any \( n \)-connected infinite domain can have only logarithmic terms to a single-valued analytic function with a given simple pole at infinity [2]. Here we generalize the solution of [3] where only impermeable boundary components were considered.

We present an analytical method for constructing the complex potential of the plane flow in any multiply connected domain with a given smooth boundary, with sources and sinks outside the flow domain by an approximate solution of the Fredholm equation. We give the following formulation of the problem: taking into account the velocity at infinity, the flux on the permeable components of the boundary and circulation around the impermeable components of the boundary, we can find an analytical function with constant imaginary parts or constant real parts on the boundary components of the region and a given simple pole at infinity. The convergence of the method is proved and some examples of its application are given. Our construction of a complex potential in an infinite multiply connected domain is based on solution of a system of Fredholm equations. A similar problem was solved in [4], but the authors of [5] solved fluxless problems by reducing them to conformal mapping onto canonical domains. The differences between our approach and that of [4] are as follows: we directly restore the Fourier coefficients of the boundary values of the unknown analytical function and then restore the function via Cauchy integral. This allows us to avoid iterations [6]. We calculate the value of the potential at the boundary of the region applying analytic continuation. The advantage of the analytical approximate solution over the numerical solutions is the possibility to apply the methods of analytical functions of complex variable for finding velocities, critical points, streamlines and other parameters of the flow.

The solution of integral equations in our method is reduced to solution of an infinite linear system over the Fourier coefficients of unknown functions. We obtain an approximate solution of the infinite system by solving a finite system with a truncated matrix.

2020 Mathematics Subject Classification. 30E20.

Key words and phrases. potential flow; multiply connected domain; Fredholm integral equation.
The computational complexity of the method is \( O((nM)^3) \), here \( M \) is the order of the corresponding system and \( n \) is the connectivity of the domain. Note that application of Fast Fourier Transform allows us to reduce the method complexity to \( O((nM)^2 \ln(nM)) \).

**Construction of the complex potential of flow in \( n \)-connected domain using the Cauchy integral**

Consider an infinite \( n \)-connected domain \( D_z \) bounded by simple smooth curves \( L_s \) given by the equations

\[
L_s = \{ z = z_s(t), z_s(0) = z_s(2\pi), t \in [0, 2\pi] \}, \ s = 1, \ldots, n.
\]

We also assume that the complex representations of the boundary curves \( L_s \) are as follows:

\[
z_s(t) = \sum_{k=-m}^{n} d_{ks} e^{ikt}, \ t \in [0, 2\pi], \ s = 1, \ldots, n.
\]

The parameterization is set so that the traversal of the contours \( L_s, \ s = 1, \ldots, n \), is counterclockwise with respect to the corresponding finite domain.

Note that the flow includes (possibly zero) circulations around the impenetrable boundary components and fluxes across the permeable boundary components of the region.

**Theorem 1.** The complex potential \( \phi(z) \) of a flow in any \( n \)-connected domain \( D_z \) can be constructed by reducing it to a solution of an infinite linear system.

**Proof.** Assume that \( v \in \mathbb{C} \) is the velocity of the flow at infinity. Consider the complex potential in the form of the function

\[
\phi(z) = \nabla z + \sum_{s=1}^{n} F_s \log(z - z_s^*) + \psi(z).
\]

Here \( z_s^* \) is a point inside the finite domain bounded by the contour \( L_s, \ s = 1, \ldots, n \), \( F_s = \frac{\Gamma_s}{2\pi i} \), for \( s = 1, \ldots, m \), \( \Gamma_s \) is the circulation around the impermeable contour \( L_{s_m}, F_s = \frac{N_s}{2\pi} \) for \( s = m + 1, \ldots, n \), \( N_s \) is the flux across the permeable contour \( L_{s_m} \), and \( \psi(z) \) is the unknown holomorphic in \( D_z \) function [2].

The complex potential is determined up to a constant term; therefore, we can assume that

\[
\psi(\infty) = 0
\]

so \( \psi(z) \) can be represented as a Cauchy-type integral.

The imaginary part of the complex potential \( \phi(z) \) from the formula (1) is constant on the contours \( L_s, \ s = 1, \ldots, m \), the real part of the complex potential \( \phi(z) \) from the formula (1) is constant on the contours \( L_s, \ s = m+1, \ldots, n \).

According to [1] the necessary and sufficient condition for analyticity of \( \psi(t) \) in \( D_z \) are the boundary relations

\[
\psi(z_s(t)) = -\sum_{\sigma=1}^{n} \frac{1}{\pi i} \int_{0}^{2\pi} \psi(z_{\sigma}(\tau)) \log(z_{\sigma}(\tau) - z_s(t)) dz d\tau.
\]

here \( t \in [0, 2\pi], \ s = 1, \ldots, n \).
Consider the new functions \( p_s(t), q_s(t) : \psi(z_s(t)) = p_s(t) + iq_s(t), s = 1, \ldots, n \). Note that
\[
q_s(t) = -\text{Im}[\nabla z_s(t) + \sum_{\sigma=1}^{n} \frac{\Gamma_{\sigma}}{2\pi i} \log(z_s(t) - z_{\sigma}^s)] + C_s,
\]
for \( s = 1, \ldots, m \), here \( C_s \) are real constants and
\[
p_s(t) = -\text{Re}[\nabla z_s(t) + \sum_{\sigma=1}^{n} \frac{N_{\sigma}}{2\pi} \log(z_s(t) - z_{\sigma}^s)] + C_s,
\]
for \( s = m + 1, \ldots, n \), here \( C_s \) are real constants for \( s = m + 1, \ldots, n \).

Consider the real part of both sides of equation (2) for \( s = 1, \ldots, m \):
\[
p_s(t) = -\sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} p_\sigma(\tau)[\arg(z_\sigma(\tau) - z_s(t))]_\tau d\tau - \sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} q_\sigma(\tau)[\log |z_\sigma(\tau) - z_s(t)|]_\tau d\tau.
\]

Consider the imaginary part of both sides of equation (2) for \( s = m + 1, \ldots, n \):
\[
q_s(t) = -\sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} q_\sigma(\tau)[\arg(z_\sigma(\tau) - z_s(t))]_\tau d\tau + \sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} p_\sigma(\tau)[\log |z_\sigma(\tau) - z_s(t)|]_\tau d\tau.
\]

After differentiating relations (5, 6) with respect to \( t \) and integrating the result by parts, we obtain the following relations on the functions \( p'_s(t), q'_s(t), s = 1, \ldots, m \) and \( q'_s(t), s = m + 1, \ldots, n \):
\[
p'_s(t) = -\sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} p'_\sigma(\tau)K_{\sigma,s}(\tau,t)d\tau + Q_s(t),
\]
\[
q'_s(t) = -\sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} q'_\sigma(\tau)K_{\sigma,s}(\tau,t)d\tau + R_s(t),
\]
here
\[
K_{\sigma,s}(\tau,t) = -[\arg(z_\sigma(\tau) - z_s(t))]'_\tau, L_{j,s}(\tau,t) = [\log |z_j(\tau) - z_s(t)|]'_\tau,
\]
\[
Q_s(t) = \sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} q_\sigma(\tau)L_{\sigma,s}(\tau,t)d\tau,
\]
\[
R_s(t) = -\sum_{\sigma=1}^{n} \frac{1}{\pi} \int_{0}^{2\pi} p'_\sigma(\tau)L_{\sigma,s}(\tau,t)d\tau,
\]
here by (3, 4) $q'_s(t) = -\text{Im}[\mathcal{V}_s'(t) + \sum_{\sigma=1}^{n} \frac{\Gamma_s(t)}{2\pi i (z_s(t) - z_{\sigma})}]$, $s = 1, \ldots, m$. $p'_s(t) = -\text{Re}[\mathcal{V}_s'(t) + \sum_{\sigma=1}^{n} \frac{N_s(t)}{2\pi i (z_s(t) - z_{\sigma})}]$, $s = m + 1, \ldots, n$.

The kernel $L_{\sigma,s}$ is singular with the singularity $\cot \frac{\tau}{2}$ for $\sigma = s$:

$$
(\log |z_\sigma(\tau) - z_s(\tau)|)_t = \text{Re} \left( \log \sum_{k=-m_s}^{n_s} d_{ks} e^{ik\tau} - e^{ikt} \right)' = \text{Re} \left( \log \sin \frac{\tau - t}{2} + 
\right.
$$

$$
+ \log \left[ \sum_{k=1}^{m_s} d_{ks} e^{ik\tau} \sum_{l=0}^{k-1} e^{il(\tau-t)} - \sum_{k=1}^{m_s} d_{(-k)s} e^{-ik\tau} \sum_{l=1}^{k-1} e^{il(\tau-t)} \right]_t
$$

$$
= -\frac{1}{2} \cot \frac{\tau - t}{2} + \left( \log \left[ \sum_{k=1}^{m_s} d_{ks} e^{ik\tau} \sum_{l=0}^{k-1} e^{il(\tau-t)} - \sum_{k=1}^{m_s} d_{(-k)s} e^{-ik\tau} \sum_{l=1}^{k-1} e^{il(\tau-t)} \right] \right)' .
$$

Cauchy integral in the sense of the principal value

$$
\frac{1}{\pi} \int_{0}^{2\pi} \log |z_\sigma(\tau)|_t \cot \frac{\tau - t}{2} d\tau
$$

Can be calculated by the Hilbert formula [14] as in [13].

As a result, we obtain the following system of Fredholm integral equations of the second kind, which can be written in operator form as follows:

$$
\begin{pmatrix}
I + K_{1,1} & K_{2,1} & \cdots & K_{m,1} & -L_{m+1,1} & \cdots & -L_{n,1} \\
K_{1,2} & I + K_{2,2} & \cdots & K_{m,2} & -L_{m+1,2} & \cdots & -L_{n,2} \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
K_{1,m} & K_{2,m} & \cdots & I + K_{m,m} & -L_{m+1,m} & \cdots & -L_{n,m} \\
L_{1,m+1} & L_{2,m+1} & \cdots & L_{m,m+1} & I + K_{m+1,m+1} & \cdots & K_{n,m+1} \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
L_{1,n} & L_{2,n} & \cdots & L_{m,n} & K_{m+1,n} & \cdots & I + K_{n,n}
\end{pmatrix}
\times
\begin{pmatrix}
p'_1 \\
p'_2 \\
\vdots \\
p'_m \\
q'_1 \\
q'_2 \\
\vdots \\
q'_n
\end{pmatrix}
= \begin{pmatrix}
Q_1 \\
Q_2 \\
\vdots \\
Q_m \\
R_{m+1} \\
\vdots \\
R_n
\end{pmatrix} .
$$

(9)

The latter system can be reduced to an infinite linear system over the Fourier coefficients of the unknown functions $p'_s(t)$, $s = 1, \ldots, m$, $q'_s(t)$, $s = m + 1, \ldots, n$ if we find the coefficients of the double Fourier expansions of the kernel of integral operators and compare the coefficients for the same trigonometric functions. An approximate solution to an infinite system with unknown Fourier coefficients is the solution of a truncated system of a finite size $M$ with unknown Fourier coefficients.
Note that the matrix at the left part of infinite system (9) consists of $n^2$ infinite block matrices. We say that $M$-truncation of the matrix of system (9) is the finite matrix obtained from the initial one after we truncate each block matrix so that it has the dimension $M \times M$.

**Theorem 2.** The solution of system (9) can be approximated by solutions of finite systems with $M$-truncated matrices when $M \to \infty$.

**Proof.** We are looking for an approximate solution of system (9) in the form of Fourier polynomials:

$$p'_s(t) = \sum_{l=1}^{M} \alpha_{ls} \cos lt + \beta_{ls} \sin lt, \ s = 1, \ldots, m,$$

$$q'_s(t) = \sum_{l=1}^{M} \alpha_{ls} \cos lt + \beta_{ls} \sin lt, \ s = m+1, \ldots, n. \quad (10)$$

The existence of an exact solution to the system (9) and the convergence of an approximate solution to an exact solution under the condition $M \to \infty$ were proved in [12] for the case of a conformal mapping of a simply connected domain. This proof can be applied to the case of a multiply connected domain if the corresponding space $l^2$ is replaced by the space $l^2 \times l^2 \times \ldots \times l^2$.

Now the Fredholm integral equations of the second kind in (4) can be reduced to the linear system with unknown Fourier coefficients $\alpha_{ls}, \beta_{ls}, s = 1, \ldots, n$:

$$(A_1 \ldots A_{1n} \ldots B_{1n}) \times \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \alpha_2 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

here $\alpha_s = (\alpha_{1s}, \ldots, \alpha_{ns})^T$, $\beta_s = (\beta_{1s}, \ldots, \beta_{ns})^T$. The vectors $a_s = (a_{1s}, \ldots, a_{ns})^T$, $b_s = (b_{1s}, \ldots, b_{ns})^T$ on the right side of the system consist of elements

$$a_{js} = \frac{1}{\pi} \int_0^{2\pi} Q_s(t) \cos jt \, dt, \quad b_{js} = \frac{1}{\pi} \int_0^{2\pi} Q_s(t) \sin jt \, dt, \quad j = 1, \ldots, n, \ s = 1, \ldots, n,$$

The matrices $A_{\sigma s}, B_{\sigma s}, C_{\sigma s}, D_{\sigma s}, \sigma, s = 1, \ldots, n$, of size $M \times M$ consist of elements

$$A_{\sigma sjk} = \delta_{\sigma s} \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \cos k \tau \int_0^{2\pi} K_{\sigma s}(\tau, t) \cos jt \, dt, \quad k = 1, \ldots, n,$$
CONSTRUCTION OF COMPLEX POTENTIALS FOR MULTIPLY CONNECTED DOMAINS WITH GIVEN BOUNDARY CIRCULATIONS AND FLUXES

\[ B_{\sigma,jk} = -\frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau,t) \cos jdt, \]

\[ C_{\sigma,jk} = -\frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \cos k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau,t) \sin jdt, \]

\[ D_{\sigma,jk} = \delta_{\sigma s} \delta_{jk} - \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \sin k\tau d\tau \int_0^{2\pi} K_{\sigma s}(\tau,t) \sin jdt, \]

Here, \( j, k = 1, \ldots, n, \delta_{rt} \) are the Kronecker delta functions.

Now this system can be reduced to an infinite system of equations for unknown coefficients \( \alpha_{k,j}, \beta_{k,j}, j = 1, \ldots, n, k = 1, \ldots, \infty \), in the form of

\[ \tilde{Y} = \tilde{P}\tilde{Y} + \tilde{Q}, \]

where \( \tilde{Y} = (\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n}, \beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,n}, \alpha_{2,1}, \ldots) \in l_2 \), infinite matrix \( \tilde{P} \) consist of elements

\[ \frac{1}{\pi^2} \int_0^{2\pi} f(mt)dt \int_0^{2\pi} g(p\tau)(\arg[z_k(\tau)])'_t d\tau - z_j(t)'_t \]

or

\[ \frac{1}{\pi^2} \int_0^{2\pi} f(mt)dt \int_0^{2\pi} g(p\tau)(|z_k(\tau) - z_j(t)|)'_t d\tau, \]

where \( f(x), g(x) \) are equal to \( \cos x \) or \( \sin x \). These elements are the Fourier coefficients of the double Fourier series \( \arg(z_k(\tau) - z_j(t))'_t \) or \( \ln|z_k(\tau) - z_j(t)|'_t \). \( \tilde{Q} \) are the sets of corresponding Fourier coefficients of functions \( Q_j(t), j = 1, \ldots, m, R_j(t), j = m + 1, \ldots, n \).

We need to construct the approximate solution \( \tilde{y}_j(t) \) of equation system (11) in the trigonometric polynomial form \( \tilde{y}_j(t) = \sum_{k=1}^{M} \alpha_{k,j} \cos kt + \beta_{k,j} \sin kt \) in order to apply truncated linear system as in [20].

So we have to find the vector \( \tilde{Y}_N \) with zero coordinates starting with the \((2M(n+1)+1)\)-th one which approximates the infinite vector \( \tilde{Y} \). Further on we identify the vector-function \( Y \), the integral operator \( P \), the vector-function \( Q \) with the sequence \( \tilde{Y} \), the infinite matrix \( \tilde{P} \) and the sequence \( \tilde{Q} \), respectively.

Evidently the kernels \( \arg(z_j(\tau) - z_k(t))'_t \) of system (9) are infinitely differentiable for \( k \neq j \). Due to Cauchy theorem we have \( \arg(z_j(\tau) - z_k(t)) = \arg(z_j(\tau - k(t))) = \arg(z_j(\tau - k(t))) = \arg(z_j(\tau - k(t)))/|k(t)|/2 \) where \( k(t) \) is the curvature of the boundary curve at the corresponding point. It can be easily verified that the kernel \( \arg(z_j(\tau) - z_k(t))'_t \) is at least twice differentiable with respect to both variables. So the double complex Fourier coefficients of \( \arg(z_k(\tau) - z_j(t))'_t \) have the following estimates: \( |c_{k,j,l,t}| < \frac{U}{|l|^2}|t|^2 \).

For \( N = (n+1)2M \) integral equation system (9) reduces to infinite linear system (11) which can be presented as follows:

\[ \begin{pmatrix} I_N - P_N & S \\ R & I_{\infty} - V \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}. \]
Here $P_N$ is an $N \times N$ block matrix $P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,N} \\ \vdots & \ddots & \vdots \\ P_{N,1} & \cdots & P_{N,N} \end{pmatrix}$, $M \times M$ matrices $P_{j,k}$ correspond to integral summands of (9), $j, k = 0, \ldots, n$. $S$ is an $N \times \infty$ matrix, $R$ is an $\infty \times N$ matrix, $V$ is an $\infty \times \infty$ matrix, $I_N$ and $I_\infty$ are the identity matrices of relative sizes. Each of the vectors $Q_1$ and $Y_1$ has $N$ coordinates, the vectors $Q_2$ and $Y_2$ have the infinite number of coordinates. The Fourier coefficients of the smooth functions tend to zero as their numbers tend to infinity, so the coefficients of the matrices $S, R$ and $V$ together with the coordinates of $Q_2$ decrease rapidly as $N \to \infty$. Due to Theorem assumptions and the Fourier coefficients speed of convergence to zero the matrix norm of $V$ and the vector norm of $Y_2$ tend to zero as $N \to \infty$.

Let us prove that there exists the number $T \in \mathbb{N}$ such that the matrix operator $I_N - P_N$ is invertible $\forall N > T$ since the limit for $P_N$ integral operator $P$ is compact and the operator $I - P$ is invertible due to the lemma assumption. Note that we do not distinguish a finitely dimensional vector and the Fourier polynomial with the corresponding finite set of coordinates in our proof. Recall first that due to chapter VI, paragraph 1 of [21] $\|P - P_N\| \to 0$ if $N \to \infty$. The operator norm that we deal with here is the usual operator norm for the Hilbert space mappings. Let us assume that $\forall T \in \mathbb{N}$ there exists $s_T > T$ such that the spectrum of $P_{s_T}$ contains 1. Then there exists an infinite sequence $(v_{s_T})_{T \in \mathbb{N}} \subset L^2$ such that $\|v_{s_T}\|_1 = 1$ and $P_{s_T} v_{s_T} = v_{s_T}$. Let us prove that then there should exist at least one limit point for the sequence $(v_{s_T})_{T \in \mathbb{N}}$. Since the operator $P$ is compact there exist both a subsequence $(v_{s_{T_j}})_{T_j \in \mathbb{N}}$ and an element $v_0 \in L^2$ so that $P v_{s_{T_j}} \to v_0$, $(j \to \infty)$. Then

$$\|P v_{s_{T_j}} v_{s_{T_j}} - v_0\| = \|P v_{s_{T_j}} - P v_{s_{T_j}}\| \leq$$

$$\leq \|P v_{s_{T_j}} - P v_{s_{T_j}}\| + \|P v_{s_{T_j}} - v_0\| \leq$$

$$\leq \|P v_{s_{T_j}} - v_0\| \to 0, (j \to \infty).$$

Thus $\|v_{s_T} - v_0\| = \|P v_{s_T} v_{s_T} - v_0\| \to 0, (j \to \infty)$. Hence $v_{s_T} \to v_0$, $(j \to \infty)$. Note that since $\|v_{s_T}\|_1 = 1, \forall j \in \mathbb{N}$, the element $v_0$ is nondegenerate. Let us show now that the relation $P v_0 = v_0$ holds true. Indeed, we have

$$\|P v_0 - v_0\| = \|P v_{s_{T_j}} - P v_{s_{T_j}} - v_0\| \leq$$

$$\leq \|P\| \|v_0 - v_{s_{T_j}}\| + \|P v_{s_{T_j}} - P v_{s_{T_j}} - v_0\| \leq$$

$$\leq \|P\| \|v_0 - v_{s_{T_j}}\| + \|P v_{s_{T_j}} - v_{s_{T_j}}\| + \|v_{s_{T_j}} - v_0\| \to 0, (j \to \infty).$$

Hence the spectrum of $P$ contains 1. A contradiction with one of the assumptions.

We now take the number $N$ so that $\|V\| < 1$ and the matrix $I_N - P_N$ possesses the inverse one. Now we have the relation

$$(I_N - P_N)[I_N - (I_N - P_N)^{-1}S(I_\infty - V)^{-1}R]Y_1 = Q_1 - S(I_\infty - V)^{-1}Q_2.$$
Obviously one can choose the value of \( N \) so large that \( \| S(I_\infty - V)^{-1} R \| = O(1/N^2) \leq r \) where \( r < 1 \) is arbitrary small. Now we estimate the norm of the difference between the solution \( x_1 \) and the solution \( \tilde{x}_1 \) of the truncated system \((I_F - P_F)\tilde{x}_1 = y_1:\)

\[
\| Y_1 - \tilde{Y}_1 \| \leq \frac{1}{1-r} \| (I_N - P_N)^{-1} \| \| S(I_\infty - V)^{-1} \| \| Q_2 \| + \frac{r}{1-r} \| (I_N - P_N)^{-1} \| \| Q_1 \| .
\]

Consider the first summand on the right-hand side of the last inequality. Recall the Jackson’s inequality: if \( f : [0, 2\pi] \to \mathbb{C} \) is an \( r \) times differentiable periodic function such that \( |f^{(r)}(x)| \leq 1 \), \( 0 \leq x \leq 2\pi \), then, for every positive integer \( n \), there exists a trigonometric polynomial \( T_{n-1} \) of degree at most \( n - 1 \) such that \( |f(x) - T_{n-1}(x)| \leq \frac{C(r)}{\pi} \) for any \( x \in [0, 2\pi] \) where \( C(r) \) depends only on \( r \) [22]. So the vector norm of \( y_2 \) can be estimated by this inequality by \( K/N^2 \). The second summand also behaves no better than \( O(1/N^2) \). So the error due to the series tail is \( O(1/N^2) \).

After we find the approximate values of Fourier coefficients

\[
(\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,n}, \beta_{1,1}, \beta_{1,2}, \ldots, \beta_{1,n}, \alpha_{2,1}, \ldots, \beta_{M,n})
\]

we restore the boundary values of the holomorphic summand of the complex potential (1) as follows.

The functions \( \psi(z_s(t)), s = 1, \ldots, m \), of formula (2) can be restored via the approximate coefficients of derivatives \( p_s(t), (10) \), and the functions \( q_s(t), (3) \), with arbitrary constant summands \( c_{0s} \):

\[
\psi(z_s(t)) = \tilde{p}_s(t) + i\tilde{q}_s(t) + c_{0s}, \text{here } \tilde{p}_s(t) = \sum_{l=1}^{M} \frac{a_{ls}}{l} \sin lt - \frac{\beta_{ls}}{l} \cos lt, s = 1, \ldots, m.
\]

The functions \( \psi(z_s(t)), s = m+1, \ldots, n \), of formula (2) can be restored via the approximate coefficients of derivatives \( q_s(t), (10) \), and the functions \( p_s(t), (4) \) with arbitrary constant summands \( c_{0s} \):

\[
\psi(z_s(t)) = p_s(t) + i\tilde{q}_s(t) + c_{0s}, \text{here } \tilde{q}_s(t) = \sum_{l=1}^{M} \frac{a_{ls}}{l} \sin lt - \frac{\beta_{ls}}{l} \cos lt, s = m+1, \ldots, n.
\]

The values of complex constants \( c_{0s}, s = 1, \ldots, n \), we get as follows: Note that the expression for the complex potential (1) includes terms of the form \( \ln(z - z_{sk}^*) \), here \( z_{sk}^*, k = 1, \ldots, n \), are points from the finite domains that do not intersect with \( D_\omega \). Since \( z_{sk}^* \) are external points of the domain \( D_\omega \), the Cauchy-type integral with density, which is the boundary value of a function analytic in \( D_\omega \), vanishes at each point \( z_{sk}^*, k = 1, \ldots, n \). Thus, to determine \( n \) complex constants, we obtain a system of \( n \) linear equations. Functions \( \psi_s(t), s = 1, \ldots, n \), are the boundary values of the analytic function in \( D_\omega \), so the Cauchy integral with the corresponding density along the boundary \( D_\omega \) vanishes at the points \( z_{sk}^*, k = 1, \ldots, n \). Therefore, we have the linear complex system

\[
\sum_{s=1}^{m} \frac{1}{2\pi i} \int_{0}^{2\pi} [\tilde{p}_s(\tau) + iq_s(\tau) + c_{0s}] [\log(z_s(\tau) - z_{sk}^*)]' d\tau +
\]

\[
\sum_{s=m+1}^{n} \frac{1}{2\pi i} \int_{0}^{2\pi} [p_s(\tau) + i\tilde{q}_s(\tau) + c_{0s}] [\log(z_s(\tau) - z_{sk}^*)]' d\tau = 0, k = 1, \ldots, n.
\]

over the unknown constants \( c_{0s}, s = 1, \ldots, n \). All coefficients of this system, except for the diagonal ones, are equal to zero.
We now have the functions $\psi_s(t), s = 1, \ldots, n, t \in [0, 2\pi]$, and therefore we can restore the complex potential. The approximate value of the complex potential (1) now has the form

$$
\phi(z) = \sum_{s=1}^{m} \frac{1}{2\pi i} \int_{0}^{2\pi} \left( \frac{\tilde{p}_s(\theta) + iq_s(\theta) + c_{0s}z'_s(\theta)}{z_s(\theta) - z} \right) d\theta + \frac{1}{2\pi i} \sum_{s=m+1}^{n} \int_{0}^{2\pi} \left( \frac{p_s(\theta) + iq_s(\theta) + c_{0s}z'_s(\theta)}{z_s(\theta) - z} \right) d\theta + \sum_{s=1}^{n} F_s \log(z - z^*_s).
$$

Note that we calculate the values of the Cauchy integral in a neighbourhood of the boundary $L = \partial D$ applying the analytical continuation in the form of Taylor series:

$$
\frac{1}{2\pi i} \int_{L} \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{l=0}^{\infty} \frac{1}{2\pi i} \int_{L} \frac{f(\zeta)}{(\zeta - z_0)^{l+1}} (z - z_0)^l,
$$

here $z_0 \in D$.

The method converges not only for the smooth boundaries but also for contours with angles greater than $\pi$ in the flux plane (see examples in [3, 26]).

3. Examples

We examine different flows in the 3-connected domain $D$, bounded by the following curves: $L_1: z_1(t) = 6 \exp(it) - \exp(-it), t \in [0, 2\pi], L_2: z_2(t) = 2 \exp(it) - \exp(-it) + 11i, t \in [0, 2\pi]$, and $L_3: z_3(t) = \exp(it) - 9, t \in [0, 2\pi]$, with a velocity at infinity equal to 1. We draw the streamlines for $\text{Im}[\phi] = \{-5, -4, -2, 0, 2, 4, 6\}$ for each considered flow.

1. Consider the region with impermeable boundaries $L_1, L_2$ and $L_3$, with circulations $\Gamma_1 = \Gamma_2 = \Gamma_3 = 0$. We present the streamlines in Fig. 1.

![Figure 1: Three holes with impermeable boundaries without circulations.](image)

2. Consider the region with impermeable boundaries $L_1, L_2$ and $L_3$, with circulations $\Gamma_1 = -0.4\pi$, $\Gamma_2 = 0$, $\Gamma_3 = -0.2\pi$. We present the streamlines in Fig. 2. Note that the streamlines along the third contour become more asymmetric and shift upwards.
CONSTRUCTION OF COMPLEX POTENTIALS FOR MULTIPLY CONNECTED DOMAINS WITH GIVEN BOUNDARY CIRCULATIONS AND FLUXES

3. Consider the region with impermeable boundaries $L_1, L_2$ and the permeable boundary $L_3$, with circulations $\Gamma_1 = \Gamma_2$, and the flux $N_3 = -0.4\pi$. We present the streamlines in Fig. 3. The streamlines intersect the third contour that represents a sink.

4. Consider the region with impermeable boundaries $L_1, L_2$ and the permeable boundary $L_3$, with circulations $\Gamma_1 = \Gamma_2$, and the flux $N_3 = 0.4\pi$. We present the streamlines in Fig. 4. The streamlines here move away from the third contour that represents a source.

References

CONSTRUCTION OF COMPLEX POTENTIALS FOR MULTIPLY CONNECTED DOMAINS WITH GIVEN BOUNDARY CIRCULATIONS AND FLUXES

Figure 4. Three holes with 2 impermeable boundaries with circulations and one permeable boundary


CONSTRUCTION OF COMPLEX POTENTIALS FOR MULTIPLY CONNECTED DOMAINS WITH GIVEN BOUNDARY CIRCULATIONS AND FLUXES


LOBACHEVSKIY INSTITUTE OF MATHEMATICS & MECHANICS, KAZAN FEDERAL UNIVERSITY, KREMLEVSKAYA ST., 35, KAZAN, 420008, RUSSIA

Email address: pivanshi@yandex.ru

LOBACHEVSKIY INSTITUTE OF MATHEMATICS & MECHANICS, KAZAN FEDERAL UNIVERSITY, KREMLEVSKAYA ST., 35, KAZAN, 420008, RUSSIA

Email address: Elena.Shirokova@kpfu.ru