Fixed point theorems for convex-power condensing operators in Banach algebras

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Abstract

In this paper, we introduce the concept of convex-power condensing mapping \( A \cdot B + C \) in a Banach algebra relative to a measure of noncompactness as a generalization of condensing and convex-power condensing mappings. We present new fixed point theorems and we apply these results to investigate the existence of solutions for a nonlinear hybrid integral equation of Volterra type.

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1 Introduction

The theory of integral equations in abstract space, developed in the first half of the twentieth century, is a very active research area. The need for a fixed point theory in Banach algebras arises out of the study of quadratic integral equations and provides a powerful tool for studying the existence of solutions. These equations have received increasing attention during recent years due to their applications in diverse fields of science and engineering (see for example [9, 15, 16, 21, 19, 27]). Some of these equations can be formulated into nonlinear operators equations of the form:

\[
A(x) \cdot B(x) + C(x) = x, \quad x \in \Omega, 
\]  

(1)

where \( \Omega \) is a nonempty subset of a Banach algebra \( X \).

In recent years, many authors have focused on the resolution of Equation (1) and obtained a lot of valuable results (see for example [3, 4, 7, 10, 11, 28] and the references therein). These studies were mainly based on the convexity of the bounded domain, the celebrate Schauder fixed point theorem [28] and properties of operators \( A, B \) and \( C \) (cf. completely continuous, \( k \)-set contractive, condensing and the potential tool of the axiomatic measures of non-compactness). In [29], Sun and Zhang introduced the definition of convex-power condensing operator with respect to
the Kuratowski measure of noncompactness and proved a fixed point theorem which extended the well-known Sadovskii’s fixed point theorem and a fixed point theorem in Liu et al. [22]. In [32], G. Zhang et al. established some fixed point theorems of Rothe and Altman types about convex-power condensing operators with respect to the Kuratowski measure of noncompactness. These results were applied to a differential equation of one order with integral boundary conditions. K. Ezzinbi and M.A. Taoudi in [14] introduced the concept of a convex-power condensing mapping $T$ with respect to another mapping $S$ as a generalization of condensing and convex-power condensing mappings. Some fixed point theorems for the sum $T + S$ where $S$ is a strict contraction and $T$ convex-power condensing with respect to $S$ are established. Ezzinbi and Taoudi also considered the case where $S$ is nonexpansive or expansive. These fixed point results extend several earlier works including [31] and many others, and offer some new tools to deal with the existence of solutions to many differential and integral equations. In [20], A. Khchine, L. Maniar and M. A. Taoudi introduced the results of Hussain and Taoudi [14] to locally convex spaces. Later, they introduced and made use of new concepts of Banach contractions in locally convex spaces. These results encompass, improve and extend the well-known Cain and Nashed [8] fixed point theorem and many others [5, 23, 24, 25, 26, 30]. To illustrate their theoretical results, they investigate the solvability of a broad class of integral equations in locally convex spaces.

In this paper, we introduce the concept of a convex-power condensing mapping $A \cdot B + C$ in a Banach algebra $X$ (see Definition 1). As a generalization of condensing and convex-power condensing mappings in a Banach space $X$. We study the existence of fixed points of this class of new operators. Then, we state some new fixed-point theorems in Banach algebras. The main results are Theorems 1 and 2. These results extend those given in [1, 3, 4, 7, 10, 11, 28, 29]. Finally, we discuss the existence of positive solutions of equation (1). So, we use the notion of positive cone which is a valuable tool in the study of nonlinear problems. In section 4, we establish existence results for nonlinear hybrid integral equations of Volterra type:

$$x(t) = f(t, x(t)) \cdot \left( \int_{0}^{t} g(s, x(s))ds \right) + h(t, x(t)),$$

for all $t \in J = [0, 1]$, where $f, g, h : J \times X \to X$, here $X$ is a Banach algebra.

2 Preliminaries

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper.
A mapping \( A : X \to X \), where \( X \) is a Banach space is called totally bounded if \( A(S) \) is relatively compact for any bounded subset \( S \) of \( X \). \( A \) is completely continuous if is continuous and totally bounded. \( A \) is called \( D \)-Lipschitzian if there exists a continuous nondecreasing function \( \phi_A : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying \( \|Ax - Ay\| \leq \phi_A(\|x - y\|) \) for all \( x, y \in X \) with \( \phi_A(0) = 0 \). Sometimes the function \( \phi_A \) is called a \( D \)-function of \( A \) on \( X \). In the special case when \( \phi_A(r) = kr, k > 0, A \) is called a Lipschitzian with a Lipschitz constant \( k \). In particular, if \( k < 1 \), \( A \) is called a contraction with a contraction constant \( k \). Further, if \( \phi_A(r) < r \) for \( r > 0 \), then \( A \) is called a nonlinear contraction on \( X \). Notice that, every \( D \)-Lipschitzian mapping \( A \) is bounded, i.e. maps bounded sets into bounded sets.

Recall Boyd and Wong [6] theorem which stats that any nonlinear contraction \( A : X \to X \), where \( X \) is a Banach space has a unique fixed point \( x^* \) and the sequence \( \{A^n(x)\} \) of successive iterations of \( A \) converges to \( x^* \) for each \( x \in X \).

We provide now some important properties of measures of noncompactness that will be used in this work. Recall that the Kuratowski measure of noncompactness is defined on each bounded subset \( C \) of a Banach space \( X \) by

\[
\alpha(C) = \inf \left\{ r > 0, \ C \subseteq \bigcup_{i=1}^{n} C_i \text{ such that } \text{diam}(C_i) \leq r \right\}.
\]

For completeness we recall some properties of \( \alpha \). Let \( C_1, C_2 \in X \) be bounded. The following properties are satisfied:

1. Monotonicity: If \( C_1 \subseteq C_2 \), then \( \alpha(C_1) \leq \alpha(C_2) \).
2. Regularity: \( \alpha(C_1) = 0 \) if and only if \( C_1 \) is relatively compact.
3. Invariant under closure: \( \alpha(C_1) = \alpha(\overline{C_1}) \), where \( \overline{C_1} \) is the closure of \( C_1 \).
4. Semi-homogeneity: \( \alpha(\lambda C_1) = |\lambda|\alpha(C_1) \) for all \( \lambda \in \mathbb{C} \).
5. Invariance under passage to the convex hull: \( \alpha(co(C_1)) = \alpha(C_1) \).
6. Algebraic semi-additivity: \( \alpha(C_1 + C_2) \leq \alpha(C_1) + \alpha(C_2) \).

By a measure of non-compactness on a Banach space \( X \) we mean a map \( \Psi : \Omega_X \to \mathbb{R}_+ \) satisfying conditions (i)-(vi), here \( \Omega_X \) stands for the collection of bounded subsets of \( X \).

Let \( X \) be a Banach space, \( D \subset X \) and \( F : D \to X \) be continuous operator. If there exists a constant \( k \geq 0 \), such that for any bounded and nonempty subsets \( S \subset D \), \( \alpha(F(S)) \leq k\alpha(S) \), then \( F \) is said to be \( k \)-set-contraction in \( D \). Notice that in view of [2], if the map \( A : X \to X \) is Lipschitz with constant \( k \), then \( A \) is \( k \)-set-contraction. Finally, if \( X \) be a Banach algebra and \( S, G \) two non-empty subsets bounded in \( X \), then \( \alpha(SG) \leq \|S\|\alpha(G) + \|G\|\alpha(S) \), (for the proof we refer to [1]).
The following results are needed in this paper.

**Lemma 1.** [2] Let $X$ be a Banach space and $C([0,a], X)$ be the space of continuous functions defined on $[0,a]$ with values in $X$. If $V \subseteq C([0,a], X)$ is bounded, then $\alpha(V(t)) \leq \alpha(V)$ for any $t \in [0,a]$, where $V(t) = \{u(t) : u \in V\}$. Furthermore if $V$ is equicontinuous, then $t \rightarrow \alpha(V(t))$ is continuous on $[0,a]$,

$$\alpha(V) = \sup\{\alpha(V(t)) : t \in [0,a]\},$$

and

$$\alpha \left( \int_0^t V(s)ds \right) \leq \int_0^t \alpha(V(s))ds,$$

for all $t \in [0,a]$, where $\int_0^t V(s)ds = \left\{ \int_0^t x(s)ds : x \in V \right\}$.

**Lemma 2.** [2] If $V \subset C([0,a], X)$ is equicontinuous and $x_0 \in C([0,a], X)$, then $co(V \cup \{x_0\})$ is also equicontinuous.

**Lemma 3.** [22] Let $f$ be bounded and uniformly continuous on $[0,a] \times \overline{B}_R$ for any $R > 0$. If $S \subset C([0,a], X)$ is bounded and equicontinuous, where $X$ is a Banach space, then $\{f(\cdot, x(\cdot)) | x \in S\}$ is bounded and equicontinuous in $C([0,a], X)$.

### 3 Fixed Point Theory

In the section, we establish some new fixed point theorems for the operator $x \mapsto Ax Bx + Cx$. Let $X$ be a Banach algebra, $\Omega$ be a nonempty subset of $X$, $A, C : X \rightarrow X$ and $B : \Omega \rightarrow X$ be three mappings and $x_0 \in \Omega$. For any $S \subset \Omega$ we set

$$F^{(1,x_0)}(AB + C, S) = \left\{ x \in \Omega : x = Ax By + Cx, \text{ for some } y \in S \right\}$$

and

$$F^{(n,x_0)}(AB + C, S) = F^{(1,x_0)}(AB + C, co(F^{(n-1,x_0)}(S) \cup \{x_0\})) \text{ for } n = 2, 3, \ldots$$

**Remark 1.** If $B \equiv 1_X$ and $C \equiv 0_X$ where $1_X$ is the unit element of the Banach algebra $X$, then

$$F^{(1,x_0)}(A, S) := A^{(1,x_0)}(S) = A(S)$$

and

$$F^{(n,x_0)}(S) := A^{(n,x_0)}(S) = A^{(1,x_0)}(co(A^{(n-1,x_0)}(S) \cup \{x_0\})) \text{ for } n = 2, 3, \ldots$$

Now, we present a definition of convex-power condensing operators in Banach algebras.

**Definition 1.** Let $\Omega$ be a nonempty closed convex subset of a Banach algebra $X$ and $\Psi$ a measure of non-compactness on $X$. Let $A, C : X \rightarrow X$ and $B : \Omega \rightarrow X$ be three bounded mappings (i.e. they take bounded sets into bounded ones) and $(n_0, x_0) \in \mathbb{N} \times \Omega$. We say that $A \cdot B + C$ is a convex-power condensing operator about $(x_0, n_0)$ with respect to $\Psi$ if

$$\Psi \left( F^{(n_0,x_0)}(S) \right) < \Psi(S),$$

for any bounded set $S \subseteq \Omega$ with $\Psi(S) > 0$. 
Remark 2. 1. We say that $A$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$ if for any bounded set $S \subseteq \Omega$ with $\Psi(S) > 0$ we have

$$\Psi \left( A^{(n_0, x_0)}(S) \right) < \Psi(S).$$

2. $A : \Omega \to \Omega$ is $\Psi$-condensing if and only if it is convex-power condensing operator about $(1, x_0)$ with respect to $\Psi$.

Recall that the concept of convex-power condensing operators was introduced by Sun and Zhang [29]. In all following, $\Psi$ is a measure of non-compactness on $X$ and $(x_0, n_0) \in \Omega \times \mathbb{N}$.

We establish the main theorem as follows:

**Theorem 1.** Let $\Omega$ be a nonempty bounded closed convex subset of a Banach algebra $X$. Let $A, C : X \to X$ and $B : \Omega \to X$ be three mappings satisfying the following conditions:

(i) $A$ and $C$ are $D$-Lipschitzians mappings with $D$-functions $\phi_A$ and $\phi_C$ respectively,

(ii) $B$ is continuous and $A \cdot B + C$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$,

(iii) the equality $x = AxBy + Cx$ with $y \in \Omega$ implies $x \in \Omega$.

Then, the operator equation $x = AxBy + Cx$ has a solution in $\Omega$ provided that $Q\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(\Omega)\| := \sup\{\|Bx\| : x \in \Omega\}$.

**Proof.** Let $y \in \Omega$ be fixed. The map $F_y$ which assigns to each $x \in X$ the value $AxBy + Cx$ defines a nonlinear contraction with a contraction function $\varphi(r) = Q\phi_A(r) + \phi_C(r), r > 0$. Indeed, for all $x_1, x_2 \in X$ we have:

$$\|F_y(x_1) - F_y(x_2)\| \leq \|Ax_1By + Cx_1 - Ax_2By - Cx_2\|$$

$$\leq \|Ax_1 - Ax_2\| \|B(y)\| + \|C_{x_1} - C_{x_2}\|$$

$$\leq \|Ax_1 - Ax_2\| \|B(\Omega)\| + \|C_{x_1} - C_{x_2}\|$$

$$\leq Q\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|)$$

$$= \varphi(\|x_1 - x_2\|).$$

Now, the Boyd and Wong fixed point theorem guarantees that there exists a unique point $x^* \in X$ such that $F_y(x^*) = x^*$, i.e. $By = (\frac{I-C}{A}) x^*$. Thus, the operator $F := (\frac{I-C}{A})^{-1} B : \Omega \to X$ is well defined. By assumption (iii) we have $F(\Omega) \subseteq \Omega$. Now we show that $F : \Omega \to \Omega$ is continuous. First notice that for each $x \in \Omega$ we have

$$Fx = A(Fx)Bx + C(Fx).$$

Let $(x_n)_n$ be a sequence in $\Omega$ converging to a point $x$. Since $\Omega$ is closed, $x \in \Omega$. Hence,
\[
\|F x_n - F x\| \leq \|A(F x_n) B x_n - A(F x) B x\| + \|C(F x_n) - C(F x)\|
\]
\[
\leq \|A(F x_n) B x_n - A(F x) B x_n\| + \|A(F x) B x_n - A(F x) B x\| + \phi_C(\|F x_n - F x\|)
\]
\[
\leq \|A(F x_n) - A(F x)\| \|B x_n\| + \|A(F x)\| \|B x_n - B x\| + \phi_C(\|F x_n - F x\|)
\]
\[
\leq \phi_A(\|F x_n - F x\|) \|B(\Omega)\| + M \|B x_n - B x\| + \phi_C(\|F x_n - F x\|)
\]
\[
\leq \phi_A(\|F x_n - F x\|)Q + M \|B x_n - B x\| + \phi_C(\|F x_n - F x\|),
\]

where
\[
\|A(F x)\| \leq \|A(F x) - A x_0\| + \|A x_0\|
\]
\[
\leq \phi_A(\|F x - x_0\|) + \|A x_0\|
\]
\[
\leq \phi_A(\text{diam}(\Omega)) + \|A x_0\| := \mathcal{M}
\]

for any \(x_0 \in \Omega\). Thus
\[
\limsup_n \|F x_n - F x\| \leq \phi_C\left( \limsup_n \|F x_n - F x\| \right) + \phi_A\left( \limsup_n \|F x_n - F x\| \right)Q.
\]

This shows that \(\limsup_n \|F x_n - F x\| = 0\) and consequently \(F\) is continuous on \(\Omega\).

Notice that
\[
F(\Omega) = \left( \frac{I - C}{A} \right)^{-1} B(\Omega) = \mathcal{F}^{(1,x_0)}(AB + C, \Omega).
\]

Let
\[
\Gamma = \left\{ M \subset \Omega, \ \overline{co}(M) = M, \ x_0 \in M \text{ and } F(M) \subseteq M \right\}.
\]

The set \(\Gamma\) is nonempty since \(\Omega \in \Gamma\). Set \(K = \bigcap_{M \in \Gamma} M\). We show that for any positive integer \(n\) we have
\[
K = \overline{co}\left( \mathcal{F}^{(n,x_0)}(AB + C, K) \cup \{x_0\} \right).
\]

To see this, we proceed by induction. Clearly \(K\) is a closed convex subset of \(\Omega\) and \(F(K) \subseteq K\). Thus \(K \in \Gamma\). This implies
\[
\overline{co}(F(K) \cup \{x_0\}) \subseteq K.
\]

Hence
\[
F(\overline{co}(F(K) \cup \{x_0\})) \subseteq F(K) \subseteq \overline{co}(F(K) \cup \{x_0\}).
\]

Consequently
\[
\overline{co}(F(K) \cup \{x_0\}) \in \Gamma.
\]

Hence
\[
K \subseteq \overline{co}(F(K) \cup \{x_0\}).
\]

As a result, we get
\[
\overline{co}(F(K) \cup \{x_0\}) = \overline{co}(\mathcal{F}^{(1,x_0)}(AB + C, K) \cup \{x_0\}) = K.
\]
This shows the property for \( n = 1 \). Suppose that the property holds for \( n \geq 2 \), then we have

\[
\mathcal{F}^{(n+1,x_0)}(AB + C, K) = \mathcal{F}^{(n,x_0)}(\overline{\mathcal{O}}(\mathcal{F}^{(n,x_0)}(AB + C, K) \cup \{x_0\}))
= F(\overline{\mathcal{O}}(\mathcal{F}^{(n,x_0)}(AB + C, K) \cup \{x_0\}))
= F(K).
\]

Consequently,

\[
\overline{\mathcal{O}}(\mathcal{F}^{(n+1,x_0)}(AB + C, K) \cup \{x_0\}) = \overline{\mathcal{O}}(F(K) \cup \{x_0\}) = K.
\]

Thus, for all \( n \geq 1 \) we have

\[
K = \overline{\mathcal{O}}(\mathcal{F}^{(n,x_0)}(AB + C, K) \cup \{x_0\})
\]

In particular, we have

\[
K = \overline{\mathcal{O}}(\mathcal{F}^{(n_0,x_0)}(AB + C, K) \cup \{x_0\})
\]

Using the properties of \( \Psi \) we get

\[
\Psi(K) = \Psi(\overline{\mathcal{O}}(\mathcal{F}^{(n_0,x_0)}(AB + C, K) \cup \{x_0\})) \leq \Psi(\mathcal{F}^{(n_0,x_0)}(AB + C, K)) < \Psi(K),
\]

which yields that \( K \) is compact. Applying the Schauder fixed point theorem we infer there exists \( x^* \in K \) such that

\[
x^* = F(x^*) = A(Fx^*)Bx^* + C(Fx^*) = Ax^*Bx^* + Cx^*.
\]

This completes the proof.

**Remark 3.** The boundedness of \( \Omega \) is not necessary if we assume that \( \mathcal{F}^{(n,x_0)}(AB + C, S) \) is bounded, whenever \( S \subset \Omega \) with \( S \) bounded.

As an easy consequence of Theorem 1 we obtain the following corollary.

**Corollary 1.** Let \( X \) be a Banach algebra, \( \Omega \) a nonempty bounded closed convex subset of \( X \). Let \( A, C : X \to X \) and \( B : \Omega \to X \) be three mappings satisfying the following conditions:

(i) \( A \) and \( C \) are Lipschitzians with the Lipschitz constants \( k_1 \) and \( k_2 \) respectively,

(ii) \( B \) is continuous and \( A \cdot B + C \) is a convex-power condensing operator about \((n_0,x_0)\) with respect to \( \Psi \),

(iii) the equality \( x = AxBy + Cx \) with \( y \in \Omega \) implies \( x \in \Omega \).

Then, the operator equation \( x = AxBy + Cx \) has a solution in the set \( \Omega \) provided that \( Qk_1 + k_2 < 1 \), where \( Q = \|B(\Omega)\| \).
On the basis of Theorem 1 we obtain the following version of Theorem 2.2 in [12].

**Corollary 2.** Let $\Omega$ be a non-empty closed convex and bounded subset of a Banach algebra $X$ and let $A, C : X \to X$ and $B : \Omega \to X$ be three operators such that,

(i) $A$ and $C$ are $D$-Lipschitzicians with $D$-functions $\phi_A$ and $\phi_C$ respectively,

(ii) $B$ is completely continuous,

(iii) $AxBx + Cx \in \Omega$ for each $x \in \Omega$.

whenever $\|\Omega\|\phi_A(r) + \phi_C(r) < r, r > 0$ where $\|\Omega\| := \sup\{\|x\| : x \in \Omega\}$.

**Proof.** Since $B : \Omega \to X$ is compact, $\overline{B}(\Omega)$ is compact and by Lemma 3.6 in [1]), the operator $(I - CA)^{-1} : \Omega \to X$ exists and is continuous, so, by composition we have $F(\Omega) := (I - CA)^{-1}B(\Omega)$ is relatively compact, as a result $A \cdot B + C$ is a convex-power condensing operator about $(1, x_0)$ with respect to $\Psi$. The result follows from Theorem 1. \hfill $\square$

Now we shall obtain the version of Theorem 1 which are the companion of the version of Leray-Schander principle.

**Theorem 2.** Let $U$ be a open bounded subset of a Banach algebra $X$ containing the origin $0$, and let $A, C : X \to X$ and $B : U \to X$ be three operators satisfying the following conditions

(a) $A$ and $C$ are $D$-Lipschitzicians with the $D$-functions $\phi_A$ and $\phi_C$ respectively,

(b) $B$ is continuous and $A \cdot B + C$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$,

(c) $Q\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(U)\|$.

(d) $\lambda A \left(\frac{u}{\lambda}\right) Bu + \lambda C \left(\frac{u}{\lambda}\right) \neq u$ for all $u \in \partial U$ and $\lambda \in (0, 1)$, where $\partial U$ in the boundary of $U$.

Then, the operator equation $AxBx + Cx = x$ has a solution in $X$.

**Proof.** According to the proof of Theorem 1, the operator $F := (I - CA)^{-1}B : U \to X$ exists and is continuous. Further, we introduce the set

$$ D = \left\{ x \in U \text{ such that } x = \lambda F(x) \text{ for some } \lambda \in [0, 1] \right\}. $$

The set $D$ is nonempty since $0 \in U$. Since $F$ is continuous, $D$ is closed. From (d) we have that $D \cap \partial U = \emptyset$, So by Uryshon’s lemma, there exists a continuous function $\eta : U \to [0, 1]$ with $\eta(x) = 0$ for $x \in \partial U$ and $\eta(x) = 1$ for $x \in D$. Define $\tilde{F} : D \to D$ by

$$ \tilde{F}(x) = \begin{cases} 
\eta(x)F(x) & \text{for } x \in U, \\
0 & \text{for } x \in X \setminus U.
\end{cases} $$
It is easy to see that \( \tilde{F} \) is continuous. Moreover, for any \( S \subset D \) we have
\[
\tilde{F}^{(1,0)}(S) = \tilde{F}(S) \subseteq \text{co} (F(S) \cup \{0\}) \subset \text{co} (\mathcal{F}^{(1,0)}(AB + C, S) \cup \{0\}).
\]
This implies that
\[
\tilde{F}^{(2,0)}(S) = \tilde{F} \left( \text{co} \left( \tilde{F}(S) \cup \{0\} \right) \right) \\
\subset \tilde{F} \left( \text{co} (F(S) \cup \{0\}) \right) \\
\subset \text{co} (F \left( \text{co} (\mathcal{F}^{(1,0)}(AB + C, S) \cup \{0\})) \cup \{0\}) \\
= \text{co} (\mathcal{F}^{(2,0)}(AB + C, S) \cup \{0\}).
\]
By induction, we have
\[
\tilde{F}^{(n,0)}(S) = \tilde{F} \left( \text{co} \left( \tilde{F}^{(n-1,0)}(S) \cup \{0\} \right) \right) \\
\subset \tilde{F} \left( \text{co} (\mathcal{F}^{(n-1,0)}(AB + C, S) \cup \{0\}) \right) \\
= \text{co} (F \left( \text{co} (\mathcal{F}^{(n-1,0)}(AB + C, S) \cup \{0\}) \cup \{0\}) \\
= \text{co} (\mathcal{F}^{(n,0)}(AB + C, S) \cup \{0\}),
\]
for each integer \( n \geq 1 \). Thus, for all \( n \geq 1 \) we have
\[
\tilde{F}^{(n,0)}(S) \subseteq \text{co} (\mathcal{F}^{(n,0)}(AB + C, S) \cup \{0\}).
\]
In particular, we have
\[
\tilde{F}^{(n,0)}(S) \subseteq \text{co} (\mathcal{F}^{(n,0)}(AB + C, S) \cup \{0\}).
\]
Using the properties of \( \Psi \) we get
\[
\Psi \left( \tilde{F}^{(n,0)}(S) \right) \leq \Psi \left( \text{co} (\mathcal{F}^{(n,0)}(AB + C, S) \cup \{0\}) \right) \leq \Psi (\mathcal{F}^{(n,0)}(AB + C, S)) < \Psi(S).
\]
Thus, \( \tilde{F} : D \rightarrow X \) is continuous, \( \tilde{F}(D) \subset D \) and \( \tilde{F} \) is convex-power condensing operator about \( (n_0, 0) \) with respect to \( \Psi \). By \cite{l29} there exists an \( x^* \in D \) such that \( \tilde{F}x^* = x^* \). The case \( x^* \in X \setminus \overline{U} \) being impossible since \( 0 \in U \), it remains that \( x^* \in \overline{U} \). So \( \eta(x^*) = 1 \) and consequently
\[
x^* = F(x^*) = A(Fx^*)Bx^* + C(Fx^*) = Ax^*Bx^* + Cx^*.
\]
This completes the proof. \( \square \)

**Corollary 3.** Let \( B(r) \) and \( \overline{B}(r) \) denote respectively the open and closed ball centered in \( 0 \) with radius \( r \) in a Banach algebra \( X \), and let \( A, C : X \rightarrow X \) and \( B : B(r) \rightarrow X \) be three operators satisfying the following conditions:

(a) \( A \) and \( C \) are \( D \)-Lipschitzians with the \( D \)-functions \( \phi_A \) and \( \phi_C \) respectively,
(b) $B$ is continuous and $A \cdot B + C$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$.

(c) $Q\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(\mathcal{E}(r))\|$. 

(d) $\lambda A \left( \frac{y}{\lambda} \right) Bu + \lambda C \left( \frac{y}{\lambda} \right) \neq u$ for all $\|u\| = r$ and $\lambda \in (0, 1)$.

Then the operator equation $AxBy + Cx = x$ has a solution in $X$.

Now, we shall discuss briefly the existence of positive solutions. Let $X_1$ and $X_2$ be two Banach algebras, with positive closed cones $X_1^+$ and $X_2^+$, respectively. An operator $F$ from $X_1$ into $X_2$ is said to be positive if it carries the positive cone $X_1^+$ into $X_2^+$ (i.e., $F(X_1^+) \subset X_2^+$). The details of cones and positive cones and their properties appear in Guo and Lakshmikantham [17] and Heikkilä and Lakshmikantham [18].

**Theorem 3.** Let $X$ be a Banach algebra, $\Omega$ a nonempty bounded closed convex subset of $X$ such that $\Omega^+ = \Omega \cap X^+ \neq \emptyset$. Let $A, C : X \to X$ and $B : \Omega \to X$ be three mappings satisfying the following conditions:

(i) $A$ and $C$ are $D$-Lipschitzians mappings with $D$-functions $\phi_A$ and $\phi_C$ respectively,

(ii) $B$ is continuous and $A \cdot B + C$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$,

(iii) the equality $x = AxBy + Cx$ with $y \in \Omega^+$ implies $x \in \Omega^+$.

Then the operator equation $x = AxBy + Cx$ has a solution in the set $\Omega^+$ provided that $Q\phi_A(r) + \phi_C(r) < r$ for $r > 0$, where $Q = \|B(\Omega^+)\|$.

**Proof.** Obviously $\Omega^+ = \Omega \cap X^+$ is a closed convex bounded subset of $X$. According to the proof of the Theorem 1, it follows that $F := \left( \frac{I-C}{A} \right)^{-1} B : \Omega^+ \to X$ is well defined. By assumption (iii) we have $F(\Omega^+) \subset \Omega^+$. Now, an application of Theorem 1 yields that $F$ has a fixed point in $\Omega^+$. As a result, by the definition of $F$, the equation $x = AxBy + Cx$ has a solution in $\Omega^+$. This completes the proof.

An interesting corollary of Theorem 3 is formulated below

**Corollary 4.** Let $X$ be a Banach algebra, $\Omega$ a nonempty bounded closed convex subset of $X$ such that $\Omega^+ = \Omega \cap X^+ \neq \emptyset$. Let $A, C : X \to X$ and $B : \Omega \to X$ be three mappings satisfying the following conditions:

1. $A$ and $C$ are Lipschitzians with the Lipschitz constants $k_1$ and $k_2$ respectively.

2. $B$ is continuous and $A \cdot B + C$ is a convex-power condensing operator about $(n_0, x_0)$ with respect to $\Psi$.

3. The equality $x = AxBy + Cx$ with $y \in \Omega^+$ implies $x \in \Omega^+$.

Then the operator equation $x = AxBy + Cx$ has a solution in the set $\Omega^+$ provided that $Qk_1 + k_2 < 1$, where $Q = \|B(\Omega^+)\|$.
4 Nonlinear Integral Equation

In this section, we consider the nonlinear hybrid integral equation of Volterra type,

\[ x(t) = f(t, x(t)) \cdot \left( \int_0^t g(s, x(s)) \, ds \right) + h(t, x(t)) \quad \text{for all } t \in J = [0, 1], \tag{3} \]

where \( f, g, h : J \times X \to X \), here \( X \) is a Banach algebra. We place the problem \( (3) \) in the function space \( C(J, X) \) of continuous functions defined on \( J \) with values in \( X \) with the norm

\[ \| x \|_\infty = \max_{t \in J} \| x(t) \|. \tag{4} \]

Clearly, \( C(J, X) \) becomes a Banach space with respect to the above norm which is also a Banach algebra with respect to the multiplication " \cdot " defined by

\[ (x \cdot y)(t) = x(t) \cdot y(t), \quad t \in J. \tag{5} \]

We consider the following set of hypotheses in the sequel.

\( (H_1) \) There exists two constants \( k_1, k_2 \in \mathbb{R}_+ \) such that

\[ \| f(t, x) - f(t, y) \| \leq k_1 \| x - y \| \quad \text{and} \quad \| h(t, x) - h(t, y) \| \leq k_2 \| x - y \| \]

for all \( t \in J \) and \( x, y \in X \).

\( (H_2) \) For almost every \( t \in J \) the map \( x \mapsto g(t, x) \) is continuous in \( X \).

\( (H_3) \) For every bounded set \( S \subset X \),

(a) there exists a positive constant \( c \) such that

\[ \alpha(g(t, S)) \leq c \alpha(S) \quad \text{for almost every } t \in J, \]

(b) \( \delta := k_1 Q + k_2 < 1 \) where \( Q := \sup \left\{ \int_0^t \| g(s, x) \| \, ds, t \in J, x \in S \right\}, \)

(c) there exists a function \( m \in L^1((0, 1), \mathbb{R}_+) \) and a nondecreasing continuous function \( \theta : \mathbb{R}_+ \to (0, +\infty) \) such that

\[ \| g(t, x) \| \leq m(t) \theta(\| x \|), \]

for all \( x \in X \) and all \( t \in J \), with

\[ \frac{M}{1 - \delta} \int_0^1 m(s) \, ds < \int_{h_0}^\infty \frac{ds}{\theta(s)}, \]

where \( h_0 = \max_{t \in J} \| h(t, 0) \| \) and \( M = \sup \{ \| f(t, x) \|, t \in J, x \in S \}. \)
Remark 4. It is worth noting that Hypothesis (H1) together with Lemma 2 (see [2]) imply that for any bounded subset $S$ of $X$ we have

$$\alpha(f(t, S)) \leq k_1\alpha(S) \quad \text{and} \quad \alpha(h(t, S)) \leq k_2\alpha(S) \quad \text{for all } t \in J.$$  

Before stating the main result of this section we prove the following auxiliary result.

Lemma 4. Let $A, B, C : C([0, a], X) \to C([0, a], X)$ be three mappings and $S$ be a bounded set of $C([0, a], X)$. Assume that

(a) $B(S)$ is equicontinuous,

(b) there exist two constants $k_1$ and $k_2$ such that

$$\|Ax - Ay\| \leq k_1\|x - y\| \quad \text{and} \quad \|Cx - Cy\| \leq k_2\|x - y\| \quad \text{for all } x, y \in X,$$

(c) the equality $x = AxBy + Cx$ with $y \in S$ implies $x \in S$.

Then, for all integer $n$ we have $F^{(n,x_0)}(AB + C, S)$ is bounded and equicontinuous, whenever $Qk_1 + k_2 < 1$, where $Q = \|B(S)\|$.

Proof. Note that for all $x \in F^{(1,x_0)}(AB + C, S)$ there is a $y \in S$ such that $x = AxBy + Cx$. By assumption (c) we have $x \in S$, so $F^{(1,x_0)}(AB + C, S)$ is bounded. Hence for $t, s \in [0, a]$ with $|t - s| < \delta$ we have

$$\|x(t) - x(s)\| = \|Ax(t)By(t) + Cx(t) - Ax(s)By(s) - Cx(s)\|
\leq \|Ax(t)By(t) - Ax(s)By(t)\| + \|Ax(s)By(t) - Ax(s)By(s)\| + \|Cx(t) - Cx(s)\|
\leq \|By(t)\|\|Ax(t) - Ax(s)\| + \|Ax(s)\|\|By(t) - By(s)\| + \|Cx(t) - Cx(s)\| \quad (6)
\leq Qk_1\|x(t) - x(s)\| + \|Ax(s)\|\|By(t) - By(s)\| + k_2\|x(t) - x(s)\|
\leq (Qk_1 + k_2)\|x(t) - x(s)\| + \|Ax(s)\|\|By(t) - By(s)\|.$$

Now, for any $x_0 \in S$, we have

$$\|Ax(s)\| \leq \|Ax(s) - Ax_0(s)\| + \|Ax_0(s)\|
\leq k_1\|x(s) - x_0(s)\| + \|Ax_0\| \leq k_1\text{diam}(S) + \|Ax_0\| := \beta < \infty.$$

Substituting this estimate in the inequality (6),

$$\|x(t) - x(s)\| \leq (Qk_1 + k_2)\|x(t) - x(s)\| + \beta\|By(t) - By(s)\|. \quad (7)$$

Now the equicontinuity of $B(S)$ implies that for any $\varepsilon > 0$, there is a $\delta > 0$ such that $|t - s| < \delta$ implies

$$\|By(t) - By(s)\| \leq \left(\frac{1 - (Qk_1 + k_2)}{\beta}\right)\varepsilon, \quad y \in S.$$
This implies that
\[ \| x(t) - x(s) \| \leq \frac{\beta}{1 - (Qk_1 + k_2)} \| By(t) - By(s) \| \leq \varepsilon. \] (8)

Consequently \( \mathcal{F}^{(1,x_0)}(AB + C, S) \) is equicontinuous. Now, since
\[ \mathcal{F}^{(2,x_0)}(AB + C, S) = \mathcal{F}^{(1,x_0)}(AB + C, co(\mathcal{F}^{(1,x_0)}(AB + C, S) \cup \{x_0\})) \],
we have \( \mathcal{F}^{(2,x_0)}(AB + C, S) \) is bounded, and the use of Lemma 2 yields \( \mathcal{F}^{(2,x_0)}(AB + C, S) \) is equicontinuous. By mathematical induction we can prove that \( \mathcal{F}^{(n,x_0)}(AB + C, S) \) is bounded and equicontinuous for all integers \( n \geq 1 \).

**Theorem 4.** Assume that the hypotheses \((H_1) - (H_3)\) hold. Then equation (3) has a solution in \( C(J,X) \).

**Proof.** Define the functions
\[ I(z) = \int_{h_0}^z \frac{ds}{\theta(s)} \quad \text{and} \quad b(t) = I^{-1}\left( \frac{M}{1 - \delta} \int_0^t m(s)ds \right). \]

First, note that
\[ b(0) = I^{-1}(0) = \frac{h_0}{1 - \delta} \quad \text{and} \quad b'(t) = \frac{M}{(1 - \delta)m(t)} \theta(b(t)) \quad \text{for all } t \in J. \]

From the second equality we deduce that \( t \mapsto b(t) \) is a nondecreasing function. We introduce now the set
\[ \Omega = \left\{ x \in C(J,X) \text{ such that } \| x(t) \| \leq b(t) \text{ and } \| x(t) - x(s) \| \leq |b(t) - b(s)| \text{ for } t, s \in J \right\}. \]

Clearly, \( \Omega \) is a closed, convex, bounded, equicontinuous subset of the Banach algebra \( C(J,X) \) with \( 0 \in \Omega \). We define the operators \( A, C : C(J,X) \to C(J,X), B : \Omega \to C(J,X) \) by
\[ Ax(t) = f(t, x(t)), \quad Bx(t) = \int_0^t g(s, x(s))ds \quad \text{and} \quad Cx(t) = h(t,x(t)) \quad t \in J. \]

Then equation (3) is equivalent to the operator equation
\[ Ax(t)Bx(t) + Cx(t) = x(t), \quad t \in J. \]

We show that the operators \( A, B \) and \( C \) satisfy all the conditions of Corollary 1. This will be shown in a series of steps.

**Step 1** The operators \( A \) and \( C \) are Lipschitzians on \( C(J,X) \) with constants \( k_1 \) and \( k_2 \) respectively. Let \( x, y \in C(J,X) \), by hypothesis \((H_1)\), we have
\[ \| Ax(t) - Ay(t) \| = \| f(t, x(t)) - f(t, y(t)) \| \leq k_1 \| x(t) - y(t) \| \leq k_1 \| x - y \|_\infty \quad \text{for all } t \in J. \]

Taking the supremum over \( t \), we obtain \( \| Ax - Ay \|_\infty \leq k_1 \| x - y \|_\infty \). This shows that \( A \) is a Lipschitzian mapping with constant \( k_1 \) on \( C(J,X) \) into itself. Similarly, \( C \) is a Lipschitzian mapping with constant \( k_2 \) on \( C(J,X) \) into itself.
Step 2 \( B \) is continuous and \( B(\Omega) \) is equicontinuous.

First we show that \( B \) is a continuous operator on \( \Omega \). Let \( \{x_n\} \) be a sequence of points in \( \Omega \) converging to a point \( x \in \Omega \). From assumption \((H_2)\) it follows that:

\[
g(t, x_n(t)) \to g(t, x(t)), \quad \text{as } n \to +\infty.
\]

Since \( \{x_n\}_n \) is bounded, then there exists \( N > 0 \) such that \( ||x_n|| \leq N \), for all \( n \in \mathbb{N} \). Thus, \( ||x|| \leq N \) and therefore

\[
||g(t, x_n(t)) - g(t, x(t))|| \leq 2m(t)\theta(N) \quad \text{from } (H_3)(c).
\]

Then, by the dominated convergence theorem, one can conclude that \( B \) is continuous.

Now, we claim that \( B(\Omega) \) is equicontinuous. Let \( t_1, t_2 \in J \) and let \( x \in \Omega \). Then, we have

\[
\|Bx(t_1) - Bx(t_2)\| = \left\| \int_0^{t_1} g(s, x(s))ds - \int_0^{t_2} g(s, x(s))ds \right\|
\]

\[
\leq \left\| \int_0^{t_2} g(s, x(s))ds \right\|
\]

\[
\leq \int_0^{t_1} \|g(s, x(s))\|ds
\]

\[
\leq \int_0^{t_1} m(s)\theta(\|x(s)\|)ds
\]

\[
\leq \int_0^{t_2} m(s)\theta(\|x(s)\|)ds
\]

\[
\leq \frac{1 - \delta}{M} \int_{t_1}^{t_2} b'(s)ds
\]

\[
\leq \frac{1 - \delta}{M} |b(t_1) - b(t_2)|.
\]

Consequently,

\[
\|Bx(t_1) - Bx(t_2)\| \to 0 \quad \text{as } t_1 \to t_2.
\]

This shows that \( B(\Omega) \) is equicontinuous.

Step 3 We shall show that \( x = AxBy +Cx \) with \( y \in \Omega \) implies \( x \in \Omega \).

Let \( y \in \Omega \) and \( t, s \in J \), we have

\[
\|x(t)\| = \|Ax(t)By(t) + Cx(t)\|
\]

\[
\leq \|Ax(t)\|\|By(t)\| + \|h(t, x(t))\|
\]

\[
\leq M \int_0^t m(s)\theta(\|y(s)\|)ds + \|h(t, x(t)) - h(t, 0)\| + \|h(t, 0)\|
\]

\[
\leq M \int_0^t m(s)\theta(b(s))ds + k_2\|x(t)\| + h_0
\]

\[
\leq (1 - \delta) \int_0^t b'(s)ds + k_2\|x(t)\| + h_0
\]

\[
\leq (1 - \delta)(b(t) - b(0)) + \delta\|x(t)\| + h_0.
\]
This implies that
\[ \| x(t) \| \leq b(t) - b(0) + \frac{h_0}{1 - \delta} = b(t). \]

Now, according to the inequality (7) we have
\[ \| x(t) - x(s) \| \leq \frac{M}{1 - \delta} \| By(t) - By(s) \|. \]  

Combining (9) and (10), we arrive at
\[ \| x(t) - x(s) \| \leq | b(t) - b(s) |. \]

Consequently, \( x \in \Omega \).

**Step 4** The operator \( A \cdot B + C \) is convex-power condensing about \((n_0, 0)\) with respect to \( \alpha \).
Let \( S \) be a subset of \( \Omega \). We have
\[
F(S) = \left( \frac{I - C}{A} \right)^{-1} B(S) = \mathcal{F}^{(1,0)}(AB + C, S)
\]
and for all \( t \in J \) we have
\[
\alpha(F(S)(t)) = \alpha(A(F(S)(t))B(S)(t) + C(F(S))(t)) \\
\leq \alpha(A(F(S)(t))B(S)(t)) + \alpha(C(F(S))(t)) \\
\leq \| B(S) \| \alpha(A(F(S)(t))) + \| A(F(S)) \| \alpha(B(S)(t)) + k_2 \alpha(F(S)(t)) \\
\leq k_1 Q \alpha(F(S)(t)) + M \alpha(B(S)(t)) + k_2 \alpha(F(S)(t)) \\
\leq \delta \alpha(F(S)(t)) + M \alpha(B(S)(t)).
\]

and therefore,
\[
\alpha(F(S)(t)) \leq \left( \frac{M}{1 - \delta} \right) \alpha(B(S)(t)) \leq \left( \frac{M}{1 - \delta} \right) \alpha \left( \int_0^t g(s, S(s))ds \right) \]  

By Lemma 3, \( \{ g(\cdot, x(\cdot)), x \in S \} \) is bounded and equicontinuous. Hence, we derive from Lemma 4 the following estimates
\[
\alpha(F(S)(t)) \leq \left( \frac{M}{1 - \delta} \right) \int_0^t \alpha(g(s, S(s))) ds \\
\leq \left( \frac{M}{1 - \delta} \right) \int_0^t \alpha(S(s)) ds \\
\leq t \left( \frac{M c}{1 - \delta} \right) \alpha(S([0, t])) \\
\leq t \left( \frac{M c}{1 - \delta} \right) \alpha(S) 
\]
for all \( t \in J \). Keeping in mind that \( \mathcal{F}^{(1,0)}(AB+C,S) \) is bounded and equicontinuous, the use of Lemma \( \text{I} \) yields
\[
\alpha (\mathcal{F}^{(1,0)}(AB+C,S)) \leq t \left( \frac{M c}{1 - \delta} \right) \alpha (S).
\]

Next, notice
\[
\alpha (\mathcal{F}^{(2,0)}(AB+C,S)(t)) = \alpha (F (co (\mathcal{F}^{(1,0)}(AB+C,S) \cup \{0\})) )
\]
\[
\leq \left( \frac{M}{1 - \delta} \right) \alpha \left( \int_{0}^{t} g(s, co (\mathcal{F}^{(1,0)}(AB+C,S) \cup \{0\})) ds \right)
\]
\[
\leq \left( \frac{M}{1 - \delta} \right) \int_{0}^{t} \alpha (g(s, co (\mathcal{F}^{(1,0)}(AB+C,S) \cup \{0\})) ds
\]
\[
\leq \left( \frac{M c}{1 - \delta} \right) \int_{0}^{t} \alpha (co (\mathcal{F}^{(1,0)}(AB+C,S) \cup \{0\})) ds
\]
\[
\leq \left( \frac{M c}{1 - \delta} \right) \int_{0}^{t} \alpha (\mathcal{F}^{(1,0)}(AB+C,S)) ds
\]
\[
\leq \left( \frac{M c}{1 - \delta} \right)^{2} \alpha (S) \int_{0}^{t} s ds
\]
\[
\leq \frac{t^{2}}{2} \left( \frac{M c}{1 - \delta} \right)^{2} \alpha (S).
\]

Since \( \mathcal{F}^{(2,0)}(AB+C,S) \) is bounded and equicontinuous. Invoking Lemma \( \text{I} \) we obtain
\[
\alpha (\mathcal{F}^{(2,0)}(AB+C,S)) \leq \frac{1}{2!} \left( \frac{M c}{1 - \delta} \right)^{2} \alpha (S).
\]

By induction, we get
\[
\alpha (\mathcal{F}^{(n,0)}(AB+C,S)) \leq \frac{1}{n!} \left( \frac{M c}{1 - \delta} \right)^{n} \alpha (S).
\]

Since \( \lim_{n \to \infty} \frac{1}{n!} \left( \frac{M c}{1 - \delta} \right)^{n} = 0 \), then there is an \( n_{0} \) with
\[
\frac{1}{n_{0}!} \left( \frac{M c}{1 - \delta} \right)^{n_{0}} < 1.
\]

This implies
\[
\alpha (\mathcal{F}^{(n_{0},0)}(AB+C,S)) < \alpha (S).
\]

Consequently, \( A \cdot B + C \) is convex-power condensing about \( (n_{0},0) \) with respect to \( \alpha \). The result follows from Corollary \( \text{I} \).  

\( \square \)
References


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