ANALYSIS OF TWO-OPERATOR BOUNDARY-DOMAIN INTEGRAL EQUATIONS FOR VARIABLE-COEFFICIENT MIXED BVP IN 2D WITH GENERAL RIGHT-HAND SIDE

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ABSTRACT. The mixed (Dirichlet-Neumann) boundary value problem (BVP) for the linear second-order scalar elliptic differential equation with variable coefficients in a bounded two-dimensional domain is considered in this paper. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$, when neither classical nor canonical co-normal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce this BVP to four systems of boundary-domain integral equations (BDIEs). Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The equivalence of the BDIE systems to the original BVP is shown. The invertibility of the associated operators is proved in the corresponding Sobolev spaces.

1. Introduction

Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modelling of inhomogeneous media (e.g. functionally graded materials or materials with damage induced inhomogeneity) in solid mechanics, electromagnetics, thermo-conductivity, fluid flows through porous media, and other areas of physics and engineering.

Generally, explicit fundamental solutions are not available if the PDE coefficients are not constant, preventing reduction of boundary value problems (BVPs) for such PDEs to explicit boundary integral equations (BIEs), which could be effectively solved numerically. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of boundary-domain integral equations (BDIEs) for further numerical solution of the latter, see e.g. [7, 9, 22, 23, 25, 29] and references therein. However this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamé system of anisotropic elasticity). To overcome this difficulty, one can apply the so-called two-operator approach, formulated in [24] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way...
way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

In [5, 6], one of the linear version of the two-operator approach is applied to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE with *square integrable right-hand side* and reduced it to four different two-operator BDIE systems. Using results in [7], a rigorous analysis of the two-operator BDIEs is given.

As described in [21, 28] for a function from the Sobolev space $H^1(\Omega)$, a classical co-normal derivative in the sense of traces may not exist. However, when this function satisfies a second order PDE with a right-hand side from $H^{-1}(\Omega)$, the generalized co-normal derivative can be defined in the weak sense, associated with the first Green identity and an extension of the PDE right-hand side to $\tilde{H}^{-1}(\Omega)$ (see, e.g., [20, Lemma 4.3], [26, Definition 3.1]). Since the extension is non-unique, the co-normal derivative appears to be a non-unique operator, which is also non-linear in $u$ unless a linear relation between $u$ and the PDE right-hand side extension is enforced. This creates some difficulties in formulating the BDIEs. These difficulties are addressed in [21, 28] presenting formulation and analysis of direct segregated BDIE systems equivalent to the Dirichlet and Neumann problems for the divergent-type PDE with a variable scalar coefficient and a general right-hand side from $H^{-1}(\Omega)$ extended when necessary to $\tilde{H}^{-1}(\Omega)$. This needed a non-trivial generalization of the third Green identity and its co-normal derivative for such functions, which extends the approach implemented in [7, 8, 9, 25, 27] for the PDE right-hand from $L_2(\Omega)$. In [1], using the two-operator approach in settings different from the one in [5, 6] and extending the results in [2], a generalization of the two-operator third Green identity and its co-normal derivative is derived and the two-operator BDIE systems for variable-coefficient mixed BVPs are investigated.

Nowadays, the theory of BDIEs in 3D is well developed, cf. [7, 8, 9, 22, 24], the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to ensure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. In this paper, the mixed BVP for the linear second-order scalar elliptic differential equation with variable coefficient in a bounded two-dimensional domain with general data is considered and a condition is set only on the domain to ensure invertibility of layer potentials. The PDE right-hand side belongs to $H^{-1}(\Omega)$ or $\tilde{H}^{-1}(\Omega)$ when neither classical nor canonical co-normal derivatives of solutions are well defined. The two-operator approach and appropriate parametrix (Levi function) are used to reduce the problem to four different systems of BDIEs. Using results in [1, 2, 3, 4] the properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is shown. The unique solvability of BDIE systems and inveribility of boundary-domain integral operators is proved in appropriate Sobolev spaces.

### 2. Co-normal derivatives and boundary value problems

Let $\Omega$ be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial \Omega$. Consider the scalar elliptic differential equation, which for sufficiently smooth function $u$ has the following strong form,

\begin{equation}
Au(x) := A(x, \partial_x)u(x) := \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x)}{\partial x_i} \right) = \tilde{f}(x), \quad x \in \Omega,
\end{equation}
where $a$ is unknown function and $\tilde{f}$ is a given function in $\Omega$. We assume that
\[
a \in C^\infty(\mathbb{R}^2), \quad 0 < a_{\min} \leq a(x) \leq a_{\max} < \infty, \quad \forall x \in \mathbb{R}^2.
\]
In what follows $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$, $H^s(\Omega) = H^s_2(\Omega)$, $H^s(\partial \Omega) = H^s_2(\partial \Omega)$ are the Bessel potential spaces, where $s \in \mathbb{R}$ is an arbitrary real number (see, e.g., [19, 20]). We recall that $H^s$ coincides with the Sobolev-Slobodetski spaces $W^s_2$ for any non-negative $s$. We denote by $\tilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$,
\[
\tilde{H}^s(\Omega) := \{ g : g \in H^s(\mathbb{R}^2), \ \text{supp}(g) \subset \overline{\Omega} \}
\]
while $H^s(\Omega)$ denotes the space of restriction on $\Omega$ of distributions from $H(\mathbb{R}^2)$,
\[
H^s(\Omega) = \{ r_\Omega g : g \in H^s(\mathbb{R}^2) \}
\]
where $r_\Omega$ denotes the restriction operator on $\Omega$. We will also use the notation $g|_\Omega := r_\Omega g$. We denote by $H^s_\partial \Omega$ the following subspace of $H(\mathbb{R}^2)$ (and $\tilde{H}(\Omega)$),
\[
H^s_\partial \Omega := \{ g : g \in H^s(\mathbb{R}^2), \ \text{supp}(g) \subset \partial \Omega \}.
\]
From the trace theorem (see, e.g., [12, 19, 20]) for $u \in H^1(\Omega)$, it follows that $\gamma^+ u \in H^{\frac{1}{2}}(\partial \Omega)$, where $\gamma^+ = \gamma^+_\partial \Omega$ is the trace operator on $\partial \Omega$ from $\Omega$. Let also $\gamma^- : H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega)$ denote a (non-unique) continuous right inverse to the trace operator $\gamma^+$, i.e., $\gamma^- \gamma^+ w = \gamma^- w = w$ for any $w \in H^{\frac{1}{2}}(\partial \Omega)$, and $(\gamma^-)^* : H^{-\frac{1}{2}}(\Omega) \to H^{\frac{1}{2}}(\partial \Omega)$ is continuous operator dual to $\gamma^- : H^{\frac{1}{2}}(\partial \Omega) \to H^1(\Omega)$, i.e., $\langle (\gamma^-)^* \tilde{f}, w \rangle_{\partial \Omega} := \langle \tilde{f}, \gamma^- w \rangle_{\Omega}$ for any $\tilde{f} \in H^{-\frac{1}{2}}(\Omega)$ and $w \in H^{\frac{1}{2}}(\partial \Omega)$.

For $u \in H^2(\Omega)$, we denote by $T_u^+$ the corresponding canonical (strong) co-normal derivative operator on $\partial \Omega$ in the sense of traces,
\[
T_u^+ := \sum_{i=1}^3 a(x) n_i(x) \gamma^+ \frac{\partial u(x)}{\partial x_i} = a(x) \gamma^+ \frac{\partial u(x)}{\partial n(x)},
\]
where $n(x)$ is the outward (to $\Omega$) unit normal vector at the point $x \in \partial \Omega$. However the classical co-normal derivative operator is generally, not well defined if $u \in H^1(\Omega)$, (see [28, Appendix A]).

For $u \in H^1(\Omega)$, the PDE $Au$ in (1) is understood in the sense of distributions,
\[
\langle Au, v \rangle_\Omega := -\mathcal{E}_a(u, v), \quad \forall v \in \mathcal{D}(\Omega),
\]
where
\[
\mathcal{E}_a(u, v) := \int_\Omega a(x) \nabla u(x) \cdot \nabla v(x) dx
\]
is a symmetric bilinear form and the duality brackets $\langle g, \cdot \rangle_\Omega$ denote the value of a linear functional (distribution) $g$, extending the usual $L_2$ inner product.

Since the set $\mathcal{D}(\Omega)$ is dense in $\tilde{H}^1(\Omega)$, the above formula defines a continuous operator $A : H^1(\Omega) \to H^{-1}(\Omega) = \left[ H^1(\Omega) \right]^*$,
\[
\langle Au, v \rangle := -\mathcal{E}_a(u, v), \quad \forall u \in H^1(\Omega), \ \forall v \in \tilde{H}^1(\Omega).
\]
Let us consider also the operator, $\tilde{A} : H^1(\Omega) \to \tilde{H}^{-1}(\Omega) = \left[ H^1(\Omega) \right]^*$,
\[
\langle \tilde{A} u, v \rangle_\Omega := -\mathcal{E}_a(u, v) = -\int_R a(x) \nabla u(x) \cdot \nabla v(x) dx = -\int_{\mathbb{R}^2} \tilde{E}[a \nabla u](x) \cdot \nabla v(x) dx
\]
\[
= \langle \nabla \cdot \tilde{E}[a \nabla u], v \rangle_{\mathbb{R}^2} = \langle \nabla \cdot \tilde{E}[a \nabla u], v \rangle_\Omega, \quad \forall u \in H^1(\Omega), \ \forall v \in H^1(\Omega)
\]
which is evidently continuous and can be written as

\[ \tilde{A}u = \nabla \cdot \tilde{E} [a \nabla u]. \]

Here \( V \in H^1(\mathbb{R}^2) \) is such that \( r_2 V = v \) and \( \tilde{E} \) denotes the operator of extension of the functions, defined in \( \Omega \), by zero outside \( \Omega \) in \( \mathbb{R}^2 \). For any \( u \in H^1(\Omega) \), the functional \( \tilde{A}u \) belongs to \( \tilde{H}^{-1}(\Omega) \) and is the extension of the functional \( Au \in H^{-1}(\Omega) \), which domain is thus extended from \( \tilde{H}^1(\Omega) \) to the domain \( H^1(\Omega) \) for \( \tilde{A}u \).

Inspired by the first Green identity for smooth functions, we can define the generalized co-normal derivative (cf., for example, [20, Lemma 4.3], [26, Definition 3.1], [17, Lemma 2.2]).

**Definition 2.1.** Let \( u \in H^1(\Omega) \) and \( Au = r_\Omega \tilde{f} \) in \( \Omega \) for some \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \). Then the generalized co-normal derivative \( T^+_a(\tilde{f}, u) \in H^{-\frac{1}{2}}(\partial \Omega) \) is defined as

\[ \langle T^+_a(\tilde{f}, u), w \rangle_{\partial \Omega} := \langle \tilde{f}, \gamma^{-1}w \rangle_\Omega + \gamma u \langle \gamma^{-1}w \rangle_\Omega, \]

\[ \forall w \in H^\frac{1}{2}(\partial \Omega), \ i.e., \ T^+_a(\tilde{f}, u) := (\gamma^{-1})^*(\tilde{f} - \tilde{A}u). \]

By [20, Lemma 4.3], [26, Theorem 5.3], we have the estimate

\[ \| T^+_a(\tilde{f}, u) \|_{H^{-\frac{1}{2}}(\partial \Omega)} \leq C_1 \| u \|_{H^1(\Omega)} + C_2 \| \tilde{f} \|_{\tilde{H}^{-1}(\Omega)}, \]

and for \( u \in H^1(\Omega) \) such that \( Au = r_\Omega \tilde{f} \) in \( \Omega \) for some \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) the first Green identity holds in the following form,

\[ \langle T^+_a(\tilde{f}, u), \gamma v \rangle_{\partial \Omega} := \langle \tilde{f}, v \rangle_\Omega + \gamma u \langle v \rangle_\Omega, \quad \forall v \in H^1(\Omega). \]

As follows from Definition 2.1, the generalised co-normal derivative is nonlinear with respect to \( u \) for a fixed \( \tilde{f} \), but linear with respect to the couple \( (\tilde{f}, u) \), i.e.,

\[ \alpha_1 T^+_a(\tilde{f}_1, u_1) + \alpha_2 T^+_a(\tilde{f}_2, u_2) = T^+_a(\alpha_1 \tilde{f}_1, \alpha_1 u_1) + T^+_a(\alpha_2 \tilde{f}_2, \alpha_2 u_2) \]

\[ = T^+_a(\alpha_1 \tilde{f}_1 + \alpha_2 \tilde{f}_2, \alpha_1 u_1 + \alpha_2 u_2) \]

for any real numbers \( \alpha_1, \alpha_2 \).

Let us also define some subspaces of \( H^s(\Omega) \), cf. [11, 14, 26, 27].

**Definition 2.2.** Let \( s \in \mathbb{R} \) and \( A_s : H^s(\Omega) \to \mathcal{D}^* \) be a linear operator. For \( t \geq -\frac{1}{2} \) we introduce the space

\[ H^{s,t}(\Omega; A_s) := \{ g \in H^s(\Omega) : \text{there exists } \tilde{f}_g \in \tilde{H}^s(\Omega) \text{ such that } A_s g|_{\Omega} = \tilde{f}_g|_{\Omega} \} \]

endowed with the norm

\[ \| g \|_{H^{s,t}(\Omega; A_s)} := \left( \| g \|_{H^s(\Omega)}^2 + \| \tilde{f}_g \|_{\tilde{H}^s(\Omega)}^2 \right)^{\frac{1}{2}} \]

and the inner product

\[ (g, h)_{H^{s,t}(\Omega; A_s)} = (g, h)_{H^s(\Omega)} + \langle \tilde{f}_g, \tilde{f}_h \rangle_{\tilde{H}^s(\Omega)} \]

The distribution \( \tilde{f}_g \in \tilde{H}^s(\Omega), t \geq -\frac{1}{2} \), in the above definition is an extension of the distribution \( A_s g|_{\Omega} \in H^s(\Omega) \), and the extension is unique (if it does exist) since any distribution from the space \( H^t(\mathbb{R}^2) \) with support in \( \partial \Omega \) is identically zero if \( t \geq -\frac{1}{2} \) (see, e.g., [20, Lemma 3.39], [26, Theorem 4.6]).
2.10]). We denote this extension as an operator $A_s$, i.e., $A_s g = \tilde{f}_g$. The uniqueness implies that the norm $\|g\|_{H^{1,0}(\Omega; A_s)}$ is well defined.

We will mostly use the operators $A, B$ or $\Delta$ as $A_s$ in the above definition. Note that since $Au - a\Delta u = \nabla a \nabla u \in L_2(\Omega)$, for $u \in H^1(\Omega)$, we have $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta)$.

**Definition 2.3.** For $u \in H^{1, -\frac{1}{2}}(\Omega; A)$, we define the canonical co-normal derivative $T^+_a u \in H^{-\frac{1}{2}}(\partial \Omega)$ as

$$
\langle T^+_a u, w \rangle_{\partial \Omega} := \langle \tilde{A}u, \gamma^{-1}w \rangle_{\Omega} + \partial_a(u, \gamma^{-1}w) = \langle \tilde{A}u - \tilde{A}u, \gamma^{-1}w \rangle_{\Omega}, \quad \forall w \in H^{\frac{1}{2}}(\partial \Omega),
$$

i.e., $T^+_a u := (\gamma^{-1})^*(\tilde{A}u - \tilde{A}u)$.

The canonical co-normal derivative $T^+_a u$ is independent of (non-unique) choice of the operator $\gamma^{-1}$, the operator $T^+_a : H^{1, -\frac{1}{2}}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial \Omega)$ is continuous, and the first Green identity holds in the following form,

$$
\langle T^+_a u, \gamma^+ v \rangle_{\partial \Omega} := \langle \tilde{A}u, v \rangle_{\Omega} + \partial_a(u, v), \quad \forall v \in H^1(\Omega).
$$

The operator $T^+_a : H^{1, t}(\Omega; A) \rightarrow H^{-\frac{1}{2}}(\partial \Omega)$ in Definition 2.3 is continuous for $t \geq -\frac{1}{2}$. The canonical co-normal derivative is defined by the function $u$ and the operator $A$ and does not depend separately on the right-hand side $\tilde{f}$ (i.e. its behavior on the boundary), unlike the generalized co-normal derivative defined in (3), and the operator $T^+_a$ is linear. Note that the canonical co-normal derivative coincides with classical co-normal derivative $T^+_a u = a \frac{\partial u}{\partial n}$ if the latter does exist in the trace sense, see, [26, Corollary 3.14 and Theorem 3.16].

Let $u \in H^{1, -\frac{1}{2}}(\Omega; A)$. Then Definitions 2.1 and 2.3 imply that the generalized co-normal derivative for arbitrary extension $\tilde{f} \in H^{-1}(\Omega)$ of the distribution $Au$ can be expressed as

$$
\langle T^+_a(\tilde{f}, u), w \rangle_{\partial \Omega} := \langle T^+_a u, w \rangle_{\partial \Omega} + \langle \tilde{f} - \tilde{A}u, \gamma^{-1}w \rangle_{\Omega}, \quad \forall w \in H^{\frac{1}{2}}(\Omega).
$$

Let us consider the auxiliary linear elliptic partial differential operator $B$ defined by

$$
Bu(x) := B(x, \partial_x)u(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( b(x) \frac{\partial u(x)}{\partial x_i} \right),
$$

where $b \in C^\infty(\mathbb{R}^2), b(x) > 0$ for $x \in \mathbb{R}^2$.

Since for $u \in H^{1,0}(\Omega, \Delta)$, $Au - Bu = (a - b)\Delta u + \nabla(a - b)\nabla u \in L_2(\Omega)$, we have, $H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)$. Let $u \in H^1(\Omega)$ and $v \in H^{1,0}(\Omega; B)$. Then we write the first Green identity for operator $B$ in the form

$$
\partial_b(u, v) + \int_\Omega u(x) Bv(x)dx = \langle T^+_b v, \gamma^+ u \rangle_{\partial \Omega}
$$

where

$$
\partial_b(u, v) = \int_\Omega b(x) \nabla u(x) \cdot \nabla v(x) dx.
$$
If, in addition, \(Au = \tilde{f}\) in \(\Omega\), where \(\tilde{f} \in \tilde{H}^{-1}(\Omega)\), then according to the definition of \(T^+_a(f, u)\), in (3), the two-operator second Green identity can be written as

\[
\langle \tilde{f}, v \rangle_{\Omega} - \int_{\Omega} u(x)Bv(x)dx + \int_{\Omega} [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx = \langle T^+_a(f, u), \gamma^+v \rangle_{\partial\Omega} - \langle T^+_b(v, \gamma^+u) \rangle_{\partial\Omega}.
\]

Moreover, for \(u, v \in H^{1,0}(\Omega; A) = H^{1,0}(\Omega; B)\) Eq. (7) becomes

\[
\int_{\Omega} [v(x)Au(x) - u(x)Bv(x)]dx + \int_{\Omega} [a(x) - b(x)]\nabla u(x) \cdot \nabla v(x)dx = \langle T^+_a(u, \gamma^+v) \rangle_{\partial\Omega} - \langle T^+_b(v, \gamma^+u) \rangle_{\partial\Omega}.
\]

### 3. Parametrix and potential type operators

**Definition 3.1.** We will say, a function \(P_b(x, y)\) of two variables \(x, y \in \Omega\) is a parametrix (Levi function) for the operator \(B(x; \partial_x)\) in \(\mathbb{R}^2\) if (see, e.g., [15, 16, 18, 22, 30, 31, 32])

\[
B(x, \partial_x)P_b(x, y) = \delta(x - y) + R_b(x, y),
\]

where \(\delta\) is the Dirac-delta distribution, while \(R(x, y)\) is a remainder possessing at most a weak singularity at \(x = y\).

For some positive constant \(r_0\), the parametrix and hence the corresponding remainder in 2D can be chosen as in [22],

\[
P_b(x, y) = \frac{1}{2\pi b(y)} \ln \left( \frac{|x - y|}{r_0} \right),
\]

\[
R_b(x, y) = \sum_{i=1}^2 \frac{x_i - y_i}{2\pi b(y)|x - y|^2} \frac{\partial b(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2.
\]

Evidently, the parametrix \(P_b(x, y)\) given by (8) is the fundamental solution to the operator \(B(y, \partial_y) := b(y)\Delta(\partial_y)\) with “frozen” coefficient \(b(x) = b(y)\), and

\[
B(y, \partial_y)P_b(x, y) = \delta(x - y).
\]

Let \(b \in C^\infty(\mathbb{R}^2)\) and \(b(x) > 0\) for \(x \in \mathbb{R}^2\). For some scalar function \(g\) the parametrix-based Newtonian and the remainder volume potential operators, corresponding to the parametrix (8) and the remainder (9) are given by

\[
P_b g(y) := \int_{\mathbb{R}^2} P_b(x, y)g(x)dx, \quad y \in \mathbb{R}^2,
\]

\[
\mathcal{P}_b g(y) := \int_{\Omega} P_b(x, y)g(x)dx, \quad y \in \Omega,
\]

\[
\mathcal{R}_b g(y) := \int_{\Omega} R_b(x, y)g(x)dx, \quad y \in \Omega.
\]

For \(g \in H^s(\mathbb{R}^2), s \in \mathbb{R}\), Eq. (10) is understood as \(P_b g = \frac{1}{b}P_\Delta g\), where the Newtonian potential operator \(P_\Delta\) for Laplacian \(\Delta\) is well defined in terms of the Fourier transform (i.e., as pseudo-differential
The direct value operators associated with (18) are
\[
\mathcal{P}_b g = \frac{1}{b r_Ω} P_Δ g, \quad \mathcal{P}_b g = \frac{a}{b r_Ω} P_a g \quad \text{and} \quad \mathcal{R}_b g = -\frac{1}{b r_Ω} \nabla P_Δ (g \nabla b),
\]
while for \( g \in H^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}, \) as (13) with \( g \) replaced by \( \tilde{E}g, \) where \( \tilde{E} : H^s(\Omega) \to \tilde{H}^s(\Omega), -\frac{1}{2} < s < \frac{1}{2}, \) is the unique extension operator related with the operator \( \tilde{E} \) of extension by zero, cf. \cite[Theorem 2.16]{26}.

For \( y \notin \partial \Omega, \) the single layer and the double layer surface potential operators, corresponding to the parametrix (8) are defined as

\[
\begin{align*}
V_b g(y) & := -\int_{\partial \Omega} P_b(x,y)g(x)\,dS_x = \frac{1}{b} V_\Delta g(y), \\
W_b g(y) & := -\int_{\partial \Omega} [T_b(x,n(x),\partial_x)P_b(x,y)]g(x)\,dS_x = \frac{1}{b} W_\Delta (bg)(y),
\end{align*}
\]

where \( g \) is some scalar density function, and the integrals are understood in the distributional sense if \( g \) is not integrable. The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential \( \mathcal{V}_b \) and the double layer potentials \( \mathcal{W}_b \) for \( y \in \partial \Omega \) are,

\[
\begin{align*}
\mathcal{V}_b g(y) & := -\int_{\partial \Omega} P_b(x,y)g(x)\,dS_x = \frac{1}{b} \mathcal{V}_\Delta g(y), \\
\mathcal{W}_b g(y) & := -\int_{\partial \Omega} T_b(x,n(x),\partial_x)P_b(x,y)g(x)\,dS_x = \frac{1}{b} \mathcal{W}_\Delta (bg)(y)
\end{align*}
\]

We can also calculate at \( y \in \partial \Omega \) the co-normal derivatives, associated with the operator \( A, \) of the single layer potential and of the double layer potential:

\[
\begin{align*}
T^\pm_a V_b g(y) & = \frac{a(y)}{b(y)} T^\pm_b V_b g(y), \\
\mathcal{L}_{ab}^\pm g(y) & := T^\pm_a W_b g(y) = \frac{a(y)}{b(y)} T^\pm_b W_b g(y) = \frac{a(y)}{b(y)} \mathcal{L}_{ab}^\pm g(y)
\end{align*}
\]

The direct value operators associated with (18) are

\[
\begin{align*}
\mathcal{W}_{ab} g(y) & := -\int_{\partial \Omega} [T_a(y,n(y),\partial_y)P_b(x,y)]g(x)\,dS_x = \frac{a(y)}{b(y)} \mathcal{W}_{ab} g(y), \\
\mathcal{W}_{ab}^\prime g(y) & := -\int_{\partial \Omega} [T_b(y,n(y),\partial_y)P_b(x,y)]g(x)\,dS_x.
\end{align*}
\]
Theorem 3.2. For \( g \) (34) \( P \) (32) \( \Delta \) (29). It is taken into account that \( b \) and its derivatives are continuous in \( \mathbb{R}^2 \) and

\[
\mathcal{L}(bg) := \mathcal{L}^+(bg) = \mathcal{L}^-(bg)
\]

by the Liapunov-Tauber theorem. Hence,

\[
\Delta(bVg) = 0, \Delta(bWg) = 0 \quad \text{in} \quad \Omega, \quad \forall g \in H^1(\partial \Omega) \quad (\forall s \in \mathbb{R}),
\]

\[
\Delta(b\mathcal{P}g) = g \quad \text{in} \quad \Omega, \quad \forall g \in \tilde{H}^1(\Omega) \quad (\forall s \in \mathbb{R}).
\]

The mapping properties of the operators (10)-(21) follow from relations (22)-(27) and are described in detail in [1, 5, Appendix A]. Particularly, we have the following jump relations.

**Theorem 3.2.** For \( g_1 \in H^{-\frac{1}{2}}(\partial \Omega) \), and \( g_2 \in H^{\frac{1}{2}}(\partial \Omega) \). Then the following relations hold on \( \partial \Omega \).

\[
(30) \quad \gamma^Vb g_1 = Vbg_1,
\]

\[
(31) \quad \gamma^Wb g_2 = \pm \frac{1}{2}g_2 + Wbg_2,
\]

\[
(32) \quad T_a^Vb g_1 = \pm \frac{1}{2}a g_1 + \gamma^Wbg_1.
\]

4. The two-operator third Green identity and integral relations

In this section applying some limiting procedures (see, e.g., [30]), we obtain the parametrix based third Green identities.

**Theorem 4.1.** (i) If \( u \in H^1(\Omega) \), then the following third Green identity holds,

\[
(33) \quad u + \mathcal{L}bu + \mathcal{P}bu + Wb\gamma^u u = \mathcal{P}b\tilde{u} \quad \text{in} \quad \Omega,
\]

where the operator \( \tilde{A} \) is defined in (2), and for \( u \in C^1(\overline{\Omega}) \),

\[
(34) \quad \mathcal{P}b\tilde{u}(y) := (\tilde{u}, P_b(.,y))_\Omega = -\mathcal{E}_a(u, P_b(.,y)) = -\int_{\Omega} a(x)\nabla u(x) \cdot \nabla x P_b(x, y) dx,
\]
and

\[ \mathcal{L}_b u = - \int_{\Omega} [a(x) - b(x)] \nabla_x P_b(x,y) \cdot \nabla u(x) dx = \frac{1}{b(y)} \sum_{j=1}^{2} \partial_j \mathcal{P}_\Delta [(a - b) \partial_j u] \quad \text{in} \quad \Omega. \]  

(ii) If \( Au = r_\Omega \tilde{f} \) in \( \Omega \), where \( \tilde{f} \in \widetilde{H}^{-1}(\Omega) \), then recalling the definition of \( T^+_a (\tilde{f}, u) \), in (3), we arrive at the generalised two-operator third Green identity in the following form,

\[ u + \mathcal{L}_b u + \mathcal{R}_b u - V_b T^+_a (\tilde{f}, u) + W_b \gamma^+ u = \mathcal{P}_b \tilde{f} \quad \text{in} \quad \Omega, \]

where it was taken into account that

\[ \langle T^+_a (\tilde{f}, u), P_b(x,y) \rangle_{\partial \Omega} = -V_b T^+_a (\tilde{f}, u), \quad \langle \tilde{f}, P_b(x,y) \rangle_{\Omega} = \mathcal{P}_b \tilde{f}. \]

**Proof.** (i) Let first \( u \in D(\Omega) \). Let \( y \in \Omega \), \( B_\varepsilon(y) \subset \Omega \) be a ball centred at \( y \) with sufficiently small radius \( \varepsilon \), and \( \Omega_\varepsilon := \Omega \setminus B_\varepsilon(y) \). For the fixed \( y \), evidently, \( P_b(.,y) \in D(\Omega_\varepsilon) \subset H^{1,0}(A; \Omega_\varepsilon) \) and has the coinciding classical and canonical co-normal derivatives on \( \partial \Omega_\varepsilon \). Then from (8) and the first Green identity (6) employed for \( \Omega_\varepsilon \) with \( v = P_b(.,y) \) we obtain

\[
- \int_{\partial B_\varepsilon(y)} T^+_x P_b(x,y) \gamma^+ u(x) ds_x - \int_{\partial \Omega} T_x P_b(x,y) \gamma^+ u(x) ds_x + \int_{\Omega_\varepsilon} u(x) R_b(x,y) dx 
= - \int_{\Omega_\varepsilon} b(x) \nabla u(x) \cdot \nabla_x P_b(x,y) dx,
\]

which we rewrite as

\[
- \int_{\partial B_\varepsilon(y)} T^+_x P_b(x,y) \gamma^+ u(x) ds_x - \int_{\partial \Omega} T_x P_b(x,y) \gamma^+ u(x) ds_x 

= - \int_{\Omega_\varepsilon} [a(x) - b(x)] \nabla u(x) \cdot \nabla_x P_b(x,y) dx + \int_{\Omega_\varepsilon} u(x) R_b(x,y) dx 
= - \int_{\Omega_\varepsilon} a(x) \nabla u(x) \cdot \nabla_x P_b(x,y) dx.
\]

Taking the limit as \( \varepsilon \to 0 \), Eq. (37) reduces to the third Green identity (33)–(34) for any \( u \in \mathcal{D}(\Omega) \).

Taking into account the density of \( \mathcal{D}(\Omega) \) in \( H^1(\Omega) \), and the mapping properties of the integral potentials, see Appendix, we obtain that (33)–(34) hold true also for any \( u \in H^1(\Omega) \).

(ii) Let \( \{ \tilde{f}_k \} \subset \mathcal{D}(\Omega) \) be a sequence of converging to \( \tilde{f} \in \widetilde{H}^{-1}(\Omega) \) as \( k \to \infty \). Then, according to [28, Theorem B.1] there exists a sequence \( \{ u_k \} \subset \mathcal{D}(\Omega) \) converging to \( u \) in \( H^1(\Omega) \) such that \( Au_k = r_\Omega \tilde{f}_k \).
and \( T_a^+(u_k) = T_a^+(\tilde{f}_k, u_k) \) converge to \( T_a^+(\tilde{f}, u) \) in \( H^{-\frac{1}{2}}(\partial \Omega) \). For such \( u_k \) by (34) and (3), we have

\[
\mathcal{P}_b^{\tilde{A}}u_k(y) = \frac{1}{b(y)} \nabla_y \cdot \int_{\Omega} a(x)P_{\Delta}(x,y)\nabla u_k(x)dx
\]

\[
= -\frac{1}{b(y)} \lim_{\epsilon \to 0} \int_{\Omega_\epsilon} a(x)\nabla u_k(x)P_{\Delta}(x,y)dx = -\lim_{\epsilon \to 0} \mathcal{E}_{\Omega_\epsilon}(u_k, P_b(.,y))
\]

\[
(38)
\]

\[
= -\lim_{\epsilon \to 0} \left[ \int_{\Omega_\epsilon} \tilde{f}_k P_b(x,y)dx - \int_{\partial B_\epsilon(y)} P_b(x,y)T_a^+u_k(x)dS(x) \right] + \lim_{\epsilon \to 0} \int_{\partial \Omega} P_b(x,y)T_a^+u_k(x)dS(x) = \mathcal{P}_b\tilde{f}_k + V_bT_a^+u_k(y).
\]

Taking the limits as \( k \to \infty \) in Eq. (38), we obtain \( \mathcal{P}_b^{\tilde{A}}u(y) = \mathcal{P}_b\tilde{f} + V_bT_a^+(\tilde{f}, u) \), which substitution to (33) gives (36).

Using the Gauss divergence theorem, we can rewrite \( \mathcal{Z}_b u(y) \) in a form that does not involve derivatives of \( u \).

\[
(39) \quad \mathcal{Z}_b u(y) := \left[ \frac{a(y)}{b(y)} - 1 \right] u(y) + \mathcal{Z}_b u(y),
\]

\[
(40) \quad \mathcal{Z}_b u(y) := \frac{a(y)}{b(y)} W_a^+u(y) - W_b u(y) + \frac{a(y)}{b(y)} \mathcal{R}_a u(y) - \mathcal{R}_b u(y),
\]

which allows to call \( \mathcal{Z}_b \) integral operator in spite of its integro-differential representation (35). Substituting Eqs. (39) and (40) into Eqs. (33) and (36), we obtain Eqs. (4.1) and (4.3) in [28, 21] respectively. Note that the operator \( \mathcal{Z}_b \) does not vanish unless operators \( A \) and \( B \) are equal. For some functions \( \tilde{f}, \Psi, \Phi \), let us consider a more general “indirect” integral relation, associated with (36).

\[
(41) \quad u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \Psi + W_b \Phi = \mathcal{P}_b \tilde{f} \quad \text{in} \quad \Omega.
\]

**Lemma 4.2.** Let \( u \in H^1(\Omega), \Psi \in H^{-\frac{1}{2}}(\partial \Omega), \Phi \in H^\frac{1}{2}(\partial \Omega) \) and \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \), satisfy (41). Then

\[
(42) \quad Au = r_2 \tilde{f} \quad \text{in} \quad \Omega,
\]

\[
(43) \quad r_2 V_b(\Psi - T_a^+(\tilde{f}, u)) - r_2 W_b(\Phi - \gamma^+ u) = 0 \quad \text{in} \quad \Omega,
\]

\[
(44) \quad \gamma^+ u + \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \gamma^+ \Psi - \frac{1}{2} \Phi + W_b \Phi = \gamma^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial \Omega,
\]

\[
(45) \quad T_a^+(\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - a \frac{a}{2b} \Psi - W_b \Phi = T_a^+(\tilde{f} + \tilde{E}_b \mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) \quad \text{on} \quad \partial \Omega,
\]

where

\[
(46) \quad \mathcal{R}_a^+ \tilde{f}(y) := -\sum_{j=1}^{2} \partial_j[(\partial_j b) \mathcal{P}_b \tilde{f}] .
\]

**Proof.** The proof is similar to the one in [28, Lemma 4.2] for \( a = b \) and [2, Lemma 1] for \( a \neq b \). Indeed, subtracting Eq. (41) from identity (33), we obtain

\[
(47) \quad V_b \Psi(y) - W_b(\Phi - \gamma^+ u)(y) = \mathcal{P}_b[\tilde{A}u(y) - \tilde{f}](y), \quad y \in \Omega.
\]
Multiplying equality (47) by \( b(y) \), applying the Laplace operator \( \Delta \) and taking into account (28), (29), we get \( r_\Omega \tilde{f} = r_\Omega (\Delta u) = Au \) in \( \Omega \). This means \( \tilde{f} \) is an extension of the distribution \( Au \in H^{-1}(\Omega) \) to \( \tilde{H}^{-1}(\Omega) \), and \( u \) satisfies Eq. (42). Then Eq. (3) implies

\[
\mathcal{P}_b[\tilde{A}u - \tilde{f}](y) = (\tilde{A}u - \tilde{f}, P_b(\ldots y))_{\Omega} = -(T_a^+ (\tilde{f}, u), P_b(y))_{\partial \Omega} = V_b T_a^+ (\tilde{f}, u), \quad y \in \Omega.
\]

Substituting (48) into (47) leads to (43). Equation (44) follows from (41) and jump relations in (30) and (31). To prove (45), let us first remark that for \( u \in H^1(\Omega) \), we have \( H^{1,0}(\Omega; A) = H^{1,0}(\Omega; \Delta) = H^{1,0}(\Omega; B) \) and

\[
B \mathcal{P}_b \tilde{f} = \tilde{f} + \mathcal{R}_b^b \tilde{f} \quad \text{in} \quad \Omega,
\]
due to (42), which implies \( B(\mathcal{P}_b \tilde{f} - u) = \mathcal{R}_b^b \tilde{f} \) in \( \Omega \), with \( \mathcal{R}_b^b \tilde{f} \) given by (46) and thus \( \mathcal{R}_b^b \tilde{f} \in L_2(\Omega) \). Then \( B(\mathcal{P}_b \tilde{f} - u) \) can be canonically extended (by zero) to \( \tilde{B}(\mathcal{P}_b \tilde{f} - u) = \tilde{E} \mathcal{R}_b^b \tilde{f} \in \tilde{H}^0(\Omega) \subset \tilde{H}^{-1}(\Omega) \). Thus there exists a canonical co-normal derivative \( T_b^+(\mathcal{P}_b \tilde{f} - u) \) written as (see, e.g., [28, Eq. (4.14)])

\[
T_b^+ (\mathcal{P}_b \tilde{f} - u) = T_b^+ (\tilde{f} + \tilde{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+ (\tilde{f}, u),
\]
and hence

\[
T_a^+ (\mathcal{P}_b \tilde{f} - u) = \frac{a}{b} T_b^+ (\mathcal{P}_b \tilde{f} - u) = \frac{a}{b} \left[ T_b^+ (\tilde{f} + \tilde{E} \mathcal{R}_a^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_b^+ (\tilde{f}, u) \right] = T_a^+ (\tilde{f} + \tilde{E} \mathcal{R}_b^b \tilde{f}, \mathcal{P}_b \tilde{f}) - T_a^+ (\tilde{f}, u).
\]

From (41) it follows that \( \mathcal{P}_b \tilde{f} - u = L_b u + R_b u - V_b \Psi + W_b \Phi \) in \( \Omega \). Substituting this on the left-hand side of (49) and taking into account (26) and the jump relation (32), we arrive at (45).

**Remark 4.3.** If \( \tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega) \subset \tilde{H}^{-1}(\Omega) \), then \( \tilde{f} + \tilde{E} \mathcal{R}_b^b \tilde{f} \in \tilde{H}^{-\frac{1}{2}}(\Omega) \) as well, which implies \( \tilde{f} + \tilde{E} \mathcal{R}_b^b \tilde{f} = \tilde{A} \mathcal{P}_b \tilde{f} \) and

\[
T_a^+ (\tilde{f} + \tilde{E} \mathcal{R}_b^b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+ (\tilde{B} \mathcal{P}_b \tilde{f}, \mathcal{P}_b \tilde{f}) = T_a^+ \mathcal{P}_b \tilde{f}.
\]
Furthermore, if the hypotheses of Lemma 4.2 are satisfied, then (42) implies \( u \in H^{1,\frac{1}{2}}(\Omega; A) \) and \( T_a^+ (\tilde{f}, u) = T_a^+ (\tilde{A}u, u) = T_a^+ u \). Henceforth (45), takes the familiar form, cf. [5, Eq. (3.23)],

\[
T_a^+ u + T_a^+ L_b u + T_a^+ R_b u - \frac{a}{2b} \Psi + \mathcal{W}_{ab}^+ \Phi + \mathcal{L}_{ab}^+ \Phi = T_a^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial \Omega.
\]

**Remark 4.4.** Let \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) and a sequence \( \{\phi_i\} \in \tilde{H}^{-1}(\Omega) \) converge to \( \tilde{f} \) in \( \tilde{H}^{-1}(\Omega) \). By the continuity of operators [5, Eqs. (A.1) and (A.2)], estimate (4) and relation (50) for \( \phi_i \), we obtain that

\[
T_a^+ (\phi_i + \tilde{E} \mathcal{R}_b^b \phi_i, \mathcal{P}_b \phi_i) = \lim_{i \to \infty} T_a^+ \phi_i + \mathcal{P}_b \phi_i.
\]
in \( \tilde{H}^{-\frac{1}{2}}(\partial \Omega) \), cf. also [28, Theorem B.1].

Lemma 4.2 and the third Green identity (36) imply, the following assertion.

**Corollary 4.5.** If \( u \in H^1(\Omega) \) and \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) are such that \( Au = r_\Omega \tilde{f} \) in \( \Omega \), then

\[
\frac{1}{2} \gamma^+ u + \gamma^+ L_b u + \gamma^+ R_b u - \gamma^+ T_a^+ (\tilde{f}, u) + \mathcal{W}_b \gamma^+ u + \mathcal{L}_b \gamma^+ u = \gamma^+ \mathcal{P}_b \tilde{f} \quad \text{on} \quad \partial \Omega,
\]
1. \[
(52) \quad \left(1 - \frac{a}{2b}\right)T_a^+ (\tilde{f}, u) + T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}_{ab} T_a^+ (\tilde{f}, u) + \mathcal{L}^+_{ab} \gamma^+ u
\]
\[
= T_a^+ (\tilde{f} + \delta \mathcal{R}_b \tilde{f}, \mathcal{P}_b \tilde{f}) \text{ on } \partial \Omega.
\]

Note that if \(\mathcal{P}_b\) is not only the parametrix but also the fundamental solution of the operator \(B\), then the remainder operator \( \mathcal{Z}_b\) vanishes in (36) and (51)-(52) (and everywhere in the paper), while the operator \( \mathcal{Z}_b\) stays unless \(A = B\). The following statement is proved in [28, Lemma 4.6].

**Theorem 4.6.** Let \(\tilde{f} \in H^{-1}(\Omega)\). A function \(u \in H^1(\Omega)\) is a solution of PDE \(Au = r_\Omega \tilde{f}\) in \(\Omega\) if and only if it is a solution of BDIDE (36).

**Proof.** If \(u \in H^1(\Omega)\) solves PDE \(Au = r_\Omega \tilde{f}\) in \(\Omega\), then it satisfies (36). On the other hand, if \(u\) solves BDIDE (36), then using Lemma 4.2 for \(\Psi = T_a^+ (\tilde{f}, u), \Phi = \gamma^+ u\) completes the proof. \(\square\)

5. Invertibility of single layer potential operator

The boundary integral operator, \(\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^\frac{1}{2}(\partial \Omega)\) is a Fredholm operator of index zero ([20, Theorem 7.6]). Thus the first relation in (24) leads to the same result for the single layer potential \(\mathcal{V}_b\). For the case of 3D, Lemma 3.2(ii) in [5] asserts that for \(\Psi^* \in H^{-\frac{1}{2}}(\partial \Omega)\), if \(V_b \Psi^* = 0\) in \(\Omega\), then \(\Psi^* = 0\) in \(\Omega\). Imposing the invertibility of single layer potential operator \(V_b\) mapping from \(H^{-\frac{1}{2}}(\partial \Omega)\) to \(H^\frac{1}{2}(\partial \Omega)\). But this is not the case for 2D. It is well-known (see, e.g., [10, Remark 1.42(ii)] and [33, Theorem 6.22]) that for some 2D domains the kernel of the operator \(\mathcal{V}_\Delta\) is nontrivial, thus due to the first relation in (24), the kernel of operator \(\mathcal{V}_b\) is nontrivial as well for the same domains.

Following [20, Theorem 8.15], there exists a unique real-valued distribution \(\psi_{eq} \in H^{-\frac{1}{2}}(\partial \Omega)\) called equilibrium density for \(\partial \Omega\) such that \(\mathcal{V}_\Delta \psi_{eq}\) is constant on \(\partial \Omega\). For \(n = 2\) the constant \(\mathcal{V}_\Delta \psi_{eq}\) is not always positive and one introduces the *logarithmic capacity*, \(\text{Cap}_{\partial \Omega}\) using the relation

\[
\mathcal{V}_\Delta \psi_{eq} = \frac{1}{2\pi} \ln \left(\frac{r_0}{\text{Cap}_{\partial \Omega}}\right),
\]

for some positive constant \(r_0\) as in equation (8). In particular \(\mathcal{V}_\Delta \psi_{eq} = 0\) if and only if \(r_0 = \text{Cap}_{\partial \Omega}\).

The following statement is proved in [20, Theorem 8.16].

**Theorem 5.1.** Let \(r_0\) be some positive constant.

(i) The operator \(\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^\frac{1}{2}(\partial \Omega)\), is \(H^{-\frac{1}{2}}(\partial \Omega)\)-elliptic, i.e., \(\langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial \Omega} \geq c \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)}^2\)

for all \(\psi \in H^{-\frac{1}{2}}(\partial \Omega)\), if and only if \(r_0 > \text{Cap}_{\partial \Omega}\).

(ii) The operator \(\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^\frac{1}{2}(\partial \Omega)\), has a bounded inverse if and only if \(r_0 \neq \text{Cap}_{\partial \Omega}\).

The following theorem insures the invertibility of the single layer potential operator \(\mathcal{V}_b\) in 2D.

**Theorem 5.2.** Let \(\Omega \subset \mathbb{R}^2\) with \(r_0 > \text{diam}(\Omega)\). Then the single layer potential \(\mathcal{V}_b : H^{-\frac{1}{2}}(\partial \Omega) \rightarrow H^\frac{1}{2}(\partial \Omega)\) is invertible.

**Proof.** Since \(\text{Cap}_{\partial \Omega} \leq \text{diam}(\Omega)\), (see, [34, p.553, properties 1 and 3]), then \(r_0 > \text{diam}(\Omega)\) implies \(r_0 > \text{Cap}_{\partial \Omega}\). For the case \(a = b\) the assertion is proved in [13, Theorem 5]. Due to the first relation in (24) and Theorem 5.1(ii) follows the invertibility of the single layer potential operator \(\mathcal{V}_b\) for the case \(a \neq b\) as well (see also [3, Theorem 2]). \(\square\)
Lemma 5.3. Let $\Gamma_1$ be a non-empty smooth piece of curve, part of boundary $\partial \Omega$. Given any $h \in H^{\frac{1}{2}}(\Gamma_1)$ and $b \in \mathbb{R}$, the system of equations
\[ r_{\Gamma_1} \nabla \tilde{\psi} + c = h \quad \text{and} \quad (1, \tilde{\psi})_{\Gamma_1} = b \]
has a unique solution $\tilde{\psi} \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1)$ and $c \in \mathbb{R}$.

Proof. Following the arguments in the proof of [20, Lemma 8.14], we introduce the Hilbert space $H = \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \times \mathbb{R}$, identify the dual space $H^*$ with $H^* = H^{\frac{1}{2}}(\Gamma_1) \times \mathbb{R}$ by writing
\[ \langle (\tilde{\psi}, c), (\varphi, b) \rangle = \langle \tilde{\psi}, \varphi \rangle_{\Gamma_1} + cb \]
and define a bounded linear operator
\[ A(\tilde{\psi}, c) = \left( r_{\Gamma_1} \nabla \tilde{\psi} + c, (1, \tilde{\psi})_{\Gamma_1} \right) . \]
Operator $A$ is self adjoint, and now we show that it has a bounded inverse.

Let $r_{\Gamma_1} \nabla = r_{\Gamma_1} \nabla_0 + r_{\Gamma_1} L$ with $r_{\Gamma_1} \nabla_0$ invertible and $r_{\Gamma_1} L$ compact operators from $\tilde{H}^{-\frac{1}{2}}(\Gamma_1)$ to $H^{\frac{1}{2}}(\Gamma_1)$. We define
\[ A_0(\tilde{\psi}, c) = (r_{\Gamma_1} \nabla_0 \tilde{\psi}, c) \quad \text{and} \quad K(\tilde{\psi}, c) = (r_{\Gamma_1} L \tilde{\psi} + c, (1, \tilde{\psi})_{\Gamma_1} - c) \]
so that $A = A_0 + K$, with $A_0 : H \to H^*$ invertible, and $K : H \to H^*$ compact. Hence $A : H \to H^*$ is Fredholm with zero index. It is invertible if the homogeneous system $A(\tilde{\psi}, c) = (0, 0)$ has only trivial solution. In fact, if
\[ r_{\Gamma_1} \nabla \tilde{\psi} + c = 0 \quad \text{and} \quad (1, \tilde{\psi})_{\Gamma_1} = 0 , \]
then $\tilde{\psi} \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1) = \{ (\tilde{\psi}, 1)_{\Gamma_1} : (\tilde{\psi}, 1)_{\Gamma_1} = 0 \}$ (cf., e.g., [33, p. 147]) and
\[ \langle (1, \tilde{\psi})_{\Gamma_1} , (r_{\Gamma_1} \nabla \tilde{\psi}, c)_{\Gamma_1} \rangle = \langle -c, \tilde{\psi} \rangle_{\Gamma_1} = -c(1, \tilde{\psi})_{\Gamma_1} = 0 , \]
so $\tilde{\psi} = 0$ by [4, Corollary 2.7(ii)] and in turn $c = -r_{\Gamma_1} \nabla \tilde{\psi} = 0$.

Theorem 5.4. Let $\Gamma_1$ be a non-empty part of the boundary curve $\partial \Omega$.

(i) The operator $r_{\Gamma_1} \nabla : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$, is $\tilde{H}^{-\frac{1}{2}}(\Gamma_1)$-elliptic if and only if $r_0 > \text{Cap}_{\Gamma_1}$.

(ii) The operators $r_{\Gamma_1} \nabla : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ and $r_{\Gamma_1} \nabla_0 : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ are continuously invertible if and only if $r_0 \neq \text{Cap}_{\Gamma_1}$.

Proof. The proof is a modification of the corresponding proof for a closed boundary in [20, Theorem 8.16]. Let $(\varphi_{eq}, c_1)$ be the unique solution to the equation $A(\varphi, c) = (0, 1)$, which exists due to Lemma 5.3. This means $r_{\Gamma_1} \nabla \varphi_{eq} = -c_1$ and $(1, \varphi_{eq})_{\Gamma_1} = 1$. Then the logarithmic capacity is defined as
\[ \text{Cap}_{\Gamma_1} := e^{-2\pi r_{\Gamma_1} \nabla \varphi_{eq}} \text{, so that} \]
\[ r_{\Gamma_1} \nabla \varphi_{eq} = \frac{1}{2\pi} \log \left( \frac{r_0}{\text{Cap}_{\Gamma_1}} \right) \]
(i) To prove the first part of Theorem 5.4, for any $\psi \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1)$ let us define $\psi_0 = \psi - (1, \psi)_{\Gamma_1} \varphi_{eq}$.
Then $\psi = \psi_0 + (1, \psi)_{\Gamma_1} \varphi_{eq}$ and $(1, \psi_0)_{\Gamma_1} = 0$. Also, since $(r_{\Gamma_1} \nabla \varphi_{eq})_{\Gamma_1} = (\psi_0, r_{\Gamma_1} \nabla \varphi_{eq})_{\Gamma_1} = \varphi_{eq}$.
\begin{align}
\langle \psi_0, -c_1 \rangle_{\Gamma_1} &= 0, \text{ we have}
\langle r_{\Gamma_1} \mathcal{A} \psi, \psi \rangle_{\Gamma_1} &= \langle r_{\Gamma_1} \mathcal{A} \psi_0, \psi_0 \rangle_{\Gamma_1} - c_1 \langle 1, \psi \rangle_{\Gamma_1} \langle 1, \psi \rangle_{\Gamma_1}
\end{align}

Now, if \( r_0 \leq \text{Cap}_{\Gamma_1} \), then \( \langle r_{\Gamma_1} \mathcal{A} \psi_{\text{eq}}, \psi_{\text{eq}} \rangle_{\Gamma_1} = -c_1 \leq 0 \), i.e., the operator \( r_{\Gamma_1} \mathcal{A} \) is not \( \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \)-elliptic.

Suppose that \( r_0 > \text{Cap}_{\Gamma_1} \) or equivalently \( c_1 < 0 \). By \cite[Theorem 8.12]{20}, both terms on the right hand side of (54) are non-negative and the first is zero if and only if \( \psi_0 = 0 \). Thus, \( \langle r_{\Gamma_1} \mathcal{A} \psi, \psi \rangle_{\Gamma_1} \geq 0 \), with equality if and only if \( \psi_0 = 0 \) and \( \langle 1, \psi \rangle_{\Gamma_1} = 0 \), i.e. if and only if \( \psi = 0 \). Hence \( r_{\Gamma_1} \mathcal{A} \) is strictly positive-definite on the whole of \( H^{-\frac{1}{2}}(\Gamma_1) \), \( \ker r_{\Gamma_1} \mathcal{A} = \emptyset \). Due to \cite[Corollary 2.7(i)]{4}, the operator \( r_{\Gamma_1} \mathcal{A} : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \rightarrow H^{\frac{1}{2}}(\Gamma_1) \) is also Fredholm with index zero and hence is invertible. Using similar arguments as in the proof of \cite[Corollary 8.13]{20}, we conclude that \( r_{\Gamma_1} \mathcal{A} \) is positive and bounded below on \( H^{\frac{1}{2}}(\Gamma_1) \). Hence the operator \( r_{\Gamma_1} \mathcal{A} \) is \( H^{-\frac{1}{2}}(\Gamma_1) \) elliptic. This completes the proof of item (i).

To prove (ii), we note that if \( \text{Cap}_{\Gamma_1} = r_0 \), then \( r_{\Gamma_1} \mathcal{A} \) cannot be invertible because \( r_{\Gamma_1} \mathcal{A} \psi_{\text{eq}} = 0 \). Thus, suppose that \( \text{Cap}_{\Gamma_1} \neq r_0 \), and \( r_{\Gamma_1} \mathcal{A} \psi = 0 \). Then we have \( r_{\Gamma_1} \mathcal{A} \psi_{\text{eq}} = c_1 \langle 1, \psi \rangle_{\Gamma_1} \), hence \( \langle r_{\Gamma_1} \mathcal{A} \psi_0, \psi_0 \rangle_{\Gamma_1} = 0 \), and therefore \( \psi_0 = 0 \) by \cite[Theorem 8.12]{20}. In turn, \( \langle 1, \psi \rangle_{\Gamma_1} = 0 \) because \( c_1 \neq 0 \), giving \( \psi = 0 \). Thus the homogeneous equation has only the trivial solution. Since due to \cite[Corollary 2.7(i)]{4} the operator \( r_{\Gamma_1} \mathcal{A} : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \rightarrow H^{\frac{1}{2}}(\Gamma_1) \) is also Fredholm with index zero, it is invertible. From the first relation in (24) then follows the invertibility of \( r_{\Gamma_1} \mathcal{A} : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \rightarrow H^{\frac{1}{2}}(\Gamma_1) \).

\begin{corollary}
Let \( \Gamma_1 \) be a non-empty part of the boundary curve and \( r_0 > \text{diam}(\Gamma_1) \). Then the operator \( r_{\Gamma_1} \mathcal{A} : \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \rightarrow H^{\frac{1}{2}}(\Gamma_1) \) has a bounded inverse.
\end{corollary}

\begin{proof}
Since \( \text{Cap}_{\Gamma_1} \leq \text{diam}(\Gamma_1) \), (see, \cite[p.553, properties 1,3 and 4]{34}), then \( r_0 > \text{diam}(\Gamma_1) \) implies \( r_0 > \text{Cap}_{\Gamma_1} \). Then the result follows from Theorem 5.4(ii).
\end{proof}

\begin{theorem}
Let \( \Gamma_2 \) be a non-empty part of the boundary curve \( \partial \Omega \). The operator
\begin{align}
(55) \quad r_{\Gamma_2} \mathcal{A}^\# : \tilde{H}^{\frac{1}{2}}(\Gamma_2) \rightarrow H^{-\frac{1}{2}}(\Gamma_2)
\end{align}

is \( \tilde{H}^{\frac{1}{2}}(\Gamma_2) \)-elliptic. Operator (55) and the operator
\begin{align}
(56) \quad r_{\Gamma_2} \mathcal{A} : \tilde{H}^{\frac{1}{2}}(\Gamma_2) \rightarrow H^{-\frac{1}{2}}(\Gamma_2)
\end{align}

are continuously invertible.
\end{theorem}

\begin{proof}
The ellipticity of operator (55) follows from inequality (6.39) in \cite{33}. The continuity of this operator and the Lax-Milgram lemma then imply its invertibility. Together with relation (27) this implies the invertibility of operator (56).
\end{proof}

\begin{lemma}
(i) Let \( \Psi^* \in H^{-\frac{1}{2}}(\partial \Omega) \) and \( r_0 > \text{diam}(\Omega) \). If \( r_\Omega V_b \Psi^* = 0 \) in \( \Omega \), then \( \Psi^* = 0 \).

(ii) If \( \Phi^* \in H^{\frac{1}{2}}(\partial \Omega) \) and \( r_\Omega W_b \Phi^* = 0 \) in \( \Omega \), then \( \Phi^* = 0 \).
\end{lemma}

\begin{proof}
The proof of (i) follows from the proof of \cite[Lemma 1]{3} and \cite[Lemma 2]{2}.
\end{proof}
Lemma 5.8. Let $\partial \Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are non-empty non-intersecting parts of the boundary curve $\partial \Omega$. Let $\Psi^* \in \tilde{H}^{\frac{1}{2}}(\Gamma_1)$ with $r_0 > \text{diam}(\Gamma_1)$ and $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\Gamma_2)$. If
\[ r_\Omega V_\Omega \Psi^*(y) - r_\Omega W_\Omega \Phi^*(y) = 0, \quad y \in \Omega, \]
then $\Psi^* = 0$ and $\Phi^* = 0$.

Proof. The proof follows from Theorem 5.4 (i) and Theorem 5.6 similar to [7, Lemma 4.2(iii)]. Indeed, if we multiply equation (5.8) by $b(y)$, we get,
\[ (57) \quad V_\Delta \Psi^* - W_\Delta (b \Phi^*) = 0, \quad \text{in } \Omega. \]
Taking the trace and the co-normal derivatives of this equation on $\Gamma_1$ and $\Gamma_2$ respectively, we obtain the following:
\[ r_\Gamma V_\Gamma \Psi^* - r_\Gamma W_\Gamma \Phi^* = 0 \]
\[ r_\Gamma W_\Gamma' \Psi^* - r_\Gamma W_\Gamma' \Phi^* = 0, \]
with $\Phi^* = b\Phi^*$. The above system of equations can be written in matrix form as
\[ \mathcal{M}_\Delta \mathcal{X} = 0 \]
where the matrix operator is given by;
\[ \mathcal{M}_\Delta = \begin{bmatrix} r_\Gamma V_\Gamma & r_\Gamma W_\Gamma \\ r_\Gamma W_\Gamma' & -r_\Gamma W_\Gamma' \end{bmatrix}, \quad \mathcal{X} = \begin{bmatrix} \Psi^* \\ \Phi^* \end{bmatrix}. \]

We shall prove that the operator $\mathcal{M}_\Delta$ is $\tilde{H}^{-\frac{1}{2}}(\Gamma_1) \times \tilde{H}^{\frac{1}{2}}(\Gamma_2)$-elliptic if $r_0 > \text{diam}(\Gamma_1)$.

In fact,
\[ \langle \mathcal{M}_\Delta \mathcal{X}, \mathcal{X} \rangle_{\partial \Omega} = \langle r_\Gamma V_\Gamma \Psi^* - r_\Gamma W_\Gamma \Phi^*, \Psi^* \rangle_{\Gamma_1} + \langle r_\Gamma W_\Gamma' \Psi^* - r_\Gamma W_\Gamma' \Phi^*, \Phi^* \rangle_{\Gamma_2} \]
\[ = \langle r_\Gamma V_\Gamma \Psi^*, \Psi^* \rangle_{\Gamma_1} + \langle -r_\Gamma W_\Gamma \Phi^*, \Psi^* \rangle_{\Gamma_1} + \langle r_\Gamma W_\Gamma' \Psi^*, \Phi^* \rangle_{\Gamma_2} + \langle -r_\Gamma W_\Gamma' \Phi^*, \Phi^* \rangle_{\Gamma_2}. \]

Suppose that $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\Gamma_2)$ and $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1)$ with $r_0 > \text{diam}(\Gamma_1)$. Due to Theorem 5.4 (i) the operator $r_\Gamma V_\Gamma: \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \to \tilde{H}^{\frac{1}{2}}(\Gamma_1)$, is $\tilde{H}^{-\frac{1}{2}}(\Gamma_1)$—elliptic, i.e., for some positive constant $c_1$ there holds
\[ \langle r_\Gamma V_\Gamma \Psi^*, \Psi^* \rangle_{\Gamma_1} = \langle \Psi^*, \Psi^* \rangle_{\partial \Omega} \geq c_1 \| \Psi^* \|^2_{\tilde{H}^{-\frac{1}{2}}(\partial \Omega)} \quad \text{for all } \Psi^* \in \tilde{H}^{-\frac{1}{2}}(\Gamma_1). \]

Due to Theorem 5.6, the operator $-L_\Gamma$ is $\tilde{H}^{\frac{1}{2}}(\Gamma_2)$-elliptic , which means, there exists some positive constant $c_2$ such that,
\[ \langle -r_\Gamma L_\Gamma \Phi^*, \Phi^* \rangle_{\Gamma_2} = \langle \Phi^*, \Phi^* \rangle_{\partial \Omega} \geq c_2 \| \Phi^* \|^2_{\tilde{H}^{\frac{1}{2}}(\partial \Omega)} \quad \text{for all } \Phi^* \in \tilde{H}(\Gamma_2). \]

Since the operators
\[ r_\Gamma W_\Gamma: \tilde{H}^{\frac{1}{2}}(\Gamma_2) \to \tilde{H}^{\frac{1}{2}}(\Gamma_1) \]
\[ r_\Gamma W_\Gamma': \tilde{H}^{-\frac{1}{2}}(\Gamma_1) \to \tilde{H}^{\frac{1}{2}}(\Gamma_2) \]
are mutually adjoint, i.e., \( \langle r_1 \mathcal{A} \Phi^*, \Psi^* \rangle_{\Gamma_1} = \langle r_2 \mathcal{A} \Psi^*, \Phi^* \rangle_{\Gamma_2} \) for arbitrary \( \Psi^* \in H^{-\frac{1}{2}}(\Gamma_1) \) and arbitrary \( \Phi^* \in H^{\frac{1}{2}}(\Gamma_2) \). Then the expressions in the middle of the r.h.s. of Eq. (58) vanish. This implies \( H^{-\frac{1}{2}}(\Gamma_1) \times H^{\frac{1}{2}}(\Gamma_2) \)-ellipticity of operator \( \mathcal{M}_\Delta \), i.e.,

\[
\langle \mathcal{M}_\Delta \mathcal{X}, \mathcal{X} \rangle \geq c \left( \| \Psi^* \|_{H^{-\frac{1}{2}}(\partial \Omega)}^2 + \| \Phi^* \|_{H^{\frac{1}{2}}(\partial \Omega)}^2 \right) = \| \mathcal{X} \|_{H^{\frac{1}{2}}(\partial \Omega)}^2
\]

Hence \( \mathcal{M}_\Delta \mathcal{X} = 0 \) implies that \( \mathcal{X} = 0 \), that is, \( \Psi^* = 0 \) and \( \Phi^* = 0 \). Since \( b(y) \neq 0 \) we have \( \Psi^* = 0 \) and \( \Phi^* = 0 \) on \( \partial \Omega \) and this completes the proof. \( \square \)

### 6. The two-operator BDIE systems for the Mixed BVP

Let \( \partial \Omega = \overline{\partial \Omega_D} \cup \overline{\partial \Omega_N} \) where \( \partial \Omega_D \) and \( \partial \Omega_N \) are nonempty and nonintersecting open subsets of \( \partial \Omega \).

We shall derive and investigate BDIEs for the following mixed BVP: Find a function \( u \in H^1(\Omega) \) satisfying conditions

(59) \[ Au = r_\Omega \tilde{f} \quad \text{in} \quad \Omega, \]

(60) \[ \gamma^+ u = \varphi_0 \quad \text{on} \quad \partial \Omega_D, \]

(61) \[ T_a^+ (\tilde{f}, u) = \psi_0 \quad \text{on} \quad \partial \Omega_N, \]

where \( \varphi_0 \in H^{\frac{1}{2}}(\partial \Omega_D) \), \( \psi_0 \in H^{-\frac{1}{2}}(\partial \Omega_N) \), \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) are given functions. Equation (59) is understood in distributional sense, Eq. (60) is understood in trace sense and Eq. (61) is understood in functional sense. The following assertion is well-known and can be proved e.g. using variational settings and the Lax-Milgram lemma.

**Theorem 6.1.** The homogeneous version of BVP (59)–(61), i.e., with \((\tilde{f}, \varphi_0, \psi_0) = (0, 0, 0)\) has only the trivial solution. Hence the nonhomogeneous problem (59)–(61) may posses at most one solution.

**Proof.** The proof follows from Green’s formula (5) with \( v = u \) as a solution of the homogeneous mixed BVP (59)–(61), see e.g., [7, Theorem 2.1]. \( \square \)

From Theorem 6.1 similar to [7, Corollary 5.16], we have the following statement, see e.g., [1, Theorem 6.2].

**Theorem 6.2.** The mixed problem (59)–(61) is uniquely solvable in \( H^1(\Omega) \). The solution is \( u = (\mathcal{A}^M)^{-1}(\tilde{f}, \varphi_0, \psi_0)^\top \), where the inverse operator, \((\mathcal{A}^M)^{-1} : H^{-\frac{1}{2}}(\partial \Omega_N) \times H^{\frac{1}{2}}(\partial \Omega_D) \times \tilde{H}^{-1}(\Omega) \to H^1(\Omega)\), to the left-hand side operator, \(\mathcal{M} : H^1(\Omega) \to \tilde{H}^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial \Omega_D) \times H^{-\frac{1}{2}}(\partial \Omega_N)\), of the mixed problem (59)–(61), is continuous.

### 7. Two-operator boundary-domain integral equations

Let \( \Phi_0 \in H^{\frac{1}{2}}(\partial \Omega) \) and \( \Psi_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) be some extensions of the given data \( \varphi_0 \in H^{\frac{1}{2}}(\partial \Omega_D) \) from \( \partial \Omega_D \) to \( \partial \Omega \) and \( \psi_0 \in H^{-\frac{1}{2}}(\partial \Omega_N) \) from \( \partial \Omega_N \) to \( \partial \Omega \), respectively. Let us also denote

(62) \[ \tilde{F}_0 := \mathcal{A}_b \tilde{f} + V_b \Psi_0 - W_b \Phi_0 \quad \text{in} \quad \Omega. \]
Due to the mapping properties of the Newtonian (volume) and layer potentials (cf. Theorems 3.1 and 3.10 in [7]), we have the inclusion \( \tilde{F}_0 \in H^1(\Omega) \), for \( \tilde{f} \in \tilde{H}^{-1}(\Omega), \Psi_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) and \( \Phi_0 \in H^\frac{1}{2}(\partial \Omega) \).

We shall use the following notations for product spaces.

\[
\mathcal{W} := \tilde{H}^{-1}(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega) \times H^\frac{1}{2}(\partial \Omega),
\]
\[
\mathcal{X} := H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial \Omega_D) \times \tilde{H}^\frac{1}{2}(\partial \Omega_N),
\]
\[
\mathcal{Y}^{11} := H^1(\Omega) \times H^\frac{1}{2}(\partial \Omega_D) \times H^{-\frac{1}{2}}(\partial \Omega_N),
\]
\[
\mathcal{Y}^{22} := H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega_D) \times H^\frac{1}{2}(\partial \Omega_N),
\]
\[
\mathcal{Y}^{21} := H^1(\Omega) \times H^{-\frac{1}{2}}(\partial \Omega).\]

To reduce BVP (59)–(61) to one or another two-operator BDIE system, we shall use equation (36) in \( \Omega \), and restrictions of Eq. (51) or (52) to appropriate parts of the boundary. We shall always substitute \( \Phi_0 + \varphi \) for \( \gamma^+ u \) and \( \Psi_0 + \psi \) for \( T_a^+(\tilde{f}, u) \), cf. [7], where \( \Phi_0 \in H^\frac{1}{2}(\partial \Omega) \) and \( \Psi_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) are considered as known, while \( \psi \) belongs to \( H^{-\frac{1}{2}}(\partial \Omega_D) \) and \( \varphi \) to \( \tilde{H}^\frac{1}{2}(\partial \Omega_N) \) due to the boundary conditions (60)–(61) and are to be found along with \( u \in H^1(\Omega) \). This will lead us to segregated BDIE systems with respect to the unknown triple

\[
\mathcal{U} := [u, \psi, \varphi]^\top \in \mathcal{X}.
\]

### 7.1. Boundary-Domain Integral Equation System M11

Let us use Eq. (36) in \( \Omega \), the restriction of Eq. (51) on \( \partial \Omega_D \) and the restriction of Eq. (52) on \( \partial \Omega_N \). Then we arrive at the following two-operator segregated system of BDIEs:

\[
\begin{aligned}
(63) & \quad u + \mathcal{Z}_b u + \mathcal{R}_b u - V_b \psi + W_b \varphi = \tilde{F}_0 & \quad \text{in} \quad \Omega, \\
(64) & \quad \gamma^+ \mathcal{Z}_b u + \gamma^+ \mathcal{R}_b u - \gamma^+ V_b \psi + \gamma^+ W_b \varphi = \gamma^+ \tilde{F}_0 - \varphi_0 & \quad \text{on} \quad \partial \Omega_D, \\
(65) & \quad T_a^+ \mathcal{Z}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}_{ab} \psi + \mathcal{L}_{ab} \varphi = T_a^+ \tilde{F}_0 - \psi_0 & \quad \text{on} \quad \partial \Omega_N,
\end{aligned}
\]

which we call BDIE system M11, where M stands for the mixed problem and 11 hints that the integral equations on the Dirichlet and Neumann parts of the boundary are of the first kind. System (63)-(65) can be written in the form

\[
\mathcal{M}^{11} \mathcal{U} = \mathcal{F}^{11},
\]

where

\[
\begin{aligned}
(66) & \quad \mathcal{F}^{11} := [\tilde{F}_0, \gamma^+ \tilde{F}_0 - \varphi_0, r_{\partial \Omega_D} T_a^+ \tilde{F}_0 - \psi_0]^\top, \\
(67) & \quad \mathcal{M}^{11} := \begin{bmatrix}
I + \mathcal{Z}_b + \mathcal{R}_b & -V_b & W_b \\
r_{\partial \Omega_D} \gamma^+ [\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial \Omega_D} \gamma^+ V_b & r_{\partial \Omega_D} W_b \\
r_{\partial \Omega_N} T_a^+ [\mathcal{Z}_b + \mathcal{R}_b] & -r_{\partial \Omega_N} \mathcal{W}_{ab} & r_{\partial \Omega_N} \mathcal{L}_{ab}
\end{bmatrix}.
\end{aligned}
\]

**Remark 7.1.** Let \( r_0 > \text{diam}(\Gamma_1) \). Then \( \mathcal{F}^{11} = 0 \) if and only if \( (\tilde{F}_0, \Phi_0, \Psi_0) = 0 \).
Proof. Indeed, the latter equality evidently implies the former. Inversely, let $\mathcal{F}^{11} = [\bar{F}_0, r_{\partial N}, \gamma^+ \bar{F}_0 - \phi_0, r_{\partial N} T_a^+ \bar{F}_0 - \psi_0]^\top = 0$. Keeping in mind Eq. (62), Lemma 4.2 with $\bar{F}_0 = 0$ for $u$ implies $r_{\partial N} \bar{f} = 0$, which means $\bar{f} \in H^{-1}_{\partial N}$ and $r_{\partial N} V_\psi \psi_0 - r_{\partial N} W_b \Phi_0 = 0$ in $\Omega$. Then [26, Theorem 2.10] and [28, Lemma 6.4] lead to $V_\psi \bar{f} = 0$ in $\Omega$ and due to Lemma 5.7, $\bar{f} = 0$ in $\Omega$. The equalities $r_{\partial N} \gamma^+ \bar{F}_0 - \phi_0 = 0$ and $r_{\partial N} T_a^+ \bar{F}_0 - \psi_0 = 0$ imply $\phi_0 = 0$ on $\partial N$ and $\psi_0 = 0$ on $\partial N$, that is, $\Phi_0 \in H^{\frac{1}{2}}(\partial N)$ and $\Psi_0 \in \tilde{H}^{\frac{1}{2}}(\partial N)$ then Lemma 5.8 gives $\Phi_0 = 0$ and $\Psi_0 = 0$ on $\partial N$.

7.2. Boundary-domain integral equation system M12. To obtain another system, we use Eq. (36) in $\Omega$ and Eq. (51) on the whole boundary $\partial \Omega$, and arrive at the two-operator segregated BDIE system:

\begin{align}
(68) \quad u + \mathcal{X}_b u + \mathcal{R}_b u - V_\psi \psi + W_b \phi = \bar{F}_0 & \quad \text{in } \Omega, \\
(69) \quad \frac{1}{2} \phi + \gamma^+ \mathcal{X}_b u + \gamma^+ \mathcal{R}_b u - V_\psi \psi + W_b \phi = \gamma^+ \bar{F}_0 - \Phi_0 & \quad \text{on } \partial \Omega.
\end{align}

System (68)-(69) can be written in the form

$$\mathcal{M}^{12} \mathcal{U} = \mathcal{F}^{12},$$

where

$$\mathcal{F}^{12} := [\bar{F}_0, \gamma^+ \bar{F}_0 - \Phi_0]^\top,$$

$$\mathcal{M}^{12} := \begin{bmatrix} I + \mathcal{X}_b + \mathcal{R}_b & -V_\psi & W_b \\
\gamma^+ [\mathcal{X}_b + \mathcal{R}_b] & -V_\psi & \frac{1}{2} I + W_b \end{bmatrix}.$$

Remark 7.2. Let $r_0 > \text{diam}(\Omega)$. Then $\mathcal{F}^{12} = 0$ if and only if $(\bar{f}, \Phi_0, \Psi_0) = 0$.

Proof. Indeed, the latter equality evidently implies the former. Conversely, let $\mathcal{F}^{12} = (\bar{F}_0, \gamma^+ \bar{F}_0 - \Phi_0) = 0$. Which implies $-V_\psi \psi_0 + W_b \Phi_0 = \mathcal{R}_b \bar{f}$. Similar argument as in Remark 7.1 gives, $\bar{f} = 0$. The equality $\gamma^+ \bar{F}_0 - \Phi_0 = 0$ implies $\Phi_0 = 0$ on $\partial N$. Thus $V_\psi \psi_0 = 0$, hence by Lemma 5.7 it follows $\Psi_0 = 0$.

7.3. Boundary-domain integral equation system M21. To obtain one more system, we use Eq. (36) in $\Omega$ and Eq. (52) on $\partial \Omega$ and arrive at the two-operator segregated BDIE system M21:

\begin{align}
(72) \quad u + \mathcal{X}_b u + \mathcal{R}_b u - V_\psi \psi + W_b \phi = \bar{F}_0 & \quad \text{in } \Omega, \\
(73) \quad \left(1 - \frac{a}{2b}\right) \psi + T_a^+ \mathcal{X}_b u + T_a^+ \mathcal{R}_b u - \mathcal{W}_b' \phi + \mathcal{Z}_b^+ \phi = T_a^+ \bar{F}_0 - \Psi_0 & \quad \text{on } \partial \Omega.
\end{align}

System (72)-(73) can be written in the form

$$\mathcal{A}^{21} \mathcal{U} = \mathcal{F}^{21},$$

where

$$\mathcal{F}^{21} := [\bar{F}_0, T_a^+ \bar{F}_0 - \Psi_0]^\top,$$

$$\mathcal{A}^{21} := \begin{bmatrix} I + \mathcal{X}_b + \mathcal{R}_b & -V_\psi & W_b \\
T_a^+ \mathcal{X}_b + \mathcal{R}_b & \left(1 - \frac{a}{2b}\right) I - \mathcal{W}_b' & \mathcal{Z}_b^+ \end{bmatrix}.$$
Remark 7.3. Let \( r_0 > \text{diam}(\Omega) \). Then \( \mathcal{F}^{21} = 0 \) if and only if \((\tilde{f}, \Phi_0, \Psi_0) = 0\).

Proof. The proof follows in the similar way as in Remark 7.2.

7.4. Boundary-domain integral equation system M22. To reduce BVP (59)-(61) to a BDIE system of almost the second kind up to the spaces (see e.g., concluding remarks [7, p.540]), we use Eq. (36) in \( \Omega \), the restriction of Eq. (52) to \( \partial \Omega_D \), and the restriction of Eq. (51) to \( \partial \Omega_N \). Then we arrive at the following two-operator segregated BDIE system M22:

\[
\begin{align*}
\mathcal{M}^{22} \mathcal{U} &= \mathcal{F}^{22}, \\
\mathcal{F}^{22} &= \begin{bmatrix} \tilde{f}_0, r_{\partial \Omega_D} (T^+_a \tilde{F}_0 - \Psi_0), r_{\partial \Omega_N} (\gamma^+ \tilde{F}_0 - \Phi_0) \end{bmatrix}^T, \\
\mathcal{M}^{22} &= \begin{bmatrix} I + \mathcal{L}^+_b + \mathcal{R}_b & -V_b & W_b \\
I_{\partial \Omega_D} T^+_a [\mathcal{L}^+_b + \mathcal{R}_b] & r_{\partial \Omega_D} \left[ \left( 1 - \frac{a}{2b} \right) I - \mathcal{W}' \right] & r_{\partial \Omega_D} \mathcal{L}^+_a \\
I_{\partial \Omega_N} \gamma^+ [\mathcal{L}^+_b + \mathcal{R}_b] & -r_{\partial \Omega_N} \mathcal{W}^b & r_{\partial \Omega_N} \left[ \frac{1}{2} I + \mathcal{W}^b \right] \end{bmatrix}
\end{align*}
\]

Remark 7.4. Let \( r_0 > \text{diam}(\Gamma_1) \). Then \( \mathcal{F}^{22} = 0 \) if and only if \((\tilde{f}, \Phi_0, \Psi_0) = 0\).

Proof. The proof follows in the similar way as in Remark 7.1.

8. Equivalence and invertibility

Now let us prove the equivalence of BVP (59)-(61) with the BDIE systems M11, M12, M21 and M22.

Theorem 8.1. Let \( \tilde{f} \in \tilde{H}^{-1}(\Omega) \) and let \( \Phi_0 \in H^\frac{1}{2}(\partial \Omega) \) and \( \Psi_0 \in H^{-\frac{1}{2}}(\partial \Omega) \) be some fixed extensions of \( \phi_0 \in H^\frac{1}{2}(\partial \Omega_D) \) and \( \psi_0 \in H^{-\frac{1}{2}}(\partial \Omega_N) \), respectively.

(i) If some \( u \in H^1(\Omega) \) solves the mixed BVP (59)-(61) in \( \Omega \), then the solution is unique and the triplet \((u, \psi, \phi) \) is \( X \), where

\[
\psi = T^+_a (\tilde{f}, u) - \Psi_0, \quad \phi = \gamma^+ u - \Phi_0 \quad \text{on} \quad \partial \Omega,
\]

solves the BDIE systems M11, M12, M21 and M22.

(ii) If \( r_0 > \text{diam}(\Gamma_1) \) and a triplet \((u, \psi, \phi) \) is \( X \) solves BDIE systems M11 or M12 or M21 or M22, then the solution \( u \) solves BVP (59)-(61), and relations in (81) hold.

Proof. Let \( u \in H^1(\Omega) \) be a solution to BVP (59)-(61). Then due to Theorem 6.1 it is unique. Set

\[
\psi := T^+_a (\tilde{f}, u) - \Psi_0, \quad \phi := \gamma^+ u - \Phi_0.
\]

Evidently, \( \psi \in \tilde{H}^{-\frac{1}{2}}(\partial \Omega_D) \) and \( \phi \in \tilde{H}^\frac{1}{2}(\partial \Omega_N) \). Then from...
Theorem 4.6 and relations (63)-(65) follows that the triplet \((u, \psi, \phi)\) satisfies the BDIE systems M11, M12, M21 and M22 with the right hand sides (66), (70), (74) and (79) respectively, which completes the proof of item (i). We give below proofs of item (ii) for the four BDIE systems M11, M12, M21 and M22 one by one.

**BDIE system M11.** Let a triplet \((u, \psi, \phi) \in X\) solve BDIE system (63)-(65). Taking the trace of Eq. (63) on \(\partial \Omega_D\) using the jump relations (30)-(31) in Theorem 3.2, and subtracting Eq. (64) from it we obtain

\[
\gamma^+ u = \phi_0 \quad \text{on} \quad \partial \Omega_D,
\]

i.e., \(u\) satisfies the Dirichlet condition (60). Taking the co-normal derivative of Eq. (63) on \(\partial \Omega_N\), using the jump relation in Eq. (32), and subtracting Eq. (65), we obtain

\[
T^+_a (f, u) = \psi_0, \quad \text{on} \quad \partial \Omega_N,
\]

i.e. \(u\) satisfies the Neumann condition (61). Hence \(u\) solves the mixed BVP (59)-(61).

Taking into account that \(\varphi = 0\), \(\Phi_0 = \phi_0\) on \(\partial \Omega_D\) and \(\psi = 0\), \(\Psi_0 = \psi_0\) on \(\partial \Omega_N\), Eqs. (82) and (83) imply that the first relation in Eq. (81) is satisfied on \(\partial \Omega_D\) and the second relation in Eq. (81) is satisfied on \(\partial \Omega_N\). Thus we have \(\Phi^* \in H^{-\frac{1}{2}}(\partial \Omega_D)\) and \(\Psi^* \in H^\frac{1}{2}(\partial \Omega_N)\).

Equation (63) and Lemma 4.2 with \(\Psi = \psi + \Psi_0\), \(\Phi = \phi + \Phi_0\) imply that \(u\) is a solution to (42) and due to (43)

\[
V_b \Psi^* - W_b \Phi^* = 0, \quad \text{in} \quad \Omega,
\]

where \(\Psi^* = \Psi_0 + \psi - T^+_a (f, u)\) and \(\Phi^* = \Phi_0 + \phi - \gamma^+ u\). Let \(\Gamma_1 = \partial \Omega_D\), \(\Gamma_2 = \partial \Omega_N\). Since \(r_0 > \text{diam}(\Gamma_1)\), Lemma 5.8 implies \(\Psi^* = \Phi^* = 0\), which completes the proof of conditions in Eq. (81).

**BDIE system M12.** Let the triplet \((u, \psi, \phi) \in X\) solve BDIE system (68)-(69). Let us consider the trace of equation (68) on \(\partial \Omega\), taking into account the jump properties and subtract it from (69) to obtain,

\[
\gamma^+ u = \phi_0 + \phi \quad \text{on} \quad \partial \Omega.
\]

This means that the second equation in (81) holds. Since \(\phi = 0\), \(\Phi_0 = \phi_0\) on \(\partial \Omega_D\) we see that the Dirichlet condition (60) is satisfied.

Equation (68) and Lemma 4.2 with \(\Psi = \psi + \Psi_0\), \(\Phi = \phi + \Phi_0\) imply that \(u\) is a solution to Eq. (42) and

\[
V_b (\Psi_0 + \psi - T^+_a (f, u)) - W_b (\Phi_0 + \phi - \gamma^+ u) = 0 \quad \text{in} \quad \Omega.
\]

Due to (84), the second term in (85) vanishes, and by Lemma 5.7 (i) we obtain

\[
\Psi_0 + \psi - T^+_a (f, u) = 0 \quad \text{on} \quad \partial \Omega,
\]

i.e., the first equation in (81) is satisfied as well. Since \(\psi = 0\), \(\Psi_0 = \psi_0\) on \(\partial \Omega_N\) equation (86) implies that \(u\) satisfies the Neumann boundary condition (61).
**BDIE system M21.** Let now a triplet \((u, \psi, \varphi) \in \mathcal{X}\) solve BDIE system (72)-(73). Taking the co-normal derivative of Eq. (72) on \(\partial \Omega\) and subtracting it from equation (73), we obtain
\[
\psi + \Psi_0 - T_a^+(\tilde{f}, u) = 0 \quad \text{on} \quad \partial \Omega. \tag{87}
\]
which proves the first equation in (81). Since \(\psi = 0\) on \(\partial \Omega_N\) and \(\Psi_0 = \psi_0\) on \(\partial \Omega_N\), we see that \(u\) satisfies the Neumann condition (61).

Equation (72) and Lemma 4.2 with \(\Psi = \psi + \Psi_0\), \(\Phi = \varphi + \Phi_0\) imply that \(u\) is a solution to equation (42) and
\[
\Psi_0 - \Psi = 0 \quad \text{on} \quad \partial \Omega. \tag{88}
\]
Due to Eq. (87) the first term vanishes in (88), and by Lemma 5.7 (ii) we obtain,
\[
\Phi_0 + \varphi - \gamma^+ u = 0 \quad \text{on} \quad \partial \Omega,
\]
which means the second condition in (81) holds as well. Taking into account \(\varphi = 0\) on \(\partial \Omega_D\) and \(\Phi_0 = \varphi\) on \(\partial \Omega_D\), that is, \(u\) satisfies the Dirichlet condition (60).

**BDIE system M22.** Let now a triplet \((u, \psi, \varphi) \in \mathcal{X}\) solve BDIE system (76)-(78). Taking the co-normal derivative of equation (81) on \(\partial \Omega_D\) and subtracting it from Eq. (77), we obtain
\[
\psi = T_a^+(\tilde{f}, u) - \Psi_0 \quad \text{on} \quad \partial \Omega_D. \tag{89}
\]
Further, take the trace of equation (76) on \(\partial \Omega_N\) and subtract it from Eq.(78). We get
\[
\varphi = \gamma^+ u - \Phi_0 \quad \text{on} \quad \partial \Omega_N. \tag{90}
\]
Equations (89) and (90) imply that the first equation (81) is satisfied on \(\partial \Omega_D\) and the second equation in (81) is satisfied on \(\partial \Omega_N\).

Equation (76) and Lemma 4.2 with \(\Psi = \psi + \Psi_0\), \(\Phi = \varphi + \Phi_0\) imply that \(u\) is a solution to Eq. (42) and \(V_b(\Psi^* - \Psi_0^*) = 0\) in \(\Omega\), where \(\Psi^* = \psi + \Psi - T_a^+(\tilde{f}, u)\) and \(\Phi^* = \Phi_0 + \varphi - \gamma^+ u\). Due to (81) and (90), we have \(\Psi^* \in \tilde{H}^{-1}(\partial \Omega_N), \Phi^* \in \tilde{H}^{-1}(\partial \Omega_D).\) Lemma 5.8 with \(\Gamma_1 = \partial \Omega_N\) and \(\Gamma_2 = \partial \Omega_D\) implies \(\Psi^* = \Phi^* = 0\) which completes the proof of conditions (81) on the whole boundary \(\partial \Omega\). Taking into account that \(\varphi = 0\) on \(\partial \Omega_D\) and \(\Phi_0 = \varphi_0\) on \(\partial \Omega_D\), and \(\psi = 0\) on \(\partial \Omega_N\) and \(\Psi_0 = \psi_0\) on \(\partial \Omega_N\), Eq. (81) imply the boundary conditions (60) and (61).

Unique solvability of the BDIE systems M11, M12, M12 and M22 then follows from the already proved relations (81) and the unique solvability of BVP (59)-(61) stated in item (i).

The mapping properties of operators in (67), (71), (75) and (80) described in [1, 5, Appendix] together with Theorem 8.1 imply the following statement.

**Corollary 8.2.** The following operators are continuous and injective
\[
\mathcal{M}^{11}: \mathcal{X} \to \mathcal{Y}^{11}, \quad \mathcal{M}^{12}: \mathcal{X} \to \mathcal{Y}^{12}, \quad \mathcal{M}^{21}: \mathcal{X} \to \mathcal{Y}^{21}, \quad \mathcal{M}^{22}: \mathcal{X} \to \mathcal{Y}^{22}.
\]
Now we are in the position to analyse the invertibility of the operators $M^{11}$, $M^{12}$, $M^{21}$ and $M^{22}$.

**Theorem 8.3.** Operators (91)-(94) are continuously invertible.

To prove the invertibility of operator (91), let us consider BDIE system $M^{11}$ with an arbitrary right hand side $\mathcal{F}^{11}_s = \{\mathcal{F}^{11}_{s_1}, \mathcal{F}^{11}_{s_2}, \mathcal{F}^{11}_{s_3}\} \in \mathbb{V}^{11}$. Taking $S_1 = \partial \Omega_N$, $S_2 = \partial \Omega_D$ and

$$ F = \mathcal{F}^{11}_{s_1}, \quad \Psi = r_{\partial \Omega_N} T_a^+ \mathcal{F}^{11}_{s_1} - \mathcal{F}^{11}_{s_3}, \quad \Phi = r_{\partial \Omega_D} \gamma^+ \mathcal{F}^{11}_{s_1} - \mathcal{F}^{11}_{s_2} $$

in [7, Lemma 5.13], extended to a wider space, presented as [1, Lemma 6], we obtain that $\mathcal{F}^{11}_s$ can be represented as

$$ \mathcal{F}^{11}_s = \mathcal{P}_b \tilde{f}_s + V_b \Psi_s - W_b \Phi_s \in \Omega, $$

$$ \mathcal{F}^{11}_{s_2} = r_{\partial \Omega_D} \gamma^+ \mathcal{F}^{11}_{s_1} - \Phi_s, $$

$$ \mathcal{F}^{11}_{s_3} = r_{\partial \Omega_N} T_a^+ \mathcal{F}^{11}_{s_1} - \Psi_s, $$

where the triple

$$ (\tilde{f}_s, \Psi_s, \Phi_s) = C_{\partial \Omega_N, \partial \Omega_D} \mathcal{F}^{11}_s \in \mathbb{V} $$

is unique and the operator

$$ C_{\partial \Omega_N, \partial \Omega_D} : \mathbb{V}^{11} \to \mathbb{V} $$

is linear and continuous.

Applying Theorem 8.1 with

$$ \tilde{f} = \tilde{f}_s, \quad \Psi_0 = \Psi_s, \quad \Phi_0 = \Phi_s, \quad \Psi_0 = r_{\partial \Omega_N} \Psi_0, \quad \Phi_0 = r_{\partial \Omega_D} \Phi_0, $$

we obtain that the system $M^{11}$ is uniquely solvable and its solution is

$$ \mathcal{U}_1 = (A^{DN})^{-1} (\tilde{f}_s, r_{\partial \Omega_D} \Phi_s, r_{\partial \Omega_N} \Psi_s)^\top, \quad \mathcal{U}_2 = T_a^+ \mathcal{U}_1 - \Psi_s, \quad \mathcal{U}_3 = \gamma^+ \mathcal{U}_1 - \Phi_s $$

while $r_{\partial \Omega_N} \mathcal{U}_2 = 0$, $r_{\partial \Omega_D} \mathcal{U}_3 = 0$. Here $(A^{DN})^{-1}$ is the continuous inverse operator to the left-hand-side operator of the mixed BVP (59)-(61), $A^{DN} : H^1(\Omega) \to \mathbb{V}$, cf. [7, Corollary 5.16]. Representation (8), and continuity of operator (95) complete the proof for $M^{11}$.

To prove invertibility of operator (94), we apply similar arguments. Let us consider the BDIE system $M^{22}$ with an arbitrary right hand side $\mathcal{F}^{22}_s = \{\mathcal{F}^{22}_{s_1}, \mathcal{F}^{22}_{s_2}, \mathcal{F}^{22}_{s_3}\} \in \mathbb{V}^{22}$. Taking now $S_1 = \partial \Omega_D$, $S_2 = \partial \Omega_N$, $S_3 = \partial \Omega_N$,

$$ F = \mathcal{F}^{22}_{s_1}, \quad \Psi = r_{\partial \Omega_N} T_a^+ \mathcal{F}^{22}_{s_1} - \mathcal{F}^{22}_{s_2}, \quad \Phi = r_{\partial \Omega_D} \gamma^+ \mathcal{F}^{22}_{s_1} - \mathcal{F}^{22}_{s_3} $$

in [1, Lemma 6], we obtain that $\mathcal{F}^{22}_s$ can be represented as

$$ \mathcal{F}^{22}_{s_1} = \mathcal{P}_b \tilde{f}_s + V_b \Psi_s - W_b \Phi_s \in \Omega, $$

$$ \mathcal{F}^{22}_{s_2} = r_{\partial \Omega_D} \gamma^+ \mathcal{F}^{22}_{s_1} - \Phi_s, $$

$$ \mathcal{F}^{22}_{s_3} = r_{\partial \Omega_N} T_a^+ \mathcal{F}^{22}_{s_1} - \Psi_s, $$

where the triple

$$ (\tilde{f}_s, \Psi_s, \Phi_s) = C_{\partial \Omega_D, \partial \Omega_N} \mathcal{F}^{22}_s \in \mathbb{V} $$
is unique and the operator
\[
C_{\partial \Omega_N, \partial \Omega_D} : Y^{22} \to W
\]
is linear and continuous.

Applying now Theorem 8.1 with the same substitutions (96), we obtain that the system M22 is uniquely solvable and its solution is given by (97). Representation (98), and continuity of operator (99) complete the proof for M22.

To prove invertibility of operator (92), let us consider the BDIE system M12 with an arbitrary right hand side \( F_{12}^* = \{F_{12}^{*1}, F_{12}^{*2}\}^T \in Y^{12} \). Taking \( F = F_{s1}^{12}, \Phi = \gamma^+ F_{s1}^{12} - F_{s2}^{12} \) on \( \partial \Omega \) in [1, Corollary 3], we obtain the representation
\[
F_{s1}^{12} = P_b \tilde{f}_s + V_b \Psi_* - W_b \Phi_* \text{ in } \Omega,
\]
\[
F_{s2}^{12} = \tilde{\gamma}^+ F_{s1}^{12} - \Phi_* \text{ on } \partial \Omega,
\]
where the triple
\[
(f_*, \Psi_*, \Phi_*)^T = \tilde{C}_{\Phi_*} F_* \in W
\]
is unique and the operator
\[
\tilde{C}_{\Phi_*} : Y^{12} \to W
\]
is linear and continuous.

Applying Theorem 8.1 with substitutions (96), we obtain that the system M12 is uniquely solvable and its solution is given by (97). Representation (100), and continuity of operator (101) complete the proof for M12.

Finally to prove invertibility of operator (93), let us consider the BDIE system M21 with an arbitrary right hand side \( F_{s2}^{21} = \{F_{s1}^{21}, F_{s2}^{21}\}^T \in Y^{21} \). Taking \( F = F_{s1}^{21}, \Psi = T_a F_{s1}^{21} - F_{s2}^{21} \) on \( \partial \Omega \) in [1, Corollary 3], we obtain that
\[
F_{s1}^{21} = P_b \tilde{f}_s + V_b \Psi_* - W_b \Phi_* \text{ in } \Omega,
\]
\[
F_{s2}^{21} = T_a F_{s1}^{21} - \Psi_* \text{ on } \partial \Omega.
\]
where the triple
\[
(f_*, \Psi_*, \Phi_*)^T = \tilde{C}_{\Psi_*} F_* \in W
\]
is unique and the operator
\[
\tilde{C}_{\Psi_*} : Y^{21} \to W
\]
is linear and continuous. Applying Theorem 8.1 with substitutions (96), we obtain that the system M21 is uniquely solvable and its solution is given by (97). Representation (102), and continuity of operator (103) complete the proof for M21.
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References


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