Carleman estimates and unique continuation property for $N$-dimensional Benjamin-Bona-Mahony equations

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Abstract

We study the unique continuation property for the $N$-dimensional BBM equations using Carleman estimates. We prove that if the solution of this equation vanishes in an open subset, then this solution is identically equal to zero in the horizontal component of the open subset.

Keywords: $N$-dimensional BBM equations, Carleman estimates, UCP, Treves' inequality

1. Introduction

The purpose of this work is to prove a unique continuation property (UCP) of the solution of $N$-dimensional version of the Benjamin-Bona-Mahony (BBM) equation:

$$u_t - L_1 u_t + u_{x_1} + \beta L_2 u_{x_1} + uu_{x_1} = 0, \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \quad (1.1)$$

where $L_1 = \sum_{j=1}^k \partial^2_{x_{r_j}}$ and $L_2 = \sum_{i=1}^m \partial^2_{x_{l_i}}$ such that $\{x_{l_1}, \ldots, x_{l_m}\}, \{x_{r_1}, \ldots, x_{r_k}\} \subset \{x_1, \ldots, x_N\}$ with $m + k = N$. Two specific two-dimensional cases of (1.1) are the Zakharov-Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation

$$u_t + u_x - u_{xxt} + uu_x + u_{xyy} = 0, \quad (1.2)$$

and the 2D-BBM equation

$$u_t + u_x - u_{xxt} - u_{yyt} + uu_x = 0. \quad (1.3)$$

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See [6] and references therein. When $N = 1$ and $\beta = 0$, Equation (1.1) is the classical BBM equation

$$u_t + u_x - u_{xxt} + uu_x = 0. \quad (1.4)$$

The BBM equation, as an improvement of the KdV equation, was considered for modeling the one-dimensional long wavelength surface waves propagating in weakly nonlinear dispersive media, as well as the evolution of weakly nonlinear ion acoustic waves in plasmas [2]. Regarding the well-posedness issue for (1.4), Bona and Tzvetkov proved in [3] that the initial value problem associated with (1.4) is globally well-posed in the $L^2$-based Sobolev class $H^s(\mathbb{R})$ if $s \geq 0$. Moreover, the map that associates the relevant solution to given initial data is shown to be smooth. On the other hand, if $s < 0$, it was demonstrated that the correspondence between initial data and putative solutions cannot be even of class $C^2$. Hence, it was concluded that the BBM equation cannot be solved by iteration of a bounded mapping leading to a fixed point in $H^s$-based spaces for $s < 0$, so that the initial value problem for the BBM equation is not even locally well-posed in $H^s$ for negative values of $s$.

Zhang and Zuazua [12] proved the UCP for the linearized BBM equation with space-dependent potential in finite domain by using the spectral analysis and the generalized eigenvector expansion of the solution. Rosier and Zhang [9] showed the UCP for small data in $H^1(\mathbb{T})$ with nonnegative zero means. We are inspired by a work of Davila and Menzala [5] who proved a unique continuation property of the one-dimensional Benjamin-Bona-Mahony equation. Mammeri [7] generalized this result to the 2-dimensional Kadomtsev-Petviashili (KP) equations. Here we extend the unique continuation property to N-dimensional dispersive equations of BBM-type, including the ZK equations, but not KP. Even if the method seems similar, we remind that the transverse dispersion for the KP equations is $\partial_{x}^{-1}u_{yy}$, while it is $u_{xty}$ for ZK, and $u_{tyy}$ for BBM.

We assume that the initial value problem associated with (1.1) is locally well-posed in $H^s(\mathbb{R}^N)$ for some $s \geq s_0$. Then, we show that if $u = u(x,t)$ is solution of the equations and $u$ vanishes in an open subset $\Omega$ of $\mathbb{R}^N \times \mathbb{R}$, then $u$ vanishes identically on the horizontal component $\Omega_h := \{(x,t) \in \mathbb{R}^N \times \mathbb{R}; \exists x_1 \text{ with } (x_1,t) \in \Omega\}$ of $\Omega$. We note that the dispersion relation of (1.1) is $i\xi_1(1-\beta \sum_{j=1}^{m} \xi_j^2).$ Carleman estimates can be used. These estimates are based on estimates with
exponential weight for the solution of the equation. More precisely, if $u$ is solution of $Lu = Vu$, with $L$ a linear operator, $V$ a well-defined potential, the Carleman estimates is written, for $\Psi$ a convex function and $\tau > 0$ to choose,

$$||e^{\tau \Psi(x)} u|| \leq C ||e^{\tau \Psi(x)} Lu||.$$ 

Such a result was shown by Saut and Scheurer [10] for a general class of dispersive equations, including the Korteweg-de Vries one. An alternative approach was suggested by Bourgain [1]. The method here is based on an analytic continuation of the Fourier transform via the theorem of Paley-Wiener.

The next section is devoted to establish a Carleman estimate for the general equation (1.1) and then the unique continuation property. Finally we apply our result to prove a unique continuation property, then deduce the approximate controllability and the uniqueness of the inverse problem for the ZK-BBM equation (1.2) and the 2D-BBM equation (1.3).

2. Unique continuation property for the $N$-dimensional BBM equation

In this section, we obtain a Carleman estimate.

2.1. Carleman estimate

The Treves inequality is reminded [11].

**Theorem 2.1.** Let $P = P(D)$ be a differential operator of order $m$ with constant coefficients. Then for all $\alpha \in \mathbb{N}^n, \xi \in \mathbb{R}^n, \Phi \in C_0^\infty(\mathbb{R}^n)$, and for $\Psi(X, \xi) = \sum_{j=1}^n X_j^2 \xi_j^2$,

$$\frac{2^{\alpha}}{\alpha!} \xi^{2\alpha} \int_{\mathbb{R}^n} |P^{(\alpha)}(D)\Phi|^2 e^{\Psi(X, \xi)} dX \leq C(m, \alpha) \int_{\mathbb{R}^n} |P(D)\Phi|^2 e^{\Psi(X, \xi)} dX,$$

with $|\alpha| = \sum_{j=1}^n \alpha_j, \alpha! = \alpha_1! \cdots \alpha_n!$ and

$$C(m, \alpha) = \begin{cases} \sup_{|r+\alpha| \leq m} \left( \frac{r+\alpha}{\alpha} \right) & \text{if } |\alpha| \leq m \smallskip \0 & \text{if } |\alpha| > m \end{cases}.$$

This inequality is applied with $X = (x - \delta, t) \in \mathbb{R}^{N+1}, x = (x_1, \cdots, x_N), \xi = (\sqrt{\tau}, \cdots, \sqrt{\tau}, \sqrt{2\tau} \delta) \in \mathbb{R}^{N+1}$ and

$$\Psi(X, \xi) = 2\tau \left( \sum_{j=1}^N (x_j - \delta)^2 + \delta^2 t^2 \right).$$
to get the following corollary.

**Corollary 2.2.** Let $P = P(\partial_{x_1}, \ldots, \partial_{x_N}, \partial_t)$ be a differential operator of order $N + 1$ with constant coefficients. Then for all $\alpha \in \mathbb{N}^{N+1}, \delta > 0, \tau > 0, \Phi \in C_0^\infty(\mathbb{R}^{N+1})$ and $\Psi(x,t) = \frac{1}{2} \sum_{j=1}^N (x_j - \delta)^2 + \delta^2 t^2$, we have

$$\frac{2|\alpha| + \alpha_N \tau |\alpha| \delta^{2N+1}}{\alpha!} \int_{\mathbb{R}^{N+1}} |P(\alpha)(D)\Phi|^2 e^{2r\Psi} dx dt \leq C(N+1, \alpha) \int_{\mathbb{R}^{N+1}} |P(D)\Phi|^2 e^{2r\Psi} dx dt.$$

This enables us to prove a Carleman estimate for the solution of (1.1).

**Proposition 2.3.** Let us define

$$\mathcal{L} := \partial_t - L_1 \partial_t + bL_2 \partial_{x_1} + f(x,t)\partial_{x_1} + g(x,t) + \sum_{j=1}^N a_j \partial_{x_j}$$

$$+ \sum_{j=1}^k b_j L_1 \partial_{x_{r_j}} + \sum_{j=1}^m c_j L_1 \partial_{x_{c_j}}$$

where $b, a_j, b_j, c_j$ are constant in $\mathbb{R}$, $f,g \in L^\infty(\mathbb{R}^{N+1})$. Suppose that $\delta > 0$ and $B_\delta := \{(x,t) \in \mathbb{R}^{N+1}; |x|^2 + t^2 < \delta^2\}$. Then, there exists $C > 0$ such that for all $\Phi \in C_0^\infty(B_\delta)$, $\Psi(x,t) = \frac{1}{2} \sum_{j=1}^N (x_j - \delta)^2 + \delta^2 t^2$ and $\tau > 0$ satisfying

$$\|f\|^2_\infty + \|g\|^2_\infty \leq 1,$$

we have

$$\tau^3 \delta^2 \int_{B_\delta} |\Phi|^2 e^{2r\Psi} dx dt + \tau^2 \delta^2 \int_{B_\delta} |\nabla_x \Phi|^2 e^{2r\Psi} dx dt \leq C \int_{B_\delta} |\mathcal{L}\Phi|^2 e^{2r\Psi} dx dt \quad (2.1)$$

**Proof.** Let $P$ be the differential operator

$$P := -i \left( \partial_t - L_1 \partial_t + bL_2 \partial_{x_1} + \sum_{j=1}^N a_j \partial_{x_j} + \sum_{j=1}^k b_j L_1 \partial_{x_{r_j}} + \sum_{j=1}^m c_j L_1 \partial_{x_{c_j}} \right),$$

for which the Fourier transform is, for $(\xi, \tau) \in \mathbb{R}^N \times \mathbb{R},$

$$\hat{P}(\xi, \tau) = \tau + \tau \sum_{j=1}^k \xi_{r_j}^2 + \sum_{j=1}^m \xi_{c_j}^2 + \sum_{j=1}^N \xi_j + \sum_{j=1}^k b_j \xi_{r_j} + \sum_{j=1}^k b_j \xi_{r_j} + \sum_{q=1}^m c_q \xi_{c_q} \xi_{r_j}. $$

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Lemma 2.4. For all $\Phi \in C_0^\infty(B_\delta)$, $\Psi(x,t) = \delta^2 t^2 + \frac{1}{2} \sum_{j=1}^N (x_j - \delta)^2$ and $\tau > 0$, we have
\[
\tau^3 \delta^2 \int_{B_\delta} |\Phi|^2 e^{2r\Psi} \, dx \, dt + \tau^2 \delta^2 \int_{B_\delta} |\nabla_x \Phi|^2 e^{2r\Psi} \, dx \, dt \lesssim \int_{B_\delta} |P\Phi|^2 e^{2r\Psi} \, dx \, dt.
\]

Proof. If $\alpha$ is in $\mathbb{N}^{N+1}$ such that $r_1$-th entry is 2, the last entry is 1 and the rest entries are zero, then we obtain
\[
\frac{\partial \hat{P}^{(\alpha)}(\xi, \tau)}{\partial \xi_{r_1}^2 \partial \tau} = 2, \quad P^{(\alpha)} \Phi = 2\Phi
\]
and $C(N+1, \alpha) = 1$. Corollary 2.2 gives
\[
2 \tau^3 \delta^2 \int_{\mathbb{R}^{N+1}} |2\Phi|^2 e^{2r\Psi} \, dx \, dt \leq \int_{\mathbb{R}^{N+1}} |P\Phi|^2 e^{2r\Psi} \, dx \, dt.
\]

We deal similarly with $\alpha \in \mathbb{N}^{N+1}$ such that the $r_j$-th and the last entries are 1 with $j = 1, \cdots, k$, and the rest ones are zero to observe that
\[
\frac{\partial \hat{P}^{(\alpha)}(\xi, \tau)}{\partial \xi_{r_j} \partial \tau} = 2\xi_{r_j}, \quad P^{(\alpha)} \Phi = -2i \Phi_{x_{r_j}},
\]
and $C(N+1, \alpha) = 2$. Corollary 2.2 again implies
\[
\tau^2 \delta^2 \int_{\mathbb{R}^{N+1}} |\Phi_{x_{r_j}}|^2 e^{2r\Psi} \, dx \, dt \leq \int_{\mathbb{R}^{N+1}} |P\Phi|^2 e^{2r\Psi} \, dx \, dt
\]
for any $j = 1, \cdots, k$. Similar consideration for the $(N+1)$-tuple $\alpha$ such that the $l_j$-th and the $r_1$-th entries are 1 for any $j = 1, \cdots, k$, and the rest ones are zero, shows that for any $j = 1, \cdots, m$
\[
\tau^2 \delta^2 \int_{\mathbb{R}^{N+1}} |\Phi_{x_{l_j}}|^2 e^{2r\Psi} \, dx \, dt \leq \int_{\mathbb{R}^{N+1}} |P\Phi|^2 e^{2r\Psi} \, dx \, dt.
\]

\[\Box\]

Lemma 2.5. We have
\[
\int_{B_\delta} (|f \Phi_{x_1}|^2 + |g \Phi|^2) e^{2r\Psi} \, dx \, dt \\
\leq \frac{\|f\|^2_{L^\infty}}{\tau^2 \delta^2} + \frac{\|g\|^2_{L^\infty}}{2^{5+r^2} \delta^2} \left( \int_{B_\delta} |\Delta \Phi|^2 e^{2r\Psi} \, dx \, dt + \int_{B_\delta} (|f \Phi_{x_1}|^2 + |g \Phi|^2) e^{2r\Psi} \, dx \, dt \right).
\]
Proof. From Lemma 2.4, one gets
\[
\int_{B_\delta} |f(x,t)\Phi_x|^2 e^{2r\Psi} \, dx \, dt \leq \|f\|_\infty^2 \int_{B_\delta} |\Phi_x|^2 e^{2r\Psi} \, dx \, dt \\
\leq \|f\|_\infty^2 \frac{\|\Phi\|^2}{\tau^2 \delta^2} \int_{B_\delta} |P\Phi|^2 e^{2r\Psi} \, dx \, dt \\
\leq \|f\|_\infty^2 \frac{\|\Phi\|^2}{\tau^2 \delta^2} \int_{B_\delta} (|\mathcal{L}\Phi|^2 + |f\Phi_x|^2 + |g\Phi|^2) e^{2r\Psi} \, dx \, dt.
\]

(2.2)

On the other hand, we have
\[
\int_{B_\delta} |g(x,t)\Phi|^2 e^{2r\Psi} \, dx \, dt \leq \|g\|_\infty^2 \int_{B_\delta} |\Phi|^2 e^{2r\Psi} \, dx \, dt \\
\leq \|g\|_\infty^2 \frac{\|\Phi\|^2}{2^5 \tau^3 \delta^2} \int_{B_\delta} |P\Phi|^2 e^{2r\Psi} \, dx \, dt \\
\leq \|g\|_\infty^2 \frac{\|\Phi\|^2}{2^5 \tau^3 \delta^2} \int_{B_\delta} (|\mathcal{L}\Phi|^2 + |f\Phi_x|^2 + |g\Phi|^2) e^{2r\Psi} \, dx \, dt.
\]

(2.3)

To conclude Proposition 2.3, it is enough to choose \(\tau > 0\) large enough with \(\frac{\|f\|_\infty^2}{\tau^2 \delta^2} + \|g\|_\infty^2 \frac{\|\Phi\|^2}{2^5 \tau^3 \delta^2} \ll 1\).

By regularization, we can generalize the preceding Carleman estimate.

**Corollary 2.6.** Let \(T > 0\). If \(\Phi \in C([-T,T];H^3(\mathbb{R}^N)), \Phi_t \in C([-T,T];H^2(\mathbb{R}^N))\) and supp \(\Phi \subseteq B_\delta\), then inequality (2.1) remains true.

**2.2. Unique continuation property**

The unique continuation property is now proven. The proof is similar to the one-dimensional case of the paper of Davila and Menzala [5].

**Lemma 2.7.** Let \(T > 0, s \geq 4, f \in L^\infty(\mathbb{R}^N \times [-T,T]), c \in \mathbb{R}\). Assume that \(u \in C^1([-T,T];H^s(\mathbb{R}^N))\) is the solution of \(\mathcal{L}u = 0\). Assume that \(u \equiv 0\) when \(x < |t|\) and \(y < |t|\) in a neighborhood of \((0,0,0)\). Then there exists a neighborhood of \((0,0,0)\) in which \(u \equiv 0\).
Remark 2.8. If \( u \in C^1([-T,T]; H^s(\mathbb{R}^N)) \) is solution of \( Lu = 0 \), since \( u_t = -(1 - L_1)^{-1}(fu_{x_1} + \beta L_2 u_{x_1}) \), then \( u_t \in C([-T,T]; Z) \), where \( Z = \{ w \in L^2(\mathbb{R}^N); (I - L_1)(I - L_2)^{-1}(I + |\partial_{x_1}|)^{-1}w \in H^s(\mathbb{R}^N) \} \). Therefore the Carleman estimate (2.1) holds if \( s \geq 4 \).

Proof. Let \( 0 < \delta < 1 \), choose \( \chi \in C_0^\infty(B_\delta) \) such that \( \chi = 1 \) in a neighborhood \( \mathcal{O}_1 \) of the origin in \( \mathbb{R}^{N+1} \) and define \( \Phi := \chi u \). It follows that \( \Phi \in C([-T,T]; H^2(\mathbb{R}^N)), \Phi_t \in C([-T,T]; H^{s-2}(\mathbb{R}^N)) \) and \( \text{supp} \Phi \subseteq B_\delta \). We deduce thanks to the preceding corollary that there exists \( C > 0 \) such that, for \( \tau > 0 \) large enough,

\[
\tau^3 \delta^2 \int_{B_\delta} |\Phi|^2 e^{2^r \Psi} dx dt \leq C \int_{B_\delta} |\mathcal{L}\Phi|^2 e^{2^r \Psi} dx dt. \tag{2.4}
\]

The right hand side integral holds on \( B_\delta \setminus \mathcal{O}_1 \), since \( \mathcal{L}\Phi = 0 \) in \( \mathcal{O}_1 \). For \( (x,t) \neq 0 \) in \( \text{supp} \Phi \), we have for any \( j = 1, \ldots, N \) that \( 0 \leq |t| \leq x_j < \delta < 1 \) and

\[
\Psi(x,t) = \frac{1}{2} \sum_{j=1}^N (x_j - \delta)^2 + \delta^2 t^2 < (|t| - \delta)^2 + \delta^2 t^2 < \delta^2, \quad \Psi(0, \ldots, 0) = \delta^2.
\]

Then for \( (x,t) \in \text{supp} \mathcal{L}\Phi \subseteq B_\delta \), there exists \( 0 < \varepsilon < \delta^2 \) such that \( \Psi(x,t) \leq \delta^2 - \varepsilon \). On the other hand, we can choose \( \mathcal{O}_2 \) a neighborhood of \( (0, \ldots, 0) \) with \( \Psi(x,t) > \delta^2 - \varepsilon \) in \( \mathcal{O}_2 \). The inequality (2.4) is then written for all \( \tau > 0 \) large enough

\[
\tau^3 \delta^2 e^{2\tau(\delta^2 - \varepsilon)} \int_{\mathcal{O}_2} |\Phi|^2 dx dt \leq C e^{2\tau(\delta^2 - \varepsilon)} \int_{B_\delta \setminus \mathcal{O}_1} |\mathcal{L}\Phi|^2 dx dt.
\]

Tending \( \tau \) to infinity implies \( \Phi \) vanishes in \( \mathcal{O}_2 \). However \( u = \Phi \) in \( \mathcal{O}_2 \subseteq \mathcal{O}_1 \) and \( u \equiv 0 \) in \( \mathcal{O}_2 \). \( \square \)

Corollary 2.9. Let \( T > 0, s \geq 4, A, C \in L^\infty(\mathbb{R}^N \times [-T,T]) \) and \( B \in \mathbb{R} \). Suppose that \( u \in C^1([-T,T]; H^s(\mathbb{R}^N)) \) be solution of

\[
u_t - L_1 u_t + A(x,t) u_{x_1} + B L_2 u_{x_1} + C(x,t) u = 0.
\]

We consider the surface \( x = \mu(t) := (\mu_1(t), \ldots, \mu_N(t)), \mu(0) = (0, \ldots, 0) \), \( \mu \) a continuously differential function in a neighborhood of the origin in \( \mathbb{R}^{N+1} \). Assume that \( u \equiv 0 \) when \( x_j < \mu_j(t) \) for any \( j = 1, \ldots, N \) in a neighborhood of the origin. Then there exists a neighborhood of \( (0, \ldots, 0) \in \mathbb{R}^{N+1} \) in which \( u \equiv 0 \).
Proof. Since \( x_j < \mu_j(t) \) in a neighborhood of \((0, \cdots, 0) \in \mathbb{R}^{N+1} \) with \( \mu \) continuously differential and \( \mu(0) = (0, \cdots, 0) \), one can find real numbers \( \widetilde{\mu}_j \) such that \( x_j < \widetilde{\mu}_jt \) for any \( j = 1, \cdots, N \) in a neighborhood of the origin in \( \mathbb{R}^{N+1} \). We consider the change of variables \((x, t) \rightarrow (X, T)\) with

\[
X_j = x_j - \widetilde{\mu}_jt + |t|, \quad T = t.
\]

This change of variables provides \( U = U(X, T) \) satisfying \( U \equiv 0 \) when \( X_j < |T| \) in a neighborhood of \((0, \cdots, 0) \in \mathbb{R}^{N+1} \) and \( \mathcal{L}U = 0 \) with

\[
\mathcal{L} := \partial_T + \sum_{j=1}^{N} (-\widetilde{\mu}_j + \text{sgn}(T))\partial_{X_j} - \partial_T L_1 - \sum_{j=1}^{k} (-\widetilde{\mu}_j + \text{sgn}(T))\partial_{X_j} L_1
\]

\[
+ \sum_{j=1}^{m} (-\widetilde{\mu}_j + \text{sgn}(T))\partial_{X_{ij}}L_1 + A\partial_{X_1} + B\partial_{X_1}L_2 + C
\]

\[
= \partial_T - L_1\partial_t + bL_2\partial_{X_1} + f\partial_{X_1} + g + \sum_{j=1}^{N} a_j\partial_{X_j}
\]

\[
+ \sum_{j=1}^{k} b_jL_1\partial_{X_{ij}} + \sum_{j=1}^{m} c_jL_1\partial_{X_{ij}}.
\]

□

Theorem 2.10. Let \( T > 0, s \geq 4, s > N/2 \) and \( u \in C^1([-T, T]; H^s(\mathbb{R}^N)) \) be solution of equation (1.1). If \( u \equiv 0 \) in an open subset \( \Omega \subseteq \mathbb{R}^N \times [-T, T] \), then \( u \equiv 0 \) in the horizontal component of \( \Omega \).

Proof. The proof follows [8] or [5] applying the preceding corollary with \( A = (1 + u), B = \beta \) and \( C = 0 \). Thanks to the Sobolev embedding \( H^s(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \) with \( s > N/2 \), \( A \) is in \( L^\infty(\mathbb{R}^N \times [-T, T]) \).

3. Unique continuation property for the ZK-BBM and 2D-BBM equations

In this section, we apply Theorem 2.10 for the ZK-BBM and 2D-BBM equations. We first consider, for \((x, y) \in \mathbb{R}^2, t \in \mathbb{R}\), the initial value problem

\[
u_t + u_x - u_{xxt} + uu_x + u_{xyy} = 0 \quad u(x, y, 0) = u_0(x, y).
\]
We recall the well-posedness result [6]. Consider the spaces $Z^s \subset H^x \subset L^2(\mathbb{R}^2)$ with the norms
\[ \|u\|_{H^s} = \|u\|_{L^2(\mathbb{R}^2)} + \|u_x\|_{L^2(\mathbb{R}^2)}, \text{ and } \|u\|_{Z^s} = \|\langle |\xi| + |\eta| \rangle^{s} \hat{u}(\xi, \eta)\|_{L^2(\mathbb{R}^2)}, \]
where for $a \in \mathbb{R}$, $\langle a \rangle = (1 + a^2)^{1/2}$ and $\hat{u}$ is the Fourier transform. In addition, for $(b, s) \in \mathbb{R}^2$, we define the space $X^{b,s}$ equipped with the norm
\[ \|u\|_{X^{b,s}} = \left\| \langle \tau - p(\xi, \eta) \rangle^b \langle |\xi| + |\eta| \rangle^s \hat{u}(\tau, \xi, \eta) \right\|_{L^2(\mathbb{R}^3)}, \]
(3.1)
where $p(\xi, \eta) = \frac{\xi + \xi \eta^2}{1 + \xi^2}$.

**Theorem 3.1.** For any $u_0 \in H_x$, there exists $u \in C(\mathbb{R}; H_x)$ which solves (1.2) with $u(0, x, y) = u_0(x, y)$ such that $\|u(t)\|_{H^x} = \|u_0\|_{H_x}$ for all $t \in \mathbb{R}$. Furthermore, there exists some $b > 1/2$ such that $u \in X^{b,0}$ and $u$ is unique in the class $X^{b,0}$. Moreover, if $s > 0$, then the map $u_0 \mapsto u(t)$ takes $Z^s$ to $X^{b,s}$ continuously.

Next, we consider, for $(x, y) \in \mathbb{R}^2, t \in \mathbb{R}$, the initial value problem
\[ u_t + u_x - u_{xxt} - u_{yy} + uu_x = 0 \]
\[ u(x, y, 0) = u_0(x, y). \]

**Theorem 3.2.** [6] For any $u_0 \in H^s(\mathbb{R}^2)$ with $s \geq 0$, there exists a $T = T(u_0) > 0$ and a unique solution $u \in C(\mathbb{R}; H^s(\mathbb{R}^2))$ of (1.3) with $u(0) = u_0$.

The following result is a consequence of Theorem 2.10, if we consider $\beta = 1$, $L_1 = \partial_x^2$ and $L_2 = \partial_y^2$ for (1.2), and $\beta = 0$ and $L_1 = \partial_x^2 + \partial_y^2$ for (1.3).

**Theorem 3.3.** Let $T > 0, s \geq 4$ and $u \in C([-T, T]; H^s(\mathbb{R}^2))$ solution of the ZK-BBM or 2D-BBM equations. If $u \equiv 0$ in an open subset $\Omega \subset \mathbb{R}^2 \times [-T, T]$, then $u \equiv 0$ in the horizontal component of $\Omega$.

4. Approximate controllability for the ZK-BBM and 2D-BBM equations

In this section, we look for an internal control $g = g(x, y, t)$ defined in the domain $\omega \subset \mathbb{R}^2$ to adjust the behavior of the solutions of the ZK-BBM equation
\[ u_t + u_x - u_{xxt} + uu_x + u_{xyy} + 1_w g = 0 \]
\[ u(x, y, 0) = u_0(x, y), \]
\[ L^2(\mathbb{R}^2) \]
and the 2D-BBM equation
\[
\begin{align*}
 u_t + u_x - u_{xxt} - u_{yyt} + uu_x + 1_w g &= 0 \\
 u(x,y,0) &= u_0(x,y).
\end{align*}
\]

The control problem of dispersive equations, including BBM, in a finite length interval has been extensively studied in the recent years (see [9] and references within). Here we are interested in a control problem associated to dispersive equation posed on the real domain. Using the unique continuation property of ZK-BBM and 2D-BBM, we can deduce their approximate controllability for any time \( T > 0 \).

**Theorem 4.1.** Let \( T > 0, s \geq 4 \) and \( u_0, u_T \in H^s(\mathbb{R}^2) \). Then for all \( \varepsilon > 0 \), there exists \( g \in L^2(\mathbb{R}^2 \times [-T,T]) \) such that the solution \( u \) of 2D-BBM, respectively of ZK-BBM equations, satisfies
\[
|| u(T) - u_T ||_{H^s} \leq \varepsilon.
\]

**Proof.** The approximate controllability is equivalent to the unique continuation property of the adjoint equation. Concerning the 2D-BBM equation, the adjoint equation reads as
\[
\begin{align*}
 v_t + v_x - v_{xxt} - v_{yyt} + \frac{u}{2} v_x &= 0 \\
 v(x,y,T) &= v_T(x,y),
\end{align*}
\]
while the adjoint equation of the ZK-BBM equation is
\[
\begin{align*}
 v_t + v_x - v_{xxt} + v_{xyy} + \frac{u}{2} v_x &= 0 \\
 v(x,y,T) &= v_T(x,y).
\end{align*}
\]
Since \( u \in C([-T,T];H^s(\mathbb{R}^2)) \) and according to Sobolev’s embedding with \( s \geq 4 \), Theorem 2.10 gives the result.

\( \Box \)

5. Uniqueness for the ZK-BBM and 2D-BBM inverse problems

Another direct consequence, of the unique continuation property of ZK-BBM and 2D-BBM, is the uniqueness of inverse problems. We aim at studying the inverse problem that consists in finding the functions \((u, g)\) solution of the inverse ZK-BBM problem
\[
\begin{align*}
 u_t + u_x - u_{xxt} + uu_x + u_{xyy} &= 0 \\
 u(x,y,t) &= g(x,y,t) \text{ in } \Omega \subseteq \mathbb{R}^2 \times [-T,T],
\end{align*}
\]
and the 2D-BBM equation respectively

\[ u_t + u_x - u_{xxt} - u_{ygt} + uu_x = 0 \]
\[ u(x, y, t) = g(x, y, t) \text{ in } \Omega \subseteq \mathbb{R}^2 \times [-T, T]. \]

**Theorem 5.1.** Let \( g, \tilde{g} \) in \( L^\infty([-T, T]; L^2(\mathbb{R}^2)) \). Assume for \( s \geq 4 \), there exist a solution \( u \) and \( \tilde{u} \in \mathcal{C}([-T, T]; H^s(\mathbb{R}^2)) \) respectively. If \( u = \tilde{u} \) in \( \Omega \subset \mathbb{R}^2 \times [-T, T] \), then \( g \equiv \tilde{g} \) and \( u \equiv \tilde{u} \).

**Proof.** Let \( g, \tilde{g} \) and \( u, \tilde{u} \) the corresponding solution. Then \( w = u - \tilde{u} \) is solution of the 2D-BBM like equation

\[ w_t + w_x - w_{xxt} - w_{ygt} + uw_x + \tilde{u}_x w = 0 \]
\[ w = 0 \text{ in } \Omega. \]

Since \( u \in \mathcal{C}([-T, T]; H^s(\mathbb{R}^2)), u_x \in \mathcal{C}([-T, T]; H^{s-1}(\mathbb{R}^2)) \) and according to Sobolev’s embedding with \( s \geq 4 \), the result follows corollary 2.9 with \( A = (1 + u), B = 1, C = u_x \).

We deal with similarly for the ZK-BBM equation where the difference \( w = u - \tilde{u} \) is solution of

\[ w_t + w_x - w_{xxt} + w_{xyy} + uw_x + \tilde{u}_x w = 0 \]
\[ w = 0 \text{ in } \Omega. \]

\[ \square \]

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**References**


