A fast multiscale Galerkin method for solving a boundary integral equation in a domain with corners

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Abstract

In this paper a fast multiscale Galerkin method is proposed for solving the boundary integral equation derived from the Dirichlet problem of Laplace equation in a domain with corners. It is well known that the integral operator in the equation can be split into two operators, one is non-compact, the other is compact. We design two truncation strategies for the representation matrices of these operators, respectively, which compress these two dense matrices to sparse ones having only \(O(2^n)\) number of nonzero entries, where \(2^n\) is the number of the wavelet basis functions used in the method. We prove that the proposed truncation strategies do not ruin the stability and convergence rate of the integral equation. Numerical experiments are presented to verify the theoretical results and demonstrate the effectiveness of the method.

Key words: fast Galerkin methods, boundary integral equation, domains with corners

AMS subject classifications: 65M38, 65R20, 45L05

1 Introduction

We consider in this paper solving the boundary integral equation for the Dirichlet problem of Laplace equation in a domain with corners. It is well known that the solution of Laplace equation can be represented as double layer potential on the boundary of a domain, where the density function of the potential satisfies the second kind of Fredholm boundary integral equation. There have been extensive researches on the boundary integral equations defined on a smoothing boundary both theoretically and computationally, see [1, 24] and the references therein. In those cases, the integral operators are compact and can be represented by sparse matrices. This leads to various efficient fast numerical methods for solving the boundary integral equations [10].

When the boundary of a domain has corners, the solution of integral equation exhibits singular behaviors, and the integral operator are not compact. This yields only poor convergence for solving the boundary integral equations numerically, even if the data of the boundary condition is enough smooth. There have been many attempts to overcome the difficulty caused by the singularity of the solution (see [3, 4, 15, 16, 19, 23, 28] and the references therein). To avoid the effects of the ill-conditioning due to the increase of the local order and the decrease of the length of the sub interval near the corner, a compression scheme with preconditioning is employed in [3, 4, 19] which produces well conditioned linear systems. This allows us to solve boundary integral equations on a large-scale domain with corners, rapidly. Alternatively, a Nyström method based on the Gauss Lobatto quadrature rule has been introduced in [15] and further developed in [16]. In this Nyström method, the singular behavior

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of the solution can be improved by a change of variables. And the resulting linear system is well conditioned.

For the sparse approximation scheme for dense matrices, there are several truncation strategies proposed for constructing sparse approximations of compact integral operators [7, 10, 14, 21]. Those strategies truncate the dense matrices to sparse ones and retain enough entries to represent critical information encoded in the original matrices such that the algorithms have the nearly linear complexity. In [7, 21], the truncation strategies were presented in developing the fast Fourier-Galerkin algorithms for solving the boundary integral equations defined on a smooth boundary. There are the truncation strategies in developing the fast multi-scale Galerkin and collocation methods for solving Fredholm integral equations (see [10, 14]). All these truncation strategies are designed for compact integral operators with smooth or weakly singular kernels, and has showed the computing advantages for solving the integral equations. However, for solving the boundary integral equation defined on a curve with corners, the integral operator may be not compact. The literature has relatively few about the truncation strategy for non-compact integral operators.

In this paper, we propose a truncation strategy for the non-compactness boundary integral operator involved in the boundary integral equation on a domain with corners. To this aim, we decompose this integral operator into a non-compact integral operator \( L \) and a compact operator \( M \), where operator \( L \) can be characterized by the Mellin convolution operator, and the operator \( M \) is a compact operator with a smooth kernel. We observe that the high order partial derivatives of the kernel function of \( L \) have algebra decay with increasing of the distance from singularity points. This means the representing matrix of operator \( L \) in wavelet bases has numerical sparsity. Meanwhile, we also prove that the kernel of \( M \) has high order continuous partial derivatives. These facts lead us to truncate the representing matrices of operators \( L \) and \( M \) into sparse matrices by introducing suitable truncation strategies such that the proposed algorithm preserving the optimal convergence order has only \( O(2^n) \) number of nonzero entries.

This paper is organized in five sections. In section 2, we describe the decomposition of the integral operator in the resulting boundary integral equation. Specifically, we write the integral operator as the sum of the Mellin convolution operator \( L \) and the compact operator \( M \). Then, we prove the existing of the solution in this section. In section 3, we propose the fast wavelet Galerkin method by introducing two truncation strategies for compressing the representing matrices of operators \( L \) and \( M \). Then, we prove that the number of nonzero entries of the compressed coefficients matrices is \( O(2^n) \). In section 4, we study the high partial derivatives of the kernels of \( L \) and \( M \), and prove proposed algorithm preserving the optimal convergence order. Numerical experiments are presented in section 5 to verify the theoretical estimates established for the proposed algorithm.

## 2 Boundary integral equations with corners

It is well known that Dirichlet boundary value problems of Laplace equation can be formulated as second kind integral equations. In the case of a simply connected bounded domain with corners in \( \mathbb{R}^2 \), the solution of the interior problem

\[
\begin{cases}
\Delta u(P) = 0, & P \in D, \\
u(P) = g(P), & P \in \partial D,
\end{cases}
\]  

(2.1)

can be presented as a double layer potential

\[
u(P) = \int_{\partial D} h(Q) \frac{\partial}{\partial n_Q} (\log |P - Q|) \, ds_Q, \quad P \in D,
\]  

(2.2)
where $g$ is a given sufficiently smooth function on the boundary $\partial D$ of domain $D$, $\partial D$ is at least twice continuously differentiable with the exception of corners, $\mathbf{n}_Q$ denotes the outer unit normal vector to the boundary $\partial D$ at point $Q$, and $h$ is the so-called double layer density function. Since the limit
\[
\lim_{A \to P, A \in D} \int_{\partial D} h(Q) \frac{\partial}{\partial \mathbf{n}_Q} (\log |A - Q|) \, ds_Q = (-2\pi + \Omega(P)) h(P) + \int_{\partial D} h(Q) \frac{\partial}{\partial \mathbf{n}_Q} (\log |P - Q|) \, ds_Q,
\]
h can be determined by solving the integral equation (see [1])
\[
(-2\pi + \Omega(P)) h(P) + \int_{\partial D} h(Q) \frac{\partial}{\partial \mathbf{n}_Q} (\log |P - Q|) \, ds_Q = g(P), \quad P \in \partial D, \tag{2.3}
\]
where $\Omega(P)$ denotes the interior angle to $\partial D$ at $P$. Here, in order to simplify the presentation, we consider the case where the boundary $\partial D$ has only one corner at point $P_0$. The value of $\beta := \Omega(P_0)$ is in $(0, 2\pi)$.

Define an operator
\[
(Kh)(P) := \left( -1 + \frac{1}{\pi} \Omega(P) \right) h(P) + \frac{1}{\pi} \int_{\partial D} h(Q) \frac{\partial}{\partial \mathbf{n}_Q} (\log |P - Q|) \, ds_Q, \quad P \in \partial D.
\]
Then, equation (2.3) can be rewritten into the form as the boundary integral equation of the second kind
\[
(I - K)h = -\frac{1}{\pi} g, \tag{2.4}
\]
where $I$ is the identity operator. Suppose the boundary $\partial D$ can be described by the arc length parametrization $\Gamma(t) := (\xi(t), \eta(t)), t \in I := [0, 1]$, where $\Gamma(0) = \Gamma(1) = P_0$ and where $\Gamma(t)$ is defined componentwise. For $k \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, denote $\Gamma^{(k)}(t) := (\xi^{(k)}(t), \eta^{(k)}(t)), t \in I$, where $\xi^{(k)}$ and $\eta^{(k)}$ are the $k$th order derivative of $\xi$ and $\eta$ respectively, and $\Gamma^{(k)}(0)$ and $\Gamma^{(k)}(1)$ are the $k$th order right and left derivatives at point $t = 0$ and $t = 1$, respectively. For a vector $(x, y) \in \mathbb{R}^2$, we define $(x, y)_{\perp} := (y, -x)$. We also define
\[
K(s, t) := \begin{cases}
\frac{1}{\pi} \left( \frac{\Gamma(t) - \Gamma(s) \cdot (\Gamma'(t))_{\perp}}{|\Gamma(t) - \Gamma(s)|^2} \right), & s \neq t \text{ and } (s, t) \notin \{(0, 1), (1, 0)\}, \\
\frac{1}{\pi} \left( \frac{\Gamma''(t) \cdot (\Gamma'(t))_{\perp}}{2|\Gamma'(t)|^2} \right), & s = t.
\end{cases}
\]
Since $\Omega(P) = \pi$ for $P \in \partial D \setminus \{P_0\}$, operator $K$ satisfies
\[
K \varphi(s) = \int_I K(s, t) \varphi(t) \, dt, \quad s \in I \text{ and } \varphi \in L^2(I),
\]
where $L^2(I)$ is used to denote the space of square integrable functions on $I$ with norm $\|\varphi\| := \left( \int_0^1 \varphi^2(t) \, dt \right)^{\frac{1}{2}}$. Let $f(t) := -\frac{1}{\pi} g(\xi(t), \eta(t))$ and $g(t) := h(\xi(t), \eta(t))$ for all $t \in I$. Then, equation (2.4) is rewritten as
\[
(I - K)g = f. \tag{2.5}
\]

It is well known (see [1, 23]) that operator $K$ is bounded from $C(\partial D)$ to $C(\partial D)$, but not compact, due to the corner in $\partial D$. Fortunately, operator $K$ can be split into a compact operator and a Mellin convolution with norm less than 1 (see [1, 23]). Hence, the existence and uniqueness of the solution
still can be established based on Fredholm alternative theorem (see Section 8.1.4 in [1]). Specifically, the kernel of the Mellin convolution is the sum of the following two functions

\[
L_0(s, t) := \begin{cases} \frac{\Gamma''(1) \cdot (\Gamma''(0))^{1/2}(s - 1)}{\pi|\Gamma'(0)t - \Gamma'(1)(s - 1)|^2}, & (s, t) \in S_0 \left( \frac{1}{2} \right), \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
L_1(s, t) := \begin{cases} \frac{\Gamma'(0) \cdot (\Gamma'(1))^1s}{\pi|\Gamma'(1)(t - 1) - \Gamma'(0)s|^2}, & (s, t) \in S_1 \left( \frac{1}{2} \right), \\ 0, & \text{otherwise}, \end{cases}
\]

where for \(0 < \varepsilon \leq 1\), \(S_0(\varepsilon) := \{(s, t) \in I^2 : 1 - \varepsilon < s \leq 1, 0 \leq t < \varepsilon\}, (s, t) \neq (1, 0)\} and \(S_1(\varepsilon) := \{(s, t) \in I^2 : 0 \leq s < \varepsilon, 1 - \varepsilon < t \leq 1, (s, t) \neq (0, 1)\}\). It was shown in [1, 23] that the operator norm of the Mellin convolution equipped kernel \(L_0 + L_1\) is less than 1, and the integral operator equipped kernel \(K - L_0 - L_1\) is compact on \(L^2(I)\).

Meanwhile, due to the corner in \(\partial D\), the solution \(q\) has a singularity (see [23]) even if function \(f\) is smooth. Specifically, \(q\) satisfies that for \(t \in (0, 1)\) and \(q \in \mathbb{N}_0\),

\[
|q^{(q)}(t)| \leq c(t(1 - t))^{\frac{p-1}{p-2}},
\]

where \(c\) is a positive constant being independent of \(t\) and \(q\). For dealing with this singularity, Kress introduced a smooth change of variable in [23], which make the solution smoother than the original one. We next show the smooth change of variable and the details of the splitting of \(K\) in this paper.

In this paper, we select \(0 < \varepsilon < \frac{1}{2}\) and \(p\) be a integer with \(p \geq 2\). Let \(\gamma\) be a smooth monotonic function with continuous derivative up to order \(p\) on \(I\), and satisfy

\[
\gamma(s) := \begin{cases} sp, & s \in [0, \varepsilon], \\ 1 - (1 - s)^p, & s \in [1 - \varepsilon, 1]. \end{cases}
\]

We also request for all \(s \in I\), \(\gamma(s) + \gamma(1-s) = 1\). For a function \(\varphi\) of two variables defined on \(I^2\), we define function \(\varphi \circ \gamma\) by \((\varphi \circ \gamma)(s, t) := \varphi(\gamma(s), \gamma(t))\). For simplicity, the same notations are employed to denote the kernels of \(K\) and Mellin convolution, that is \(K(s, t) := \gamma'(t)(K \circ \gamma)(s, t)\), \(L_0(s, t) := \gamma'(t)\gamma(0)(L_0 \circ \gamma)(s, t)\) and \(L_1(s, t) := \gamma'(t)(L_1 \circ \gamma)(s, t)\). Also denote \(f(t) := f(\gamma(t))\) and define \(\rho(t) := g(\gamma(t))\) for \(t \in I\).

To ensure the smoothness of kernel \(K - L_0 - L_1\), a smooth cut-off function \(\chi\) defined on \(I\) is required. In this paper, \(\chi\) is requested to have continuous derivatives up to order \(p\) on \(I\). Moreover, \(\chi\) is required to satisfy \(0 \leq \chi(t) \leq 1\) and

\[
\chi(t) = \begin{cases} 1, & t \in [0, \varepsilon/2], \\ 0, & t \in [\varepsilon, 1]. \end{cases}
\]

Let \(\chi_0(t) := \chi(t)\) and \(\chi_1(t) := \chi(1-t)\) for \(t \in I\). For \(s, t \in I\), define \(\tilde{L}_0(s, t) := \chi_1(s)L_0(s, t)\chi_0(t)\), and \(\tilde{L}_1(s, t) := \chi_0(s)L_1(s, t)\chi_1(t)\). Let \(\mathcal{L}^0\) and \(\mathcal{L}^1\) be two integral operators defined by

\[
(\mathcal{L}^0 \varphi)(s) := \int_0^1 \tilde{L}_0(s, t)\varphi(t)dt, \quad (\mathcal{L}^1 \varphi)(s) := \int_0^1 \tilde{L}_1(s, t)\varphi(t)dt, \quad \varphi \in L^2(I) \text{ and } s \in I.
\]

Define \(\mathcal{L} := \mathcal{L}^0 + \mathcal{L}^1\). Also define

\[
M(s, t) := \begin{cases} K(s, t) - \tilde{L}_0(s, t) - \tilde{L}_1(s, t), & (s, t) \in I^2 \setminus \{(0, 1), (1, 0)\}, \\ 0, & (s, t) \in \{(0, 1), (1, 0)\}.
\end{cases}
\]
and $\mathcal{M}$ be an integral operator with kernel $M$, i.e.,

$$(\mathcal{M}\varphi)(s) := \int_0^1 M(s,t)\varphi(t)dt, \quad \varphi \in L^2(I) \text{ and } s \in I.$$  

Let $\mathcal{S} := \mathcal{I} - \mathcal{L}$. The equation (2.5) can be rewritten into the following form

$$(\mathcal{S} - \mathcal{M}) \rho = f.$$  

It is well-known that the solution of equation (2.5) exists and is unique. To ensure the change of variable $\gamma$ and the cut-off function $\chi$ do not ruin the existence and uniqueness of the solution, we next establish the existence and uniqueness of the solution of (2.6). To this aim, we show here that the norm of $\mathcal{L}$ is less than 1 by the following technical lemmas. This ensures that operator $\mathcal{S}$ is invertible. Then, by noting that operator $\mathcal{M}$ is compact (see [1, 23]), we establish the existence and uniqueness of the solution of (2.6) by Fredholm alternative theorem.

The $\gamma(s) + \gamma(1-s) = 1$ and monotonicity of $\gamma$ ensure that $\gamma$ is a bijective from $[0, \frac{1}{2}]$ to $[0, \frac{1}{2}]$, and also a bijective from $[\frac{1}{2}, 1]$ to $[\frac{1}{2}, 1]$. Then, by noting that $\Gamma'(0) \cdot \Gamma'(1) = -\cos \beta$ and $\Gamma'(0) \cdot (\Gamma'(1))^\perp = \Gamma'(0) \cdot (\Gamma'(1))^\perp = \sin \beta$, we can rewritten functions $L_0$ and $L_1$ as

$$L_0(s,t) = \frac{\gamma'(t)}{\gamma(t)} k \left( \frac{(1-s)}{\gamma(t)} \right),$$  

and

$$L_1(s,t) = \frac{\gamma'(1-t)}{\gamma(1-t)} k \left( \frac{\gamma(s)}{\gamma(1-t)} \right),$$  

where $k(t) := \frac{ts \sin \beta}{\pi(1-2t \cos \beta + t^2)}$. This leads us to study the property of function $k$ in order to estimate the up boundary of $\mathcal{L}$. As a preparation, we compute an integration in the following lemma.

**Lemma 2.1** There holds

$$\int_0^{+\infty} \frac{pk(y^p)}{y^{1/2}} \, dy = \frac{\sin \frac{\pi - \beta}{2p}}{\sin \frac{\pi}{2p}}.$$  

**Proof:** In this lemma, we denote imaginary unit by $i$. Changing variable $y = t^2$ yields to

$$\int_0^{+\infty} \frac{pk(y^p)}{y^{1/2}} \, dy = \frac{2p \sin \beta}{\pi} \int_0^{+\infty} \frac{t^{2p}}{1 - 2t^{2p} \cos \beta + t^{4p}} \, dt.$$  

It is well-known that polynomial $1 - 2t^{2p} \cos \beta + t^{4p}$ has $4p$ complex roots, $t_{1,j} := e^{i(t_{2,j} + \frac{\beta}{2p})}$ and $t_{2,j} := e^{i(t_{2,j} + \frac{2\pi - \beta}{2p})}$, $j = 0, \ldots, 2p-1$. Note that $\text{Im}(t_{1,j}) > 0$ and $\text{Im}(t_{2,j}) > 0$ if and only if $0 \leq j \leq p-1$. Thus, according to residue theorem, there holds that

$$\int_0^{+\infty} \frac{t^{2p}}{1 - 2t^{2p} \cos \beta + t^{4p}} \, dt = \frac{\pi i}{4p} \sum_{j=0}^{p-1} \left[ t_{1,j} \left( 1 - \cos \beta \right) + t_{2,j} \left( 1 - \cos \beta \right) \right]$$  

$$= \frac{\pi i}{4p} \sum_{j=0}^{p-1} e^{i\frac{\pi}{2p}} \left[ e^{i\beta} - \cos \beta + e^{i\frac{2\pi - \beta}{2p}} - \cos \beta \right]$$  

$$= \frac{-\pi i \sin \frac{\pi - \beta}{2p} e^{i\frac{\pi}{2p}}}{2p \sin \beta} \sum_{j=0}^{p-1} e^{i\frac{\pi}{2p}}$$  

$$= \frac{-\pi i \sin \frac{\pi - \beta}{2p} e^{i\frac{\pi}{2p}}}{p \sin \beta (1 - e^{i\frac{\pi}{2p}})} = \frac{\pi \sin \frac{\pi - \beta}{2p}}{2p \sin \beta \sin \frac{\pi}{2p}}.$$  

(2.10)
Combining (2.9) with (2.10), we obtain the desired result.

\[\Box\]

**Lemma 2.2** Operator \(L\) is bounded from \(L^2(I)\) to \(L^2(I)\), and \(\|L\| < 1\).

**Proof:** We first estimate the norm of operator \(L^0\). Note that \(\text{supp}(\chi_0) \subset [0, \epsilon], \text{supp}(\chi_1) \subset [1 - \epsilon, 1]\) and \(0 \leq \chi_1(s) \leq 1\) for all \(s \in [1 - \epsilon, 1]\). From (2.7) and the definition of \(\gamma\), we have that for all \(\varphi \in L^2(I)\),

\[
\|L^0\varphi\| \leq \left( \int_1^{r_0} \left( \int_0^1 \left| p \frac{1}{t} \left( \frac{(1 - s)^p}{y^p} \right) \chi_0(t) \varphi(t) dt \right|^2 dt \right)^{1/2} ds \right)^{1/2}.
\]

Changing variable \(y := \frac{1 - s}{t}\) in the above inequality and using Minkowski inequality lead to

\[
\|L^0\varphi\| \leq \left( \int_0^{1+\infty} \left| p \frac{k(y^p)}{y^{1/2}} \right| \left( \int_0^{1/y} \left( \frac{1 - s}{y} \right) \varphi \left( \frac{1 - s}{y} \right) \right)^2 ds \right)^{1/2} dy,
\]

where the domain of functions \(\chi_0\) and \(\varphi\) are extended to \([0, +\infty)\), and \(\chi_0 \left( \frac{1 - s}{y} \right) = \varphi \left( \frac{1 - s}{y} \right) = 0\) if \(\frac{1 - s}{y} > 1\). Changing variable \(t := \frac{1 - s}{y}\) in the above inequality leads to

\[
\|L^0\varphi\| \leq \left( \int_0^{1+\infty} \left| p \frac{k(y^p)}{y^{1/2}} \right| \left( \int_0^{1/y} \chi_0^2(t) \varphi^2(t) dt \right)^{1/2} dy.
\]

Since \(\epsilon < 1/2\) and \(\text{supp}(\chi_0) \subset [0, \epsilon]\), there holds that \(\left( \int_0^{1/y} \chi_0^2(t) \varphi^2(t) dt \right)^{1/2} \leq \|\chi_0\varphi\|\). Thus, from the above inequality, we have that

\[
\|L^0\varphi\| \leq \|\chi_0\varphi\| \int_0^{1+\infty} \frac{p|k(y^p)|}{y^{1/2}} dy. \quad (2.11)
\]

Equalities (2.7) and (2.8) implies that \(L_1(s, t) = L_0(1 - s, 1 - t)\). Thus, there holds that for \(s \in I, (L^1\varphi)(s) = (L^0\varphi)(1 - s)\), where \(\varphi(\cdot) = \varphi(1 - \cdot)\). Thus, from (2.11) we obtain that

\[
\|L^1\varphi\| \leq \|\chi_1\varphi\| \int_0^{1+\infty} \frac{p|k(y^p)|}{y^{1/2}} dy. \quad (2.12)
\]

Since \(\epsilon < 1/2\), there have \(\text{supp}(\chi_0) \cap \text{supp}(\chi_1) = \emptyset\) and \(\text{supp}(L^0\varphi) \cap \text{supp}(L^1\varphi) = \emptyset\). This means that \(\|L\varphi\|^2 = \|L^0\varphi\|^2 + \|L^1\varphi\|^2\) and \(\|\chi_0\varphi + \chi_1\varphi\|^2 = \|\chi_0\varphi\|^2 + \|\chi_1\varphi\|^2\). Hence, from (2.11), (2.12) and the definition of \(L\), we know that

\[
\|L\varphi\|^2 \leq \|\chi_0\varphi + \chi_1\varphi\|^2 \left( \int_0^{1+\infty} \frac{p|k(y^p)|}{y^{1/2}} dy \right)^2. \quad (2.13)
\]

Further, noting for all \(t \in [0, 1], 0 \leq \chi_0(t) + \chi_1(t) \leq 1\), we have that \(\|\chi_0\varphi + \chi_1\varphi\|^2 \leq \|\varphi\|^2\). Therefore, we obtained the inequality of this lemma from (2.13) and Lemma 2.1. 

With the result shown in Lemma 2.2, we investigate the existence and uniqueness of the solution of (2.6) in the following theorem.
Theorem 2.3 If $S - \mathcal{M}$ is injective from $L^2(I)$ to $L^2(I)$, then operator $I - S^{-1}\mathcal{M}$ has a bounded inverse on $L^2(I)$. Further, for all $f \in L^2(I)$, boundary integral equation (2.6) has a unique solution in $L^2(I)$ and the solution depends continuously on $f$.

Proof: Lemma 2.2 ensures that the inverse of operator $S$ exists and is bounded on $L^2(I)$ by Theorem 2.14 in [22]. Thus, solving equation (2.6) equals to solve

$$ (I - S^{-1}\mathcal{M}) \rho = S^{-1}f. \quad (2.14) $$

Since the kernel of $\mathcal{M}$ is continuous (see Theorem 4.10), operator $S^{-1}\mathcal{M}$ is compact. Thus, by the Fredholm alternative theorem, operator $I - S^{-1}\mathcal{M}$ has a bounded inverse on $L^2(I)$ since operator $S - \mathcal{M}$ is injective. Therefore, equation (2.14) is uniquely solvable for all $f \in L^2(I)$, and the solution depends continuously on $f$. \hfill $\square$

3 The fast multiscale wavelet Galerkin method

Since the kernel of $\mathcal{L}$ is smooth enough except at points $(0,1)$ and $(1,0)$, and the kernel of $\mathcal{M}$ is continuous, the representation matrices of operators $\mathcal{L}$ and $\mathcal{M}$ in wavelet bases can be compressed to sparse matrices. This observation leads us to develop a fast Galerkin method via wavelet basis. In this section, the wavelet basis proposed in [10] are employed to generate these representation matrices, and two truncation strategies are designed to compress the representation matrices of operators $\mathcal{L}$ and $\mathcal{M}$, respectively. Both truncation strategies, introduced in this section, produce the sparse representation matrices which have only $O(2^n)$ nonzero entries, and ensure the optimal convergence rate of solutions.

As a preparation, we recall the multiscale wavelet basis in [10]. For $n \in \mathbb{N}_0$, let $\pi_n$ be the uniform mesh which divides the interval $I$ into $2^n$ pieces, and $\mathbb{X}_n$ be the space constructed by the piecewise polynomials of order $r$ with respect to $\pi_n$. It is easily observed that the sequence $\mathbb{X}_n$, $n \in \mathbb{N}_0$, is nested, that is $\mathbb{X}_n \subset \mathbb{X}_{n+1}$. Since $\mathbb{X}_n$, $n \in \mathbb{N}_0$, is ultimately dense in $L^2(I)$ in the sense that $\bigcup_{n \in \mathbb{N}_0} \mathbb{X}_n = L^2(I)$, there has an orthogonal decomposition of space $L^2(I)$,

$$ L^2(I) = \bigoplus_{n \in \mathbb{N}_0} \mathbb{W}_n, $$

where $\mathbb{W}_0 := \mathbb{X}_0$, and $\mathbb{X}_{i+1} = \mathbb{X}_i \oplus \perp \mathbb{W}_i$ for $i \geq 0$. Let $w(0) := \dim(\mathbb{X}_0) = r$ and $w(i) := \dim(\mathbb{W}_i) = r2^{i-1}$ for $i > 0$. For $n \in \mathbb{N}_0$, define $\mathbb{Z}_{n+1} := \{0,1,\ldots,n\}$. Suppose that for $i \geq 0$,

$$ \mathbb{W}_i = \text{span}\{w_{ij} : j \in \mathbb{Z}_{w(i)}\}, $$

where $\{w_{ij} : j \in \mathbb{Z}_{w(i)}\}$ is an orthonormal basis of $\mathbb{W}_i$. For each $n \in \mathbb{N}_0$, define the index set $\mathbb{U}_n := \{(i,j) : i \in \mathbb{Z}_{n+1}, j \in \mathbb{Z}_{w(i)}\}$. Thus, the property $\mathbb{X}_n = \bigoplus_{i \in \mathbb{Z}_{n+1}} \mathbb{W}_i$ shows that

$$ \mathbb{X}_n = \text{span}\{w_{ij} : (i,j) \in \mathbb{U}_n\}. $$

Let $\bigcup := \{(i,j) : j \in \mathbb{Z}_{w(i)}, i \in \mathbb{N}_0\}$. Let $\mathcal{P}_n$ be the orthogonal projection from $L^2(I)$ onto $\mathbb{X}_n$.

The Galerkin method for solving (2.6) is to find $\rho_n \in \mathbb{X}_n$ such that

$$ (I - \mathcal{L}_n - \mathcal{M}_n) \rho_n = \mathcal{P}_n f, \quad (3.15) $$

where $\mathcal{L}_n := \mathcal{P}_n \mathcal{L}$ and $\mathcal{M}_n := \mathcal{P}_n \mathcal{M}$. Using the multiscale wavelet basis for space $\mathbb{X}_n$, the above Galerkin method (3.15) is to seek

$$ \rho_n := \sum_{(i,j) \in \mathbb{U}_n} \rho_{ij} w_{ij} \in \mathbb{X}_n, $$

where $\rho_{ij} w_{ij} \in \mathbb{X}_n$. 7
such that
\[
\sum_{(i,j)\in\mathbb{U}_n} \rho_{ij} \langle w_{i',j'}, w_{ij} - \mathcal{L}w_{ij} - \mathcal{M}w_{ij} \rangle = \langle w_{i',j'}, f \rangle, \quad (i', j') \in \mathbb{U}_n,
\]
(3.16)
where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(L^2(I)\). Let \(L_{i',j',ij} := \langle w_{i',j'}, \mathcal{L}w_{ij} \rangle\) and \(M_{i',j',ij} := \langle w_{i',j'}, \mathcal{M}w_{ij} \rangle\) for \((i,j), (i',j') \in \mathbb{U}_n\). Define the matrix
\[
\mathbf{L}_n := [L_{i',j',ij} : (i', j'), (i, j) \in \mathbb{U}_n], \quad \mathbf{M}_n := [M_{i',j',ij} : (i', j'), (i, j) \in \mathbb{U}_n],
\]
and vectors
\[
f_n := [\langle w_{i',j'}, f \rangle : (i', j') \in \mathbb{U}_n], \quad \mathbf{\rho}_n := [\rho_{ij} : (i, j) \in \mathbb{U}_n].
\]
With these notations, equation (3.16) takes the equivalent form
\[
(\mathbf{I}_n - \mathbf{L}_n - \mathbf{M}_n)\mathbf{\rho}_n = f_n,
\]
(3.17)
where \(\mathbf{I}_n\) is the identity matrix.

It is clear that the number of nonzero entries of \(\mathbf{L}_n\) is \((r2^n)^2\), so is \(\mathbf{M}_n\). Computing these entries is very costly. To handle this computational issue, we design sparse representation matrices of operators \(\mathcal{L}\) and \(\mathcal{M}\), respectively, by developing two matrix truncation strategies. To this aim, we recall the properties of the multiscale wavelet basis as follows (see [10]),

(I) For \(j \in \mathbb{Z}_w(0)\), supp\((w_{0,j}) = [0, 1]\). And for any \((i,j) \in \mathbb{U}\) with \(i \geq 1\),
\[
\text{supp}(w_{i,j}) = \left[2^{-i+1}\left\lfloor \frac{j}{r} \right\rfloor, 2^{-i+1}\left\lfloor \frac{j}{r} \right\rfloor + 1 \right],
\]
where \(\left\lfloor \frac{j}{r} \right\rfloor\) is the greatest integer less than or equal to \(\frac{j}{r}\).

(II) For any \(i',i \in \mathbb{N}\), there holds
\[
\langle w_{i',j'}, w_{ij} \rangle = \delta_{i'i}\delta_{jj'}, \quad j' \in \mathbb{Z}_w(i'), j \in \mathbb{Z}_w(i),
\]
where \(\delta_{ij}\) is the Kronecker delta.

(III) The vanish moment of the multiscale wavelet basis is \(r\). This means for any \(i \geq 1\) and polynomial \(P\) of order not greater than \(r\),
\[
\langle w_{ij}, P \rangle = 0, \quad j \in \mathbb{Z}_w(i).
\]

(IV) There exists a positive constant \(\theta_0\) such that \(||w_{ij}||_\infty \leq \theta_0 2^{i/2}\) for all \((i,j) \in \mathbb{U}\).

(V) For given \(q \in \mathbb{N}\) with \(q \leq r\), there exists a positive constant \(c\) such that for any \(n \in \mathbb{N}_0\) and \(u \in H^q\),
\[
||u - \mathcal{P}_n u|| \leq c 2^{-nq} ||u||_{H^q}.
\]

We let dist\((\cdot, \cdot)\) be the distance between two sets, and \(S_{ij} := \text{supp}(w_{ij})\) for \((i,j) \in \mathbb{U}\). Define \(L_{i',j',ij}^\kappa := \langle w_{i',j'}, \mathcal{L}^\kappa w_{ij} \rangle\) for \(\kappa \in \mathbb{Z}_2\) and \((i,j), (i',j') \in \mathbb{U}\). The truncation strategies (T1) and (T2) are presented as follows.

(T1) For \(\kappa \in \mathbb{Z}_2, n \in \mathbb{N}_0\) and \((i',j'), (i,j) \in \mathbb{U}_n\), define
\[
\widehat{L}_{i',j',ij}^{\kappa,n} := \begin{cases} L_{i',j',ij}^\kappa, & \text{dist}(S_{i',j'}, 1 - \kappa) \leq \tau_{i'i}^n \text{ and dist}(S_{ij}, \kappa) \leq \tau_{i'i}^n, \\ 0, & \text{otherwise}, \end{cases}
\]
(3.18)
in which the truncation parameters \(\tau_{i'i}^n\) are identified by
\[
\tau_{i'i}^n := \min\{2^{-i+a(n-i')}, 1\},
\]
(3.19)
Thus, the truncation strategy (T1) ensures that for all $n \in \mathbb{N}_0$, the truncated matrix of $L_n$ is defined by

$$L_n := \left[ \tilde{L}_{n,i}^{0} : i' \in \mathbb{Z}_{n+1} \right],$$

where $\tilde{L}_{n,i}^{0} := \left[ \tilde{L}_{i,j}^{0} \right]$, where $\tilde{L}_{i,j}^{0} = \left[ \tilde{L}_{n,i,j}^{0} : j' \in \mathbb{Z}_{w(i')}, j \in \mathbb{Z}_{w(i)} \right]$. The truncation strategies lead to the fast multiscale Galerkin method which is to find $\tilde{\rho}_n = [\tilde{\rho}_{i,j} \in \mathbb{R} : (i, j) \in \mathcal{U}_n]$ such that

$$L_n - \tilde{L}_n - \tilde{M}_n \tilde{\rho}_n = f_n. \tag{3.21}$$

Denote by $N(A)$ the number of nonzero entries of matrix $A$. Let $\tilde{L}_{i'}^{n} := \left[ \tilde{L}_{i',j}^{n} : j' \in \mathbb{Z}_{w(i')}, j \in \mathbb{Z}_{w(i)} \right]$ for $\kappa \in \mathbb{Z}_2$ and $i', i \in \mathbb{Z}_{n+1}$. We next estimate $N(L_n)$ and $N(M_n)$.

**Theorem 3.1** There exists a positive constant $c$ such that for all $n \in \mathbb{N}_0$,

$$N(L_n) \leq c2^n, \tag{3.22}$$

and

$$N(M_n) \leq c2^n, \tag{3.23}$$

where $c$ only depends on the parameters $a$ and $b$ inserted in (3.19) and (3.20).

**Proof:** For any $\kappa \in \mathbb{Z}_2$, $i, j \in \mathbb{Z}_{n+1}$ and $s > 0$, let $N_\kappa(i, s)$ be the cardinality of the set $\{ j \in \mathbb{Z}_{w(i)} : \text{dist}(S_{ij}, \kappa) \leq s \}$. According to property (I), there have that for all $s > 0$ and $i \in \mathbb{N}_0$,

$$N_0(i, s) \leq r(2^i - 1)s + 1 \quad \text{and} \quad N_1(i, s) \leq r(2^{i+1} - 1)s + 1.$$

Thus, the truncation strategy (T1) ensures that for all $n \in \mathbb{N}_0$, $\kappa \in \mathbb{Z}_2$ and $i, i' \in \mathbb{Z}_{n+1}$,

$$N(L_{i'}^{n}) = N_{i'-n}(i', \tau_{i'}^{n})N_\kappa(i, \tau_{i'}^{n}) \leq r^2(2^i \tau_{i'}^{n} + 1)(2^{i'} \tau_{i'}^{n} + 1).$$

Since $\tilde{L}_{i'}^{n} = \tilde{L}_{i'}^{0} + \tilde{L}_{i'}^{1}$, there has $N(L_{i'}^{n}) = N(L_{i'}^{0}) + N(L_{i'}^{1})$. Thus, there holds that for all $n \in \mathbb{N}_0$,

$$N(L_n) \leq 2r^2 \sum_{i \in \mathbb{Z}_{n+1}} \sum_{i' \in \mathbb{Z}_{n+1}} (2^i \tau_{i'}^{n} + 1)(2^{i'} \tau_{i'}^{n} + 1). \tag{3.24}$$

Combining (3.19) with (3.24), we have that for all $n \in \mathbb{N}_0$,

$$N(L_n) \leq 8r^2 \left( \sum_{i \in \mathbb{Z}_{n+1}} \sum_{0 \leq i' < n - \frac{n}{2}} 2^{i+i'} \right) + 2r^2 \left( \sum_{i \in \mathbb{Z}_{n+1}} \sum_{n - \frac{n}{2} \leq i' \leq n} (2^{a(n-i')} + 1)(2^{i+a(n-i')} + 1) \right). \tag{3.25}$$
Since
\[
\sum_{i \in \mathbb{Z}_{n+1}} \sum_{0 \leq i' < n - \frac{1}{a}} 2^{i+i'} < \sum_{i \in \mathbb{Z}_{n+1}} \left(2^{n+i(1-\frac{1}{a})+1} - 2^i\right) = 2^{n+1} \frac{1 - 2^{(n+1)(1-\frac{1}{a})}}{1 - 2(1-\frac{1}{a})} - 2^{n+1} + 1,
\]
and \(a < 1\), there exists a positive constant \(c_1\) such that for all \(n \in \mathbb{N}_0\),
\[
\sum_{i \in \mathbb{Z}_{n+1}} \sum_{0 \leq i' < n - \frac{1}{a}} 2^{i+i'} < c_1 2^n.
\]
(3.26)

Then, we obtain a up-boundary of the term in the first bracket of (3.25).

We now estimate the term in the second bracket of (3.25). Since \(2^{-a} < 1\) and \(a < 1\), we obtain that for all \(n \in \mathbb{N}_0\),
\[
\sum_{i \in \mathbb{Z}_{n+1}} \sum_{n - \frac{1}{a} \leq i' \leq n} 2^{-i+i'+2a(n-i')} \leq 2^n \sum_{i \in \mathbb{Z}_{n+1}} 2^{-i} \left(\frac{2^{(2a-1)(\frac{i}{n})+1}}{2^{2a-1} - 1}\right)
\leq \frac{2^{n+2a-1}}{2^{2a-1} - 1} \sum_{i \in \mathbb{Z}_{n+1}} 2^{(1-\frac{1}{a})i}.
\]

Thus, by noting that \(a < 1\), we know there exists a positive constant \(c_2\) such that for all \(n \in \mathbb{N}_0\),
\[
\sum_{i \in \mathbb{Z}_{n+1}} \sum_{n - \frac{1}{a} \leq i' \leq n} 2^{-i+i'+2a(n-i')} < c_2 2^n.
\]
(3.27)

Hence, by noting \(2^{-i+i'+a(n-i')} < 2^{-i+i'+2a(n-i')}\), and \(2a(n-i') \leq 2i'-i+2a(n-i')\), from (3.27) we know that there is a positive constant \(c_3\) such that for all \(n \in \mathbb{N}_0\),
\[
\sum_{i \in \mathbb{Z}_{n+1}} \sum_{n - \frac{1}{a} \leq i' \leq n} (2^{a(n-i')} + 1)(2^{i'-i+a(n-i')} + 1) \leq c_3 2^n.
\]
(3.28)

Substituting (3.26) and (3.28) into (3.25) leads to the desired result (3.22).

We now estimate (3.23). The truncation strategy (T2) leads that for all \(n \in \mathbb{N}_0\),
\[
\mathcal{N}(\widetilde{M}_n) = \sum_{i \in \mathbb{Z}_{n+1}} \sum_{0 \leq i' \leq n-(a+1)i} \mathcal{N}(M_{i,i'}).
\]
(3.29)

Since \(w(0) = r\) and \(w(i) = r^{2^{i-1}}\), the definition of \(M_{i,i'}\) shows that there exists a positive constant \(c_4\) such that for all \(i, i' \in \mathbb{N}_0\), \(\mathcal{N}(M_{i,i'}) \leq c_4 2^{i+i'}\). Thus, from (3.29), we know that for all \(n \in \mathbb{N}_0\),
\[
\mathcal{N}(\widetilde{M}_n) \leq c_4 \sum_{i \in \mathbb{Z}_{n+1}} \sum_{0 \leq i' \leq n-(a+1)i} 2^{i+i'}.
\]
(3.30)

By noting \(b > 0\) and conducting some computing, from (3.30), we obtain the desired result (3.23) of this theorem. \(\blacksquare\)
4 Stability and Convergence Analysis

We analyze in this section the stability of the proposed fast multiscale wavelet Galerkin method and its convergence order. It will be shown that the truncation strategies do not ruin the stability and optimal convergence order of the conventional wavelet Galerkin method. This aim can be done by estimating the difference between $L_n$ and $L_{n}$, and between $M_n$ and $M_{n}$. These differences are determined naturally by the smoothness of kernels of operators $L$ and $M$. This leads us to study the smoothness of the kernels of $L$ and $M$ in Subsections 4.1 and 4.2, respectively. Then, we investigate in Subsection 4.3 the stability and convergence order of the proposed fast multiscale wavelet Galerkin method.

4.1 Smoothness of the kernel of $L$

Faà di Bruno’s formulas play important roles in studying the smoothness of the kernels of $L$ and $M$. For the convenience, we recall the Faà di Bruno’s formulas as follow. For all $q \in \mathbb{N}$, define $N_q := \{1, 2, \ldots, q\}$. Also define $N_0 := \emptyset$. For all $q \in \mathbb{N}$, let $\Lambda_q$ be the set of all partitions of the set $N_q$. We also let $\Delta_q := \{\emptyset\}$. For any set $A$, we use $|A|$ to denote its cardinality. So that for all $\lambda \in \Lambda_q$, $|\lambda|$ is the number of blocks in the partition $\lambda$, and for all $B \in \lambda$, $|B|$ is the number of element in the block $B$. Let $\psi$ be a univariate smooth function defined on $I$, and $\varphi$ be a smooth function defined on $I^2$. For all $s, t \in I$, we define function $\psi \circ \varphi$ by $(\psi \circ \varphi)(s, t) := \psi(\varphi(s, t))$. For all $\alpha_1, \alpha_2 \in \mathbb{N}_0$, denote $\varphi^{(\alpha_1, \alpha_2)} := \varphi^{(\alpha_1) \circ \alpha_2 \circ \varphi}$. For $s \in I$, we set $\prod_{B \in \emptyset} \psi(B)(s) = 1$. Faà di Bruno’s formulas used in this subsection are shown as follows. For all $\alpha_1, \alpha_2 \in \mathbb{N}_0$,

\[
(\psi \circ \varphi)^{(\alpha_1, \alpha_2)}(s, t) = \sum_{\lambda \in \Lambda_{\alpha_1 + \alpha_2}} \left(\psi^{(\lambda)} \circ \varphi\right)(s, t) \prod_{B \in \lambda} \varphi^{(B_1 \cup B_2)}(s, t),
\]

and

\[
(\varphi \circ \gamma)^{(\alpha_1, \alpha_2)}(s, t) = \sum_{\lambda_1 \in \Lambda_{\alpha_1}} \sum_{\lambda_2 \in \Lambda_{\alpha_2}} \left(\varphi^{(\lambda_1 \cup \lambda_2)} \circ \gamma\right)(s, t) \prod_{B \in \lambda_1} \gamma^{(B)}(s) \prod_{B' \in \lambda_2} \gamma^{(B')} (t),
\]

where $B_1 := B \cap N_{\alpha_1}$ and $B_2 := B \cap (N_{\alpha_1 + \alpha_2} \setminus N_{\alpha_1})$.

We first investigate the high order partial derivatives of $\tilde{L}_0$. The high order partial derivatives of $\tilde{L}_1$ can be obtained directly by equality $\tilde{L}_1(s, t) = \tilde{L}_0(1 - s, 1 - t)$. To this aim, we rewrite $\tilde{L}_0$ as follows. For $(s, t) \in I^2$, we define

\[
g_1(s, t) := |\Gamma'(0)t - \Gamma'(1)(s - 1)|^2,
\]

and

\[
f_1(s, t) := -\Gamma'(1) \cdot \left[\Gamma'(0)\right]^{-1}(s - 1).
\]

Define $H := \left(\frac{\partial}{\partial t}\right) \circ \gamma$. It can be checked that $\tilde{L}_0(s, t) = \frac{1}{2} \chi_1(s)\gamma'(t)H(s, t)\chi_0(t)$ for $(s, t) \in \mathbb{S}_0\left(\frac{1}{2}\right)$. This leads us to study the high order partial derivatives of $H$ by the following three lemmas. As a preparation, we define the notations as follows. For all $k_1, k_2 \in \mathbb{N}_0$, let

\[
\mathbb{N}_0^{k_1, k_2} := \begin{cases}
\{(m_1, m_2) \in \mathbb{N}_0 : m_1 + m_2 = l, m_1 \geq \min\{k_1, 1\}, m_2 \geq \min\{k_2, 1\}\}, & \text{if } l > 1, \\
\{(0, 1), (1, 0),\} & \text{if } l = 1, \\
\{(0, 0),\} & \text{if } l = 0.
\end{cases}
\]

For all $k, j \in \mathbb{N}_0$, we also let $C_j := \frac{k!}{j!(k-j)!}$. For $k \in \mathbb{N}$, we denote by $C^k(E)$ the space of functions with $k$-order derivatives that are continuous in $E$. We remind that the parameters $p$ and $\epsilon$ are inserted in the definition of function $\gamma$.

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Lemma 4.1 Let \( \varphi \) be a function defined on \( S_0(\frac{1}{2}) \), \( \psi \) be defined on \( \mathbb{R} \setminus \{0\} \) and \( l \in \mathbb{N} \). If there is a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \), and all \( (s, t) \in S_0(\frac{1}{2}) \),

\[
|\varphi^{(k_1,k_2)}(s,t)(1-s)^{k_1}t^{k_2}| \leq c \sum_{(m_1,m_2) \in \mathbb{Z}^{k_1,k_2}} (1-s)^{m_1}t^{m_2},
\]

then there is a positive constant \( \tilde{c} \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and all \((s, t) \in S_0(\frac{1}{2}) \),

\[
|\psi \circ \varphi^{(k_1,k_2)}(s,t)(1-s)^{k_1}t^{k_2}| \leq \tilde{c} \sum_{\lambda \in \Lambda_{k_1+k_2}} \left| \left( \psi^{(|\lambda|)} \circ \varphi \right)(s,t) \right| \sum_{(m_1,m_2) \in \mathbb{Z}^{k_1,k_2}} (1-s)^{m_1}t^{m_2}.
\]

Proof: By using Faà di Bruno’s formulas (4.31), we have that for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and \((s, t) \in S_0(\frac{1}{2}) \),

\[
|\psi \circ \varphi^{(k_1,k_2)}(s,t)(1-s)^{k_1}t^{k_2}| \leq \sum_{\lambda \in \Lambda_{k_1+k_2}} \left| \left( \psi^{(|\lambda|)} \circ \varphi \right)(s,t) \right| \prod_{B \in \Lambda} \left| \varphi^{(|B|)}(1-s)|B|(s,t) \right|,
\]

where \( B_1 := B \cap \Lambda_{k_1} \) and \( B_2 := B \cap (\Lambda_{k_1+k_2} \setminus \Lambda_{k_1}) \). Note that for each \( \lambda \in \Lambda_{k_1+k_2} \), \( \sum_{B \in \Lambda} |B_1| = k_1 \) and \( \sum_{B \in \Lambda} |B_2| = k_2 \). From inequality (4.35), there holds that for each \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and all \((s, t) \in S_0(\frac{1}{2}) \),

\[
|\psi \circ \varphi^{(k_1,k_2)}(s,t)(1-s)^{k_1}t^{k_2}| \leq \sum_{\lambda \in \Lambda_{k_1+k_2}} \left| \left( \psi^{(|\lambda|)} \circ \varphi \right)(s,t) \right| \prod_{B \in \Lambda} \varphi^{(|B|)}(1-s)|B|(s,t)(1-s)|B_1|\prod_{B \in \Lambda} \varphi^{(|B_1|)}(s,t).
\]

Substituting the condition (4.33) into the right hand side of the above inequality and using Lemma A.1, we obtain the result of this lemma. □

Lemma 4.2 Let \( \varphi \) and \( \varphi_j, j \in \mathbb{Z}_{2p+1}, \) be functions defined on \( S_0(\frac{1}{2}) \). Also let \( l \in \mathbb{N} \) and \( l_0 \in \mathbb{N}_0 \). If there is a positive constant \( c_1 \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and all \((s, t) \in S_0(\frac{1}{2}) \),

\[
|\varphi^{(k_1,k_2)}(s,t)(1-s)^{k_1}t^{k_2}| \leq c_1 \sum_{\lambda \in \Lambda_{k_1+k_2}} \varphi^{(|\lambda|)}(s,t) \sum_{(m_1,m_2) \in \mathbb{Z}^{k_1,k_2}} (1-s)^{m_1}t^{m_2},
\]

then there is a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and all \((s, t) \in S_0(\epsilon) \),

\[
|\varphi \circ \gamma^{(k_1,k_2)}(s,t)| \leq c \sum_{\lambda \in \Lambda_{k_1+k_2}} \varphi^{(|\lambda|)}((1-s)^p, t^p) \sum_{(m_1,m_2) \in \mathbb{Z}^{k_1,k_2}} (1-s)^{m_1-p}t^{m_2-p}.
\]

Proof: By Faà di Bruno’s formula (4.32), we have that for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, \) and all \((s, t) \in S_0(\frac{1}{2}) \),

\[
(\varphi \circ \gamma)^{(k_1,k_2)}(s,t) = \sum_{\lambda_1 \in \Lambda_{k_1}} \sum_{\lambda_2 \in \Lambda_{k_2}} \varphi^{(|\lambda_1,\lambda_2|)}(\gamma(s), \gamma(t)) \prod_{B \in \Lambda_1} \gamma^{(|B|)}(s) \prod_{B \in \Lambda_2} \gamma^{(|B|)}(t).
\]

Note that if \( k_1 > 0 \), then for each \( \lambda_1 \in \Lambda_{k_1}, \) \( \sum_{B \in \Lambda_1} |B| = k_1 \), and for each \( B \in \Lambda_1 \) and \( s \in [1-\epsilon, 1] \), \( |\gamma^{(|B|)}(s)| = \frac{pl}{(p-|B|)!} |1-s|^{p-|B|} \). Thus, when \( k_1 > 0 \), there holds that for each \( \lambda_1 \in \Lambda_{k_1} \) and \( s \in [1-\epsilon, 1] \),

\[
\prod_{B \in \Lambda_1} \gamma^{(|B|)}(s) = (1-s)^{p|\lambda_1|-k_1} \prod_{B \in \Lambda_1} \frac{pl}{(p-|B|)!}.
\]
Similarly, when \( k_2 > 0 \), there holds that for each \( \lambda \in \Lambda_{k_2} \) and \( t \in [0, e] \),

\[
\prod_{B^t \in \Lambda_{k_2}} \gamma^{(|B^t|)}(t) = \sum_{B^t \in \Lambda_{k_2}} \frac{p!}{(p - |B^t|)!}.
\]

Note that \( \Lambda_0 = \{\emptyset\} \) and \( \prod_{B \in \emptyset} \gamma^{(|B|)}(s) = 1 \) and \( \mathbb{Z}_{p+1} \) is a finite set. Thus, combining the above equalities shows that there is a constant \( c_2 \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( (s, t) \in \mathcal{S}_0(\epsilon) \),

\[
\left| (\varphi \circ \gamma)^{(k_1, k_2)}(s, t) \right| \leq c_2 (1 - s)^{-k_1} t^{-k_2} \sum_{\lambda_1 \in \Lambda_{k_1}} \sum_{\lambda_2 \in \Lambda_{k_2}} \left| \varphi^{(|\lambda_1|, |\lambda_2|)}(\gamma(s), \gamma(t))(1 - s)^{|\lambda_1|} t^{|\lambda_2|} \right|.
\]

Then, by noting that for all \( (s, t) \in \mathcal{S}_0(\epsilon) \), \( (1 - s)^p = 1 - \gamma(s) \) and \( t^p = \gamma(t) \), we obtain that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( (s, t) \in \mathcal{S}_0(\epsilon) \),

\[
\left| (\varphi \circ \gamma)^{(k_1, k_2)}(s, t) \right| \leq c_2 (1 - s)^{-k_1} t^{-k_2} \sum_{\lambda_1 \in \Lambda_{k_1}} \sum_{\lambda_2 \in \Lambda_{k_2}} \left| \varphi^{(|\lambda_1|, |\lambda_2|)}(\gamma(s), \gamma(t))(1 - s)^{|\lambda_1|} |\gamma|^{|\lambda_2|}(t) \right|.
\]

Since for all \( (s, t) \in \mathcal{S}_0(\epsilon) \), \( 1 - \gamma(s) = (1 - s)^p \) and \( \gamma(t) = t^p \), substituting the condition (4.36) into the above inequality shows that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( (s, t) \in \mathcal{S}_0(\epsilon) \),

\[
\left| (\varphi \circ \gamma)^{(k_1, k_2)}(s, t) \right| \leq c_1 c_2 (1 - s)^{-k_1} t^{-k_2} \sum_{\lambda_1 \in \Lambda_{k_1}} \sum_{\lambda_2 \in \Lambda_{k_2}} \sum_{\lambda \in \Lambda_{|\lambda_1| + |\lambda_2|}} \varphi|\lambda| ((1 - s)^p, t^p) \sum_{(m_1, m_2) \in \mathbb{Z}_{|\lambda_1| + |\lambda_2|}} (1 - s)^{m_1} t^{m_2}. \tag{4.38}
\]

Note that for all \( \lambda_1 \in \Lambda_{k_1} \) and \( \lambda_2 \in \Lambda_{k_2} \), \( \mathbb{Z}_{|\lambda_1| + |\lambda_2|} = \mathbb{Z}_{|\lambda_1| + |\lambda_2|} \) and \( |\lambda_1| + |\lambda_2| \leq k_1 + k_2 \). From (4.38), we obtain

\[
\left| (\varphi \circ \gamma)^{(k_1, k_2)}(s, t) \right| \leq c_1 c_2 (1 - s)^{-k_1} t^{-k_2} \sum_{\lambda_1 \in \Lambda_{k_1}} \sum_{\lambda_2 \in \Lambda_{k_2}} \sum_{\lambda \in \Lambda_{k_1 + k_2}} \varphi|\lambda| ((1 - s)^p, t^p) \sum_{(m_1, m_2) \in \mathbb{Z}_{|\lambda_1| + |\lambda_2|}} (1 - s)^{m_1} t^{m_2}.
\]

Since the cardinalities of sets \( \Lambda_{k_1} \) and \( \Lambda_{k_2} \) are constants depending on \( k_1 \) and \( k_2 \) respectively, from the above inequality we can obtain the result of this lemma. \( \square \)

**Lemma 4.3** Let \( \varphi, \varphi_j \), \( j \in \mathbb{Z}_{p+1} \), and \( \phi \) be functions defined on \( \mathcal{S}_0(\frac{1}{2}) \). Also let \( l, l' \in \mathbb{N} \) and \( l_0 \in \mathbb{N}_0 \). If there is a positive constant \( c_1 \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( (s, t) \in \mathcal{S}_0(\frac{1}{2}) \),

\[
\left| \varphi^{(k_1, k_2)}(s, t)(1 - s)^{k_1} t^{k_2} \right| \leq c_1 \sum_{\lambda \in \Lambda_{k_1 + k_2}} \varphi|\lambda| (1 - s, t) \sum_{(m_1, m_2) \in \mathbb{Z}_{|\lambda_1| + |\lambda_2|}} (1 - s)^{m_1} t^{m_2}, \tag{4.39}
\]

and

\[
\left| \varphi^{(k_1, k_2)}(s, t)(1 - s)^{k_1} t^{k_2} \right| \leq c_1 \sum_{(m_1, m_2) \in \mathbb{Z}_{|\lambda_1| + |\lambda_2|}} (1 - s)^{m_1} t^{m_2}, \tag{4.40}
\]

then there is a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( (s, t) \in \mathcal{S}_0(\frac{1}{2}) \),

\[
\left| (\varphi \circ \gamma)^{(k_1, k_2)}(s, t)(1 - s)^{k_1} t^{k_2} \right| \leq c \sum_{\lambda \in \Lambda_{k_1 + k_2}} \varphi|\lambda| ((1 - s), t) \sum_{(m_1, m_2) \in \mathbb{Z}_{|\lambda_1| + |\lambda_2|}} (1 - s)^{m_1} t^{m_2}. \tag{4.41}
\]
Proof: For each \( k_1, k_2 \in \mathbb{Z}_{p+1} \), there holds that

\[
(\varphi \phi)^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2} = \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} C_{j_1}^{k_1} C_{j_2}^{k_2} \varphi^{(k_1-j_1, k_2-j_2)}(s, t) \varphi^{(j_1, j_2)}(s, t)(1 - s)^{k_1}t^{k_2}.
\]

Substituting (4.39) and (4.40) into the above inequality, and using Lemma A.1, we know that there exists a constant \( c_2 \) such that for all \((s, t) \in S_0(\frac{1}{2})\),

\[
|\varphi^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2}| \leq c_2 \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \sum_{\lambda \in \Lambda_{j_1+j_2}} \varphi^{(s, t)}(1 - s, t) \sum_{(m_1, m_2) \in S_0^{k_1, k_2}} (1 - s)^{m_1}t^{m_2}.
\]

Note that \( j_1 + j_2 \leq k_1 + k_2 \). Then, we have

\[
|\varphi^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2}| \leq c_2 \sum_{j_1=0}^{k_1} \sum_{j_2=0}^{k_2} \sum_{\lambda \in \Lambda_{k_1+k_2}} \varphi^{(s, t)}(1 - s, t) \sum_{(m_1, m_2) \in S_0^{k_1, k_2}} (1 - s)^{m_1}t^{m_2}.
\]

Since \( k_1, k_2 \in \mathbb{Z}_{p+1} \) and \( \mathbb{Z}_{p+1} \) is a finite set, then we then obtain the result of this lemma. \( \square \)

**Lemma 4.4** There is a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \), and all \((s, t) \in I^2\),

\[
|g_1^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2}| \leq c \sum_{(m_1, m_2) \in S_0^{k_1, k_2}} (1 - s)^{m_1}t^{m_2}. \tag{4.42}
\]

**Proof:** It can be checked that for all \((s, t) \in I^2 \) and \( j_1, j_2 \in \mathbb{N}_0\),

\[
g_1^{(j_1, j_2)}(s, t) = \begin{cases} 
|\Gamma'(0)t - \Gamma'(1)(s - 1)|^2, & j_1 = 0, j_2 = 0, \\
-2\Gamma'(0) t - \Gamma'(1)(s - 1), & j_1 = 1, j_2 = 0, \\
2\Gamma'(1) \cdot \Gamma'(1), & j_1 = 2, j_2 = 0, \\
2\Gamma'(0) \cdot (\Gamma'(0)t - \Gamma'(1)(s - 1)), & j_1 = 0, j_2 = 1, \\
2\Gamma'(0) \cdot \Gamma'(0), & j_1 = 0, j_2 = 2, \\
-2\Gamma'(0) \cdot \Gamma'(1), & j_1 = 1, j_2 = 1, \\
0, & \text{others.}
\end{cases} \tag{4.43}
\]

This ensures the result of lemma. \( \square \)

**Lemma 4.5** There is a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \), and all \((s, t) \in S_0(\frac{1}{2})\),

\[
\left| \left( \frac{1}{g_1} \right)^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2} \right| \leq c \sum_{\lambda \in \Lambda_{k_1+k_2}} \sum_{(m_1, m_2) \in S_0^{k_1, k_2}} (1 - s)^{m_1}t^{m_2}. \tag{4.44}
\]

There exists a positive constant \( \tilde{c} \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \), and all \((s, t) \in S_0(\epsilon)\),

\[
\left| H^{(k_1, k_2)}(s, t) \right| \leq \tilde{c} \sum_{\lambda \in \Lambda_{k_1+k_2}} \sum_{(m_1, m_2) \in S_0^{k_1, k_2}} (1 - s)^{m_1}t^{m_2}. \tag{4.45}
\]
Proof: From Lemma 4.4, we know $g_1$ satisfies the condition (4.33). Then by setting $\varphi(s, t) = g_1(s, t)$ and $\psi(s) = \frac{1}{s}$, from Lemma 4.1, we know that there exists a constant $c_1 > 0$ such that for all $k_1, k_2 \in \mathbb{Z}_{p+1}$, and all $(s, t) \in S_0(\frac{1}{2})$,

$$\left| \frac{1}{g_1} \right|^{(j_1, j_2)}(s, t)(1 - s)^{j_1}t^{j_2} \leq c_1 \sum_{\lambda \in \Lambda_{j_1+j_2}} \frac{(-1)^{\lambda}}{g_1^{1+(\lambda)}(s, t)} \sum_{(m_1, m_2) \in \mathbb{Z}_0^{j_1+j_2}} (1 - s)^{m_1}t^{m_2}.$$  

It is easy to check that there is a positive constant $c_2$ such that for all $(s, t) \in S_0(\frac{1}{2})$,

$$g_1(s, t) \geq c_2((1 - s)^2 + t^2). \tag{4.46}$$

Substituted (4.46) into above inequality, we obtain (4.44).

Note that for all $(s, t) \in S_0(\epsilon)$, $(1 - s)^p = 1 - \gamma(s)$ and $t^p = \gamma(t)$. Thus, from the definition of $f_1$ and (4.46), we know (4.45) holds when $k_1 + k_2 = 0$. We next consider the cases with $k_1 + k_2 > 0$. By the definitions of $f_1$, it shows that for all $j_1, j_2 \in \mathbb{N}_0$ and $(s, t) \in S_0(\frac{1}{2})$, 

$$|f_1^{(j_1, j_2)}(s, t)| = \begin{cases} \left| \Gamma'(1) \cdot \left[ \Gamma'(0) \right] (s - 1) \right|, & j_1 = 0, j_2 = 0, \\ \left| \Gamma'(1) \cdot \left[ \Gamma'(0) \right] (s - 1) \right|, & j_1 = 1, j_2 = 0, \\ 0, & \text{others}. \end{cases}$$

Thus, there exists a constant $c_3 > 0$ such that for all $j_1, j_2 \in \mathbb{N}_0$, and all $(s, t) \in S_0(\frac{1}{2})$

$$|f_1^{(j_1, j_2)}(s, t)(1 - s)^{j_1}t^{j_2}| \leq c_3 \sum_{(m_1, m_2) \in \mathbb{Z}_0^{j_1+j_2}} (1 - s)^{m_1}t^{m_2}. $$

This means $f_1$ satisfies the condition (4.40) in Lemma 4.3. Meanwhile, (4.44) shows $g_1$ satisfies (4.39) with $l = 2$, $l_0 = 0$, and $\varphi_j(s, t) = \frac{1}{((1 - s)^2 + t^2)^{\gamma_j}}, j \in \mathbb{Z}_{2p+1}$. Thus, from Lemma 4.3 we know that there is a positive constant $c_4$ such that for all $k_1, k_2 \in \mathbb{Z}_{p+1}$ with $k_1 + k_2 > 0$, and all $(s, t) \in S_0(\frac{1}{2})$,

$$\left| \frac{f_1}{g_1} \right|^{(k_1, k_2)}(s, t)(1 - s)^{k_1}t^{k_2} \leq c_4 \sum_{\lambda \in \Lambda_{k_1+k_2}} \frac{\sum_{(m_1, m_2) \in \mathbb{Z}_0^{k_1+k_2}} (1 - s)^{m_1}t^{m_2}}{((1 - s)^2 + t^2)^{\lambda+1}}.$$ 

Hence, we can say that function $\frac{f_1}{g_1}$ satisfies the condition (4.36) in Lemma 4.2 with $l = 2$, $l_0 = 1$, and $\varphi_j(s, t) = \frac{1}{((1 - s)^2 + t^2)^{\gamma}}, j \in \mathbb{Z}_{2p+1}$. By noting $H = (\frac{f_1}{g_1}) \circ \gamma$, from Lemma 4.2 we obtain the result of this lemma.

Using Lemmas A.1 and 4.5, we obtain the estimate of the high order partial derivatives of $L_0$ as follows.

Lemma 4.6 There exists a positive constant $c$ such that for all $k_1, k_2 \in \mathbb{Z}_{p+1}$, and all $(s, t) \in S_0(\epsilon)$,

$$\left| L_0^{(k_1, k_2)}(s, t) \right| \leq c \min \left\{ \frac{1}{t^{k_1+k_2+1}}, \frac{1}{(1 - s)^{k_1+k_2+1}} \right\}.$$
Proof: By the product rule of derivatives,
\[ L^{(k_1,k_2)}_0(s,t) = \frac{1}{\pi} \sum_{j \in \mathbb{Z}_{k_2+1}} C_{k_2}^j \gamma^{(k_2-j+1)}(t) H^{(k_1,j)}(s,t). \]

Then, by noting that for all \( t \in (0,\epsilon) \), \( k_2 \in \mathbb{Z}_{p+1} \), and \( 0 \leq j \leq k_2 \), \( \gamma^{(k_2-j+1)}(t) = \frac{p!}{(p-k_2+j-1)!} t^{p-k_2+j-1} \), from (4.45) of Lemma 4.5, we obtain that there exists a positive constant \( c \) such that for all \( k_1, k_2 \in \mathbb{Z}_{p+1} \), and \( (s,t) \in S_0(\epsilon) \),
\[ \left| L^{(k_1,k_2)}_0(s,t) \right| \leq c_1 \sum_{j \in \mathbb{Z}_{k_2+1}} \sum_{\lambda \in \Lambda_{k_1+j}} \sum_{(m_1,m_2) \in \mathbb{Z}_{k_1+j}} \frac{(1-s)^{m_1-p-k_1}t^{m_2p} \lambda^{-p-k_2-1}}{((1-s)^{2p} + t^{2p})^{\lambda+1}}. \] (4.47)

Since for all \( k_1, k_2 \in \mathbb{Z}_{p+1}, j \in \mathbb{Z}_{k_2+1}, \lambda \in \Lambda_{k_1+j} \), and \((m_1,m_2) \in \mathbb{Z}_{k_1+j} \),
\[ \frac{(1-s)^{m_1-p-k_1}t^{m_2p}}{((1-s)^{2p} + t^{2p})^{\lambda+1}} \leq \begin{cases} \frac{t^{k_1-k_2-1}}{(1-s)^{2p} + t^{2p})^{\lambda+1}}, & 0 < (1-s) \leq t, \\ (1-s)^{k_1-k_2-1} & 0 < t \leq 1 - s, \end{cases} \]

we obtain the desired result from inequality (4.47). \( \Box \)

With Lemma 4.6, we obtain the estimate of the smoothness of \( \tilde{L}_0 \) and \( \tilde{L}_1 \) which characterize the smoothness of the kernel of operator \( L \). These estimates are shown in the next corollary.

**Corollary 4.7** Let \( q \in \mathbb{N}_0 \). If \( q \leq p \), then there exists a positive constant \( c \) such that for \( (s,t) \in S_0(1) \),
\[ \left| \tilde{L}^{(q,q)}_0(s,t) \right| \leq c \min \left\{ \frac{1}{t^{2q+1}}, \frac{1}{(1-s)^{2q+1}} \right\}. \] (4.48)

and for \( (s,t) \in S_1(1) \),
\[ \left| \tilde{L}^{(q,q)}_1(s,t) \right| \leq c \min \left\{ \frac{1}{s^{2q+1}}, \frac{1}{(1-t)^{2q+1}} \right\}. \] (4.49)

**Proof:** Since \( \chi(t) = 0 \), \( t \in [0,1] \), the definition of \( \tilde{L}_0 \) shows that for all \( (s,t) \in S_0(1) \setminus S_0(\epsilon) \), \( \tilde{L}_0(s,t) = 0 \). Thus, (4.48) holds when \( (s,t) \in S_0(1) \setminus S_0(\epsilon) \). For \( (s,t) \in S_0(\epsilon) \) and \( q \leq p \),
\[ \tilde{L}^{(q,q)}_0(s,t) = \sum_{k_1=0}^q \sum_{k_2=0}^q (-1)^{q-k_1} C_q^{k_1} C_q^{k_2} \chi^{(q-k_1)}(1-s) \chi^{(q-k_2)}(1-s) \chi^{(q-k_1)}(1-s) \chi^{(q-k_2)}(1-s) L^{(k_1,k_2)}_0(s,t). \] (4.50)

Since for all \( 0 \leq k \leq q \), function \( \chi^{(q-k)} \) is bounded on \((0,\epsilon]\), combining Lemma 4.6 with equality (4.50) shows there exists a positive constant \( c_1 \) such that for all \( (s,t) \in S_0(\epsilon) \),
\[ \left| \tilde{L}^{(q,q)}_0(s,t) \right| \leq c_1 \sum_{k_1=0}^q \sum_{k_2=0}^q \min \left\{ \frac{1}{t^{k_1+k_2+1}}, \frac{1}{(1-s)^{k_1+k_2+1}} \right\}. \]

This leads to (4.48). Since \( \tilde{L}_1(s,t) = \tilde{L}_0(1-s,1-t) \), we obtain (4.49) from (4.48). \( \Box \)
4.2 Smoothness of the kernel of $M$

In this subsection, we investigate the smoothness of the kernel $M$ of $M$. Here, we focus on illustrating the smoothness of $M$ on $S_{0}(\varepsilon/2)$. The smoothness of $M$ on $S_{1}(\varepsilon/2)$ can be obtained similarly. For $(s, t) \in I^{2}$, define

$$f_{2}(s, t) := (\Gamma(t) - \Gamma(s)) \cdot [\Gamma'(t)]^{\perp},$$

and

$$g_{2}(s, t) := |\Gamma(t) - \Gamma(s)|^{2}.$$

Note that for $(s, t) \in S_{0}(\varepsilon/2)$, there have $K(s, t) = \frac{1}{\pi} \gamma'(t) \left( \frac{f_{2}}{g_{2}} \circ \gamma \right) (s, t)$, $\tilde{L}(s, t) = \frac{1}{\pi} \gamma'(t) \left( \frac{f_{2}}{g_{2}} \circ \gamma \right) (s, t)$ and $M(s, t) = K(s, t) - \tilde{L}(s, t)$. Thus, we have that for $(s, t) \in S_{0}(\varepsilon/2)$,

$$M(s, t) = \frac{1}{\pi} \gamma'(t) \left( \frac{f_{2}}{g_{2}} \circ \gamma - \frac{f_{1}}{g_{1}} \circ \gamma \right) (s, t).$$

Define $U := f_{2} - f_{1}$, $V := g_{2} - g_{1}$, $J := V / g_{1}$ and $G := f_{2}V / g_{1}g_{2}$. Then, from the above equality, we know that for $(s, t) \in S_{0}(\varepsilon/2)$,

$$M(s, t) = \frac{1}{\pi} \gamma'(t)(J \circ \gamma - G \circ \gamma)(s, t).$$

This equality leads us to investigate the smoothness of functions $G \circ \gamma$ and $J \circ \gamma$. As a preparation, we analyze the high order derivatives of $J$ and $G$ by the following two lemmas. To this aim, we need the following two assumptions about the smoothness of the boundary $\partial D$.

(A1) $\partial D$ is continuously differentiable up to order $p + 1$, with the exception of point $P_{0}$, i.e., for $k \in \mathbb{Z}_{p+2}$, $\xi^{(k)}$ and $\eta^{(k)}$ are continuous on $I$.

(A2) $\Gamma''(0) = (0, 0)$ and $\Gamma'''(1) = (0, 0)$.

**Lemma 4.8** If assumptions (A1) and (A2) hold, then there is a positive constant $c$ such that for all $k_{1}, k_{2} \in \mathbb{Z}_{p+1}$ and $(s, t) \in I^{2}$,

$$|f_{2}^{(k_{1}, k_{2})}(s, t)(1 - s)^{k_{1}}t^{k_{2}}| \leq c \sum_{(m_{1}, m_{2}) \in \mathbb{Z}_{1}^{k_{1}, k_{2}}}(1 - s)^{m_{1}}t^{m_{2}}, \quad (4.51)$$

$$|g_{2}^{(k_{1}, k_{2})}(s, t)(1 - s)^{k_{1}}t^{k_{2}}| \leq c \sum_{(m_{1}, m_{2}) \in \mathbb{Z}_{2}^{k_{1}, k_{2}}}(1 - s)^{m_{1}}t^{m_{2}}, \quad (4.52)$$

$$|U^{(k_{1}, k_{2})}(s, t)(1 - s)^{k_{1}}t^{k_{2}}| \leq c \sum_{(m_{1}, m_{2}) \in \mathbb{Z}_{2}^{k_{1}, k_{2}}}(1 - s)^{m_{1}}t^{m_{2}}, \quad (4.53)$$

and

$$|V^{(k_{1}, k_{2})}(s, t)(1 - s)^{k_{1}}t^{k_{2}}| \leq c \sum_{(m_{1}, m_{2}) \in \mathbb{Z}_{2}^{k_{1}, k_{2}}}(1 - s)^{m_{1}}t^{m_{2}}. \quad (4.54)$$

**Proof:** From assumption (A2), we know that there has a constant $c_{1} > 0$ such that for all $(s, t) \in I^{2}$,

$$|\Gamma(t) - \Gamma(s)| = |\Gamma'(0)t - \Gamma'(1)(s - 1) + O(t^{3}) + O((1 - s)^{3})| \leq c_{1} \sum_{(m_{1}, m_{2}) \in \mathbb{Z}_{1}^{k_{1}, k_{2}}}(1 - s)^{m_{1}}t^{m_{2}}.$$

Thus, by noting that $|((\Gamma'(t))^{\perp})|$ is uniformly bounded on $I$, from the definition of $f_{2}$ we know that inequality (4.51) holds when $k_{1} = k_{2} = 0$. Since $\xi$ and $\eta$ have continuous derivatives up to order
Thus, for each $k$, since assumption (A1) holds, we know that for all $s, t \in I$, we obtain that for each $k_1, k_2 \in \mathbb{Z}_{p+1}$ with $k_1 + k_2 \leq 2$, there exists a constant $c_1$ such that for all $s, t \in I$,

$$|U^{(k_1, k_2)}(s, t)(1-s)^{k_1}t^{k_2}| \leq c_1 \sum_{(m_1, m_2) \in \mathbb{Z}_3^{k_1, k_2}} (1-s)^{m_1}t^{m_2}.$$  

Since assumption (A1) holds, we can obtain that for each $k_1, k_2 \in \mathbb{Z}_{p+1}$, $U^{(k_1, k_2)}$ is bounded on $I^2$. Thus, for each $k_1, k_2 \in \mathbb{Z}_{p+1}$ with $k_1 + k_2 \geq 3$, there is a constant $c_2$ such that for all $s, t \in I$,

$$|U^{(k_1, k_2)}(s, t)(1-s)^{k_1}t^{k_2}| \leq c_2(1-s)^{k_1}t^{k_2} \leq c_2 \sum_{(m_1, m_2) \in \mathbb{Z}_3^{k_1, k_2}} (1-s)^{m_1}t^{m_2}.$$  

Then, we prove the inequality (4.53).

We next prove (4.54). It is easy to check that for all $s, t \in I$,

$$g_2^{(k_1, k_2)}(s, t) = \begin{cases} |\Gamma(t) - \Gamma(s)|^2, & k_1 = k_2 = 0, \\ 2 \sum_{j=1}^{k_1-1} \Gamma^{(k_1-j)}(s) \cdot \Gamma^{(j)}(s) - 2\Gamma^{(k_1)}(s) \cdot (\Gamma(t) - \Gamma(s)), & k_1 \geq 1, k_2 = 0, \\ 2 \sum_{j=1}^{k_2-1} \Gamma^{(k_2-j)}(t) \cdot \Gamma^{(j)}(t) + \Gamma^{(k_2)}(t) \cdot (\Gamma(t) - \Gamma(s)), & k_1 = 0, k_2 \geq 1, \\ -2\Gamma^{(k_1+1)}(s) \cdot \Gamma^{(k_2+1)}(t), & k_1 \geq 1, k_2 \geq 1. \end{cases}$$

Due to $\Gamma''(0) = (0, 0)$ and $\Gamma''(1) = (0, 0)$, we have $\Gamma(t) - \Gamma(s) = \Gamma'(0)t - \Gamma'(1)(s-1) + O(t^3) + O((1-s)^3)$, $\Gamma'(t) = \Gamma'(0) + O(t^2)$, $\Gamma'(s) = \Gamma'(1) + O((1-s)^2)$, $\Gamma''(t) = O(t)$ and $\Gamma''(s) = O(1-s)$. Then we obtain (4.52). From (4.43), (4.56) and assumption (A1), we can see that for $k_1, k_2 \in \mathbb{Z}_{p+1}$ with $k_1 + k_2 \leq 3$, there exists a constant $c_3$ such that for all $s, t \in I$,

$$|V^{(k_1, k_2)}(s, t)(1-s)^{k_1}t^{k_2}| \leq c_3 \sum_{(m_1, m_2) \in \mathbb{Z}_4^{k_1, k_2}} (1-s)^{m_1}t^{m_2}.$$  

Since assumption (A1) holds, we know that for all $k_1, k_2 \in \mathbb{Z}_{p+1}$, $V^{(k_1, k_2)}$ is bounded on $I^2$. Thus, for $k_1, k_2 \in \mathbb{Z}_{p+1}$ with $k_1 + k_2 \geq 4$, there is a constant $c_4$ such that for all $s, t \in I$,

$$|V^{(k_1, k_2)}(s, t)(1-s)^{k_1}t^{k_2}| \leq c_4(1-s)^{k_1}t^{k_2} \leq c_4 \sum_{(m_1, m_2) \in \mathbb{Z}_4^{k_1, k_2}} (1-s)^{m_1}t^{m_2}.$$
Combining the above two inequality yields to the desired inequality (4.54). □

We next analyze the regularity of function $G$ and $J$ by the following lemmas.

**Lemma 4.9** If assumptions (A1) and (A2) hold, then there is a positive constant $c$ such that for all $k_1, k_2 \in \mathbb{Z}_{p+1}$ and $(s, t) \in S_0(\frac{1}{2})$,

$$
J^{(k_1, k_2)}(s, t)(1 - s)^{k_1} t^{k_2} \leq c \sum_{\lambda \in \Lambda_{k_1+k_2}} \frac{\sum_{(m_1, m_2) \in Z_{k_1+k_2}^1} (1 - s)^{m_1} t^{m_2}}{((1 - s)^2 + t^2)^{|\lambda|+1}}. \quad (4.57)
$$

and

$$
G^{(k_1, k_2)}(s, t)(1 - s)^{k_1} t^{k_2} \leq c \sum_{\lambda \in \Lambda_{k_1+k_2}} \frac{\sum_{(m_1, m_2) \in Z_{k_1+k_2}^1} (1 - s)^{m_1} t^{m_2}}{((1 - s)^2 + t^2)^{|\lambda|+2}}. \quad (4.58)
$$

**Proof:** Note that $J = \frac{1}{g_1} U$. Thus, from (4.44), (4.53) and Lemma 4.3, we obtain the result (4.57) of this lemma.

From (4.51), (4.54) and Lemma 4.3, we know that there is a positive constant $c_1$ such that for $j_1, j_2 \in \mathbb{Z}_{p+1}$ and all $(s, t) \in S_0(\frac{1}{2})$,

$$
|f_2 V^{(j_1, j_2)}(s, t)(1 - s)^{j_1} t^{j_2}| \leq c_1 \sum_{(m_1, m_2) \in Z_{j_1+j_2}^1} (1 - s)^{m_1} t^{m_2}. \quad (4.59)
$$

Meanwhile, from (4.42), (4.52), and Lemma 4.3, we know that there is a positive constant $c_2$ such that for all $j_1, j_2 \in \mathbb{Z}_{p+1}$, and all $(s, t) \in S_0(\frac{1}{2})$,

$$
|g_1 g_2^{(j_1, j_2)}(s, t)(1 - s)^{j_1} t^{j_2}| \leq c_2 \sum_{(m_1, m_2) \in Z_{j_1+j_2}^1} (1 - s)^{m_1} t^{m_2}. \quad (4.60)
$$

Then, by using Lemma 4.1, we obtain that there is a positive constant $c_3$ such that for all $j_1, j_2 \in \mathbb{Z}_{p+1}$, and all $(s, t) \in S_0(\frac{1}{2})$,

$$
\left| \left( \frac{1}{g_1 g_2} \right)^{j_1,j_2} (s, t)(1 - s)^{j_1} t^{j_2} \right| \leq c_3 \sum_{\lambda \in \Lambda_{j_1+j_2}} (-1)^{|\lambda|} |\lambda|! \frac{\sum_{(m_1, m_2) \in Z_{4|\lambda|+2}} (1 - s)^{m_1} t^{m_2}}{(g_1 g_2)^{|\lambda|+1}(s, t)}. \quad (4.61)
$$

Note that for all $(s, t) \in S_0(\frac{1}{2})$, further, by noting that $g_2 = V + g_1$, using (4.46) and (4.54), we can claim that there is a positive constant $c_4$ such that for all $(s, t) \in S_0(\frac{1}{2})$,

$$
|g_2(s, t)| \geq c_4 ((1 - s)^2 + t^2).
$$

Thus, substituting (4.46) and the above inequality into (4.60) , we obtain that there is a constant $c_5$ such that for all $j_1, j_2 \in \mathbb{Z}_{p+1}$, and all $(s, t) \in S_0(\frac{1}{2})$,

$$
\left| \left( \frac{1}{g_1 g_2} \right)^{j_1,j_2} (s, t)(1 - s)^{j_1} t^{j_2} \right| \leq c_5 \sum_{\lambda \in \Lambda_{j_1+j_2}} \frac{\sum_{(m_1, m_2) \in Z_{4|\lambda|+2}} (1 - s)^{m_1} t^{m_2}}{((1 - s)^2 + t^2)^{|\lambda|+2}}. \quad (4.61)
$$

Then, replacing $\varphi$ and $\phi$ in Lemma 4.3 by $\frac{1}{g_1 g_2}$ and $f_2 V$, respectively, from Lemma 4.3, we obtain the result (4.58) of this lemma. □

Combining Lemmas 4.2 and 4.9, we obtain the regularity of $M$ on $S_0(\epsilon/2)$.
Lemma A.2 also ensures that for all \((k, k)\), \(k, k \in \mathbb{Z}_p\), exist and are continuous on \(S_0(\epsilon/2)\).

**Proof:** Since the assumptions (A1) holds, we know that \(M^{(k, k)}\) exists, and is continuous on \(S_0(\epsilon/2)\). We next show that \(M^{(k, k)}\) exists, and is continuous at point \((1, 0)\). For all \((s, t) \in S_0(\epsilon/2)\), by the definition of \(M\) and the derivative formula, it is easy to see that

\[
M^{(k, k)}(s, t) = \frac{1}{\pi} \sum_{j=0}^{k} C^j_k \gamma^{(k_2-j+1)}(t) \left( (J \circ \gamma)^{(k_1, j)} - (G \circ \gamma)^{(k_1, j)} \right)(s, t).
\]

Then we have

\[
\left| M^{(k, k)}(s, t) \right| \leq \frac{1}{\pi} \sum_{j=0}^{k} C^j_k \left| \gamma^{(k_2-j+1)}(t) \right| \left( \left| (J \circ \gamma)^{(k_1, j)}(s, t) \right| + \left| (G \circ \gamma)^{(k_1, j)}(s, t) \right| \right).
\]

Note that (4.57) shows \(J\) satisfies (4.36) with \(l = 2\), \(l_0 = 3\), and \(\varphi_j(s, t) = \frac{1}{((1-s)^2+t^{2p})^{j+1}}\), \(j \in \mathbb{Z}_{2p+1}\). Meanwhile, (4.58) shows \(G\) satisfies (4.36) with \(l = 4\), \(l_0 = 5\), and \(\varphi_j(s, t) = \frac{1}{((1-s)^2+t^{2p})^{j+2}}\), \(j \in \mathbb{Z}_{2p+1}\). Thus, from Lemma 4.2 and the definition of \(\gamma\), we obtain that there is a positive constant \(c_1\) such that for all \((s, t) \in S_0(\epsilon/2)\),

\[
\left| M^{(k, k)}(s, t) \right| \leq c_1 \sum_{j=0}^{k} \sum_{\lambda \in \Lambda_{k+j}} \left( \sum_{(m_1, m_2) \in \mathbb{Z}_{2|\lambda|+5}^{k_1, j}} \frac{(1-s)^{m_1p-k_1}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+2}} + \sum_{(m_1, m_2) \in \mathbb{Z}_{2|\lambda|+3}^{k_1, j}} \frac{(1-s)^{m_1p-k_1}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+3}} \right)^{p-k_1-1}.
\]

(4.62)

Noting that

\[
\frac{(1-s)^{m_1p-k_1}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+2}} = \frac{(1-s)^{m_1p-min\{k_1,1\}p}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+2}} \frac{(1-s)^{min\{k_1,1\}p-k_1}}{(1-s)^{min\{k_1,1\}p-k_1}}.
\]

When \((m_1, m_2) \in \mathbb{Z}_{2|\lambda|+5}^{k_1, j}\), there holds \(m_1 + m_2 = p - min\{k_1,1\}p - 4p|\lambda| - 4p = p - min\{k_1,1\}p\). Thus, Lemma A.2 ensures that for all \((m_1, m_2) \in \mathbb{Z}_{2|\lambda|+5}^{k_1, j}\) and \((s, t) \in S_0(\epsilon/2)\),

\[
\frac{(1-s)^{m_1p-k_1}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+2}} \leq \frac{((1-s)^2p+t^{2p})^{p-min\{k_1,1\}p}}{(1-s)^{min\{k_1,1\}p-k_1}} \frac{(1-s)^{min\{k_1,1\}p-k_1}}{(1-s)^{min\{k_1,1\}p-k_1}} \leq (1-s)^{min\{k_1,1\}p-k_1}.
\]

Similarly, when \((m_1, m_2) \in \mathbb{Z}_{2|\lambda|+3}^{k_1, j}\), there holds \((m_1 + m_2)p - min\{k_1,1\}p - 2p|\lambda| - 2p = p - min\{k_1,1\}p\). Lemma A.2 also ensures that for all \((m_1, m_2) \in \mathbb{Z}_{2|\lambda|+3}^{k_1, j}\) and \((s, t) \in S_0(\epsilon/2)\),

\[
\frac{(1-s)^{m_1p-k_1}t^{m_2p}}{((1-s)^2p+t^{2p})^{2|\lambda|+1}} \leq (1-s)^{min\{k_1,1\}p-k_1}.
\]

Then, substituting the above two inequality into (4.62) shows that there is a positive constant \(c_2\) such that for all \((s, t) \in S_0(\epsilon/2)\),

\[
M^{(k, k)}(s, t) \leq c_2(1-s)^{min\{k_1,1\}p-k_1}t^{p-k_2-1}.
\]
This implies $M^{(k_1,k_2)}$ exists, and is continuous at point $(1,0)$. \hfill \Box

Similarly, we can show that $M^{(k_1,k_2)}$ is continuous on $\mathbb{S}_1(\epsilon/2)$, for all $k_1, k_2 \in \mathbb{Z}_p$. Then, we get directly the regularity of $M$ on $I^2$ which is claimed in the following corollary.

**Corollary 4.11** If assumptions (A1) and (A2) hold, then $M^{(k_1,k_2)}$, $k_1, k_2 \in \mathbb{Z}_p$, exist and are continuous on $I^2$.

**Proof:** The definition of $M$ shows that $M^{(k_1,k_2)}$, $k_1, k_2 \in \mathbb{Z}_p$, exist and are continuous on $I^2 \setminus \{(1,0), (0,1)\}$. Theorem 4.10 shows that for each $k_1, k_2 \in \mathbb{Z}_p$, $M^{(k_1,k_2)}$ exist and is continuous at $(1,0)$. By the proof similar to Theorem 4.10, we know that for each $k_1, k_2 \in \mathbb{Z}_p$, $M^{(k_1,k_2)}$ exist and is continuous at $(0,1)$ Thus, this corollary holds. \hfill \Box

### 4.3 Stability and Convergence Analysis

In this subsection, we investigate the stability of the proposed fast multiscale wavelet Galerkin method and estimate the resulting approximate solution. The unique solvability of equation (3.21) is a direct consequence of the stability of the fast multiscale wavelet Galerkin method.

The analysis of this method requires the operator form of the equation (3.21). To meet this end, we convert linear system (3.21) to an abstract operator equation form. For $\varphi \in L^2(I)$ and $s \in I$, define

$$
(\mathcal{L}_n \varphi)(s) := \int_I \tilde{L}_n(s,t) \varphi(t) dt,
$$

where

$$
\tilde{L}_n(s,t) := \sum_{(i,j),(i',j') \in U_n} (\tilde{L}_{ij,n^0}^{0,0} + \tilde{L}_{ij,n^1}^{1,0} w_{ij})(t) w_{i'j'}(s).
$$

Then solving the linear system (3.21) is equivalent to finding $\tilde{\rho}_n = \sum_{(i,j) \in U_n} \tilde{p}_{ij} w_{ij} \in X_n$ such that

$$
(\mathcal{I} - \mathcal{L}_n - \mathcal{M}_n) \tilde{\rho}_n = f_n. \tag{4.63}
$$

To analyze the stability and convergence of the proposed method, we first estimate the convergence rate of the solution of equation (3.15). For $q \in \mathbb{N}$, denote by $H^q(I)$ the usual Sobolev space of order $q$ with the norm

$$
\|\varphi\|_{H^q} := \|\varphi\| + \|\varphi^{(q)}\|.
$$

For operators $A$ and $A_n$, $n \in \mathbb{N}_0$, we call operators $A_n$, $n \in \mathbb{N}_0$, point-wisely convergent to $A$ on $L^2(I)$ if and only if for all $\varphi \in L^2(I),

$$
\lim_{n \to \infty} \| (A - A_n) \varphi \| = 0.
$$

For all $n \in \mathbb{N}_0$, define $S_n := \mathcal{I} - \mathcal{L}_n$.

**Theorem 4.12** Let $q \in \mathbb{N}$. If the vanish moment $r$ of the multiscale wavelet basis is less than $p$, $q \leq r$, and $S - M$ is injective from $L^2(I)$ to $L^2(I)$, then the inverse operators of $S_n - M_n$, $n \in \mathbb{N}_0$, exist and are uniformly bounded for sufficiently large $n$. Further, if the solution $\rho$ of (2.6) is in $H^q(I)$, then there exists a positive constant $c$ such that for sufficiently large $n$,

$$
\|\rho_n - \rho\| \leq c 2^{-nq} \|\rho\|_{H^q}. \tag{4.64}
$$

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Proof: Equation (3.15) can be rewritten as

\[(S_n - M_n) \rho_n = P_n f.\]

Lemma 2.2 ensures that there exists a positive constant \(c_1 < 1\) such that \(\|L\| \leq c_1\). Further, since \(\|P_n\| = 1\) and \(L_n = P_n L\), there holds that for all \(n \in \mathbb{N}_0\), \(\|L_n\| \leq c_1 < 1\). This implies that the inverse operators of \(S_n\), \(n \in \mathbb{N}_0\), exist and are uniformly bounded. Since \(S_n^{-1} P_n - S_n^{-1} = S_n^{-1} (P_n S - S_n) S_n^{-1} = S_n^{-1} (P_n - I) S_n^{-1}\), and operators \(P_n, n \in \mathbb{N}_0\), point-wisely convergent to \(I\) on \(L^2(I)\), operators \(S_n^{-1} P_n, n \in \mathbb{N}_0\), point-wisely convergent to \(S^{-1}\) on \(L^2(I)\). Thus, by noting that \(M_n = P_n M\), from Theorem 10.6 of [22] and the compactness of \(M\), we know that

\[\lim_{n \to \infty} \|S^{-1} M - S_n^{-1} M_n\| = 0.\]

Therefore, by Theorem 2.6 of [22] and Theorem 2.3, the inverse operators of \(I - S_n^{-1} M_n, n \in \mathbb{N}_0\), exist and are uniformly bounded for all sufficiently large \(n\). So do the inverse operators of \(S_n - M_n, n \in \mathbb{N}_0\).

The standard argumentation for projection methods shows that for all \(\rho \in H^q(I)\) and \(n \in \mathbb{N}_0\),

\[\rho_n - \rho = ((S_n - M_n)^{-1} P_n (S - M) - I) \rho.\]

Since for all \(\varphi \in X_n, P_n (S - M) \varphi = (S_n - M_n) \varphi\), from the above equation we have

\[\rho_n - \rho = ((S_n - M_n)^{-1} P_n (S - M) - I) (\rho - \varphi).\]

Thus, by noting that the inverse operators of \(S_n - M_n, n \in \mathbb{N}_0\), are uniformly bounded for sufficiently large \(n\), from (4.65) we have that there exists a positive constant \(c\) such that for all \(\rho \in H^q(I)\) and sufficiently large \(n\),

\[\|\rho_n - \rho\| \leq c \|\rho - P_n \rho\|.
\]

Since \(q \leq r\), from the property (V) of the multiscale wavelet basis, we obtain inequality (4.64).

We next turn to establish the stability and convergence analysis of the proposed fast multiscale Galerkin method, by estimating the difference between matrices \(L_n\) and \(\tilde{L}_n\), and between matrices \(M_n\) and \(\tilde{M}_n\).

Let \(L_{n,i}^\kappa := [L_{n,i,j}^\kappa : j' \in \mathbb{Z}_{w(i')}, j \in \mathbb{Z}_{w(i)}]\) for \(\kappa \in \mathbb{Z}_2\) and \(i', i \in \mathbb{Z}_{n+1}\). Define \(L_{n,i}^\kappa = L_{n,i}^0 + L_{n,i}^1\). It is clear that

\[L_n = [L_{n,i}^\kappa : i', i \in \mathbb{Z}_{n+1}].\]

To study the difference between \(L_n\) and \(\tilde{L}_n\), we investigate the difference between \(L_{n,i}^\kappa\) and \(\tilde{L}_{n,i}^\kappa\) for all \(\kappa \in \mathbb{Z}_2\) and \(i', i \in \mathbb{Z}_{n+1}\). As a preparation, we show the boundary of \(L_{n,i'}^{0,j},ij\) and \(L_{n,i'}^{1,j},ij\) in the following lemma.

**Lemma 4.13** Let \(\kappa \in \mathbb{Z}_2\). If \(r < p\), then there has a positive constant \(c\) such that for all \((i, j), (i', j') \in U\) with \(\text{dist}(S_{ij}, \kappa) > 0\),

\[|L_{n,i'}^{\kappa,j},ij| \leq c 2^{-(i+i')(r-1)/2} \int_{S_{i'}^{ij}} \int_{S_{ij}} \frac{1}{|\kappa - t|^{2r+1}} dt ds,\]

(4.66)

and for all \((i, j), (i', j') \in U\) with \(\text{dist}(S_{i'j'}, 1 - \kappa) > 0\),

\[|L_{n,i'}^{\kappa,j'},ij| \leq c 2^{-(i+i')(r-1)/2} \int_{S_{i'j'}} \int_{S_{ij}} \frac{1}{|1 - \kappa - s|^{2r+1}} dt ds.\]

(4.67)
Lemma 4.14

(i) for all \(Z\) condition dist(\(S\)) hence, combining (4.68) with (4.70), we obtain (4.66). By the similar process of argument, from the above inequality into (4.69) leads to

Then, substituting above inequality into (4.69) leads to

This with the properties (I) and (IV) shows that for \(s \in S'_{i,j'}\) and \(t \in S_{i,j}\),

where \(s' \in S'_{i,j'}\) and \(t' \in S_{i,j}\). Thus, combining the above equality with Corollary 4.7, we know that

where \(c_1\) is the constant in Lemma 4.7.

When \(\text{dist}(S_{i,j}, \kappa) > 0\), from the property (I) we have \(\text{dist}(S_{i,j}, \kappa) \geq \text{meas}(\text{supp}(w_{i,j}))\). This implies that for any \(t', t \in S_{i,j}\),

Then, substituting above inequality into (4.69) leads to

Hence, combining (4.68) with (4.70), we obtain (4.66). By the similar process of argument, from the condition dist(\(S'_{i,j'}, 1 - \kappa\)) > 0 and (4.68), we obtain inequality (4.67).

We next investigate the difference between \(L_{i,j'}^\kappa\) and \(\tilde{L}_{i,j'}^{\kappa,n}\), by the following two lemmas. Define \(Z_{w_i}(\delta) := \{j \in \mathbb{Z}_{w(i)} : \text{dist}(S_{i,j}, \kappa) \leq \delta\}\).

Lemma 4.14 Let \(\kappa \in \mathbb{Z}_2\). If \(r < p\), then there exists a positive constant \(c\) such that

(i) for all \(i, i' \in \mathbb{N}_0\) and \(j' \in \mathbb{Z}_{w(i')}\),

\[
\sum_{j \in \mathbb{Z}_{w(i)} \setminus Z_w(\tau_{i,j'})^\kappa} \leq c 2^{-i(r-1/2)-i'(r+1/2)} (\tau_{i,j'}^n)^{-2r},
\]
(ii) For all \(i, i' \in \mathbb{N}_0\) and \(j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii}', \tau_{ii}')\),
\[
\sum_{j \in Z_{w(i)}} |L_{ij}^n| \leq c2^{-i(r-1/2)-i'(r-1/2)}(\tau_{ii}')^{-2r}, \tag{4.72}
\]

(iii) For all \(i, i' \in \mathbb{N}_0\) and \(j \in Z_{w(i)}\),
\[
\sum_{j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii})} |L_{ij}^n| \leq c2^{-i(r-1/2)-i'(r-1/2)}(\tau_{ii})^{-2r}, \tag{4.73}
\]

(iv) For all \(i, i' \in \mathbb{N}_0\) and \(j \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii}')\),
\[
\sum_{j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii})} |L_{ij}^n| \leq c2^{-i(r+1/2)-i'(r-1/2)}(\tau_{ii}')^{-2r}. \tag{4.74}
\]

**Proof:** From inequality (4.66) of Lemma 4.13, we know that there exists a constant \(c_1\) such that for all \(i, i' \in \mathbb{N}_0\) and \(j' \in Z_{w(i')}\),
\[
\sum_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} |L_{ij}^n| \leq c_12^{-i(i+i')(r-1/2)} \int_{S_{ij}} \left( \sum_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} \int_{S_{ij}} \frac{1}{|t-\kappa|^{2r+1}} dt \right) ds. \tag{4.75}
\]

The property (I) and the definition of \(Z_{v_1}(\tau_{ii}')\) show that
\[
\bigcup_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} S_{ij} \subset [\min\{|\kappa-\tau_{ii}'|, 1-\kappa\}, \max\{|\kappa-\tau_{ii}'|, 1-\kappa\}].
\]

Thus, the term in the bracket of (4.75) satisfies
\[
\sum_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} \int_{S_{ij}} \frac{1}{|t-\kappa|^{2r+1}} dt \leq r \int_{\min\{|\kappa-\tau_{ii}'|, 1-\kappa\}}^{\max\{|\kappa-\tau_{ii}'|, 1-\kappa\}} \frac{1}{|t-\kappa|^{2r+1}} dt.
\]

Therefore, by substituting the above inequality into (4.75) and conducting some computation, we obtain (4.71).

Now, we prove (4.72). For all \(i, i' \in \mathbb{N}_0\) and \(j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii}')\),
\[
\sum_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} |L_{ij}^n| = \sum_{j \in Z_{v_1}(\tau_{ii})} |L_{ij}^n| + \sum_{j \in Z_{w(i)} \setminus Z_{p_1}(\tau_{ii})} |L_{ij}^n| \tag{4.76}
\]

From (4.67) of Lemma 4.13, we obtain that there exists a constant \(c_2\) for all \(i, i' \in \mathbb{N}_0\) and \(j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii}')\),
\[
\sum_{j \in Z_{v_1}(\tau_{ii})} |L_{ij}^n| \leq c_22^{-i(i+i')(r-1/2)} \sum_{j \in Z_{v_1}(\tau_{ii})} \int_{S_{ij}} \int_{S_{ij}} \frac{1}{|1-\kappa-s|^{2r+1}} dsdt. \tag{4.77}
\]

The definition of \(Z_{v_1}(\tau_{ii}')\) shows that for \(j' \in Z_{w(i')} \setminus Z_{v_1}(\tau_{ii}')\) and \(s \in S_{ij}'\),
\[
|1-\kappa-s| > \tau_{ii}'.
\]
Thus, by property (I) we have that for \( j' \in Z_w(i') \setminus Z_{i'}^{1-\kappa}(\tau_{i'}^n) \) and \( j \in Z_i^n(\tau_{i'}^n) \),

\[
\int_{S_{i',j'}} \frac{1}{1 - \kappa - s|2r+1|} ds \leq 2^{-i'+1}(\tau_{i'}^n)^{-2r-1}.
\]  

From property (I), we also know that \( \sum_{j \in Z_i^n(\tau_{i'}^n)} \text{meas}(S_{i,j}) \leq c r \tau_{i'}^n \). Hence, combining (4.77) with (4.78), we have that for all \( i, i' \in \mathbb{N}_0 \) and \( j' \in Z_w(i') \setminus Z_{i'}^{1-\kappa}(\tau_{i'}^n) \),

\[
\sum_{j \in Z_i^n(\tau_{i'}^n)} |L_{i',j',ij}^n| \leq c_2 2^{2r-2(r-\frac{1}{2})-i'2(r+1)}(\tau_{i'}^n)^{-2r}.
\]  

Substituting (4.71) and (4.79) into (4.76) leads to (4.72). By the symmetry between \( (i, j) \) and \( (i', j') \), we obtain the inequalities (4.73) and (4.74) from (4.71) and (4.72).

To measure the difference between \( L_{i',j'}^n \) and \( \tilde{L}_{i',j'}^n \), we define the following norms of matrix. For matrix \( A = [a_{ij} \in \mathbb{R}]_{n \times n} \), let

\[
\|A\|_1 := \max_{0 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}, \quad \|A\|_\infty := \max_{0 \leq j \leq n} \left\{ \sum_{i=1}^{n} |a_{ij}| \right\}
\]

and

\[
\|A\| := \left( \max \{\Pi(A^T A)\} \right)^{\frac{1}{2}},
\]

where \( \Pi(A^T A) \) is the set of the eigenvalues of matrix \( A^T A \).

**Lemma 4.15** Let \( \kappa \in \mathbb{Z}_2 \). If \( r < p \), then there exists a positive constant \( c \) such that for all \( n \in \mathbb{N}_0 \) and \( i, i' \in Z_{n+1} \),

\[
\|L_{i,i'}^n - \tilde{L}_{i,i'}^{0,n} \|_\infty \leq c 2^{-i(r-\frac{1}{2})-i'(r+\frac{1}{2})}(\tau_{i,i'}^n)^{-2r},
\]  

(4.80)

\[
\|L_{i,i'}^n - \tilde{L}_{i,i'}^{n,n} \|_1 \leq c 2^{-i(r-\frac{1}{2})-i'(r-\frac{1}{2})}(\tau_{i,i'}^n)^{-2r},
\]  

(4.81)

and

\[
\|L_{i,i'}^n - \tilde{L}_{i,i'}^{n,n} \| \leq c 2^{-(i+i')r}(\tau_{i,i'}^n)^{-2r}.
\]  

(4.82)

**Proof:** The truncation strategy (T1) ensures that

\[
\sum_{j \in Z_w(i)} |L_{i',j',ij}^n - \tilde{L}_{i',j',ij}^n| = \left\{ \begin{array}{ll}
\sum_{j \in Z_w(i) \setminus Z_{i,i'}^{1-\kappa}(\tau_{i,i'}^n)} |L_{i',j',ij}^n|, & \quad j' \in Z_{i'}^{1-\kappa}(\tau_{i'}^n),
\sum_{j \in Z_w(i)} |L_{i',j',ij}^n|, & \quad j' \in Z_w(i') \setminus Z_{i'}^{1-\kappa}(\tau_{i'}^n),
\end{array} \right.
\]  

(4.83)

and

\[
\sum_{j' \in Z_w(i')} |L_{i',j',ij}^n - \tilde{L}_{i',j',ij}^n| = \left\{ \begin{array}{ll}
\sum_{j' \in Z_w(i') \setminus Z_{i,i'}^{1-\kappa}(\tau_{i,i'}^n)} |L_{i',j',ij}^n|, & \quad j \in Z_{i,i'}^{1-\kappa}(\tau_{i,i'}^n),
\sum_{j' \in Z_w(i')} |L_{i',j',ij}^n|, & \quad j \in Z_w(i) \setminus Z_{i,i'}^{1-\kappa}(\tau_{i,i'}^n).
\end{array} \right.
\]  

(4.84)

According to the definition of \( \| \cdot \|_\infty \), we obtain (4.80) by substituting inequalities (4.71) and (4.72) into (4.83). According to the definition of \( \| \cdot \|_1 \), we obtain (4.81) by substituting inequalities (4.73) and (4.74) into (4.84).

Since \( L_{i,i'}^n = L_{i,i'}^0 + L_{i,i'}^1 \) and \( \tilde{L}_{i,i'}^n = \tilde{L}_{i,i'}^{0,n} + \tilde{L}_{i,i'}^{1,n} \), inequalities (4.80) and (4.81) ensure that there exists a positive constant \( c_1 \) such that for all \( n \in \mathbb{N}_0 \) and \( i, i' \in Z_{n+1} \),

\[
\|L_{i,i'}^n - \tilde{L}_{i,i'}^{n,n} \|_\infty \leq c_1 2^{-i(r-\frac{1}{2})-i'(r+\frac{1}{2})}(\tau_{i,i'}^n)^{-2r},
\]  

(4.85)
and
\[
\|L_{i'i} - \tilde{L}_{i'i}^n\|_1 \leq c_1 2^{-i(r+1/2) - i'(r-1/2)} (\tau_{i'i}^n)^{-2r}.
\]
(4.86)

Note that \(\|L_{i'i} - \tilde{L}_{i'i}^n\|_1^2 \leq \|L_{i'i} - \tilde{L}_{i'i}^n\|_1 \|L_{i'i} - \tilde{L}_{i'i}^n\|_\infty\). Thus, from (4.85) and (4.86), we obtain inequality (4.82).

Now, we are ready to investigate the difference between operators \(L_n\) and \(\tilde{L}_n\).

**Lemma 4.16** Let \(q \in \mathbb{N}\) with \(q \leq r\). If \(r < p\), then there exists a positive constant \(c\) such that for all \(n \in \mathbb{N}_0\) and \(u \in H^q(I)\),
\[
\|(L_n - \tilde{L}_n)P_n u\| \leq cn 2^{-qn} \|u\|_{H^q}.
\]

**Proof:** Let \(e_n(u) := \|(L_n - \tilde{L}_n)P_n u\|\). The definitions of \(L_n\) and \(\tilde{L}_n\) show that for \(u \in L^2(I)\),
\[
e_n(u) \leq \sum_{i, i' \in \mathbb{Z}_{n+1}} \|L_{i'i} - \tilde{L}_{i'i}^n\| \|(P_i - P_{i-1})u\|.
\]

Thus, from property (V) and Lemma 4.15, we have that there exists a positive constant \(c_1\) such that for \(u \in H^q(I)\),
\[
e_n(u) \leq c_1 \|u\|_{H^q} \sum_{i, i' \in \mathbb{Z}_{n+1}} 2^{-(i+i')r-(i-1)q} (\tau_{i'i}^n)^{-2r}.
\]

Then, by the definition of \(\tau_{i'i}^n\), we have
\[
e_n(u) \leq c_1 \|u\|_{H^q} \sum_{i, i' \in \mathbb{Z}_{n+1}} 2^{-(i+i')r} (2^{-i+a(n-i')} - 2^{-r} 2^{-q} \sum_{i, i' \in \mathbb{Z}_{n+1}} 2^{(n-1)(q-r)+(n-i')(1-2a)r},
\]
(4.87)

where \(a\) appears in the definition of \(\tau_{i'i}^n\). Because of \(q \leq r\) and \(a > \frac{1}{2}\), applying the sum formula of geometric progression to (4.87), we obtain the desired result. \(\square\)

We next turn to analyze the difference between \(M_n\) and \(\tilde{M}_n\). The analysis is based on the following lemma which can be proved by applying the result of Lemma 2.1 of [9] to function \(M\).

**Lemma 4.17** Let \(q \in \mathbb{N}\). If the assumptions (A1) and (A2) hold, and \(q < p\), then there exists a constant \(c\) such that for all \(i, i' \in \mathbb{N}_0\),
\[
\|M_{1i'}\| \leq c 2^{-(i+i')q}.
\]

**Proof:** Since the assumptions (A1) and (A2) hold, Corollary 4.11 shows that for \(k_1, k_2 \in \mathbb{Z}_p\), \(M^{(k_1,k_2)}\) exists and is continuous on \(I^2\). Then, from Lemma 2.1 of [9], we obtain the result of this lemma. \(\square\)

With the smoothness assumptions (A1) and (A2) and Lemma 4.17, we obtain the estimate of the truncation error of operator \(M\).

**Lemma 4.18** Let \(q \in \mathbb{N}\) with \(q \leq r\). If \(r < p\), and the assumptions (A1) and (A2) hold, then there exists a positive constant \(c\) such that for all \(u \in H^q(I)\) and \(n \in \mathbb{N}_0\),
\[
\|(M_n - \tilde{M}_n)P_n u\| \leq c 2^{-qn} \|u\|_{H^q}.
\]

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Proof: Let \( d_n(u) := \| (\mathcal{M}_n - \mathcal{M}_n) \mathcal{P}_n u \| \) for \( u \in L^2(I) \). The truncation strategy (T2) ensures that for all \( u \in L^2(I) \) and \( n \in \mathbb{N}_0 \),

\[
d_n(u) \leq \sum_{i'+(1+b)i > n} \| M_{i'} \| \| (\mathcal{P}_i - \mathcal{P}_{i-1}) u \|.
\]

Combining the above inequality with property (V), we obtain that there exists constant \( c_1 \) such that for \( u \in H^q(I) \) and \( n \in \mathbb{N}_0 \),

\[
d_n(u) \leq c_1 \| u \|_{H^q} \sum_{i'+(1+b)i > n} 2^{-(i+i')q-(i-1)q} = c_1 2^{-nq+q} \| u \|_{H^q} \sum_{i\in\mathbb{Z}_{n+1}} \sum_{n-i'>(1+b)i} 2^{(n-i')q-2qi} \leq c_1 2^{-nq+q} \| u \|_{H^q} \sum_{i\in\mathbb{Z}_{n+1}} \frac{2^{(b-1)iq}}{2^q - 1}.
\]

Because of \( 0 < b < 1 \), we obtained the desired result from the above inequality. \( \square \)

Now, we are ready to establish the stability and convergence analysis of the proposed fast multiscale wavelet Galerkin method. These results are presented in the next theorem.

**Theorem 4.19** Let \( q \in \mathbb{N} \) with \( q \leq r \). If \( r < p \), the assumptions (A1) and (A2) hold, \( \mathcal{S} - \mathcal{M} \) is injective from \( L^2(I) \) to \( L^2(I) \), and \( \rho \in H^q(I) \), then there exist a positive constant \( c \) and a positive integer \( n_0 \) such that for \( n \geq n_0 \) and \( \varphi \in H^q(I) \),

\[
\|(I - \tilde{\mathcal{L}}_n - \tilde{\mathcal{M}}_n) \mathcal{P}_n \varphi\| \geq c \| \varphi \|,
\]

and

\[
\|\rho - \tilde{\rho}_n\| \leq cn2^{-qn}\|\rho\|_{H^q}.
\]

Proof: According to proof of Theorem 4.12, we obtain that the operator \( (\mathcal{S}_n - \mathcal{M}_n)^{-1} \) exists and uniform boundedness for sufficiently large \( n \). Thus, by noting that

\[
\|(I - \tilde{\mathcal{L}}_n - \tilde{\mathcal{M}}_n) \mathcal{P}_n \varphi\| \geq \|(I - \mathcal{L}_n - \mathcal{M}_n) \mathcal{P}_n \varphi\| - \|(\tilde{\mathcal{L}}_n - \mathcal{L}_n) \mathcal{P}_n \varphi\| - \|(\mathcal{M}_n - \tilde{\mathcal{M}}_n) \mathcal{P}_n \varphi\|,
\]

from Lemmas 4.16 and 4.18, we obtain inequality (4.88). Inequality (4.88) implies that for sufficiently large \( n \),

\[
\|\rho - \tilde{\rho}_n\| \leq \|\rho - \mathcal{P}_n \rho\| + \frac{1}{c} \|(I - \tilde{\mathcal{L}}_n - \tilde{\mathcal{M}}_n) (\mathcal{P}_n \rho - \tilde{\rho}_n)\|.
\]

Since

\[
\mathcal{P}_n (I - \mathcal{L} - \mathcal{M}) \rho = (I - \tilde{\mathcal{L}}_n - \tilde{\mathcal{M}}_n) \tilde{\rho}_n,
\]

we find that

\[
(I - \tilde{\mathcal{L}}_n - \tilde{\mathcal{M}}_n) (\mathcal{P}_n \rho - \tilde{\rho}_n) = (\mathcal{P}_n (I - \mathcal{L} - \mathcal{M})) (\mathcal{P}_n \rho - \rho) + (\tilde{\mathcal{L}}_n - \mathcal{L}_n) \mathcal{P}_n \rho + (\mathcal{M}_n - \tilde{\mathcal{M}}_n) \mathcal{P}_n \rho.
\]

(4.91)

Substituting (4.91) into (4.90) and using the Cauchy-Schwarz inequality, we obtain (4.89) from Lemmas 4.16 and 4.18. \( \square \)

It is clear that \( \tilde{\rho}_n \circ \gamma^{-1} \) is an approximation of the solution \( \varrho \) of equation (2.5), where \( \gamma^{-1}(s) \) is the inverse of \( \gamma(s) \) on \( I \). There is still a question: how good does \( \tilde{\rho}_n \circ \gamma^{-1} \) approximation \( \varrho \)? In order
Lemma 4.20 There exists a positive constant $c$ such that for all $\varphi \in L^2(I)$,
\[
\|\varphi\| \leq c\|\varphi \circ \gamma\|.
\] (4.92)

Let $\alpha > 0$. If the assumption (A1) holds and $k \in \mathbb{N}$ with $\max \left\{ \frac{k-1/2}{\alpha}, k \right\} < p$, then there is a positive constant $c$ such that for all $\varphi \in C_{k,\alpha}$,
\[
\|\varphi \circ \gamma\|_{H^k} \leq c\|\varphi\|_{k,\alpha},
\] (4.93)
where $p$ appears in the definition of $\gamma$.

Proof: Since $\|\varphi\| = \left( \int_0^1 \varphi^2(s)ds \right)^{1/2}$, by the change of variable $s = \gamma(t)$ in the right term of the above equality and $\gamma'(t)$ is bounded on $[0, 1]$, we have
\[
\|\varphi\| = \left\{ \int_0^1 \varphi^2(\gamma(t))\gamma'(t)dt \right\}^{1/2} \leq \|\gamma\|^{1/2}_{\infty} \left\{ \int_0^1 \varphi^2(\gamma(t))dt \right\}^{1/2}.
\]

The first result of this lemma is obtained.

We next prove the second inequality of this lemma. Since $k < p$ and the assumption (A1) holds, $(\varphi \circ \gamma)^{(k)}$ exists and is continuous on $I$. It is clear that
\[
\left\| (\varphi \circ \gamma)^{(k)} \right\|^2 = \int_{[0, \frac{1}{2}]} \left( (\varphi \circ \gamma)^{(k)} \right)^2 (t)dt + \int_{[\frac{1}{2}, 1]} \left( (\varphi \circ \gamma)^{(k)} \right)^2 (t)dt.
\] (4.94)

From Faà di Bruno’s formula (4.31) and the definition of $\gamma$, we have that for $t \in [0, \frac{1}{2}]$,
\[
(\varphi \circ \gamma)^{(k)}(t) = \sum_{\lambda \in \Lambda_k} \varphi^{(\lambda)}(t^p)p^{\lambda|-k} \prod_{B \in \lambda} \frac{p!}{(p - |B|)!},
\] (4.95)
and for $t \in \left[ 1 - \frac{1}{2}, 1 \right]$,
\[
(\varphi \circ \gamma)^{(k)}(t) = \sum_{\lambda \in \Lambda_k} (-1)^{|\lambda|+k} \varphi^{(\lambda)}(\gamma(t))(1-t)^{p|\lambda|-k} \prod_{B \in \lambda} \frac{p!}{(p - |B|)!}.
\] (4.96)

Since $\frac{k-1/2}{\alpha} < p$, i.e. $k < p\alpha + 1/2$, from the definition of $|\varphi|_{\lambda,\alpha}$, we know that there is a positive constant $c_1$ such that for all $\varphi \in C_{k,\alpha}$ and $\lambda \in \Lambda_k$,
\[
\int_0^{\frac{1}{2}} \left| \varphi^{(\lambda)}(t^p)p^{\lambda|-k} \right|^2 dt \leq c_1|\varphi|_{\lambda,\alpha}^2 \quad \text{and} \quad \int_{1-\frac{1}{2}}^1 \left| \varphi^{(\lambda)}(\gamma(t))(1-t)^{p|\lambda|-k} \right|^2 dt \leq c_1|\varphi|_{\lambda,\alpha}^2.
\]
Thus, from (4.95) and (4.96), we can conclude that there is a constant $c_2$ depending on $p, \alpha$ and $k$ such that for all $\varphi \in C_{k,\alpha}$,
\[
\int_{[0,\frac{1}{2}]}[(\varphi \circ \gamma)^{(k)}]^2(t)\,dt \leq c_2 \|\varphi\|_{k,\alpha}^2.
\] 
(4.97)

Due to the boundedness of $\gamma^{(j)}(t)$ on $I$ for $1 \leq j \leq k$, we can obtain that there is a constant $c_3$ such that for all $\varphi \in C_{k,\alpha}$,
\[
\int_{[\frac{1}{2},1-\frac{1}{2}]}[(\varphi \circ \gamma)^{(k)}]^2(t)\,dt \leq c_3 \|\varphi\|_{k,\alpha}^2,
\] 
(4.98)

where constant $c_3$ depends on $\gamma, \alpha$ and $k$. Substituting (4.97) and (4.98) into (4.94), we have that there exists a constant $c$ depending on $\gamma, \alpha$ and $k$ such that for all $\varphi \in C_{k,\alpha}$,
\[
\| (\varphi \circ \gamma)^{(k)} \| \leq c \|\varphi\|_{k,\alpha}.
\]

Therefore, by noting that for all $\varphi \in C_{k,\alpha}$, $\|\varphi\| \leq \|\varphi \circ \gamma\|_\infty \leq \|\varphi\|_{k,\alpha}$, we obtain the second inequality of this lemma. \qed

With Lemma 4.20, we estimate $\|g - \tilde{\rho}_n \circ \gamma^{-1}\|$ in the following proposition.

**Proposition 4.21** Let $\alpha := \frac{1}{1+|1-\frac{1}{2}|}$. If $\mathcal{S} : \mathcal{M}$ is injective from $L^2(I)$ to $L^2(I)$, $\max \left\{ \frac{r-1/2}{\alpha}, r \right\} < p$, $\varrho \in C_{r,\alpha}$ and the assumptions (A1) and (A2) hold, then there exists a positive constant $c$ such that for sufficiently large $n$,
\[
\|g - \tilde{\rho}_n \circ \gamma^{-1}\| \leq c n^{2^{-\alpha n}} \|\varrho\|_{r,\alpha}.
\]

**Proof:** Note that $\rho = g \circ \gamma$. From Lemma 4.20, we know that there exists a positive constant $c_1$ such that for all $n \in \mathbb{N}$,
\[
\|g - \tilde{\rho}_n \circ \gamma^{-1}\| \leq c_1 \|\rho - \tilde{\rho}_n\|,
\]
and $\|\rho\|_{H^r} \leq c_1 \|\varrho\|_{r,\alpha}$. Then, we obtain the result from Theorem 4.19. \qed

5 **Numerical Experiments**

In this section, we demonstrate the effectiveness and accuracy of the fast wavelet Galerkin method proposed in this paper. These examples also confirm the theoretical results of the proposed method. For each example, we employ two wavelet basis, one is piecewise linear, the other is piecewise cubic. We list both of them as follows.

The piecewise linear wavelet basis is shown as below. The orthonormal basis functions for $X_0$ are given by
\[
w_{00}(t) := 1, \quad w_{01}(t) := \sqrt{3}(2t - 1), \quad t \in [0, 1],
\]
and the orthonormal basis functions for $W_1$ are given by
\[
w_{10}(t) := \begin{cases} 1 - 6t, & t \in [0, 1/2), \\ 5 - 6t, & t \in [1/2, 1], \end{cases} \quad w_{11}(t) := \begin{cases} \sqrt{3}(1 - 4t), & t \in [0, 1/2), \\ \sqrt{3}(4t - 3), & t \in [1/2, 1]. \end{cases}
\]
The basis functions $w_{ij}$ for $i > 1, j \in \mathbb{Z}_{w(i)}$ are constructed recursively from $w_{10}$ and $w_{11}$. We denote “mod” by module operation. For $i > 1, j \in \mathbb{Z}_{w(i)}$, we define
\[
w_{ij}(t) := 2^{-i} \cdot \left(2^{i-1}t - \left\lfloor \frac{j}{2} \right\rfloor \right),
\]
where \( j' = j \mod 2 \) and \( \left\lfloor \frac{j}{2} \right\rfloor \) is the greatest integer less than or equal to \( \frac{j}{2} \). For details of the concrete construction, see [10]. In this case, the dimension \( d_n \) of space \( X_n \) is \( 2^{n+1} \), parameter \( r \) equals to 2, and we set parameter \( p = 3 \).

We next show the piecewise cubic wavelet basis. The orthonormal basis functions for \( X_0 \) are given by

\[
w_{00}(t) := 1, \quad w_{01}(t) := \sqrt{3}(2t - 1), \quad w_{02}(t) := \sqrt{5}(6t^2 - 6t + 1), \quad t \in [0, 1].
\]

and

\[
w_{03}(t) := \sqrt{7}(20t^3 - 30t^2 + 12t - 1), \quad t \in [0, 1].
\]

The orthonormal basis functions for \( W_1 \) are given by

\[
w_{10}(t) := \begin{cases} \frac{\sqrt{5}}{10}(240t^2 - 90t + 5), & t \in [0, 1/2), \\ -\frac{\sqrt{5}}{10}(240t^2 - 390t + 155), & t \in [1/2, 1], \end{cases} \quad w_{11}(t) := \begin{cases} \sqrt{3}(30t^2 - 14t + 1), & t \in [0, 1/2), \\ \sqrt{3}(30t^2 - 46t + 17), & t \in [1/2, 1], \end{cases}
\]

\[
w_{12}(t) := \begin{cases} \sqrt{7}(160t^3 - 120t^2 + 24t - 1), & t \in [0, 1/2), \\ -\sqrt{7}(160t^3 - 360t^2 + 264t - 63), & t \in [1/2, 1], \end{cases}
\]

and

\[
w_{13}(t) := \begin{cases} \frac{14\sqrt{5}}{21}(160t^3 - 120t^2 + \frac{165}{2}t - \frac{13}{14}), & t \in [0, 1/2), \\ \frac{14\sqrt{5}}{21}(160t^3 - 360t^2 + \frac{1845}{2}t - \frac{877}{14}), & t \in [1/2, 1]. \end{cases}
\]

For \( i > 1, j \in \mathbb{Z}_{w(i)} \), we define

\[
w_{ij}(t) := 2^{i-1}w_{ij'} \left( 2^{i-1}t - \left\lfloor \frac{j}{4} \right\rfloor \right).
\]

Here, the dimension \( d_n \) of space \( X_n \) is \( 2^{n+2} \), parameter \( r \) equals to 4, and we set parameter \( p = 5 \).

The solution \( u \) of the problem (2.1) is approximated by the following form

\[
\tilde{u}_n(x, y) := \int_0^1 \gamma(t)\gamma'(t)(\xi(\gamma(t)) - x) - \xi'(\gamma(t))(\eta(\gamma(t)) - y)) \frac{\tilde{\rho}_n(t)}{(\xi(\gamma(t)) - x)^2 + (\eta(\gamma(t)) - y)^2}dt, \quad (x, y) \in D,
\]

(5.99)

where \( \tilde{\rho}_n \) is the solution of equation (4.63). Function \( u_n \) is obtained by replacing \( \tilde{\rho}_n \) with \( \rho_n \) in (5.99), where \( \rho_n \) is obtained by solving the equation (3.17). The absolute errors \( e_n(x, y) \) and \( \tilde{e}_n(x, y) \) at point \((x, y) \in D \) are defined by \( e_n(x, y) := |u(x, y) - u_n(x, y)| \) and \( \tilde{e}_n(x, y) := |u(x, y) - \tilde{u}_n(x, y)| \).

Due to the lack of analytic solutions of equation (2.6), in order to estimate errors \( \|\rho_n - \rho\| \) and \( \|\tilde{\rho}_n - \rho\| \), we replace analytic solution \( \rho \) by \( \rho_{10} \) and \( \rho_8 \) for the piecewise linear and cubic basis respectively. In this section, we denote “C.O.” as the the convergence order of the algorithm, which is computed by the following form: the convergence order of index \( n \) is

\[
\log_2 \frac{\|\rho_{10} - \tilde{\rho}_{n-1}\|}{\|\rho_{10} - \rho_{n}\|} \quad \text{or} \quad \log_2 \frac{\|\rho_8 - \tilde{\rho}_{n-1}\|}{\|\rho_8 - \rho_{n}\|}
\]

(5.100)

**Example 1.** In this example, we consider solving the boundary integral equation (2.1) on the drop-shaped domain with the boundary curve \( \partial D \) showed in left of Figure 1 given by the parametric presentation

\[
\Gamma(t) = \left( \frac{2}{\sqrt{3}} \sin(\pi t), -\sin(2\pi t) \right), \quad 0 \leq t \leq 1.
\]

(5.101)

which has a corner at \( t = 0 \) with interior angle \( \beta = \frac{2\pi}{3} \). We suppose the solution of (2.1) is the harmonic function

\[
u(x, y) = r^\frac{3}{2} \cos \left( \frac{3\theta}{2} \right),
\]

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in polar coordinates $r$ and $\theta$, and the boundary condition is given by setting $g = u$ on $\partial D$.

Now we choose the piecewise linear and cubic wavelet basis to verify the theoretical estimate of the convergence order in Theorem 4.19. In Table 1, we list the numerical results obtained by the non-truncated wavelet Galerkin method and the fast wavelet Galerkin method, where the piecewise linear wavelet basis is employed. The numerical results obtained by employing the piecewise cubic wavelet basis are listed in Table 2. We can see that the results listed in the column titled “C.O.” are close to 2 and 4 in Tables 1 and 2, respectively. These results confirm the theoretical estimate of the convergence order in Theorem 4.19.

Table 1: Numerical results of algorithm based on the piecewise linear polynomials

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$|\rho_{10} - \rho_n|$</th>
<th>C.O.</th>
<th>$|\rho_{10} - \hat{\rho}_n|$</th>
<th>C.O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.43e-2</td>
<td></td>
<td>1.72e-2</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>6.14e-3</td>
<td>1.9833</td>
<td>4.36e-3</td>
<td>1.9771</td>
</tr>
<tr>
<td>128</td>
<td>1.54e-3</td>
<td>1.9957</td>
<td>1.09e-3</td>
<td>1.9945</td>
</tr>
<tr>
<td>256</td>
<td>3.85e-4</td>
<td>1.9988</td>
<td>2.74e-4</td>
<td>1.9988</td>
</tr>
<tr>
<td>512</td>
<td>9.64e-5</td>
<td>1.9979</td>
<td>6.83e-5</td>
<td>2.0023</td>
</tr>
<tr>
<td>1024</td>
<td>2.45e-5</td>
<td>1.9741</td>
<td>1.66e-5</td>
<td>2.0436</td>
</tr>
</tbody>
</table>

Table 2: Numerical results of algorithm based on piecewise cubic polynomials

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$|\rho_8 - \rho_n|$</th>
<th>C.O.</th>
<th>$|\rho_8 - \hat{\rho}_n|$</th>
<th>C.O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.0692e-3</td>
<td></td>
<td>2.0692e-3</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1.7626e-4</td>
<td>3.5533</td>
<td>1.7684e-4</td>
<td>3.5485</td>
</tr>
<tr>
<td>128</td>
<td>1.1162e-5</td>
<td>3.9810</td>
<td>1.1230e-5</td>
<td>3.9770</td>
</tr>
<tr>
<td>256</td>
<td>7.0005e-7</td>
<td>3.9950</td>
<td>7.0140e-7</td>
<td>4.0009</td>
</tr>
<tr>
<td>512</td>
<td>4.3705e-8</td>
<td>4.0016</td>
<td>4.3719e-8</td>
<td>4.0039</td>
</tr>
</tbody>
</table>

In Tables 3 and 4, we show respectively the errors $e_n(x,y)$ and $\tilde{e}_n(x,y)$ at point $(x,y)$ for the piecewise linear and cubic wavelet basis. In the second and fourth columns, we show the errors $e_n(0.2,0)$ and $e_n(0.2,2)$ between $u$ and $u_n$ at points $(0.2,0)$ and $(0.2,2)$, respectively. In the third
and fifth columns, we list the errors $\hat{e}_n(0.2, 0)$ and $\hat{e}_n(0.2, 2)$ between $u$ and $\tilde{u}_n$ at points $(0.2, 0)$ and $(0.2, 2)$, respectively.

Table 3: The errors of the solution of equation (2.1) based on the piecewise linear wavelet basis

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$e_n(0.2, 0)$</th>
<th>$\tilde{e}_n(0.2, 0)$</th>
<th>$e_n(0.2, 2)$</th>
<th>$\tilde{e}_n(0.2, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.72e-4</td>
<td>2.02e-4</td>
<td>3.04e-4</td>
<td>4.67e-4</td>
</tr>
<tr>
<td>64</td>
<td>1.23e-5</td>
<td>3.43e-5</td>
<td>8.68e-6</td>
<td>8.96e-6</td>
</tr>
<tr>
<td>128</td>
<td>7.78e-7</td>
<td>1.46e-6</td>
<td>7.53e-7</td>
<td>6.84e-7</td>
</tr>
<tr>
<td>256</td>
<td>4.88e-8</td>
<td>2.65e-8</td>
<td>5.34e-8</td>
<td>8.32e-8</td>
</tr>
<tr>
<td>512</td>
<td>3.14e-9</td>
<td>2.25e-9</td>
<td>3.55e-9</td>
<td>6.89e-9</td>
</tr>
<tr>
<td>1024</td>
<td>2.91e-10</td>
<td>4.93e-10</td>
<td>4.37e-10</td>
<td>6.57e-10</td>
</tr>
</tbody>
</table>

Table 4: The errors of the solution of equation (2.1) based on piecewise cubic wavelet basis

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$e_n(0.2, 0)$</th>
<th>$\tilde{e}_n(0.2, 0)$</th>
<th>$e_n(0.2, 2)$</th>
<th>$\tilde{e}_n(0.2, 2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>2.8716e-5</td>
<td>2.8134e-5</td>
<td>7.0394e-5</td>
<td>7.0478e-5</td>
</tr>
<tr>
<td>64</td>
<td>2.0826e-7</td>
<td>7.4722e-7</td>
<td>5.1384e-6</td>
<td>5.1335e-6</td>
</tr>
<tr>
<td>256</td>
<td>6.3172e-14</td>
<td>2.4258e-14</td>
<td>1.0201e-10</td>
<td>1.3979e-10</td>
</tr>
<tr>
<td>512</td>
<td>5.0238e-15</td>
<td>4.6907e-15</td>
<td>2.4370e-14</td>
<td>5.5150e-14</td>
</tr>
</tbody>
</table>

We also illustrate the sparsity of the coefficient matrix of the proposed method in this paper. In Table 5, we list the number $N(M_n + L_n)$ of nonzero entries of $M_n + L_n$, and the compression ratio (C.R.) of $M_n + L_n$, which is computed by

$$C.R. := \frac{N(M_n + L_n)}{N(M_n + L_n)}$$

We plot in Figure 2 the distribution of the nonzero entries of $\tilde{M}_7$ and $\tilde{L}_7$, in which the blue zone identifies the position where nonzero entries are located. Table 5 and Figure 2 show us the sparseness of $M_n$ and $L_n$.

Table 5: The results of matrix truncation

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>C.R.</th>
<th>$N(M_n + L_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>0.757812</td>
<td>776</td>
</tr>
<tr>
<td>64</td>
<td>0.445312</td>
<td>1824</td>
</tr>
<tr>
<td>128</td>
<td>0.254394</td>
<td>4168</td>
</tr>
<tr>
<td>256</td>
<td>0.142822</td>
<td>9360</td>
</tr>
<tr>
<td>512</td>
<td>0.079254</td>
<td>20776</td>
</tr>
<tr>
<td>1024</td>
<td>0.043548</td>
<td>45664</td>
</tr>
</tbody>
</table>

In Figure 3, we plot the value of error $\|\rho_{10} - \rho_n\|$ versus number $N(M_n + L_n)$ by the blue solid line. Also, we plot the value of $\|\rho_{10} - \tilde{\rho}_n\|$ versus number $N(M_n + \tilde{L}_n)$ by the red dotted line. These values are listed in Tables 1 and 5. From Figure 3, we can see that the proposed fast method can obtain...
Figure 2: The distribution of nonzero entries of $\tilde{M}_7$ (left) and $\tilde{L}_7$ (right)

Figure 3: Comparing errors and the numbers of nonzero entries of Example 1

higher accuracy than the Galerkin method when the numbers of nonzero entries of the coefficient matrices involved in these methods are equal.

**Example 2:** In this example we consider solving the boundary integral equation (2.1) on the domain showed in left of Figure 1 which has a reentrant corner $P_0 = (0,0)$ with interior angle $\beta = \frac{3\pi}{2}$, and the boundary curve given by the parametric presentation

$$\Gamma(t) := \left(\frac{2}{3} \sin(3\pi t), -\sin(2\pi t)\right), \quad 0 \leq t \leq 1.$$ 

Moreover, we assume the exact solution of (2.1) is harmonic function

$$u(r, \theta) = r^\frac{2}{3} \cos\left(\frac{2}{3} \theta\right),$$

in polar coordinates $r$ and $\theta$, then the boundary condition is given by setting $g = u$ on $\partial D$.

Here, we still employ the piecewise linear and cubic wavelet basis to illustrate the theoretical estimate of convergence orders in Theorem 4.19. The results are listed in Tables 6 and 7. The values listed in the column titled “C.O.” are close to 2 in Table 6 and the values listed in the column titled “C.O.” are close to 4 in Table 7, which confirm the theoretical estimate of convergence order.

In Tables 8 and 9, we respectively show the errors $e_n(x,y)$ and $\tilde{e}_n(x,y)$ at point $(x,y)$ for the piecewise linear and cubic wavelet basis. In the second and fourth columns, we show the errors $e_k(0.3,0.6)$ and $e_k(0.2,0.4)$ between $u$ and $u_n$ at points $(0.3,0.6)$ and $(0.2,0.4)$, respectively. In the
Table 6: Numerical results of algorithm based on piecewise linear wavelet basis

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$|\rho_{10} - \rho_n|$</th>
<th>C.O.</th>
<th>$|\rho_{10} - \tilde{\rho}_n|$</th>
<th>C.O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.05e-2</td>
<td></td>
<td>1.06e-2</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>2.66e-3</td>
<td>1.983</td>
<td>2.70e-3</td>
<td>1.970</td>
</tr>
<tr>
<td>128</td>
<td>6.70e-4</td>
<td>1.999</td>
<td>6.71e-4</td>
<td>2.006</td>
</tr>
<tr>
<td>256</td>
<td>1.69e-4</td>
<td>1.999</td>
<td>1.70e-4</td>
<td>1.983</td>
</tr>
<tr>
<td>512</td>
<td>4.26e-5</td>
<td>1.987</td>
<td>4.31e-5</td>
<td>1.978</td>
</tr>
<tr>
<td>1024</td>
<td>1.06e-5</td>
<td>2.000</td>
<td>1.10e-5</td>
<td>1.980</td>
</tr>
</tbody>
</table>

Table 7: Numerical results of algorithm based on piecewise cubic wavelet basis

<table>
<thead>
<tr>
<th>$d_n$</th>
<th>$|\rho_8 - \rho_n|$</th>
<th>C.O.</th>
<th>$|\rho_8 - \tilde{\rho}_n|$</th>
<th>C.O.</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>6.5665e-4</td>
<td></td>
<td>6.5659e-4</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>5.0212e-5</td>
<td>3.7090</td>
<td>5.0915e-5</td>
<td>3.5533</td>
</tr>
<tr>
<td>128</td>
<td>3.2126e-6</td>
<td>3.9662</td>
<td>3.2262e-6</td>
<td>3.9810</td>
</tr>
<tr>
<td>256</td>
<td>2.0645e-7</td>
<td>3.9599</td>
<td>2.0675e-7</td>
<td>3.9949</td>
</tr>
<tr>
<td>512</td>
<td>1.3874e-8</td>
<td>3.8952</td>
<td>1.3985e-8</td>
<td>4.0015</td>
</tr>
</tbody>
</table>

third and fifth columns, we list the errors $\tilde{e}_n(0.3, 0.6)$ and $\tilde{e}_n(0.2, 0.4)$ between $u$ and $\tilde{u}_n$ at points $(0.3, 0.6)$ and $(0.2, 0.4)$, respectively.

In Figure 4, we plot the value of error $\|\rho_{10} - \rho_n\|$ versus number $N(M_n + L_n)$ by the blue solid line. Also, we plot the value of error $\|\rho_{10} - \tilde{\rho}_n\|$ versus number $N(M_n + \tilde{L}_n)$ by the red dotted line. These values are listed in Tables 5 and 6.

![Figure 4: Comparing errors and the numbers of nonzero entries of Example 2](image)

Figure 4: Comparing errors and the numbers of nonzero entries of Example 2

6 Conclusion

We present a fast wavelet Garlekin method for solving the boundary integral equation deduced from the Dirichlet problem of Laplace equations in domains with corners. The high order derivatives of the kernel function is analysed very carefully. This helps us describe the decay rate of the wavelet
Lemma A.2 Let \( l, l' \in \mathbb{N} \). Then there is a constant \( c \geq 0 \) such that for each \( k_1, k_2, k_1', k_2' \in \mathbb{N}_0 \) and all \( s, t > 0 \),

\[
\left( \sum_{(m_1, m_2) \in \mathbb{Z}_{k_1}^2} s^{m_1}t^{m_2} \right) \left( \sum_{(m_1', m_2') \in \mathbb{Z}_{k_1'}^2} s^{m_1'}t^{m_2'} \right) \leq c \sum_{(m_1, m_2) \in \mathbb{Z}_{k_1+k_1'}^2} s^{m_1}t^{m_2}.
\]

**Proof:** Note that for all \( (m_1, m_2) \in \mathbb{Z}_{k_1}^2 \) and \( (m_1', m_2') \in \mathbb{Z}_{k_1'}^2 \), there have \( m_1+m_1' \geq \min\{1, k_1+k_1'\} \), \( m_2+m_2' \geq \min\{1, k_2+k_2'\} \) and \( m_1 + m_1' + m_2 + m_2' = l + l' \). Thus, from the following equality

\[
\left( \sum_{(m_1, m_2) \in \mathbb{Z}_{k_1}^2} s^{m_1}t^{m_2} \right) \left( \sum_{(m_1', m_2') \in \mathbb{Z}_{k_1'}^2} s^{m_1'}t^{m_2'} \right) = \sum_{(m_1, m_2) \in \mathbb{Z}_{k_1+k_1'}^2} \sum_{(m_1', m_2') \in \mathbb{Z}_{k_1'}^2} s^{m_1+m_1'}t^{m_2+m_2'},
\]

we obtain this lemma .

**Lemma A.2** Let \( l \in \mathbb{N} \) with \( l < p \). Then for all \( s, t > 0 \) and \( m_1, m_2 \in \mathbb{N}_0 \) with \( m_1 + m_2 > 2pl \),

\[
\frac{s^{m_1}t^{m_2}}{(s^{2p}+t^{2p})^{\frac{m_1+m_2-2pl}{2p}}} \leq (s^{2p}+t^{2p})^{\frac{m_1+m_2-2pl}{2p}}. \tag{1.102}
\]
Proof: It is easy to check that
\[
\frac{s_{m1}t_{m2}}{(s^{2p} + t^{2p})^{\frac{m1}{2p}}} = \frac{s_{m1}}{(s^{2p} + t^{2p})^{\frac{m1}{2p}}} \frac{t_{m2}}{(s^{2p} + t^{2p})^{\frac{m2}{2p}}} (s^{2p} + t^{2p})^{\frac{m1+m2-2p}{2p}}.
\]
Then, by noting that \( \frac{s_{m1}}{(s^{2p} + t^{2p})^{\frac{m1}{2p}}} \leq 1 \) and \( \frac{t_{m2}}{(s^{2p} + t^{2p})^{\frac{m2}{2p}}} \leq 1 \), we obtain this lemma. \( \square \)

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