MULTIPLICITY OF NONTRIVIAL SOLUTIONS FOR A CLASS OF FRACTIONAL ELLIPTIC EQUATIONS

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ABSTRACT. In this paper, we are concerned with the following fractional Laplacian equation

\[
(-\Delta)^s u = a(x)|u(x)|^{q-2}u(x) + \lambda b(x)|u(x)|^{p-2}u(x)
\]

in \((-1,1),
\]

\[
u > 0 \quad \text{in} \quad (-1,1),
\]

\[
u = 0 \quad \text{in} \quad \mathbb{R} \setminus (-1,1),
\]

where \(s \in (0,1), \lambda > 0\). Using variational methods, we show existence and multiplicity of positive solutions.

Keywords: Fractional Laplacian operator, Multiple solutions, Nehari manifold.

1. Introduction

The problems with the fractional Laplacian operator has attracted many attention in recent years. This type of operator appears in probabilistic framework as well as in mathematical finance as infinitesimal operators of stable Lévy process [1, 6, 7]. Also it has been a classical topic in Fourier analysis and nonlinear partial differential equations. The critical exponent problems for fractional Laplacian have been studied in [9, 21, 22, 24, 25]. For more about aspect of applications of the fractional Laplacian we refer the reader to [13]. A prototype of application in biology is illustrated in [20].

In the local setting, the semilinear elliptic systems involving Laplace operator with exponential nonlinearity have been investigated in [14, 18]. However, there is a substantial difference between the Laplacian and the fractional Laplacian. Whereas it is known that the first one is local and therefore suitable for describing diffusion problems, the second one is nonlocal and commonly used for describing super-diffusion (Lévy flights). These differences are reflected in the way of computing for both operators (Green formula, integration by part, Leibnitz formula, ...).

The inspiring point for this work was [2, 3, 11] where the case of polynomial nonlinearities involving linear and quasilinear operators has been studied. Furthermore, these results for sign changing nonlinearities with polynomial type subcritical and critical growth have been obtained in [8, 10, 15, 16]
using Nahari manifold and fibering map analysis. Recently semilinear equations involving fractional Laplacian with exponential nonlinearities have been studied by many authors. Among them we cite [4, 17, 19, 23] and the references therein.

Inspired by the above mentioned papers, in this article, we study the following fractional Laplacian problem

\[
\begin{aligned}
(\mathbf{P}_\lambda) \quad & \begin{cases} 
(\Delta)^su = a(x)|u(x)|^{q-2}u(x) + \lambda b(x)|u(x)|^{p-2}u(x) & \text{in } (-1, 1), \\
u > 0 & \text{in } (-1, 1), \\
u = 0 & \text{in } \mathbb{R} \setminus (-1, 1),
\end{cases}
\end{aligned}
\]

where \( s \in (0, 1), \lambda > 0 \). Here \( p, q \) and the functions \( a \) and \( b \) satisfies the following assumptions:

- (H1) \( 1 < q < 2 < p < 2^*, \ 2^* = \frac{2n}{n-2s} \).
- (H2) \( a, b \in L^\infty((0, 1)) \).

Our main result in this paper is the following

**Theorem 1.1.** Under the assumptions (H1)-(H2). There exist \( \lambda_0 > 0 \) such that for all \( 0 < \lambda < \lambda_0 \), problem \((\mathbf{P}_\lambda)\) has at least two nontrivial solutions.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and variational setting. Section 3 is devoted to proof of our main result.

2. Preliminaries and variational setting

In this section, we introduce the variational setting of our problem, also we present some preliminaries results that will be needed in the proof of our main result.

We denote the classical fractional sobolev space

\[
H^s((-1, 1)) = \left\{ u \in L^2((-1, 1)) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{1}{2} + s}} \in L^2((-1, 1) \times (-1, 1)) \right\},
\]

with the Gagliardo norm

\[
\|u\|_{H^s((-1, 1))} = \|u\|_2 + \left( \int_{-1}^{1} \int_{-1}^{1} \frac{|u(x) - u(y)|^2}{|x - y|^{\frac{1}{2} + s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

We shall work in the following space

\[
E = \{ u \in H^s(\mathbb{R}) : u = 0 \text{ a.e. in } \mathbb{R} \setminus (-1, 1) \},
\]

endowed with the norm

\[
\|u\| = \left( \int_{-1}^{1} \int_{-1}^{1} \frac{|u(x) - u(y)|^2}{|x - y|^{\frac{1}{2} + s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]

We recall from [5], that \((E, \| \cdot \|)\) is a Hilbert space. Moreover, \( C^2_0((-1, 1)) \subset E, X \subset H^s((-1, 1)) \) and \( E \subset H^s(\mathbb{R}) \). Also the embedding \( E \hookrightarrow L^{2^*}((-1, 1)) \) is continuous where \( 2^* = \frac{2n}{n-2s} \).
Associated to the problem \((P_\lambda)\) we define the functional \(J_\lambda : E \to \mathbb{R}\) given by

\[
J_\lambda(u) = \frac{1}{2}||u||^2 - \frac{1}{q} \int_{-1}^{1} a(x)|u(x)|^q \, dx - \frac{\lambda}{p} \int_{-1}^{1} b(x)|u(x)|^p \, dx, \quad u \in E.
\]

We say that \(u \in E\) is a weak solution of problem \((P_\lambda)\) if for every \(v \in E\) we have:

\[
\int_{\mathbb{R}^2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{\frac{1}{2} + s}} \, dx \, dy = \int_{-1}^{1} a(x)|u(x)|^{q-2}u(x)v(x) \, dx + \lambda \int_{-1}^{1} b(x)|u(x)|^{p-2}u(x)v(x) \, dx.
\]

Note that \(u\) is a positive solution of problem \((P_\lambda)\), if \(u\) is positive and

\[
||u||^2 - \int_{-1}^{1} a(x)u(x)^q \, dx - \lambda \int_{-1}^{1} b(x)u(x)^p \, dx = 0. \tag{2.1}
\]

It is easy to see that the energy functional \(J_\lambda\) is not bounded below on the space \(E\), but is bounded below on an appropriate subset of \(E\) and a minimizer on subsets of this set gives rise to solutions of \((P_\lambda)\). In order to obtain the existence results, we introduce the Nehari manifold

\[
N_\lambda := \{ u \in E \setminus \{0\} : \langle J_\lambda'(u), u \rangle = 0 \}.
\]

Then, \(u \in N_\lambda\) if and only if

\[
||u||^2 - \int_{-1}^{1} a(x)u(x)^q \, dx - \lambda \int_{-1}^{1} b(x)u(x)^p \, dx = 0. \tag{2.2}
\]

We note that from (2.1), we can see that \(N_\lambda\) contains every non zero solution of problem \((P_\lambda)\).

**Lemma 2.1.** Under conditions (H1)-(H2), \(J_\lambda\) is coercive and bounded bellow on \(N_\lambda\).

**Proof.** let \(u \in N_\lambda\), then, by the Sobolev embedding, we obtain

\[
J_\lambda(u) = \left( \frac{1}{2} - \frac{1}{p + 1} \right)||u||^2 - \left( \frac{1}{q + 1} - \frac{1}{p + 1} \right) \int_{-1}^{1} a(x)u(x)^{q+1} \, dx \\
\geq c_1||u||^2 - c_2||u||^q.
\]

Since \(q > 1\), then \(J_\lambda\) is bounded bellow and coercive on \(N_\lambda\). \(\square\)

Now as we know that the Nehari manifold is closely related to the behavior of the functions \(\Phi_u : [0, \infty) \to \mathbb{R}\) defined as

\[
\Phi_u(t) = J_\lambda(tu).
\]

Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [12].
For $u \in E$, we have
\[
\Phi_u(t) = \frac{t^2}{2}||u||^2 - \frac{t^{q+1}}{q+1} \int_{-1}^{1} a(x)u(x)^q dx - \lambda \frac{t^{p+1}}{p+1} \int_{-1}^{1} b(x)u(x)^p dx,
\]
\[
\Phi'_u(t) = t||u||^2 - t^q \int_{-1}^{1} a(x)u(x)^q dx - \lambda t^p \int_{-1}^{1} b(x)u(x)^p dx,
\]
\[
\Phi''_u(t) = ||u||^2 - qt^{q-1} \int_{-1}^{1} a(x)u(x)^q dx - \lambda pt^{p-1} \int_{-1}^{1} b(x)u(x)^p dx.
\]

Then it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\Phi'_u(1) = 0$. Thus it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection.

For this we set
\[
\mathcal{N}_\lambda^+ = \{ u \in \mathcal{N}_\lambda : \Phi''_u(1) > 0 \} = \{ tu \in E : \Phi'_u(t) = 0, \Phi''_u(t) > 0 \},
\]
\[
\mathcal{N}_\lambda^- = \{ u \in \mathcal{N}_\lambda : \Phi''_u(1) < 0 \} = \{ tu \in E : \Phi'_u(t) = 0, \Phi''_u(t) < 0 \},
\]
\[
\mathcal{N}_\lambda^0 = \{ u \in \mathcal{N}_\lambda : \Phi''_u(1) = 0 \} = \{ tu \in E : \Phi'_u(t) = 0, \Phi''_u(t) = 0 \}.
\]

Before studying the behavior of Nehari manifold using fibering maps, we introduce some notations
\[
\mathcal{A}^+ = \{ u \in E : \int_{-1}^{1} a(x)u(x)^q dx > 0 \}; \quad \mathcal{A}^- = \{ u \in E : \int_{-1}^{1} a(x)u(x)^q dx < 0 \};
\]
\[
\mathcal{B}^+ = \{ u \in E : \int_{-1}^{1} b(x)u(x)^p dx > 0 \}; \quad \mathcal{B}^- = \{ u \in E : \int_{-1}^{1} b(x)u(x)^p dx < 0 \}.
\]

Now we study the fiber map $\Phi_u$ according to the sign of $\int_{-1}^{1} a(x)u(x)^q dx$ and $\int_{-1}^{1} b(x)u(x)^p dx$.

**Case 1**: $u \in \mathcal{A}^- \cap \mathcal{B}^-$.  
In this case $\Phi_u(0) = 0$ and $\Phi'_u(t) > 0, \forall t > 0$ which implies that $\Phi_u$ is strictly increasing and hence no critical point.

**Case 2**: $u \in \mathcal{A}^- \cap \mathcal{B}^+$.  
Let $m_u : [0, \infty) \to \mathbb{R}$ be the function defined as
\[
m_u(t) = t^{1-q}||u||^2 - \lambda t^{p-1} \int_{-1}^{1} b(x)u(x)^p dx.
\]
Clearly, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if $t$ is a solution of
\[
m_u(t) = \int_{-1}^{1} a(x)|u(x)|^q dx.
\]
Moreover, it is easy to see that $m'_u(t) = 0$ if and only if
\[
t := T = \left( \frac{(1-q)||u||^2}{(p-q) \int_{-1}^{1} b(x)u(x)^p dx} \right)^{\frac{1}{p-1}}.
\]  (2.3)
On the other hand, let \( \lambda_0 = \frac{p - q}{1 - q} \), then for all \( \lambda \in (0, \lambda_0) \) we obtain

\[
m_u(T) = \left( \frac{(1 - q)\|u\|^2}{(p - q) \int_{-1}^{1} b(x)u(x)^{p+1}dx} \right)^{\frac{1-q}{p-1}} \|u\|^2 \\
- \lambda \left( \frac{(1 - q)\|u\|^2}{(p - q) \int_{-1}^{1} b(x)u(x)^{p+1}dx} \right)^{\frac{p-q}{p-1}} \int_{-1}^{1} b(x)u(x)^{p+1}dx
\]

\[
= \frac{\|u\|^2}{(\int_{-1}^{1} b(x)u(x)^{p+1}dx)^{\frac{1-q}{p-1}}} \left( \frac{1 - q}{p - q} \right)^{\frac{p-q}{p-1}} \left( \frac{p - q - \lambda}{1 - q} \right) > 0,
\]

where \( T \) is given by equation (2.3).

Since \( \int_{-1}^{1} b(x)u(x)^{p+1}dx > 0 \), then, the Table of variations of the function \( m_u \) is as follows

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>( T )</th>
<th>( \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_u )</td>
<td>+</td>
<td>0</td>
<td>−</td>
</tr>
<tr>
<td>( m_u(T) )</td>
<td>0</td>
<td>( m_u(T) &gt; 0 )</td>
<td>−∞</td>
</tr>
</tbody>
</table>

Hence, there exists a unique \( t_0 \) such that \( m_u(t_0) = \int_{-1}^{1} a(x)u(x)^{q+1}dx \), moreover if \( t < t_1 \), then \( m_u(t) > \int_{-1}^{1} a(x)u(x)^{q+1}dx \) and if \( t > t_1 \), then \( m_u(t) < \int_{-1}^{1} a(x)u(x)^{q+1}dx \).

On the other hand, by the fact that

\[
\Phi_u(t) = t^q \left( m_u(t) - \int_{-1}^{1} a(x)u(x)^q dx \right),
\]

(2.4)

it follows immediately that \( \Phi_u \) has a global maximum at \( t = t_0 \).

**Case 3:** \( u \in \mathcal{A}^+ \cap \mathcal{B}^- \).

Since \( \int_{-1}^{1} b(x)u(x)^{p+1}dx < 0 \), then, the Table of variations of the function \( m_u \) is as follows

<table>
<thead>
<tr>
<th>( x )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( m_u )</td>
<td>+</td>
<td>+∞</td>
</tr>
</tbody>
</table>

Therefore, there exists a unique \( t_1 \) such that \( m_u(t_1) = \int_{-1}^{1} a(x)u(x)^{q+1}dx \), moreover if \( t < t_1 \), then \( m_u(t) < \int_{-1}^{1} a(x)u(x)^{q+1}dx \) and if \( t > t_1 \) then
\[ m_u(t) > \int_{-1}^{1} a(x)u(x)^{q+1} dx. \] So, from (2.4), \( \Phi_u \) has a global minimum at \( t = t_1. \)

**Case 4:** \( u \in \mathcal{A}^+ \cap \mathcal{B}^+. \)

In this case, we claim that there exists \( \mu_0 > 0 \) such that for \( \lambda \in (0, \mu_0) \), \( \Phi_u \) has exactly two critical points \( t_2 \) and \( t_3 \). Moreover, \( t_2 \) is a local minimum point and \( t_3 \) is a local maximum point. Thus \( t_2 u \in \mathcal{N}_{\lambda}^+ \) and \( t_3 u \in \mathcal{N}_{\lambda}^- \).

We prove this claim in the following Lemmas:

**Lemma 2.2.** There exists \( \mu_0 > 0 \) such that for \( \lambda \in (0, \mu_0) \), \( \Phi_u \) take positive value for all non-zero \( u \in E \). Moreover, Let \( \lambda \in (0, \lambda_0) \). If \( u \in \mathcal{F}^+ \cap \mathcal{A}^+ \), then, \( \Phi_u \) has exactly two critical points.

**Proof.** Since \( u \in \mathcal{B}^+ \), \( m_u \) has the same table of variations as in cases 2.

Since \( \int_{-1}^{1} a(x)u(x)^{q+1} dx > 0 \), there is \( t_2 \) and \( t_3 \) such that

\[ m_u(t_2) = m_u(t_3) = \int_{-1}^{1} a(x)u(x)^{q+1} dx, \]

moreover, if \( t \in (-\infty, t_2) \cup (t_3, \infty) \) then, \( m_u(t) < \int_{-1}^{1} a(x)u(x)^{q+1} dx \) and if \( t \in (t_2, t_3) \), then \( m_u(t) > \int_{-1}^{1} a(x)u(x)^{q+1} dx \). So, from (2.4), \( \Phi_u \) has a local minima at \( t_2 \) and a local maxima at \( t_3 \). \( \square \)

**Lemma 2.3.** There exists \( \mu_0 > 0 \) such that for \( \lambda \in (0, \mu_0) \), \( \Phi_u \) take positive value for all non-zero \( u \in E \).

**Proof.**

Let \( u \in E \), define

\[ M_u(t) = \frac{t^2}{2} ||u||^2 - \frac{t^{p+1}}{p+1} \int_{-1}^{1} b(x)u(x)^{p+1} dx. \]

Then,

\[ M_u'(t) = t ||u||^2 - t^p \int_{-1}^{1} b(x)u(x)^{p+1} dx, \]

and

\[ M_u'(t) = 0 \iff t = T := \left( \frac{||u||^2}{\int_{-1}^{1} b(x)u(x)^{p+1} dx} \right)^{\frac{1}{p-1}}. \]

Moreover,

\[ M_u(T) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left( \frac{||u||^{p+1}}{\int_{-1}^{1} b(x)u(x)^{p+1} dx} \right)^{\frac{2}{p-1}} \]

and

\[ M_u''(T) = (1-p)||u||^2 < 0. \]
Thus, $M_u$ attains its maximum value at $T$.

For $1 \leq \nu < p^*_a$, we denoted by $S_\nu$ the Sobolev constant of embedding $E \hookrightarrow L^\nu((-1,1))$, then, by (2.2) we have

$$M_u(T) \geq \frac{p-1}{2(p+1)(\|b\|_{\infty} S_{p+1}^{p+1})} = \delta. \quad (2.5)$$

On the other hand

$$\frac{T^{q+1}}{q+1} \int_{-1}^{1} a(x)u(x)^{q+1}dx \leq \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \|u\|^{q+1} T^{q+1}$$

$$= \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \left( \frac{\|u\|^2}{\int_{-1}^{1} b(x)u(x)^{p+1}dx} \right)^{\frac{q+1}{p-1}}$$

$$= \left( \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \left( \frac{\|u\|^p}{\int_{-1}^{1} b(x)u(x)^{p+1}dx} \right)^{\frac{q+1}{p-1}} \right)^{\frac{q+1}{q}}$$

Thus

$$\Phi_u(T) = M_u(T) - \lambda \frac{T^{q+1}}{q+1} \int_{-1}^{1} a(x)|u(x)|^{q+1}dx$$

$$\geq M_u(T) - \lambda \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \left( \frac{2p+1}{p-1} M_u(T) \right)^{\frac{q+1}{q}}$$

$$= \left( \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \left( \frac{2p+1}{p-1} M_u(T) \right)^{\frac{q+1}{q}} \right)^{\frac{q+1}{q}}$$

$$\geq \delta^{\frac{q+1}{q}} \left[ \delta^{\frac{1-q}{2}} - \lambda \frac{\|a\|_{\infty} S_{q+1}^{q+1}}{q+1} \left( \frac{2p+1}{p-1} \right)^{\frac{q+1}{q}} \right],$$

where $\delta$ is the constant given in (2.5).

Let

$$\mu_0 = \frac{(q+1)\delta^{\frac{1-q}{2}}}{\|a\|_{\infty} S_{q+1}^{q+1}} \left( \frac{p-1}{2p+2} \right)^{\frac{q+1}{2}}. \quad (2.6)$$

Then, this completes the proof.

**Corollary 2.1.** If $\lambda < \mu_0$, then, there exists $\delta_1 > 0$ such that $J_\lambda(u) > \delta_1$ for all $u \in \mathcal{N}_\lambda$.

**Proof.** Let $u \in \mathcal{N}_\lambda$, then $\Phi_u$ has a positive global maximum at $T = 1$ and

$$\int_{-1}^{1} a(x)|u(x)|^qdx > 0.$$ Thus, if $\lambda < \mu_0$, we have

$$J_\lambda(u) = \Phi_u(1) = \Phi_u(T)$$
\[ \geq \delta^2 \left( \delta^{p-q} - \lambda c \right) > 0, \]

where \( \delta \) is same as in Lemma 2.3. \hfill \Box

**Lemma 2.4.** There exists \( \mu_1 \) such that if \( 0 < \lambda < \mu_1 \), then \( N_\lambda^0 = \emptyset \).

**Proof.** Let
\[ \mu_1 = \frac{1 - q}{\|b\|_\infty S_{q+1}^{p+1}(p-q)} \left( \frac{p-1}{\|a\|_\infty S_{p+1}^{q+1}(p-q)} \right)^{\frac{1}{p-q}}. \]

Suppose otherwise, \( 0 < \lambda < \mu_1 \) such that \( N_\lambda^0 \neq \emptyset \). Then, for \( 0 \neq u \in N_\lambda^0 \), we have
\[ 0 = \Phi_u''(1) = \|u\|^2 - q \int_{-1}^{1} a(x)|u(x)|^{q+1}dx - \lambda p \int_{-1}^{1} b(x)|u(x)|^{p+1}dx. \]

So, it follows from (2.2) that
\[ (1-q)\|u\|^2 = \lambda(p-q) \int_{-1}^{1} b(x)|u|^{p+1}dx \leq \lambda(p-q)\|b\|_\infty S_{q+1}^{p+1}|u|^{q+1}, \]

which yields to
\[ \|u\| \leq \left( \frac{\lambda\|b\|_\infty S_{q+1}^{p+1}(p-q)}{1-q} \right)^{\frac{1}{p-q}}. \tag{2.7} \]

On the other hand, by (2.2) we get
\[ (p-1)\|u\|^2 = (p-q) \int_{-1}^{1} a(x)|u(x)|^{q+1}dx \leq (p-q)\|a\|_\infty S_{p+1}^{q+1}|u|^{p+1}, \]

then
\[ \|u\| \geq \left( \frac{p-1}{\|a\|_\infty S_{p+1}^{q+1}(p-q)} \right)^{\frac{1}{p-q}}. \tag{2.8} \]

Combining (2.7) and (2.8) we obtain \( \lambda \geq \mu_1 \), which is a contradiction. \hfill \Box

Here and always, we define \( \lambda_0 \) as
\[ \lambda_0 = \min(\mu_0, \mu_1), \]

where \( \mu_0 \) is given by equation (2.6) and \( \mu_1 \) is defined in the proof of Lemma 2.4.

We remark that if \( 0 < \lambda < \lambda_0 \), then all the above Lemmas hold true.

**Lemma 2.5.** Let \( u \) be a local minimizer for \( J_\lambda \) on subsets \( N_\lambda^+ \) or \( N_\lambda^- \) of \( N_\lambda \) such that \( u \notin N_\lambda^0 \), then \( u \) is a critical point of \( J_\lambda \).
Proof. Since \( u \) is a minimizer for \( J_\lambda \) under the constraint \( I_\lambda(u) := \langle J'_\lambda(u), u \rangle = 0 \), by the theory of Lagrange multipliers, there exists \( \mu \in \mathbb{R} \) such that \( J'_\lambda(u) = \mu I'_\lambda(u) \). Thus
\[
\langle J'_\lambda(u), u \rangle = \mu \langle I'_\lambda(u), u \rangle = \mu \Phi'_u(1) = 0,
\]
but \( u \notin \mathcal{N}_\lambda^0 \) and so \( \Phi'_u(1) \neq 0 \). Hence \( \mu = 0 \) completes the proof. \( \square \)

3. PROOF OF THE MAIN RESULT

Throughout this section, we assume that the parameter \( \lambda \) satisfies \( 0 < \lambda < \lambda_0 \), and we assume that hypothesis (H1)-(H2) are satisfying. The proof is divided in several lemmas.

Lemma 3.1. If \( 0 < \lambda < \lambda_0 \), then, \( J_\lambda \) achieves its minimum on \( \mathcal{N}_\lambda^+ \)

Proof. Since \( J_\lambda \) is bounded below on \( \mathcal{N}_\lambda \) and so on \( \mathcal{N}_\lambda^+ \), there exists a minimizing sequence \( \{u_k\} \subset \mathcal{N}_\lambda^+ \) such that
\[
\lim_{k \to \infty} J_\lambda(u_k) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u).
\]

As \( J_\lambda \) is coercive on \( \mathcal{N}_\lambda \), \( \{u_k\} \) is a bounded sequence in \( E \). Therefore, for all \( 1 \leq \nu < p^*_s \) we have
\[
\begin{cases}
  u_k \rightharpoonup u_\lambda \text{ weakly in } E,
  u_k \to u_\lambda \text{ strongly in } E'(\mathbb{R}^n).
\end{cases}
\]

If we choose \( u \in E \) such that \( \int_{-1}^1 a(x)|u(x)|^q dx > 0 \), then, there exists \( t_1 > 0 \) such that \( t_1 u \in \mathcal{N}_\lambda^+ \) and \( J_\lambda(t_1 u) < 0 \). Hence, \( \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u) < 0 \).

On the other hand, since \( \{u_k\} \subset \mathcal{N}_\lambda \), then we have
\[
J_\lambda(u_k) = \left( \frac{1}{p} - \frac{1}{r} \right)||u_k||^p - \lambda \left( \frac{1}{q} - \frac{1}{r} \right) \int_{-1}^1 a(x)|u_k(x)|^q dx,
\]

and so
\[
\lambda \left( \frac{1}{q} - \frac{1}{r} \right) \int_{-1}^1 a(x)|u_k(x)|^q dx = \left( \frac{1}{p} - \frac{1}{r} \right)||u_k||^p - J_\lambda(u_k).
\]
Letting \( k \) to infinity, we get
\[
\int_{-1}^1 a(x)|u_\lambda(x)|^q dx > 0.
\]

Next we claim that \( u_k \rightharpoonup u_\lambda \). Suppose this is not true, then
\[
||u_\lambda||^p < \lim_{k \to \infty} ||u_k||^p.
\]

Since \( \Phi'_u(t_1) = 0 \) it follows that \( \Phi'_u(t_1) > 0 \) for sufficiently large \( k \). So, we must have \( t_1 > 1 \) but \( t_1 u_\lambda \notin \mathcal{N}_\lambda^+ \) and so
\[
J_\lambda(t_1 u_\lambda) < J_\lambda(u_\lambda) \leq \lim_{k \to \infty} J_\lambda(u_k) = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u),
\]

Proof of the Main result

Proof.

Since \( u \) is a minimizer for \( J_\lambda \) under the constraint \( I_\lambda(u) := \langle J'_\lambda(u), u \rangle = 0 \), by the theory of Lagrange multipliers, there exists \( \mu \in \mathbb{R} \) such that \( J'_\lambda(u) = \mu I'_\lambda(u) \). Thus
\[
\langle J'_\lambda(u), u \rangle = \mu \langle I'_\lambda(u), u \rangle = \mu \Phi'_u(1) = 0,
\]
but \( u \notin \mathcal{N}_\lambda^0 \) and so \( \Phi'_u(1) \neq 0 \). Hence \( \mu = 0 \) completes the proof. \( \square \)
which is a contradiction. It leads to \( u_k \to u_\lambda \) and so \( u_\lambda \in \mathcal{N}_\lambda^+ \), since \( \mathcal{N}_\lambda^0 = \emptyset \). Finally, \( u_\lambda \) is a minimizer for \( J_\lambda \) on \( \mathcal{N}_\lambda^+ \).

**Lemma 3.2.** If \( 0 < \lambda < \lambda_0 \), then, \( J_\lambda \) achieves its minimum on \( \mathcal{N}_\lambda^- \).

**Proof.** Let \( u \in \mathcal{N}_\lambda^- \), then from corollary 2.4, there exists \( \delta_1 > 0 \) such that \( J_\lambda(u) \geq \delta_1 \). So, there exists a minimizing sequence \( \{ u_k \} \subset \mathcal{N}_\lambda^- \) such that

\[
\lim_{k \to \infty} J_\lambda(u_k) = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) > 0. \tag{3.2}
\]

On the other hand, since \( J_\lambda \) is coercive, \( \{ u_k \} \) is a bounded sequence in \( E \). Therefore, for all \( 1 \leq \nu < p^* \) we have

\[
\begin{cases}
  u_k \rightharpoonup v_\lambda & \text{weakly in } E, \\
  u_k \to v_\lambda & \text{strongly in } L^\nu(\mathbb{R}^n). 
\end{cases}
\]

Since \( u \in \mathcal{N}_\lambda \), then

\[
J_\lambda(u_k) = \left( \frac{1}{p} - \frac{1}{q} \right) ||u_k||^p + \left( \frac{1}{q} - \frac{1}{r} \right) \int_{-1}^1 F(x,u_k)dx. \tag{3.3}
\]

Letting \( k \) to infinity, it follows from (3.2) and (3.3) that

\[
\int_{-1}^1 F(x,v_\lambda)dx > 0. \tag{3.4}
\]

Hence, \( v_\lambda \in \mathcal{F}^+ \) and so \( \Phi_{v_\lambda} \) has a global maximum at some point \( T \) and consequently, \( Tv_\lambda \in \mathcal{N}_\lambda^- \). on the other hand, \( u_k \in \mathcal{N}_\lambda^- \) implies that 1 is a global maximum point for \( \Phi_{u_k} \), i.e.

\[
J_\lambda(tu_k) = \Phi_{u_k}(t) \leq \Phi_{u_k}(1) = J_\lambda(u_k). \tag{3.5}
\]

Next we claim that \( u_k \to u_\lambda \). Suppose this is not true, then

\[
||u_\lambda||^p < \lim_{k \to \infty} ||u_k||^p,
\]

it follows from (3.5) that

\[
J_\lambda(Tv_\lambda) = \frac{T^p}{p} ||v_\lambda||^p - \frac{T^r}{r} \int_{-1}^1 F(x,v_\lambda)dx - \lambda \frac{T^q}{q} \int_{-1}^1 a(x)||v_\lambda||^qdx
\]

\[
< \inf_{k \to \infty} \left( \frac{T^p}{p} ||u_k||^p - \frac{T^r}{r} \int_{-1}^1 F(x,u_k)dx - \lambda \frac{T^q}{q} \int_{-1}^1 a(x)||u_k||^qdx \right)
\]

\[
\leq \lim_{k \to \infty} J_\lambda(Tu_k) \leq \lim_{k \to \infty} J_\lambda(u_k) = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u),
\]

which is a contradiction. Hence, \( u_k \to v_\lambda \) and so \( v_\lambda \in \mathcal{N}_\lambda^- \), since \( \mathcal{N}_\lambda^0 = \emptyset \).

Now, Let us proof Theorem 1.1: By Lemma 3.1 and Lemma 3.2 problem (\( P_\lambda \)) has tow weak solution \( u_\lambda \in \mathcal{N}_\lambda^+ \) and \( v_\lambda \in \mathcal{N}_\lambda^- \). On the other hand, from (3.1) and (3.4), this solutions are nontrivial. Since \( \mathcal{N}_\lambda^- \cap \mathcal{N}_\lambda^+ = \emptyset \), then \( u_\lambda \) and \( v_\lambda \) are distinct.
REFERENCES


