NEW GENERAL DECAY RESULT OF THE LAMINATED BEAM SYSTEM WITH INFINITE HISTORY

ABSTRACT. This paper is concerned with the asymptotic behavior of the solution of a Laminated Timoshenko beam system with viscoelastic damping. We extend the work known for this system with finite memory to the case of infinite memory. We use minimal and general conditions on the relaxation function and establish explicit energy decay formula which gives the best decay rates expected under this level of generality. We assume that the relaxation function $g$ satisfies for some nonnegative functions $\xi$ and $H$, $g'(t) \leq -\xi(t)H(g(t)), \forall t \geq 0$. Our decay results generalize and improve many earlier results in the literature. Moreover, we remove some assumptions on the boundedness of initial data used in many earlier papers in the literature.

1. Introduction. Laminated beam is an up-to-date research subject and a hot subject due to the high applicability of such materials in the industry. It was first introduced by Hansen and Spies see [1, 2]. In [2], Hansen and Spies derived the mathematical model for two-layered beams with structural damping due to the interface slip. The system is given by the following equations:

\[
\begin{aligned}
\rho \varphi_{tt} + G(\psi - \varphi_x)_x &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
I_p(3w - \psi)_tt - D(3w - \psi)_{xx} - G(\psi - \varphi_x) &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
3I_p w_{tt} - 3Dw_{xx} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t &= 0, & (x, t) &\in (0, 1) \times (0, +\infty),
\end{aligned}
\]

where $\varphi$ denotes the transverse displacement of the beam which departs from its equilibrium position, $\psi$ denotes the rotation angle, $3w - \psi$ denotes the effective rotation angle, $w$ is proportional to the amount of slip along the interface at time $t$ and longitudinal spatial variable $x$. The equation (1.1), describes the dynamics of the slip; $\rho, G, I_p, D, \gamma, \beta$ are the density of the beams, the shear stiffness, the mass moment of inertia, the flexural rigidity, the adhesive stiffness of the beams, the adhesive damping parameter, respectively. If $\beta = 0$, then (1.1) describes the coupled laminated beams without a structural damping at the interface. If $\beta \neq 0$, the adhesion at the interface supplies a restoring force proportion to the interfacial slip. In the recent years, an increasing interest has been developed to determine the asymptotic behavior of several laminated beam problems. Raposo [3] considered the following system when a frictional damping acts on the effective rotation and transverse displacement:

\[
\begin{aligned}
\rho \varphi_{tt} + G(\psi - \varphi_x)_x + k_1 \varphi_t &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
I_p(3w - \psi)_tt - D(3w - \psi)_{xx} - G(\psi - \varphi_x) + k_2(3w - \psi)_t &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
3I_p w_{tt} - 3Dw_{xx} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t &= 0, & (x, t) &\in (0, 1) \times (0, +\infty).
\end{aligned}
\]

He obtained the exponential decay result under appropriate initial data and boundary conditions. Lo and Tatar [4] considered the following system when a viscoelastic damping acts on the effective rotation and in the slip:

\[
\begin{aligned}
\rho \varphi_{tt} + G(\psi - \varphi_x)_x &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
I_p(3w - \psi)_tt - (3w - \psi)_{xx} + \int_0^t h(t - \tau)(3w - \psi)_{xx}(\tau)d\tau &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
I_p w_{tt} - w_{xx} + G(\psi - \varphi_x) + \frac{4}{3} \gamma w + \int_0^t g(t - \tau)w_{xx}(\tau)d\tau &= 0, & (x, t) &\in (0, 1) \times (0, +\infty), \\
\varphi_x(0, t) &= \varphi(1, t) = \psi(0, t) = \psi_x(1, t) = w(0, t) = w_x(1, t) = 0, & t &\in [0, +\infty), \\
\varphi(x, 0) = \varphi_0(x), & x &\in [0, 1], \\
\psi(x, 0) = \psi_0(x), & x &\in [0, 1], \\
w(x, 0) = w_0(x), & x &\in [0, 1], \\
\theta(x, 0) = \theta_0(x), & x &\in [0, 1].
\end{aligned}
\]

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They obtained the exponential stability when the wave speeds are equal. Apalara [5] considered a laminated beam with structural damping and Cattaneo law

\[
\begin{aligned}
\rho \varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\
I_\rho (3w - \psi)_{tt} - D(3w - \psi)_{xx} - G(\psi - \varphi_x) + \delta \theta_x = 0, \\
I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3} \gamma w + \frac{4}{3} \beta w_t = 0,
\end{aligned}
\]

(1.4)

He obtained the exponential stability and the polynomial stability depending on a stability number. Mustafa [6] considered the following viscoelastic laminated beam system with structural damping

\[
\begin{aligned}
\rho \varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\
I_\rho (3w - \psi)_{tt} - D(3w - \psi)_{xx} - G(\psi - \varphi_x) + \int_0^t g(t - r)w_{xx}(s)ds = 0, \\
3I_\rho w_{tt} - 3Dw_{xx} + 3G(\psi - \varphi_x) + 4\gamma w + 4\beta w_t = 0,
\end{aligned}
\]

(1.5)

and established the general decay result under the equal-speed wave propagation case. Z. Chen et. al investigate the general decay rate of the solutions for problem (1.5) with structural damping in the case of non-equal wave speeds. For more results in this direction, we refer to see [6] and the references therein. For laminated beam system with infinite history, Liu and Zhao [7] considered the following laminated beam system

\[
\begin{aligned}
\rho \phi_{tt} - G(\psi - \phi_x)_x + \theta_x = 0, \\
I_\rho (3w - \psi)_{tt} - fG(\psi - \phi_x) - D(3w - \psi)_{xx} + \int_0^\infty g(s)(3w - \psi)_{xx}(s)ds - \theta = 0, \\
I_\rho w_{tt} + G(\psi - \phi_x) + \frac{4}{3} \gamma w + \frac{4}{3} \beta w_t - Dw_{xx} = 0, \\
k \psi_t - \tau \psi_{xx} + \phi_x + (3w - \psi) = 0.
\end{aligned}
\]

(1.6)

They proved the global well posedness of the system (1.6). Further, by using the perturbed energy method, they proved, for $\beta \neq 0$, the exponential and polynomial stabilities. However, $\beta = 0$, by using the perturbed energy method and Gearhart Herbst Prüss Huang theorem, they established exponential stability in case of equal wave speeds assumption and lack of exponential stability in case of nonequal wave speeds. For more results in this direction, we refer the reader to [8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. Motivated by the above results, we consider the following viscoelastic-type laminated beam system:

\[
\begin{aligned}
\rho \phi_{tt} - G(\psi - \phi_x)_x = 0, \\
I_\rho (3w - \psi)_{tt} - G(\psi - \phi_x) - D(3w - \psi)_{xx} + \int_0^\infty g(s)(3w - \psi)_{xx}(t - s)ds = 0, \\
I_\rho w_{tt} + G(\psi - \phi_x) + \frac{4}{3} \gamma w + \frac{4}{3} \beta w_t - Dw_{xx} = 0, \\
\phi(x, 0) = \phi_0(x), \quad \psi(x, -t) = \psi_0(x, t), \quad w(x, -t) = w_0(x, t), \\
\phi_x(0, t) = \phi_1(t), \quad \psi(0, t) = \psi_1(t), \quad w(0, t) = w_1(1, t) = 0, \\
\phi_0(x, 0) = \phi_1(x), \quad \psi_1(x, 0) = \psi_1(x), \quad w_1(x, 0) = w_1(x).
\end{aligned}
\]

(1.7)

where $(x, t) \in (0, 1) \times (0, +\infty)$, $\rho, G, I_\rho, D, \gamma, \beta$ are positive physical constants, $\phi_0, \phi_1, \psi_0, \psi_1, w_0, w_1$ are given data. We intend to study the asymptotic behavior of solutions of (1.7) under the general assumption (2.2) (below) instead of the one used in many papers in the literature. Furthermore, our class of admissible initial data is larger than the one considered in the literature because we do not assume any boundedness condition on $(3w - \psi)_{xx}$. The rest of this paper is organized as follows. In section 2, we present some assumptions and material needed for our work. Some technical lemmas are presented and proved in section 3. Finally, we state and prove our main decay results and provide some examples in section 4 and Section 5.
2. Preliminaries. In this section, we present some materials needed for the proof of our results and state the existence result of the problem. We use the standard Lebesgue and Sobolev spaces with their usual scalar products and norms. Throughout this paper, $c$ is used to denote a generic positive constant. We assume the following hypotheses

\[(A) \ g : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a } C^1 \text{ nonincreasing function satisfying, for some } \beta_0 > 0, \]
\[
(2.1) \quad -\beta_0 g(s) \leq g'(s), \ g(t) > 0 \quad \text{and} \quad D - \int_0^{+\infty} g(s)ds := \ell > 0, \]

and there exists a $C^1$ function $H : \mathbb{R}_+ \to \mathbb{R}_+$ which is linear or it is strictly increasing and strictly convex $C^2$ function on $(0, r]$ for some $r > 0$ with $H'(0) = H''(0) = 0$, $\lim_{s \to +\infty} H'(s) = +\infty$, $s \mapsto sH'(s)$ and $s \mapsto s(H')^{-1}(s)$ are convex on $(0, r]$. Moreover, there exists a positive nonincreasing differentiable function $\xi$ such that

\[
(2.2) \quad g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0, \]

and $\int_0^{+\infty} \xi(s)ds = +\infty$.

**Remark 2.1.** [18] If $H$ is a strictly increasing and strictly convex $C^2$ function on $(0, r]$, with $H(0) = H'(0) = 0$, then it has an extension $\overline{H}$, which is strictly increasing and strictly convex $C^2$ function on $\mathbb{R}_+$. For instance, if $H(r) = a, H'(r) = b$ and $H''(r) = c$, we can define $\overline{H}$, for $t > r$, by

\[
(2.3) \quad \overline{H}(t) = \frac{c}{2}t^2 + (b - cr)t + \left(a + \frac{c}{2}r^2 - br\right). \]

For simplicity, in the rest of this paper, we use $H$ instead of $\overline{H}$.

**Remark 2.2.** [18] Since $H$ is strictly convex on $(0, r]$ and $H(0) = 0$, then

\[
(2.4) \quad H(\theta z) \leq \theta H(z), \quad 0 \leq \theta \leq 1 \text{ and } z \in (0, r]. \]

We consider the following spaces

\[
(2.5) \quad H^1_0 := \{u \in H^1(0, 1) : u(0) = 0\} \quad H^1 := \{u \in H^1(0, 1) : u(1) = 0\}. \]

For completeness, we state, without proof, the global existence and regularity result which can be established by the semigroup theory (see [15, 19, 20, 21]).

**Proposition 2.1.** Let $(\phi_0, \phi_1) \in H^1_0(0, 1) \times L^2(0, 1)$ and $(\psi_0(., 0), \psi_1), (w_0(., 0), w_1) \in H^1_0(0, 1) \times L^2(0, 1)$ be given. Assume that (A) holds. Then, problem (1.7) has a unique local (weak) solution

\[
\phi \in C(\mathbb{R}_+; H^1_0(0, 1)) \cap C^1(\mathbb{R}_+; L^2(0, 1)), \quad \psi, w \in C(\mathbb{R}_+; H^1_0(0, 1)) \cap C^1(\mathbb{R}_+; L^2(0, 1)) \]

Moreover, if $(\phi_0, \phi_1)(H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1)$ and $(\psi_0(., 0), \psi_1), (w_0, w_1) \in (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1)$, then the solution satisfies

\[
\phi \in C(\mathbb{R}_+; H^2(0, 1) \cap H^1_0(0, 1)) \cap C^1(\mathbb{R}_+; H^1_0(0, 1)) \cap C^2(\mathbb{R}_+; L^2(0, 1)), \]

\[
\psi, w \in C(\mathbb{R}_+; H^2(0, 1) \cap H^1_0(0, 1)) \cap C^1(\mathbb{R}_+; H^1_0(0, 1)) \cap C^2(\mathbb{R}_+; L^2(0, 1)). \]

We introduce the "modified" energy associated to problem (1.7)

\[
E(t) := \frac{1}{2} \int_0^1 \left(\rho \phi^2 + I_\rho (3w - \psi)^2 + 3I_\rho w^2 + G(\psi - \phi_x)^2 + 4\gamma w^2 + 3Dw_x^2\right) dx \]
\[
(2.6) \quad \frac{\ell}{2} \int_0^1 (3w - \psi)^2 + \frac{1}{2} g \circ (3w - \psi)_x, \]

and for any $u \in L^2(\mathbb{R}_+; L^2(0, 1))$,\n
\[
(gou)(t) = \int_0^{+\infty} g(s)||u(t) - u(t-s)||^2 ds. \]
Using system (1.7), integration by parts and using the fact that
\[
\frac{d}{dt}g \circ (3w - \psi)_x = g' \circ (3w - \psi)_x + 2 \left( \int_0^l g(s) ds \right) \int_0^t (3w - \psi)_x (3w - \psi)_x dx + 2 \int_0^l (3w - \psi)_x(t) \int_0^t g(t - s)(3w - \psi)_{xx}(s) ds dx
\]
(2.7)
one can obtain
\[
E'(t) = -4\beta \int_0^l w_t^2 + \frac{1}{2} g' \circ (3w - \psi)_x \leq 0.
\]

Next, we introduce the second energy functional
\[
E_*(t) := \frac{1}{2} \int_0^l \left( \rho \phi_{tt}^2 + L(3w_{tt} - \psi_{tt})^2 + 3I_\rho w_{tx}^2 + G(\psi_t - \phi_{xt})^2 + 4\gamma w_t^2 + 3Dw_{xx}^2 \right) dx
\]
(2.9)
\[
\frac{\ell}{2} \int_0^t (3w_{xt} - \psi_{xt})^2 dx + \frac{1}{2} g \circ (3w - \psi)_{xt}(t).
\]

3. Technical lemmas. In this section, we state and establish several lemmas needed for the proofs of our main results.

Lemma 3.1 (22). Let \( (\phi, (3w-\psi), w) \) be the strong solution of (1.7). Then, the second energy functional satisfies, for all \( t \geq 0 \),
\[
E'_*(t) = -4\beta \|w_{tt}\|_2^2 + \frac{1}{2} (g' \circ (3w_{xt} - \psi_{xt})) \leq 0
\]
and, hence,
\[
E_*(t) \leq E_*(0).
\]

Lemma 3.2. There exist positive constants \( M_0 \) and \( M_3 \) such that
\[
\int_t^{+\infty} g(s) \|(3w - \psi)_x(t) - (3w - \psi)_x(t - s)\|_2^2 ds \leq M_0 h_0(t),
\]
(3.3)
\[
\int_0^{+\infty} g(s) \|3w + \psi_{xt}(t) + (3w + \psi)(t + s)\|_2^2 ds \leq M_3 h_3(t),
\]
(3.4)
where \( h_0(t) = \int_0^{+\infty} g(t + s) \left( 1 + \|(3w - \psi)_{ox}(s)\| \right) ds \) and \( h_3(t) = \int_0^{+\infty} g(t + s) \left( 1 + \|(3w - \psi)_{ox}(s)\| \right) ds \).

Proof. The proof of (3.3) is identical to the one in [23]. Indeed, we have
\[
\int_t^{+\infty} g(s) \|(3w - \psi)_x(t) - (3w - \psi)_x(t - s)\|_2^2 ds \leq 2\|(3w - \psi)_x(t)\|^2 \int_t^{+\infty} g(s) ds
\]
\[
+ 2 \int_t^{+\infty} g(s) \|(3w - \psi)_x(t - s)\|_2^2 ds \leq 2 \sup_{s \geq 0} \|(3w - \psi)_x(s)\|^2 \int_0^{+\infty} g(t + s) ds + 2 \int_0^{+\infty} g(t + s) \|(3w - \psi)_x(s)\|^2 ds
\]
(3.5)
\[
\leq \frac{4E(s)}{\ell} \int_0^{+\infty} g(t + s) ds + 2 \int_0^{+\infty} g(t + s) \|(3w - \psi)_{ox}(s)\|^2 ds
\]
\[
\leq \frac{4E(0)}{\ell} \int_0^{+\infty} g(t + s) ds + 2 \int_0^{+\infty} g(t + s) \|(3w - \psi)_{ox}(s)\|^2 ds
\]
\[
\int_0^{+\infty} g(t + s) (1 + \|(3w - \psi)_{ox}(s)\|) ds.
\]
where \( M_0 = \max \{ 2, \frac{4E(0)}{\varrho} \} \). To prove \( (3.4) \), we use the same arguments in the proof of \( (3.3) \) with equations \( (3.1) \) and \( (3.2) \) as follows:

\[
\int_{t}^{+\infty} g(s) \left\| \left( 3w - \psi \right)_x(t) \right\|^2_2 ds \leq 2 \left\| (3w - \psi)_x(t) \right\|^2_2 \int_{t}^{+\infty} g(s) ds \\
+ 2 \int_{t}^{+\infty} g(s) \left\| (3w - \psi)_x((t-s)) \right\|^2_2 ds \\
\leq 2 \sup_{s \geq 0} \left\| (3w - \psi)_x(s) \right\|^2_2 \int_{0}^{+\infty} g(t+s) ds + 2 \int_{0}^{+\infty} g(t+s) \left\| (3w - \psi)_x(s) \right\|^2_2 ds \\
\leq 2E_*(0) \int_{0}^{+\infty} g(t+s) ds + 2 \int_{0}^{+\infty} g(t+s) \left\| (3w - \psi)_x(s) \right\|^2_2 ds \\
\leq 2E_*(0) \int_{0}^{+\infty} g(t+s) ds + 2 \int_{0}^{+\infty} g(t+s) \left\| (3w - \psi)_x(s) \right\|^2_2 ds \\
\leq M_1 \int_{0}^{+\infty} g(t+s) \left( 1 + \left\| (3w - \psi)_x(s) \right\|^2_2 \right) ds.
\]

where \( M_1 = \max \{ 2, 2E_*(0) \} \).

Now, we define the following functionals:

\[
F_1(t) := -\varrho \int_{0}^{1} \phi \psi dx, \quad F_2(t) := I_\varrho \int_{0}^{1} w \psi dx, \\
F_3(t) := I_\varrho \int_{0}^{1} (3w - \psi)_x \left( 3w - \psi \right) dx, \\
F_4(t) := -I_\varrho \int_{0}^{1} (3w - \psi)(3w - \psi)_x dx, \\
F_5(t) := \frac{-4\varrho_0}{G} \int_{0}^{1} \phi \left( \int_{0}^{1} w(z, t) dz \right) dx - \frac{3\varrho D}{G} \int_{0}^{1} \phi w x dx + 3I_\varrho \int_{0}^{1} w \left( \psi - \phi \right) dx.
\]

**Lemma 3.3.** The functional defined by

\[
L(t) = NE(t) + \sum_{i=1}^{5} N_i F_i(t)
\]

satisfies, for all \( t \in \mathbb{R}_+ \) and a suitable choice of the constant \( N, N_i(i = 1, ..., 5) \geq 0 \)

\[
(3.7) \quad L \sim E,
\]

and the estimate

\[
L'(t) \leq -c_1 \left[ \left\| \psi \right\|_2^2 + \left\| 3w - \psi \right\|_2^2 + \left\| w \right\|_2^2 + \left\| w_x \right\|_2^2 + \left\| 3w_x - \psi \right\|_2^2 \right] \\
\quad + \frac{1}{4} \left( g \circ (3w - \psi)_x \right) + N_5 \left( \frac{D\varrho}{G} - I_\varrho \right) \int_{0}^{1} \phi (3w x - \psi x) dx, \quad \forall t \geq 0,
\]

**Proof.** The proof is similar to the ones in [6, 24] with replacing the finite memory by infinite memory.

**Lemma 3.4.** The functional

\[
F_0(t) = \int_{0}^{t} g(s) \left\| (3w - \psi_x(s)) \right\|^2_2 ds,
\]

satisfies, along the solution, the estimate

\[
(3.9) \quad F_0(t) \leq \frac{1}{2} \left( g \circ (3w - \psi)_x \right) + 3(D - \ell) \left\| (3w - \psi_x) \right\|^2_2 dx + \frac{1}{4} \int_{0}^{+\infty} g(s) \left\| (3w - \psi_x)(t) - (3w - \psi_x)(t-s) \right\|^2_2 ds,
\]
Thanks to (3.11), we have for all $t \in \mathbb{R}^+$,

$$
(3.15)
$$

**Corollary 3.1.** There exists $G$ such that $\mu H$.

**Lemma 3.5.** Assume that $\frac{G}{\rho} = \frac{D}{I}$.

Then, the energy functional satisfies, for all $t \in \mathbb{R}^+$, the following estimate

$$
(3.11)
$$

$$
\int_0^t E(s)ds < \bar{m} f(t),
$$

where $f(t) = 1 + \int_0^t h_0(s)ds$ and $h_0$ is defined in Lemma 3.2.

**Proof.** As in [23], we have $F(t) = L(t) + F_0(t)$, then using (3.10) and (3.11), we obtain, for all $t \in \mathbb{R}^+$,

$$
F'(t) \leq -mE(t) + \frac{1}{2} \int_t^{+\infty} g(s) \| (3w - \psi_x)(t) - (3w - \psi_x)(t - s) \|_2^2 ds,
$$

where $m$ is some positive constant. Therefore, using (3.3), we obtain

$$
(3.12)
$$

$$
\begin{align*}
&\leq F(0) + \frac{M_0}{2} \int_0^t \int_0^{+\infty} g(\tau + s) \left( 1 + \| (3w - \psi_x)(s) \|_2^2 \right) d\tau ds \\
&\leq F(0) + \frac{M_0}{2} \int_0^t h_0(s)ds.
\end{align*}
$$

Hence, we get

$$
(3.13)
$$

$$
\int_0^t E(s)ds \leq \frac{F(0)}{m} + \frac{M_0}{2m} \int_0^t h_0(s)ds \leq \bar{m} \left( 1 + \int_0^t h_0(s)ds \right),
$$

where $\bar{m} = \max \left\{ \frac{F(0)}{m}, \frac{M_0}{2m} \right\}$.

**Corollary 3.1.** There exists $0 < q_0 < 1$ such that, for all $t \geq 0$, we have the following estimate:

$$
(3.14)
$$

$$
\int_0^t g(s) \| (3w - \psi_x)(t) - (3w - \psi_x)(t - s) \|_2^2 ds \leq \frac{1}{q(t)} H^{-1} \left( \frac{q(t) \mu(t)}{\xi(t)} \right)
$$

where $H$ is defined in Remark 2.1.

**Lemma 3.6.** Assume that $\mu(t) := -\int_0^t g^\prime(s) \| (3w - \psi_x)(t) - (3w - \psi_x)(t - s) \|_2^2 ds \leq -\epsilon E'(t)$

and

$$
(3.16)
$$

$$
q(t) := \frac{q_0}{f(t)}.
$$

**Proof.** As in [23], using (2.6) and (3.11), we have

$$
(3.17)
$$

$$
\begin{align*}
&\int_0^t \| (3w - \psi_x)(t) - (3w - \psi_x)(t - s) \|_2^2 ds \\
&\leq 2 \int_0^t \int_0^t \left( |(3w - \psi_x)(t)|^2 + |(3w - \psi_x)(t - s)|^2 \right) dsdx \\
&\leq \frac{4}{\ell} \int_0^t (E(t) + E(t - s)) dsdx \\
&\leq \frac{8}{\ell} \int_0^t E(s)dsdx \leq \frac{8}{\ell} \bar{m} f(t), \forall t \in \mathbb{R}^+.
\end{align*}
$$

Thanks to (3.11), we have for all $t \geq 0$ and for $0 < q_0 < \min \left\{ 1, \frac{\ell}{8m^2} \right\}$, $q(t) < 1$ and

$$
q(t) \int_0^t \| (3w - \psi_x)(t) - (3w - \psi_x)(t - s) \|_2^2 ds < 1.
$$

\[\square\]
4. A decay result for equal speeds of wave propagation. In this section, we state and prove a new general decay result in the case of equal speeds of wave propagation (3.10). As in [23], we introduce the following functions:

\[(4.1)\] \[G_1(t) := \int_t^1 \frac{1}{sH'(s)} ds,\]

\[(4.2)\] \[G_2(t) = tH'(t), \quad G_3(t) = t(H')^{-1}(t), \quad G_4(t) = G_3^+(t).\]

Further, we introduce the class \(S\) of functions \(\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+^*\) satisfying \(c_1, c_2 > 0\) (should be selected carefully in (4.16)):

\[(4.3)\] \[\chi \in \mathcal{C}^1(\mathbb{R}_+), \quad \chi \leq 1, \quad \chi' \leq 0,\]

and

\[(4.4)\] \[c_2G_4 \left[ \frac{c}{d}q(t)h_0(t) \right] \leq c_1 \left( G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - \frac{G_2(G_5(t))}{\chi(t)} \right),\]

where \(d > 0, c\) is a generic positive constant which may change from line to line, \(h_0\) and \(q\) are defined in Lemma 3.2 and Lemma 3.5 and

\[(4.5)\] \[G_5(t) = G_1^{-1} \left( c_1 \int_0^t \chi(s) ds \right).\]

**Remark 4.1.** [26] According to the properties of \(H\) introduced in (A), \(G_2\) is convex increasing and defines a bijection from \(\mathbb{R}_+\) to \(\mathbb{R}_+^*, \) \(G_1\) is decreasing defines a bijection from \([0,1]\) to \(\mathbb{R}_+^*,\) and \(G_3\) and \(G_4\) are convex and increasing functions on \([0,\varepsilon_0].\) Then the set \(S\) is not empty because it contains \(\chi(s) = \varepsilon\Psi_5(s)\) for any \(0 < \varepsilon \leq 1,\) small enough. Indeed, (4.3) is satisfied since (4.11) and (4.15). On the other hand, we have \(q(t)h_0(t)\) is nonincreasing, \(0 \leq G_5 \leq 1,\) and \(H'\) and \(G_4\) are increasing, then (4.4) is satisfied if

\[(4.6)\] \[c_2G_4 \left[ \frac{c}{d}q(t)h_0(0) \right] \leq c_1 \left( H' \left( \frac{1}{\varepsilon} \right) - H'(1) \right),\]

which holds for \(0 < \varepsilon \leq 1,\) small enough, since \(\lim_{t \to +\infty} H'(1) = +\infty.\) But with the choice \(\varepsilon = \varepsilon_5,\)

(4.6) (below) does not lead to any stability estimate. The idea is to choose \(\chi\) satisfy (4.3) and (4.4) such that (4.6) gives the best possible decay rate for \(E.\)

**Theorem 4.1.** Assume that (A) and (3.10) hold, then there exists a strictly positive constant \(C\) such the solution of (1.7) satisfies, for all \(t \geq 0,\)

\[(4.6)\] \[E(t) \leq CG_5(t) \frac{1}{\chi(t)q(t)}.\]

**Proof.** We combine (2.8), (3.3), (3.8), (3.14), and (2.8), to have, for some \(m > 0\) and for any \(t \geq 0,\) we have

\[(4.7)\] \[L'(t) \leq -mE(t) + \frac{c}{q(t)} H^{-1} \left( \frac{q(t)\mu(t)}{\chi(t)} \right) + c\theta_0(t).\]

Without loss of generality, one can assume that \(E(0) > 0.\) For \(\varepsilon_0 < r,\) let the functional \(\mathcal{F}\) defined by

\[(4.8)\] \[\mathcal{F}(t) := H' \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) \frac{H(t)}{L(t)},\]

which satisfies \(\mathcal{F} \sim E.\) By noting that \(H'' \geq 0, q \leq 0\) and \(E' \leq 0,\) we get

\[\mathcal{F}'(t) = \varepsilon_0 \frac{qE'(t)}{E(0)} H'' \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) L(t) + H' \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) L'(t)\]

\[\leq -mE(t)H' \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) + \frac{c}{q(t)} H' \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) H^{-1} \left( \frac{q(t)\mu(t)}{\chi(t)} \right)\]

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Let \( H^* \) be the convex conjugate of \( H \) in the sense of Young (see [27]), then
\[
H^*(s) = s(H')^{-1}(s) - H \left[ (H')^{-1}(s) \right], \text{ if } s \in (0, H'(r])
\]
and satisfies the following generalized Young inequality
\[
AB \leq H^*(A) + H(B), \text{ if } A \in (0, H'(r)], B \in (0, r].
\]
So, with \( A = H^* \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) \) and \( B = H^{-1} \left( \frac{\mu(t)}{\xi(t)} \right) \) and using (2.8) and (4.8)-(4.10), we arrive at
\[
\mathcal{F}'(t) \leq -m E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c\xi(t)\varepsilon_0 E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c \left( \frac{\mu(t)}{\xi(t)} \right)
\]
\[
+ ch_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right)
\]
\[
\leq -m E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c\varepsilon_0 E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c \left( \frac{\mu(t)}{\xi(t)} \right)
\]
\[
+ ch_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right).
\]
So, multiplying (4.11) by \( \xi(t) \) and using (3.15) and the fact that \( \frac{\varepsilon_0 E(t)q(t)}{E(0)} < r \) gives
\[
\xi(t)\mathcal{F}'(t) \leq -m \xi(t)E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c\xi(t)\varepsilon_0 E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right)
\]
\[
+ cp(t)q(t) + c\xi(t)h_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right)
\]
\[
\leq -\varepsilon_0 \left( \frac{m E(0)}{\varepsilon_0} - c \right) \xi(t) E(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) - cE' + c\xi(t)h_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right).
\]
Consequently, recalling the definition of \( G_2 \) and choosing \( \varepsilon_0 \) small enough so that \( k = \left( \frac{m E(0)}{\varepsilon_0} - c \right) > 0 \),
we obtain, for all \( t \in \mathbb{R}_+ \),
\[
\mathcal{F}_1(t) \leq -k \xi(t) \left( \frac{E(t)}{E(0)} \right)^2H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + c\xi(t)h_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right)
\]
\[
= -k \frac{\xi(t)}{q(t)} G_2 \left( \frac{E(t)q(t)}{E(0)} \right) + c \xi(t)h_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right),
\]
where \( \mathcal{F}_1 = \xi \mathcal{F} + cE \sim E \) and satisfies for some \( \alpha_1, \alpha_2 > 0 \).
\[
\alpha_1 \mathcal{F}_1(t) \leq E(t) \leq \alpha_2 \mathcal{F}_1(t).
\]
Since \( G_2(t) = H'(t) + tH''(t) \), then, using the strict convexity of \( H \) on \((0,r)\], we find that \( G_2'(t), G_2(t) > 0 \) on \((0,r)\]. Let \( d > 0 \), use the general Young inequality (4.10) on the last term in (4.12) with \( A = H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) \) and \( B = \left[ \frac{\xi}{h_0(t)} \right] \), to get
\[
ch_0(t)H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) = \frac{d}{q(t)} \left[ \frac{c}{d} q(t)h_0(t) \right] \left( H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) \right)
\]
\[
\leq \frac{d}{q(t)} G_3 \left( H' \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) \right) + \frac{d}{q(t)} G_3 \left[ \frac{c}{d} q(t)h_0(t) \right]
\]
\[
\leq \frac{d}{q(t)} G_2 \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + \frac{d}{q(t)} G_4 \left[ \frac{c}{d} q(t)h_0(t) \right].
\]
Now, combining (4.12) and (4.14) and choosing \( d \) small enough so that \( k_1 = (k - d) > 0 \), we arrive at
\[
\mathcal{F}_1(t) \leq -k \frac{\xi(t)}{q(t)} G_2 \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + \frac{d \xi(t)}{q(t)} G_2 \left( \frac{\varepsilon_0 E(t)q(t)}{E(0)} \right) + \frac{d \xi(t)}{q(t)} G_2 \left[ \frac{c}{d} q(t)h_0(t) \right]
\]
Using the equivalent property in (4.13) and since $G_2$ is increasing, we have, for some $d_0 = \frac{a_2}{E(0)} > 0$,

$$G_2 \left( \varepsilon_0 \frac{E(t)q(t)}{E(0)} \right) \geq G_2 \left( d_0 F_1(t)q(t) \right).$$

Letting $F_2(t) := d_0 F_1(t)q(t)$ and recalling that $q' \leq 0$, then we obtain, for some constant $\xi_1 = d_0k_1 > 0$ and $\xi_2 = d_0d > 0$,

$$F_2(t) \leq -\xi_1 \xi(t)G_2(F_2(t)) + c_2 \xi(t)G_4 \left[ \frac{c}{q(t)} q(t)h_0(t) \right].$$

Since $d_0q(t)$ is nonincreasing, then use of the equivalent property $F_1 \sim E$ implies that there exists $b_0 > 0$ such that $F_2(t) \geq b_0 E(t)q(t)$. Let $t \in \mathbb{R}_+$ and $\chi(t)$ satisfy (4.3) and (4.4). If

$$b_0 q(t) E(t) \leq 2 \frac{G_5(t)}{\chi(t)},$$

then, we have

$$E(t) \leq \frac{2}{b_0} \frac{G_5(t)}{\chi(t)}.$$ 

If

$$b_0 q(t) E(t) > 2 \frac{G_5(t)}{\chi(t)},$$

then, for any $0 \leq s \leq t$, we get

$$b_0 q(s) E(s) > 2 \frac{G_5(t)}{\chi(t)},$$

since, $q(t) E(t)$ is nonincreasing function. Therefore, we have for any $0 \leq s \leq t$,

$$F_2(s) > 2 \frac{G_5(t)}{\chi(t)}.$$ 

Using (4.21), the definition of $G_2$, the fact that $G_2$ is convex and $0 < \chi \leq 1$, we have, for any $0 \leq s \leq t$ and $0 < \xi_1 \leq 1$,

$$G_2 \left( \xi_1 \chi(s) F_2(s) - \xi_1 G_5(s) \right) \leq \xi_1 \chi(s) G_2 \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right)$$

$$\leq \xi_1 \chi(s) F_2(s) H' \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right) - \xi_1 \chi(s) \frac{G_5(s)}{\chi(s)} H' \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right)$$

$$\leq \xi_1 \chi(s) F_2(s) H' \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right) - \xi_1 \chi(s) \frac{G_5(s)}{\chi(s)} H' \left( F_2(s) - \frac{G_5(s)}{\chi(s)} \right).$$

Now, we let

$$F_3(t) = \xi_1 \chi(t) F_2(t) - \xi_1 G_5(t),$$

where $\xi_1$ is small enough so that $F_3(0) \leq 1$. Then (4.22) becomes, for any $0 \leq s \leq t$,

$$G_2 \left( F_3(s) \right) \leq \xi_1 \chi(s) G_2 \left( F_2(s) - \xi_1 \chi(s) G_2 \left( \frac{G_5(s)}{\chi(s)} \right) \right).$$

Further, we have

$$F_3'(t) = \xi_1 \chi'(t) F_2(t) + \xi_1 \chi(t) F_2'(t) - \xi_1 G_5'(t).$$

Since $\chi' \leq 0$ and using (4.16), then for any $0 \leq s \leq t$, $0 < \xi_1 \leq 1$, we obtain

$$F_3'(t) \leq \xi_1 \chi(t) F_2'(t) - \xi_1 G_5'(t)$$

$$\leq -\xi_1 \xi(t) \chi(t) G_2(F_2(t)) + c_2 \xi_1 \xi(t) \chi(t) G_4 \left[ \frac{c}{q(t)} q(t)h_0(t) \right] - \xi_1 G_5'(t).$$
Then, using (4.4) and (4.24), we get
\[ \mathcal{F}_3(t) \leq -c_1 \xi(t) G_2(\mathcal{F}_3(t)) + c_2 \epsilon_1 G_3 \left( \frac{\int_0^t g(\tau) h_0(\tau) \, d\tau}{\chi(t)} \right) - c_1 G'_5(t). \] (4.27)

From the definition of \( G_1 \) and \( G_5 \), we have
\[ G_1(G_5(s)) = c_1 \int_{-\infty}^s \xi(\tau) \, d\tau, \]
hence,
\[ G'_5(s) = -c_1 \xi(s) G_2(G_5(s)). \] (4.28)

Now, we have
\[ c_2 \epsilon_1 \chi(t) G_4 \left( \frac{\int_0^t g(\tau) h_0(\tau) \, d\tau}{\chi(t)} \right) - c_1 c_2 \epsilon_1 \chi(t) G_2 \left( \frac{G_5(t)}{\chi(t)} \right) = c_1 G'_5(t) \]
(4.29)

Then, according to (4.4), we get
\[ \epsilon_1 \chi(t) G_4 \left( \frac{\int_0^t g(\tau) h_0(\tau) \, d\tau}{\chi(t)} \right) = c_1 G_2 \left( \frac{G_5(t)}{\chi(t)} \right) - c_3 G_2(G_5(t)) \leq 0 \]

Then (4.27) gives
\[ \mathcal{F}_3(t) \leq -c_1 \xi(t) G_2(\mathcal{F}_3(t)). \] (4.30)

Thus from (4.30) and the definition of \( G_1 \) and \( G_2 \) in (4.1) and (4.2), we obtain
\[ \left( G_1(\mathcal{F}_3(t)) \right)' \geq c_1 \xi(t). \] (4.31)

Integrating (4.31) over \([0,t]\), we get
\[ G_1(\mathcal{F}_3(t)) \geq c_1 \int_0^t \xi(s) \, ds + G_1(\mathcal{F}_3(0)). \] (4.32)

Since \( G_1 \) is decreasing, \( \mathcal{F}_3(0) \leq 1 \) and \( G_1(1) = 0 \), then
\[ \mathcal{F}_3(t) \leq G_1^{-1} \left( c_1 \int_0^t \xi(s) \, ds \right) = G_5(t). \] (4.33)

Recalling that \( \mathcal{F}_3(t) = \epsilon_1 \chi(t) \mathcal{F}_2(t) - \epsilon_1 G_5(t) \), we have
\[ \mathcal{F}_2(t) \leq \frac{1 + \epsilon_1}{\epsilon_1} \frac{G_5(t)}{\chi(t)}. \] (4.34)

Similarly, recall that \( \mathcal{F}_2(t) := d_0 \mathcal{F}_1(t) q(t) \), then
\[ \mathcal{F}_1(t) \leq \frac{1 + \epsilon_1}{d_0 \epsilon_1} \frac{G_5(t)}{\chi(t) q(t)}. \] (4.35)

Since \( \mathcal{F}_1 \sim E \), then for some \( b > 0 \), we have \( E(t) \leq b \mathcal{F}_1 \); which gives
From (4.18) and (4.36), we obtain the following estimate

\begin{equation}
E(t) \leq c_3 \left( \frac{G_2(t)}{\chi(t)q(t)} \right),
\end{equation}

where \( c_4 = \max\{ \frac{2}{b_1}, \frac{b_1^{(1+c_1)}}{a_0i_1} \} \). This completes the proof of Theorem 4.1. \( \square \)

**Example 1 [23]:** Let \( g(t) = \frac{a}{(1+t)^{r}} \), where \( \nu > 1 \) and \( 0 < a < \nu - 1 \) so that (A1) and (3.10) are satisfied. In this case \( \xi(t) = \nu a \frac{t^r}{r} \) and \( H(t) = t^{\frac{r}{r+1}} \). Then, there exist positive constants \( a_i (i = 0, ..., 3) \) depending only on \( a, \nu \) such that

\begin{equation}
G_4(t) = a_0 t^{\frac{r}{r+1}} , \quad G_2(t) = a_1 t^{\frac{r}{r+1}}, \quad G_1(t) = a_2 (t^{\frac{r}{r+1}} - 1), \quad G_5(t) = (a_3 t + 1)^{-\nu}.
\end{equation}

We will discuss two cases:

**Case 1:** if

\begin{equation}
m_0(1 + t)^r \leq 1 + \| (3w - \psi)_x \|^{2} \leq m_1(1 + t)^r
\end{equation}

where \( 0 < r < \nu - 1 \) and \( m_0, m_1 > 0 \), then we have, for some positive constants \( a_i (i = 4, ..., 7) \) depending only on \( a, \nu, m_0, m_1, r \), the following:

\begin{equation}
a_4(1 + t)^{-\nu + 1 + r} \leq h_0(t) \leq a_5(1 + t)^{-\nu + 1 + r},
\end{equation}

\begin{equation}
\frac{q_0}{q(t)} \geq a_6 \begin{cases} 
1 + \ln(1 + t), & \nu - r = 0; \\
2, & 1 < \nu - r < 2; \\
(1 + t)^{-\nu + r + 2}, & 1 < \nu - r < 2.
\end{cases}
\end{equation}

\begin{equation}
\frac{q_0}{q(t)} \leq a_7 \begin{cases} 
1 + \ln(1 + t), & \nu - r = 0; \\
2, & 1 < \nu - r < 2; \\
(1 + t)^{-\nu + r + 2}, & 1 < \nu - r < 2.
\end{cases}
\end{equation}

We notice that condition (4.4) is satisfied if

\begin{equation}
(t + 1)^{\nu} q(t) h_0(t) \chi(t) \leq a_8 \left(1 - (\chi(t))^{\frac{r}{r+1}}\right).
\end{equation}

where \( a_8 > 0 \) depending only on \( a, \nu, c_1 \) and \( c_2 \). Choosing \( \chi(t) \) as follows

\begin{equation}
\chi(t) = \lambda \begin{cases} 
(1 + t)^{-p}, & p = r + 1, \nu - r \geq 2; \\
(1 + t)^{-p}, & p = \nu - 1, 1 < \nu - r < 2.
\end{cases}
\end{equation}

with \( 0 < \lambda \leq 1 \), so that (4.3) is valid. Moreover, using (4.40) and (4.41), we see that if \( 0 < \lambda \leq 1 \) is small enough, then (4.4) is satisfied. Hence (4.6) and (4.42) imply that, for any \( t \in \mathbb{R} \)

\begin{equation}
E(t) \leq a_9 \begin{cases} 
1 + \ln(1 + t), & \nu - r \geq 2; \\
(1 + t)^{-\nu + r - 1}, & \nu - r > 2 \text{ or } 2 < \nu - r < 2.
\end{cases}
\end{equation}

Thus, the estimate (4.45) gives \( \lim_{t \to +\infty} E(T) = 0 \). Case 2: if \( m_0 \leq 1 + \| (3w - \psi)_x \|^{2} \leq m_1 \). That is \( r = 0 \) in (4.40) (as it was assumed in many papers in the literature). Then, the estimate (4.45) gives \( \lim_{t \to +\infty} E(T) = 0 \) when \( r = 0 \).

5. A decay result for non-equal speeds of wave propagation. In this section, we give an estimate to the decay rate in the case of non-equal speeds of wave propagation. We start by stating, under assumption (A) and under non-equal speeds of wave propagation, some Lemmas that are necessary for the proof of our second main result. First, we have the following estimate for the last term in the right-hand side of (3.8).

**Lemma 5.1 [22]:** Let \( \phi, (3w - \psi), w \) be the strong solution of (1.7). Then, for any \( \varepsilon > 0 \), we have

\begin{equation}
(\phi D_t^i D_t^j)^{1} \phi_t (3w_{xt} - \psi_{xt})dx \leq \varepsilon E(t) + \frac{C}{\varepsilon} \left( (g \circ (3w_{xt} - \psi_{xt})(t) - E'(t) \right), \quad \forall t \geq 0.
\end{equation}
Lemma 5.2. We have, for any \( t > 0 \), the following estimate:

\[
(5.2) \quad \int_0^t g(s) \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds \\
\leq \frac{1}{\gamma(t)} H^{-1} \left( \frac{\gamma(t)\theta(t)}{\xi(t)} \right),
\]

where \( \gamma(t) := \frac{\gamma_0}{t} \), \( \gamma_0 \in (0,1) \), and

\[
\theta(t) := -\int_0^t g'(s) \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds \\
+ \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 ds
\leq -c(E'(t) + E_*(t)).
\]

Proof. Let us define the following functional:

\[
(5.4) \quad \eta(t) := \gamma(t) \int_0^t \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds.
\]

The use of (2.6), (2.8), (2.9) and (3.2) gives for any \( t \geq 0 \),

\[
\gamma(t) \int_0^t \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds \\
\leq 2\gamma(t) \int_0^t \left( \|(3w_x - \psi_x)(t)\|^2_2 + \|(3w_x - \psi_x)(t-s)\|^2_2 \right) ds \\
+ \|(3w_{xt} - \psi_{xt})(t)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t-s)\|^2_2 ds
\leq \frac{4\gamma(t)}{\ell} \int_0^t (E(t) + E(t-s) + E_*(t) + E_*(t-s)) ds
\leq \frac{8\gamma(t)}{\ell} \int_0^t [E(0) + c(E_*(0))] ds
\leq \frac{8\gamma_0}{\ell} [E(0) + c(E_*(0))] < +\infty, \quad \forall t > 0.
\]

This allows us to pick \( 0 < \gamma_0 < 1 \) such that \( \eta < 1 \). Thus using Jensen’s inequality and (5.4), we obtain, for any \( t > 0 \),

\[
\theta(t) = -\frac{1}{\eta(t)} \int_0^t \eta(t) g'(s) \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 \\
+ \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds
\geq \frac{1}{\eta(t)} \int_0^t \eta(t) \xi(s) (g(s)) \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 \\
+ \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds
\geq \frac{\xi(t)}{\gamma(t)} H \left( \gamma(t) \int_0^t g(s) \left( \|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|^2_2 \\
+ \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds \right)
\geq \frac{\xi(t)}{\gamma(t)} H \left( \gamma(t) \int_0^t g(s) \left( \|(3w_x - \psi_x)(t-s)\|^2_2 + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|^2_2 \right) ds \right),
\]

Then (5.2) is established. \( \square \)

Theorem 5.1. Let \((\phi, (3w - \psi), w)\) be the solution of (1.7). Assume that (A) holds and the relation \( (3.10) \) is not satisfied. Then, there exist positive constants \( C_1’S \) such that the solution of (1.7) satisfies,
for all $t \geq 0$,  
\begin{equation}
E(t) \leq \frac{C_1}{(t+1)G_2^{-1}} \left[ C_2 + \int_0^t \xi(s)G_4 \left( \frac{C_3 h_2(t)}{s+1} \right) ds \right],
\end{equation}

where $h_2 = h_0 + h_1$ and $h_0$, $h_1$, $G_2$ and $G_4$ are functions defined earlier in this paper.

**Proof.** Combining (3.8) and (5.1), we have, for some $m > 0$,  
\begin{align*}
L'(t) &\leq -mE(t) + c \left( g \circ (3w_x - \psi_x) \right)(t) + \left( \frac{pD}{G} - I_\rho \right) \int_0^t \varphi_1(3w_{xt} - \psi_{xt})dx \\
&\leq -(m - \varepsilon)E(t) + c \left( g \circ (3w_x - \psi_x) \right)(t) + \frac{c}{\varepsilon} \left( g \circ (3w_{xt} - \psi_{xt}) \right)(t) - E'(t), \quad \forall t \geq 0.
\end{align*}

After fixing $\varepsilon$ small enough, we arrive at  
\begin{equation}
L'(t) \leq -m_1E(t) + c \left( g \circ (3w_x - \psi_x) + g \circ (3w_{xt} - \psi_{xt}) \right)(t) - cE'(t), \quad \forall t \geq 0,
\end{equation}

where $m_1$ is a fixed positive constant. By setting $F := L + cE \sim E$, we obtain, for any $t \geq 0$,  
\begin{equation}
F'(t) \leq -m_1E(t) + c \left( g \circ (3w_x - \psi_x) + g \circ (3w_{xt} - \psi_{xt}) \right)(t).
\end{equation}

Combining (3.3), (5.2) and (5.8), we have  
\begin{equation}
F'(t) \leq -m_1E(t) + \frac{c}{\gamma(t)} H^{-1} \left( \frac{\gamma(t)\theta(t)}{\xi(t)} \right) + ch_2(t), \quad \forall t > 0,
\end{equation}

where $h_2(t) = h_0(t) + h_1(t)$ and $h_0$ and $h_1$ are defined in Lemma 3.2. Let $0 < \varepsilon_1 < r$, then define a functional $F_1$ by  
\begin{equation}
F_1(t) := H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) F(t), \quad \forall t > 0.
\end{equation}

Then, estimate (5.9) together with the facts that $E' \leq 0$, $H' > 0$ and $H'' > 0$ leads to  
\begin{align}
F_1'(t) &\leq -m_1E(t)H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) + cH' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) h_2(t) \\
&\quad + \frac{c}{\varepsilon_1 H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) H^{-1} \left( \frac{\gamma(t)\theta(t)}{\xi(t)} \right)}, \quad \forall t > 0.
\end{align}

Let $H^*$ be the convex conjugate of $H$ as in (4.9), set  
\begin{equation*}
A = H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) \quad \text{and} \quad B = H^{-1} \left( \frac{\gamma(t)\theta(t)}{\xi(t)} \right).
\end{equation*}

Combining (4.9), (4.10) and (5.10) and selecting $\varepsilon_1$ small enough, we obtain, $\forall t > 0$ and $m_2 > 0$,  
\begin{equation}
F_1'(t) \leq -m_2 \frac{E(t)}{E(0)} H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) + cH' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) h_2(t).
\end{equation}

Multiplying both sides of (5.11) by $\xi(t)$ and using $\varepsilon_1 \frac{E(t)}{E(0)} < r$ and inequality (5.3), we arrive at  
\begin{align}
\xi(t)F_1'(t) &\leq -m_2 \xi(t) \frac{E(t)}{E(0)} H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) + c\theta(t) + cH' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) \xi(t)h_2(t) \\
&\leq -m_2 \xi(t) \frac{E(t)}{E(0)} H' \left( \varepsilon_1 \frac{E(t)\gamma(t)}{E(0)} \right) \xi(t)h_2(t) - c(E'(t) + E'_*(t)) \quad \forall t > 0.
\end{align}
Thus, by setting $F_2 = \xi F_1 + c(E + E_*)$ and noting that $0 \leq H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) \leq H' \left( \xi_1 \right)$ and $0 \leq \xi(t) \leq \xi(0)$, we deduce that $F_2 \geq cE_\ast \geq cE$ and because $\xi$ is nonincreasing, the estimate \[5.12\] becomes,

$$F'_2(t) \leq -m_2\xi(t) \frac{E(t)}{E(0)} H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + c\xi(t)H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) h_2(t)$$

$$= \frac{m_2}{\gamma(t)}G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + c\xi(t)H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) h_2(t), \quad \forall t > 0.$$  

(5.13)

Since $G_2'(t) = H'(\xi_1 t) + \xi_1 tH''(\xi_1 t)$, then, using the strict convexity of $H$ on $(0, r)$, we find that $G_2'(t), G_2(t) > 0$ on $(0, 1]$. Using the generalized Young inequality \[4.10\] on the last term in \[5.13\] with $B = \frac{1}{\gamma}h_2(t)$ and $A = H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right)$, we have

$$c h_2(t) H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) = \frac{d}{\gamma(t)} \left[ \frac{c}{d} \gamma(t) h_2(t) \right] \left[ H' \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) \right]$$

$$\leq \frac{d}{\gamma(t)} G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + \frac{d}{\gamma(t)} G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) \frac{c}{d} \gamma(t) h_2(t)$$

$$\leq \frac{d}{\gamma(t)} G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + \frac{d}{\gamma(t)} G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right)$$

(5.14)

Now, combining \[5.13\] and \[5.14\] and choosing $d$ small enough, we arrive at

$$F'_2(t) \leq -m_2\xi(t) \frac{E(t)}{E(0)} G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + \xi(t)G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right)$$

(5.15)

$$\leq -c\xi(t)G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right) + \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right).$$

Since $E' > 0$ and $\gamma' < 0$, then $(E\gamma)(t)$ is decreasing functions. Using this fact and since $G_2$ is increasing, then, for $0 \leq t \leq T$, we have

$$G_2 \left( \xi_1 \frac{E(T)\gamma(T)}{E(0)} \right) \leq G_2 \left( \xi_1 \frac{E(t)\gamma(t)}{E(0)} \right).$$

(5.16)

Combining \[5.15\] with \[5.16\] and multiplying by $\gamma(t)$, we get

$$\gamma(t) F'_2(t) + c\xi(t)G_2 \left( \xi_1 \frac{E(T)\gamma(T)}{E(0)} \right) \leq c\xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right),$$

(5.17)

since $\gamma' < 0$, then

$$\left( \gamma(t) F_2(t) \right)' + c\xi(t)G_2 \left( \xi_1 \frac{E(T)\gamma(T)}{E(0)} \right) \leq c\xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right).$$

(5.18)

Integrating \[5.18\] over $[0, T]$, we have

$$G_2 \left( \xi_1 \frac{E(T)\gamma(T)}{E(0)} \right) \int_0^T \xi(t)dt \leq \frac{F_2(0)\gamma(0)}{c} + \int_0^T \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right) dt,$$

(5.19)

and then

$$G_2 \left( \xi_1 \frac{E(T)\gamma(T)}{E(0)} \right) \leq \left[ \frac{F_2(0)}{c} + \int_0^T \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right) dt \right].$$

(5.20)

Thus

$$\xi_1 \frac{E(T)\gamma(T)}{E(0)} \leq G_2^{-1} \left[ \frac{F_2(0)}{c} + \int_0^T \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right) dt \right].$$

(5.21)

Then, we obtain

$$\xi_1 \frac{E(T)\gamma(T)}{E(0)} \leq G_2^{-1} \left[ \frac{F_2(0)}{c} + \int_0^T \xi(t)G_4 \left( \frac{c}{d} \gamma(t) h_2(t) \right) dt \right].$$

(5.22)
where $C = \max \left\{ 1, \frac{F_2(0)}{c} \right\}$. This finishes the proof of Theorem (5.1).

Example 2 [23]: We consider the same function $g(t) = \frac{a}{(1+t)^r}$ as in Example 1, where $\nu > 1$ and $0 < a < \nu - 1$. Thus, we have

$$
\int_0^T \xi(t)G_4 \left( \frac{\gamma(t)}{\tau} b_2(t) \right) dt < +\infty,
$$

and then in case

$$
m_0(1+t)^r \leq 1 + \|[[3w-\psi]]_{0,T}\|^2 \leq m_1(1+t)^r
$$

where $0 < r < \nu - 1$ and $m_0, m_1 > 0$, then we have, for any $t \in \mathbb{R}^+$

$$
E(T) \leq a_0 \left\{ \begin{array}{ll}
1 + \ln(1 + T) & \nu - r = 2; \\
T^{-\left(\frac{1}{\nu} \right)} & \nu - r > 2; \\
1 + T & 1 < \nu - r < 2.
\end{array} \right.
$$

Thus for $\nu - r \geq 2$ or $\frac{1}{2}(r + \sqrt{r^2 + 4r + 8}) < \nu < r + 2$, the estimate (5.25) gives $\lim_{T \to +\infty} E(T) = 0$. If $m_0 \leq 1 + \|[[3w-\psi]]_{0,T}\|^2 \leq m_1$. Then, estimate (5.25) holds for $r = 0$ as it was assumed in many papers in the literature. That is

$$
E(T) \leq a_7 \left\{ \begin{array}{ll}
1 + \ln(1 + T) & \nu = 2; \\
T^{-\left(\frac{1}{\nu} \right)} & \nu > 2; \\
1 + T & 1 < \nu < 2.
\end{array} \right.
$$

Thus for $\nu \geq 2$ or $\sqrt{2} < \nu < 2$, the estimate (5.26) gives $\lim_{T \to +\infty} E(T) = 0$.

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REFERENCES

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