Bohl Theorem for Volterra Equation

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Abstract

In this paper we are concerned with the robust stability of Volterra equations. We consider the conditions preserving the stability of these systems under perturbations. Also, we study the so-called Bohl-Perron type stability theorems.

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1 Introduction

Studying the robust stability of systems plays an important role both in theory and practice since the system always operates under the effect of uncertain perturbations. The designers want to have systems working stably under small perturbations. If the system is described by mathematical models, the study of its robust stability via analyzing parameters is an interesting problem. There are many works dealing with conditions imposed on coefficients under which the system is robustly stable. For example, one can measure the robust stability by using the so-called stability radii for linear systems [6,12]. However, it is difficult to compute the stability radius of a time-varying system, which leads to some estimates of the perturbation ensuring the stability of perturbed systems (see [7,15]).

The other problem in studying robust stability is to deal with the Bohl-Perron Theorem, which says that, for a differential/difference equation, if the good input implies the acceptable output then the system must be exponentially stable. The earliest work in this topic belongs to Perron [13] (1930). He proved his celebrated theorem which says that if the solution of the equation \( x'(t) = A(t)x(t) + f(t), t \geq 0 \) with the initial condition \( x(0) = 0 \) is bounded for every continuous function \( f \) bounded on \([0, \infty)\), then the trivial solution of the corresponding homogeneous equation \( \dot{x}(t) = A(t)x(t), t \geq 0 \) is uniformly asymptotically stable. Later, one continues to study this problem for delay equation of the form \( x'(t) = \sum_{k=1}^{m} A_k(t)x(t-\tau_k) + f(t) \) or \( \dot{x}(t) = Lx(t) + f(t), t \geq 0 \) where \( L \) is an operator acting on \( C([-r,0], \mathbb{R}^n) \) (see [14] and therein). Discrete versions of Bohl-Perron Theorem can be found in [3,5,7].

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The aim of this paper is to continue the study of these problems by considering the robust stability for the Volterra system

\[ \dot{x}(t) = A(t)x(t) + \int_0^t H(t,s)x(s)ds + f(t), \quad t \geq 0, \]  

(1.1)

where \( A(\cdot) \) and \( H(\cdot, \cdot) \) are specified later. First, we deal with the preservation of stability for the linear Volterra equation (1.1) under small perturbations. Since the derivative of state process \( x(t) \) depends on all past path \( x(s), 0 \leq s \leq t \), we have to use a more general inequality of Gronwall-Bellman type to obtain the upper bound of perturbations. Next, Bohl-Perron type theorems are established for the equation (1.1). The most difficulty we have to face here is that the Cauchy operator of the corresponding homogeneous equation does not have the semi-group property, which implies that the classical argument to solve this problem is no longer valid. To overcome it, we define weighed spaces \( L_\gamma([0, \infty); X) \) and \( C_\gamma([0, \infty); X) \) (see Definitions below) and consider operators acting between these spaces. In case of the Volterra equation having damped memory, we obtain a classical result. That is, the input \( f \) in \( L_1([0, \infty); X) \) implying the output in \( L_1([0, \infty); X) \) is equivalent to exponential stability of homogeneous systems.

The paper is organized as follows. In the next section we recall some basic properties of linear Volterra equation. In Section 3, we prove that if the linear Volterra equations are exponentially stable, then under small Lipschitz perturbations they are still exponentially stable. Section 4 presents the famous Bohl-Perron Theorem for (1.1). We introduce some weighted spaces and consider the solutions of (1.1) as elements of these spaces. Hence, we show that the exponential stability is equivalent to the surjectivity of certain operators. Some examples are introduced to illustrate the results.

2 Linear Volterra differential equations

Let \( X \) be a Banach space and \( \mathcal{L}(X) \) be the space of the continuous linear transformations on \( X \). We consider the linear Volterra system

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + \int_0^t H(t,s)x(s)ds + q(t), \quad t \geq 0, \\
x(0) &= x_0 \in X,
\end{align*}
\]

(2.1)

where \( A(\cdot) : [0, \infty) \to \mathcal{L}(X) \) is a continuous function; \( H(\cdot, \cdot) \) is a two variable continuous function defined on the set \( \{(t,s) : 0 \leq s \leq t < \infty\} \), valued in \( \mathcal{L}(X) \) and \( q : [0, \infty) \to X \) is a continuous function. The existence and uniqueness of solutions to (2.1) can be referred to [2].

The homogeneous equation corresponding with (2.1), i.e., \( q \equiv 0 \), is

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + \int_0^t H(t,s)x(s)ds, \quad t \geq 0, \\
x(0) &= x_0.
\end{align*}
\]

(2.2)

We define the Cauchy operator \( \Phi(t,s), t \geq s \geq 0 \) generated by the system (2.2) as the solution of the equation

\[
\begin{align*}
\dot{\Phi}(t,s) &= A(t)\Phi(t,s) + \int_s^t H(t,\tau)\Phi(\tau,s)d\tau, \quad t \geq s \geq 0, \\
\Phi(s,s) &= I.
\end{align*}
\]

(2.3)
We have the following useful lemma, called the variation of constants formula,

**Lemma 2.1.** The solution of the Volterra equation (2.1) can be expressed as

\[ x(t) = \Phi(t, 0)x_0 + \int_0^t \Phi(t, \rho)q(\rho)d\rho. \]  

\((2.4)\)

**Proof.** By directly differentiating both sides of (2.4) it follows that

\[ \dot{x}(t) = \left( A(t)\Phi(t, 0) + \int_0^t H(t, \tau)\Phi(\tau, 0)d\tau \right)x_0 + q(t) + \int_0^t \left[ A(t)\Phi(t, \rho) + \int_\rho^t H(t, \tau)\Phi(\tau, \rho)d\tau \right]q(\rho)d\rho \]

\[ = A(t)\Phi(t, 0)x_0 + \int_0^t \Phi(t, \rho)q(\rho)d\rho + \int_0^t H(t, \tau)\Phi(\tau, 0)x_0d\tau + \int_0^t q(\rho)d\rho \int_\rho^t H(t, \tau)\Phi(\tau, \rho)d\tau + q(t) \]

\[ = A(t)x(t) + \int_0^t H(t, \tau)(\Phi(\tau, 0)x_0 + \int_0^\tau \Phi(\tau, \rho)q(\rho)d\rho)d\tau + q(t) \]

\[ = A(t)x(t) + \int_0^t H(t, \tau)x(\tau)d\tau + q(t). \]

Thus, we have the proof. □

We note that for the Volterra equation (2.2) the semi-group property of the Cauchy operator

\[ \Phi(t, s) = \Phi(t, u)\Phi(u, s), \quad 0 \leq s \leq u \leq t, \]  

\((2.5)\)

in general is not true. Indeed, by definition

\[ \dot{\Phi}(t, s) = A(t)\Phi(t, s) + \int_s^t H(t, \tau)\Phi(\tau, s)d\tau = A(t)\Phi(t, s) + \int_u^t H(t, \tau)\Phi(\tau, s)d\tau + q_s(t), \]

where

\[ q_s(t) = \int_s^u H(t, \tau)\Phi(\tau, s)d\tau. \]

Therefore, by applying (2.4) it follows that

\[ \Phi(t, s) = \Phi(t, u)\Phi(u, s) + \int_u^t \Phi(t, h)q_s(h)dh \]

\[ = \Phi(t, u)\Phi(u, s) + \int_u^t \Phi(t, h) \int_s^u H(h, \tau)\Phi(\tau, s)d\tau dh. \]

Thus, (2.5) is true for all \(0 \leq s \leq u \leq t\) if and only if

\[ \int_u^t \Phi(t, h) \int_s^u H(h, \tau)\Phi(\tau, s)d\tau dh = 0 \quad \text{for all } 0 \leq s \leq u \leq t, \]

which implies that \(H(t, s) = 0\) for all \(0 \leq s \leq t.\)

This fact tells us that the classical method using semi-group property to study the Bohl-Perron Theorem for Volterra equations is no longer valid because this method profits the semi-group property to obtain an inequality by which we can prove the exponential stability of the unperturbed equation (see [7,8,13] for examples).
Definition 2.2.

i) The Volterra equation (2.2) is uniformly bounded if there exists a positive number $M_0$ such that

$$\|\Phi(t, s)\| \leq M_0, \quad t \geq s \geq 0. \quad (2.6)$$

ii) Let $\omega > 0$. The Volterra equation (2.2) is $\omega$-exponentially stable if there exists a positive number $M$ such that

$$\|\Phi(t, s)\| \leq Me^{-\omega(t-s)}, \quad t \geq s \geq 0. \quad (2.7)$$

The conditions ensuring the boundedness or stability of the equation (2.2) can be referred to [1,5,16] and references therein.

3 Stability of Volterra equation under small perturbations

In this section, we consider the effect of small perturbations to the stability of the Volterra equation (2.2). Let $H(\cdot, \cdot)$ be a continuous kernel defined on the set $\{(t, s) : 0 \leq s \leq t \leq \infty\}$. Suppose that for every $s \leq t$ and $x \in X$, the coefficients $H(t, s)x$ and $A(t)x$ of the equation (2.2) are perturbed by noise and they become $H(t, s)x \mapsto H(t, s)x + f(t, s, x)$ and $A(t)x \mapsto A(t)x + g(t, x)$. Thus, for any $t_0 \geq 0$, the Cauchy problem for the perturbed equation (2.2) has the form

$$\begin{cases}
\dot{x}(t) = A(t)x(t) + \int_{t_0}^t H(t, s)x(s)ds + \int_{t_0}^t f(t, s, x(s))ds + g(t, x(t)), & t \geq t_0 \\
x(t_0) = x_0 \in X,
\end{cases} \quad (3.1)$$

where $f(t, s, x)$ and $g(t, x)$ are two continuous functions, Lipschitz in $x$ with Lipschitz coefficients $k_{t,s}$ and $l_t$ respectively. We suppose further that $k_{t,s}, t \geq s \geq 0$ and $l_t, t \geq 0$ are continuous.

For any $x_0 \in X$ and $t_0 \geq 0$, the equation (3.1) has a unique solution, namely $x(\cdot, t_0, x_0)$, with the initial condition $x(t_0, t_0, x_0) = x_0$ and this solution is defined on $t \geq t_0$ (see [2, Theorem 3]). Suppose further that

$$f(t, s, 0) = 0; \quad g(t, 0) = 0, \quad \text{for all } t \geq s \geq 0.$$ 

With these assumptions, the equation (3.1) has the trivial solution $x(\cdot) \equiv 0$.

In the following, we write simply $x(\cdot)$ or $x(\cdot, t_0)$ for $x(\cdot, t_0, x_0)$ if there is no confusion.

The robust stability for the system (3.1) under small perturbations has been studied by T.A. Burton in [4] and R. Grimmer et al. in [10] via Lyapunov functions. S. I. Grossman et al. in [11] considered the uniform stability of (3.1) with the functions $f$ and $g$ to be “high order” by direct estimates.

In this paper, we approach the robust stability in an other manner. We use the general Gronwall-Bellman inequality to give conditions under which the solution of the system (3.1) is either bounded or exponentially stable.

To proceed, we need the following lemma (see [9]).
Lemma 3.1. Let the functions \( u(t), \sigma(t), v(t), w(t,r) \) be nonnegative and continuous for \( a \leq r \leq t \), and let \( c_1 \) and \( c_2 \) be nonnegative. If for \( t \in [a, \infty) \)

\[
u(t) \leq \sigma(t) \left\{ c_1 + c_2 \int_{a}^{t} \left[ v(s)u(s) + \int_{a}^{s} w(s,r)u(r)dr \right] ds \right\},
\]

then for \( t \geq a \),

\[
u(t) \leq c_1 \sigma(t) \exp \left\{ c_2 \int_{a}^{t} \left( v(s)\sigma(s) + \int_{a}^{s} w(s,r)\sigma(r)dr \right) ds \right\}.
\]

Firstly, we consider the boundedness of solutions of the equation (2.2) under small perturbations. For convenience, we denote \( \gamma_{t,s} = \int_{s}^{t} k_{t,u}du, \ t \geq s \geq 0. \)

Theorem 3.2. Assume that the equation (2.2) is uniformly bounded and

\[
N = \int_{t_0}^{\infty} (l_t + \gamma_{t,t_0}) dt < \infty.
\]

Then, there exists a constant \( M_1 > 0 \) such that the solution \( x(\cdot) \) of (3.1) satisfies

\[
\| x(t) \| \leq M_1 \| x(t_0) \|, \ t \geq t_0.
\]

Proof. From the variation of constants formula (2.4), it follows that

\[
x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,\tau) \left( g(\tau, x(\tau)) + \int_{t_0}^{\tau} f(\tau, u, x(u))du \right) d\tau, \ t \geq t_0.
\]

By virtue of Lipschitz condition of \( f(t,s,x), g(t,x) \) in \( x \) and the boundedness assumption of solutions (see Definition 2.2), we get

\[
\| x(t) \| \leq M_0 \| x(t_0) \| + M_0 \int_{t_0}^{t} \left( l_{\tau} \| x(\tau) \| + \int_{t_0}^{\tau} k_{\tau,u} \| x(u) \| du \right) d\tau, \ t \geq t_0.
\]

By using generalized Gronwall-Bellman inequality in Lemma 3.1 with \( \sigma = 1, \ c_1 = M_0 \| x(t_0) \| \) and \( c_2 = M_0 \) we have

\[
\| x(t) \| \leq M_0 \| x(t_0) \| \exp \left\{ M_0 \int_{t_0}^{t} (l_{\tau} + \int_{t_0}^{\tau} k_{\tau,u}du) d\tau \right\} = M_0 \| x(t_0) \| \exp \left\{ M_0 \int_{t_0}^{t} (l_{\tau} + \gamma_{\tau,t_0}) d\tau \right\}.
\]

Therefore,

\[
\| x(t) \| \leq M_0 e^{M_0 N} \| x(t_0) \|, \ t \geq t_0.
\]

The proof is complete. \( \square \)
Example 3.3. Consider the equation
\[ \dot{x}(t) = -\int_0^t x(s)ds + \int_0^t \frac{s}{1+t^4} \sin(x(s))ds, \quad t \geq 0. \] (3.4)

The homogeneous equation corresponding with (3.4) is
\[ \dot{x}(t) = -\int_0^t x(s)ds. \] (3.5)

It is easy to compute the Cauchy operator of (3.5)
\[ \Phi(t, s) = \cos(t - s). \]

Moreover, the function \( f(t, s, x) = \frac{s}{1+t^4} \sin(x) \) is Lipschitz continuous with the Lipschitz coefficient \( k_{t,s} = \frac{s}{1+t^4} \). Hence, \( \gamma_{t,s} = \frac{t^2-s^2}{2(1+t^4)} \) and \( N = \int_0^\infty \gamma_{t,0}dt = \frac{\sqrt{2\pi}}{8} \). Thus, from Theorem 3.2, it follows that the solution of (3.4) is bounded by \( \exp\left\{ \frac{\sqrt{2\pi}}{8} \right\} \).

Next, we consider the robust exponential stability of (2.2). We will show that the Volterra equation (2.2) preserves the exponential stability under small perturbations.

For any \( \omega > 0 \), denote
\[ \phi_{t,s} = \int_s^t e^{\omega(t-u)} k_{t,u}du, \quad t \geq s \geq 0. \]

Then, we have the following theorem

Theorem 3.4. Assume that the equation (2.2) is \( \omega \)-exponentially stable, i.e.,
\[ \|\Phi(t, s)\| \leq Me^{-\omega(t-s)}, \quad \text{for all} \quad t \geq s \geq t_0, \]
and
\[ \limsup_{u \rightarrow \infty} \frac{1}{u} \int_u^v (l_\tau + \phi_{\tau,0}) \, d\tau = \delta < \frac{\omega}{M}. \] (3.6)

Then, there exist positive constants \( K \) and \( \omega_1 \) such that
\[ \|x(t)\| \leq Ke^{-\omega_1(t-s)}\|x(s)\|, \quad \text{for all} \quad t \geq s \geq t_0, \]
where \( x(\cdot) = x(\cdot, s) \) is the solution of (3.1), with the initial condition \( x(s) \). That is, the perturbed equation (3.1) is \( \omega_1 \)-exponentially stable.

Proof. Let \( \varepsilon_0 \) be a positive number such that \( \delta + \varepsilon_0 \leq \frac{\omega}{M} \). Then, from (3.6), there exists a number \( T_0 > 0 \) such that
\[ \int_u^v (l_\tau + \phi_{\tau,0}) \, d\tau < (\delta + \varepsilon_0)(v - u), \quad v \geq u \geq T_0. \] (3.7)
By the continuity of solutions of (3.1) on the initial condition we can find a positive constant $M_{T_0}$, depending only on $T_0$ such that

$$\|x(t)\| \leq M_{T_0}\|x(s)\|, \quad t_0 \leq s \leq t \leq T_0. \quad (3.8)$$

By formula (2.4), estimate (2.7) we get

$$\|x(t)\| \leq \|\Phi(t,s)x(s)\| + \int_s^t \|\Phi(t,\tau)\| \left(\|g(\tau, x(\tau))\| + \int_s^\tau \|f(\tau, u, x(u))\| du\right) d\tau$$

$$\leq Me^{-\omega(t-s)}\|x(s)\| + M\int_s^t e^{-\omega(t-\tau)} (l_{\tau} \|x(\tau)\| + \int_s^\tau k_{\tau,u} \|x(u)\| du) d\tau$$

$$\leq Me^{-\omega(t-s)}\|x(s)\| + Me^{-\omega(t-s)} \int_s^t (e^{\omega(\tau-s)} l_{\tau} \|x(\tau)\| + \int_s^\tau e^{\omega(\tau-s)} k_{\tau,u} \|x(u)\| du) d\tau.$$  

Using the generalized Gronwall-Bellman inequality in Lemma 3.1 with $\sigma(t) = Me^{-\omega(t-s)}$, $c_1 = M\|x(s)\|$, $c_2 = 1$, and $v(t) = e^{\omega(t-s)} t$, it follows that

$$\|x(t)\| \leq Me^{-\omega(t-s)}\|x(s)\| \exp \left\{ M \int_s^t \left( l_{\tau} + \int_s^\tau e^{\omega(\tau-u)} k_{\tau,u} du \right) d\tau \right\}$$

$$= Me^{-\omega(t-s)}\|x(s)\| \exp \left\{ M \int_s^t \left( l_{\tau} + \phi_{\tau,s} \right) d\tau \right\}$$

$$\leq Me^{-\omega(t-s)}\|x(s)\| \exp \left\{ M \int_s^t \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau \right\}. \quad (3.9)$$

First, we consider the case $t_0 \leq s \leq T_0 < t$. From (3.7) and (3.9) we have

$$\|x(t)\| \leq Me^{-\omega(t-s)}\|x(s)\| \exp \left\{ M \int_{T_0}^t \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau + M \int_s^{T_0} \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau \right\}$$

$$\leq M \exp \left\{ M \int_s^{T_0} \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau \right\} \|x(s)\| e^{-\omega(M(\delta + \varepsilon_0))(t-s)}$$

$$\leq M \exp \left\{ M \int_s^{T_0} \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau \right\} \|x(s)\| e^{-\omega_1(t-s)},$$

where $\omega_1 := \omega - M(\delta + \varepsilon_0) > 0$. Thus,

$$\|x(t)\| \leq K_1\|x(s)\| \exp \left\{ -\omega_1(t-s) \right\} \quad t_0 \leq s \leq T_0 < t,$$

with $K_1 = M \exp \left\{ M \int_s^{T_0} \left( l_{\tau} + \phi_{\tau,t_0} \right) d\tau \right\}$.

Next, when $t_0 < T_0 \leq s \leq t$ we use a similar argument as above to get the estimate

$$\|x(t)\| \leq M\|x(s)\| \exp \left\{ -\omega_1(t-s) \right\} \quad.$$  

For the remaining case $t_0 \leq s \leq t \leq T_0$, with $\omega_1 > 0$ defined above and the inequality (3.8), we have

$$\|x(t)\| \leq M_{T_0}\|x(s)\| \leq M_{T_0} e^{\omega_1 T_0} e^{-\omega_1(t-s)}\|x(s)\|.$$
Combining the above estimates yields
\[ \|x(t)\| \leq Ke^{-\omega(t-s)}\|x(s)\| \text{ for all } t \geq s \geq t_0, \]
where \( K = \max\{M, K_1, M_{T_0}e^{\omega T_0}\} \). The proof is complete.

For the Volterra equations with bounded memory we have the following assessment.

**Corollary 3.5.** Suppose that the equation (2.2) is \( \omega \)-exponentially stable and there exists a positive constant \( \beta \) such that \( k_{t,s} = 0 \) when \( t - s > \beta \). Then, the inequality
\[ \lim_{t \to \infty} \sup_{t > s, s \to \infty} \left( l_t + e^{\omega \beta} \int_{0 \vee (t-\beta)}^t k_{t,u} du \right) = \delta < \frac{\omega}{M} \]
implies the exponential stability of the equation (3.1).

**Proof.** Since \( k_{t,s} = 0 \) when \( t - s > \beta \),
\[ \int_s^t e^{\omega(t-u)} k_{t,u} du \leq e^{\omega \beta} \int_{0 \vee (t-\beta)}^t k_{t,u} du. \]
Therefore,
\[ \lim_{t \to \infty} \sup_{t > s, s \to \infty} \frac{1}{t-s} \int_s^t (l_t + \phi_{t,s}) d\tau \leq \lim_{t \to \infty} \sup_{t > s, s \to \infty} \frac{1}{t-s} \int_s^t \left( l_t + e^{\omega \beta} \int_{0 \vee (\tau-\beta)}^\tau k_{\tau,u} du \right) d\tau \leq \delta. \]
The proof is complete.

**Remark 3.6.** In case there is only outer force perturbation intervening into the equation (2.2), i.e., \( k_{t,s} = 0 \), the condition (3.6) becomes
\[ \lim_{t \to \infty} \sup_{t > s} l_t = \delta < \frac{\omega}{M}. \]

**Example 3.7.** Consider the Volterra system
\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) - \int_0^t [e^{-(t-s)}x_1(s) - e^{-4(t-s)}\cos t + \cos s(-4 + \sin s)x_2(s) + f_1(t, s, x(s))] \, ds, \\
\dot{x}_2(t) &= x_1(t) + \int_0^t e^{-4(t-s)}\cos t + \cos s(-4 + \sin s)x_2(s) \, ds + \int_0^t f_2(t, s, x(s)) \, ds, \\
x_1(0) &= x_1^0, \quad x_2(0) = x_2^0.
\end{align*}
\]
We show that the homogeneous equation corresponding with (3.10)
\[
\begin{align*}
\dot{y}_1(t) &= -y_1(t) - \int_0^t [e^{-(t-s)}y_1(s) + e^{-4(t-s)}\cos t + \cos s(-4 + \sin s)y_2(s)] \, ds, \\
\dot{y}_2(t) &= y_1(t) + \int_0^t e^{-4(t-s)}\cos t + \cos s(-4 + \sin s)y_2(s) \, ds,
\end{align*}
\]
is exponentially stable. Indeed, by putting
\[ y_3(t) = \int_0^t e^{-(t-s)}y_1(s) \, ds, \quad y_4(t) = \int_0^t e^{-4(t-s)}\cos t + \cos s(-4 + \sin s)y_2(s) \, ds, \]
we obtain
\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2 \\
\dot{y}_3 \\
\dot{y}_4
\end{pmatrix} = 
\begin{pmatrix}
-1 & 0 & -1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & -1 & 0 \\
0 & -4 + \sin t & 0 & -4 + \sin t
\end{pmatrix} 
\begin{pmatrix}
y_1 \\
y_2 \\
y_3 \\
y_4
\end{pmatrix} = Ay
\]

with the initial condition \( y_1(0) = y_1^0, y_2(0) = y_2^0, y_3(0) = 0, y_4(0) = 0 \) and \( y = (y_1, y_2, y_3, y_4) \).

Denote
\[
D = 
\begin{pmatrix}
-4 & 2 & 0 & 3 \\
2 & -5.5 & 0 & -7 \\
0 & 0 & -4 & 0 \\
3 & -7 & 0 & -9.5
\end{pmatrix},
\]

\[
P = 
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 \\
0 & 0 & 2 & 0 \\
0 & 1 & 0 & 2
\end{pmatrix}.
\]

It can be checked that \( D \) is a negative definite matrix and \( P \) is a positive definite one. Let \( V(y) = y^\top Py \). By direct calculation we see that the following Lyapunov inequality
\[
A^\top P + PA = 
\begin{pmatrix}
-4 & 2 & 0 & 3 \\
2 & -8 + 2 \sin(t) & 0 & -10 + 3 \sin(t) \\
0 & 0 & -4 & 0 \\
3 & -10 + 3 \sin(t) & 0 & -14 + 4 \sin(t)
\end{pmatrix} \preceq D,
\]

where \( E \preceq F \) means that \( F - E \) is a positive definite matrix.

Further, \( \|y\|^2 \leq V(y) \leq 3 \|y\|^2 \). Therefore,
\[
\dot{V}(y) \leq y^\top Dy \leq -0.21 \|y\|^2 \leq -0.07V(y).
\]

Thus, \( \|y(t, s)\| \leq \sqrt{3} \|y(s)\| e^{-0.035(t-s)}, t \geq s \). In the other word, the system (3.11) is asymptotically stable with \( \omega = 0.035 \) and \( M = \sqrt{3} \).

Consider the perturbed system (3.10) with
\[
f_1(t, s, x) = ke^{-t+s} \sin(x_1) \quad \text{and} \quad f_2(t, s, x) = ke^{-t+s} (x_1 + \sin(x_2)).
\]

In this case, \( k_{t,s} = 2ke^{-t+s} \). Therefore, \( \phi_{t,s} = \frac{2k}{1 - e^{-0.035(t-s)}} \), which follows that
\[
\limsup_{t>s, s \to \infty} \frac{1}{t-s} \int_s^t \phi_{r,s} d\tau = \limsup_{t>s, s \to \infty} \frac{2k}{0.965} \left(1 + \frac{e^{-0.965(t-s)} - 1}{0.965(t-s)}\right) < \frac{2k}{0.965}.
\]

Thus, the system (3.10) is exponentially stable if \( k < \frac{0.035 \times 0.965}{2\sqrt{3}} \).

## 4 Bohl-Perron Theorem

In this section, we extend the Bohl-Perron Theorem to a class of Volterra equations. As is mentioned above, the most difficulty that we face here is that the semi-group property of the Cauchy operator is no longer valid, which implies we have to find a suitable technique to solve the problem. The Bohl-Perron type theorem for the Volterra equation (2.1) is studied in [11] by using the topology space \( LL_1(\mathbb{R}_+) \) of all measurable functions \( f \) on \( \mathbb{R}_+ \) such that the seminorms \( \int_S \|f(t)\| dt \) are finite for all compact subsets \( S \subset \mathbb{R}_+ \). Let \( \mathcal{F} \) be a certain
subspace of $LL_1(\mathbb{R}+)$. It is shown in [11] that if the operator $\rho$ defined in [11, formula (10)] maps the space $\mathcal{F}$ to space $\mathcal{F}$, then $\rho$ is continuous as a mapping from $\mathcal{F} \to \mathcal{F}$.

In this paper, we follow this idea to study the Bohl-Perron Theorem by considering the exponent stability to (2.1) via weighted spaces $L^\gamma(t_0)$ and $C^\gamma(t_0)$ defined below. We construct an operator $L$, similar to $\rho$ in [11], and show that the exponential stability of (2.2) is equivalent the fact that $L$ is surjective.

4.1 Bohl-Perron Theorem with unbounded memory

Let $\gamma \geq 0$. We introduce two families of Banach spaces $L^\gamma$ and $C^\gamma$ as

$$L^\gamma(t_0) = \left\{ f : [t_0, \infty) \to X, f \text{ is measurable and } \int_{t_0}^{\infty} e^{\gamma t} \| f(t) \| dt < \infty \right\},$$

$$C^\gamma(t_0) = \left\{ x : [t_0, \infty) \to X, x \text{ is continuous, } x(t_0) = 0 \text{ and } \sup_{t \geq t_0} e^{\gamma t} \| x(t) \| < \infty \right\},$$

with the norms defined as follows

$$\| f \|_{L^\gamma(t_0)} = \int_{t_0}^{\infty} e^{\gamma t} \| f(t) \| dt$$

and respectively

$$\| x \|_{C^\gamma(t_0)} = \sup_{t_0 \leq t < \infty} e^{\gamma t} \| x(t) \|.$$

It is noted that when $\gamma = 0$ we have

$$L^0(t_0) = L_1([t_0, \infty), X) = \left\{ f : [t_0, \infty) \to X, f \text{ is measurable and } \int_{t_0}^{\infty} \| f(t) \| dt < \infty \right\},$$

$$C^0(t_0) = C_b([t_0, \infty), X) = \left\{ x : [t_0, \infty) \to X, x(t_0) = 0, x \text{ is continuous and bounded} \right\}.$$

To simplify notations, we write $L^\gamma(0), C^\gamma(0)$ by $L^\gamma$ and $C^\gamma$ if there is no confusion.

For any $f \in L^\gamma$ we consider equation

$$\dot{x}(t) = A(t)x(t) + \int_{t_0}^{t} H(t, s)x(s)ds + f(t), \quad t \geq t_0, \tag{4.1}$$

with initial condition $x(t_0) = 0$.

Since $f$ may not be continuous, the equation (4.1) perhaps does not have the classical solution whose derivative exists every where. Therefore, we come to the concept of mild solutions as the following definition.

**Definition 4.1.** The function $x(t), t \geq t_0$ is said to be a (mild) solution of (4.1) if

$$x(t) = \int_{t_0}^{t} \left( A(\tau)x(\tau) + \int_{t_0}^{\tau} H(\tau, s)x(s)ds + f(\tau) \right) d\tau, \quad t \geq t_0. \tag{4.2}$$
It is easy to see that if \( x(t) \) is a mild solution of (4.1) then \( x(t) \) is a.e differentiable in \( t \) and its derivative satisfies the equation (4.1), where a.e means “almost everywhere”.

By a similar argument as in the proof of Lemma 2.4 it is seen that the solution \( x(t) \) of (4.1) is given by

\[
x(t) = \int_{t_0}^{t} \Phi(t, s)f(s)ds, \quad t \geq t_0. \tag{4.3}
\]

Indeed,

\[
\int_{t_0}^{t} \left[ A(\tau) \int_{t_0}^{\tau} \Phi(\tau, u)f(u)du + \int_{t_0}^{\tau} H(\tau, s) \left( \int_{t_0}^{s} \Phi(s, u)f(u)du \right) ds + f(\tau) \right] d\tau
\]

\[
= \int_{t_0}^{t} \left[ A(\tau) \int_{t_0}^{\tau} \Phi(\tau, u)f(u)du + \int_{t}^{\tau} \left( \int_{u}^{\tau} H(\tau, s)\Phi(s, u)ds \right) f(u)du + f(\tau) \right] d\tau
\]

\[
= \int_{t_0}^{t} \left[ \int_{t_0}^{\tau} d\tau \Phi(\tau, u)f(u)du + f(\tau) \right] d\tau = \int_{t_0}^{t} f(u) \left( \int_{t}^{t} \frac{d}{d\tau} \Phi(\tau, u)d\tau \right) du + \int_{t_0}^{t} f(\tau)d\tau
\]

\[
= \int_{t_0}^{t} \Phi(t, u)f(u)du = x(t).
\]

Based on the formula (4.3) we consider the operator \( \mathcal{L}_{t_0} \) defined on \( L^\gamma(t_0) \) associated with the equation (4.1) as follows:

\[
\mathcal{L}_{t_0} f(t) = \int_{t_0}^{t} \Phi(t, s)f(s)ds, \quad t \geq t_0, \quad f \in L^\gamma(t_0). \tag{4.4}
\]

We write simply \( \mathcal{L} \) for \( \mathcal{L}_{0} \).

**Theorem 4.2.** For any \( \gamma \geq 0 \), if \( \mathcal{L} \) maps \( L^\gamma \) to \( C^\gamma \), then there exists a positive constant \( K \) such that for all \( t_0 \geq 0 \),

\[
\| \mathcal{L}_{t_0} \| \leq K. \tag{4.5}
\]

**Proof.** First, we prove (4.5) when \( t_0 = 0 \). For every \( t > 0 \) we define an operator \( F_t : L^\gamma \to X \) by

\[
F_t(f) = e^{\gamma t} \int_{0}^{t} \Phi(t, s)f(s)ds = e^{\gamma t} \mathcal{L} f(t).
\]

Since \( \mathcal{L} \) maps \( L^\gamma \) to \( C^\gamma \),

\[
\sup_{t \geq 0} \| F_t(f) \| = \sup_{t \geq 0} e^{\gamma t} \| \mathcal{L} f(t) \| < \infty, \quad f \in L^\gamma.
\]

Therefore, by the Uniform Boundedness Principle

\[
\sup_{t \geq 0} \| F_t \| = K < \infty.
\]

It is noted that,

\[
\| \mathcal{L} \| = \sup_{f \in L^\gamma} \frac{\| \mathcal{L} f \|_{C^\gamma}}{\| f \|} = \sup_{f \in L^\gamma} \frac{\sup_{t \geq 0} \| F_t(f) \|}{\| f \|} = \sup_{t \geq 0} \| F_t \| = K. \tag{4.6}
\]
We now prove (4.5) with arbitrary $t_0 > 0$. Let $f(t)$ be an arbitrary function in $L^\gamma(t_0)$. We define the function $\bar{f}$ as follows: $\bar{f}(t) = 0$ if $t < t_0$, else $\bar{f}(t) = f(t)$. It is seen that

$$\mathcal{L}\bar{f}(t) = \int_0^t \Phi(t, s)\bar{f}(s)ds = \int_{t_0}^t \Phi(t, s)f(s)ds = \mathcal{L}_{t_0}f(t), \quad t \geq t_0.$$ 

Therefore, from (4.6) we get

$$\|\mathcal{L}_{t_0}f\|_{C^\gamma(t_0)} = \sup_{t \geq t_0} e^{\gamma t} \|\mathcal{L}_{t_0}f(t)\| = \sup_{t \geq t_0} e^{\gamma t} \|\mathcal{L}\bar{f}(t)\| = \|\mathcal{L}\bar{f}\|_{C^\gamma} \leq K\|\mathcal{L}\|L^\gamma = K\|f\|_{L^\gamma(t_0)}.$$ 

The proof is complete.

**Theorem 4.3.** Let $\gamma > 0$ be a positive number. The operator $\mathcal{L}$ maps $L^\gamma$ to $C^\gamma$ if and only if (2.2) is $\gamma$-exponentially stable.

**Proof.** The proof contains two parts.

**Necessity.** First, we prove that if $\mathcal{L}$ maps $L^\gamma$ to $C^\gamma$ then (2.2) is $\gamma$-exponentially stable.

By virtue of Theorem 4.2, $\mathcal{L}$ is a bounded operator from $L^\gamma(s)$ to $C^\gamma(s)$ with $\|\mathcal{L}\| = K$.

For all $f \in L^\gamma(s)$ and $0 \leq s \leq t$, we have

$$e^{\gamma t} \left\| \int_s^t \Phi(t, u)f(u)du \right\| \leq \|\mathcal{L}f\|_{C^\gamma(s)} \leq K \|f\|_{L^\gamma(s)}.$$ 

(4.7)

For any $\sigma > 0$ and $v \in X$, we consider the function

$$f_\sigma(u) = \begin{cases} \frac{1}{\sigma} e^{-\frac{u-s}{\sigma}} - \gamma u, & \text{if } u \geq s \\ 0 & \text{if } u < s. \end{cases}$$

It is seen that

$$\int_0^\infty e^{\gamma u} \|f_\sigma(u)\|du = \frac{1}{\sigma} \int_s^\infty e^{-\frac{u-s}{\sigma}} \|v\|du = \|v\|.$$ 

This means that $f_\sigma \in L^\gamma$ and $\|f_\sigma\|_{L^\gamma} = \|v\|$. Furthermore,

$$\lim_{\sigma \to 0} \int_s^t \Phi(t, u)f_\sigma(u)du = \lim_{\sigma \to 0} \int_s^t \Phi(t, u)\frac{1}{\sigma} e^{-\frac{u-s}{\sigma}} - \gamma u vdu = \lim_{\sigma \to 0} \int_{s+\sigma h}^{t+\sigma h} \Phi(t, s + \sigma h)e^{-\gamma s} \|v\| du = e^{-\gamma s} \Phi(t, s)v.$$ 

Combining with (4.7) obtains the desired estimate

$$\|\Phi(t, s)\| \leq Ke^{-\gamma(t-s)}, \quad t, s \geq 0.$$ 

Thus, (2.2) is uniformly asymptotically stable.

**Sufficiency.** We will show that if (2.2) is $\gamma$-exponentially stable then $\mathcal{L}$ maps $L^\gamma$ to $C^\gamma$.

Let $f \in L^\gamma$, from (4.4) we see that

$$e^{\gamma t} \|\mathcal{L}f(t)\| \leq e^{\gamma t} \int_0^t \|\Phi(t, s)\| \|f(s)\| ds \leq M e^{\gamma t} \int_0^t e^{-\gamma(t-s)} \|f(s)\| ds = M \int_0^t e^{\gamma s} \|f(s)\| ds \leq M \|f\|_{L^\gamma} < \infty.$$ 

Thus, $\mathcal{L}f \in C^\gamma$. The proof is complete.
Remark 4.4. The argument dealt with in the proof of Theorem 4.3 is still valid for \( \gamma = 0 \). Thus, if \( \mathcal{L} \) maps \( L_1 \) to \( C_b \) then the solution of (2.2) with the initial condition \( x(0) = 0 \) is bounded.

Corollary 4.5. The equation (2.2) is \( \gamma \)-exponentially stable if and only if the solution of

\[
\dot{y}(t) = A(t)y(t) + \gamma y(t) + \int_0^t H(t, s)e^{\gamma(t-s)}y(s)ds + f(t), \quad t \geq 0, \quad (4.8)
\]

is bounded for all \( f \in L^\gamma \).

Proof. Denote by \( \Psi(t,s) \) the Cauchy operator of the homogeneous equation corresponding to (4.8), i.e., \( \Psi(s,s) = I \) and

\[
\dot{\Psi}(t,s) = A(t)\Psi(t,s) + \gamma \Psi(t,s) + \int_s^t H(t, \tau)e^{\gamma(t-\tau)}\Psi(\tau,s)d\tau.
\]

From (2.3) we get

\[
\frac{d}{dt}(e^{\gamma t}\Phi(t,s)) = A(t)e^{\gamma t}\Phi(t,s) + \gamma e^{\gamma t}\Phi(t,s) + \int_s^t H(t, \tau)e^{\gamma(t-\tau)}e^{\gamma \tau}\Phi(\tau,s)d\tau.
\]

The uniqueness of solutions says that

\[
\Psi(t,s) = e^{\gamma t}\Phi(t,s). \quad (4.9)
\]

Hence, the \( \gamma \)-exponential stability of (2.2) implies that the solution of (4.8) is bounded.

Let \( y(t) \) be the solution of (4.8) with the initial condition \( y(0) = 0 \). By (4.4), this solution can be expressed as

\[
y(t) = \int_0^t \Psi(t, \tau)f(\tau)d\tau = e^{\gamma t}\int_0^t \Phi(t, \tau)f(\tau)d\tau = e^{\gamma t}\mathcal{L}f(t). \quad (4.10)
\]

The boundedness of \( y(t) \) says that \( \mathcal{L} \) maps \( L^\gamma \) to \( C^\gamma \). Therefore, by Theorem 4.3, the equation (2.2) is exponentially stable. The proof is complete. \( \square \)

### 4.2 Bohl-Perron Theorem with damped memory

We consider the equation (2.1) with the assumption

**Assumption 4.6.** \( A(t) \) is bounded on \( [0, \infty) \) by a constant \( A \) and \( H(t,s) \) is bounded on the set \( 0 \leq t - s \leq 1 \) by \( N_1 \). Further, there is a \( \beta > 0 \) such that

\[
\overline{H} = \sup_{s>0} \int_s^\infty e^{\beta(t-s)}\|H(t,s)\|(t-s)dt < \infty.
\]

It follows from this assumption that

\[
H_1 = \sup_{s \geq 0} \int_s^\infty \|H(t,s)\|dt < \infty.
\]
Denote
\[ C_{1,1}([0, \infty); X) = \{ x : [0, \infty) \to X \text{ such that } x \text{ is a.e differentiable, } x(0) = 0 \} \]
and both \( \dot{x}, x \in L_1([0, \infty); X) \).

We endow \( C_{1,1}([0, \infty); X) \) with the norm of \( L_1([0, \infty); X) \). Then, it becomes an (incomplete) normed vector space.

Consider the operator \( \mathcal{N} \) associated to (4.1) given by
\[ \mathcal{N} x(t) = \dot{x}(t) - A(t)x(t) - \int_0^t H(t, s)x(s)ds, \quad x \in L_1([0, \infty), X). \]

For any \( x \in L_1 \) we have
\[
\left\| \int_0^t H(\cdot, s)x(s)ds \right\|_{L_1} = \int_0^\infty \left\| \int_0^t H(t, s)x(s)ds \right\| dt \leq \int_0^\infty \int_0^t \|H(t, s)\| \|x(s)\| ds \leq H_1 \|x\|_{L_1}. \tag{4.12}
\]
Thus, \( \mathcal{N} \) maps from \( C_{1,1} \) to \( L_1([0, \infty); X) \). By uniqueness of solution of (2.2), it is clear that \( \mathcal{N} \) is an injective map.

**Theorem 4.7.** Let Assumption 4.6 holds. Then, the equation (2.2) is \( \omega \)-exponentially stable for an \( \omega > 0 \) if and only if \( \mathcal{N} \) is surjective.

**Proof.** Suppose that the system (2.2) is \( \omega \)-exponentially stable for a certain \( \omega > 0 \). This means that there is a positive constant \( M \) such that \( \|\Phi(t, s)\| \leq Me^{-\omega(t-s)} \) for any \( t \geq s \geq 0 \). For any \( f \in L_1([0, \infty), X) \) we put
\[ x(t) = Lf(t) = \int_0^t \Phi(t, s)f(s)ds. \]
It is seen that \( x(t) \) is a.e differentiable and \( \mathcal{N} x = f \). Further,
\[
\int_0^\infty \|x(t)\| dt = \int_0^\infty \left\| \int_0^t \Phi(t, s)f(s)ds \right\| dt \leq M \int_0^\infty \left( \int_0^t e^{-\omega(t-s)} \|f(s)\| ds \right) dt \leq M \int_0^{\infty} \left( \|f(s)\| \int_s^\infty e^{-\omega(t-s)} dt \right) ds = \frac{M}{\omega} \|f(\cdot)\|_{L_1}.
\]
Therefore, \( x \in L_1([0, \infty), X) \), which implies \( A(\cdot)x(\cdot) \in L_1([0, \infty), X) \) by virtue of boundedness of \( A(\cdot) \) and
\[ \int_0^\infty H(\cdot, s)x(s)ds \in L_1([0, \infty), X) \]
by (4.12). These relations say that \( \dot{x} \in L_1([0, \infty), X) \). Thus, \( x \in C_{1,1}([0, \infty); X) \). This means that \( \mathcal{N} \) is surjective.

Conversely, assume that \( \mathcal{N} \) is surjective, we will show that (2.2) is \( \omega \)-exponentially stable, where \( 0 < \omega < \min \left\{ \beta, \frac{1}{2(1+H\|x\|)} \right\} \) and \( \beta, H \) defined in Assumption 4.6. Indeed, since \( \mathcal{N} \) is
injective, we can define $N^{-1}$ acting $L_1([0, \infty), X)$ to $C_{1,1}([0, \infty), X)$. It is clear $N^{-1} = L$. Moreover, by a similar way as in the proof of Theorem 4.2, we imply the boundedness of $L$. Putting $x(t) = e^{-\omega t} y(t)$ gets

$$\mathcal{N}x(t) = Ne^{-\omega t} y(t) = e^{-\omega t} \dot{y}(t) - \omega e^{-\omega t} y(t) - A(t)e^{\omega t} y(t) - \int_0^t H(t, s)e^{-\omega s} y(s)ds$$

$$= e^{-\omega t} \left( \dot{y}(t) - \omega y(t) - A(t)y(t) - \int_0^t H(t, s)e^{\omega(t-s)} y(s)ds \right)$$

$$= e^{-\omega t} \left( \dot{y}(t) - A(t)y(t) - \int_0^t H(t, s)y(s)ds - \omega y(t) - \int_0^t H(t, s) \left( e^{\omega(t-s)} - 1 \right) y(s)ds \right)$$

$$= e^{-\omega t} (\mathcal{N}y(t) + Gy(t)),$$

where

$$Gy(t) = -\omega y(t) - \int_0^t H(t, s) \left( e^{\omega(t-s)} - 1 \right) y(s)ds.$$

Thus,

$$\mathcal{N}x(t) = e^{-\omega t} \mathcal{M}y(t), \quad (4.13)$$

where $\mathcal{M} = \mathcal{N} + G$. Further, by Assumption 4.6, for any $f \in L_1([0, \infty), X)$ we have

$$\int_0^\infty \|G(\mathcal{L}f)(t)\| dt \leq \omega \|\mathcal{L}f\| + \int_0^\infty \left( \int_0^t \left( e^{\omega(t-s)} - 1 \right) H(t, s)\mathcal{L}f(s)ds \right) dt$$

$$\leq \omega \|\mathcal{L}f\| + \int_0^\infty \left( \int_0^t \omega e^{\omega(t-s)} (t-s) \|H(t, s)\|\mathcal{L}f(s)ds \right) dt$$

$$\leq \omega \|\mathcal{L}\| \|f\| + \omega \|\mathcal{L}\| \int_0^\infty \|f(s)\| \left( \int_s^\infty e^{\beta(t-s)} (t-s) \|H(t, s)\| dt \right) ds$$

$$\leq \omega (1 + \beta) \|\mathcal{L}\| \|f\|.$$ 

Therefore, $G\mathcal{L}f \in L_1([0, \infty), X)$ and with chosen $\omega$ as above, we obtain

$$\|G\mathcal{L}f\| = \left\| \int_0^\infty \left( 1 - e^{\omega(t-s)} \right) H(\cdot, s)\mathcal{L}f(s)ds - \omega \mathcal{L}f(\cdot) \right\| \leq \frac{\|f\|}{2},$$

which implies that $\mathcal{ML} = I + G\mathcal{L}$ is invertible. Thus, $\mathcal{M}$ is a surjective, i.e., for any $f \in L_1([0, \infty), X)$, the equation

$$\mathcal{M}y = f \quad (4.14)$$

has a solution in $C_{1,1}([0, \infty), X)$. Using the same argument as in the proof of Theorem 4.3 we can prove that $\mathcal{M}^{-1}$ is bounded. Let $\Psi(t, s)$ be the Cauchy operator of the equation $\mathcal{M}y = 0$ with the initial condition $\Psi(s, s) = I$. Then, the solution $y(t) = \mathcal{M}^{-1}f(t)$ with the initial condition $y(0) = 0$ of the equation (4.14) has the expression

$$y(t) = \int_0^t \Psi(t, s)f(s)ds, \quad t > 0.$$
The boundedness of $\mathcal{M}^{-1}$ says that there is a $K_1 > 0$ such that

$$
\int_0^\infty \left\| \int_0^t \Psi(t, s)f(s)ds \right\| dt \leq K_1 \| f \|_{L_1}, \quad \text{for all } f \in L_1.
$$  \hspace{1cm} (4.15)

For any $v \in X$ and $\sigma > 0$ put $f_\sigma(s) = \frac{1}{\sigma} e^{-\frac{s}{\sigma}} 1_{[0, \infty)}(s)v$. From (4.15) we have

$$
\int_0^\infty \left\| \int_0^t \Psi(t, s)\sigma v e^{-s}ds \right\| dt \leq K_1 \| v \|.
$$

Letting $\sigma \to 0$ obtains

$$
\int_0^\infty \| \Psi(t, 0)v \| dt \leq K_1 \| v \|. \hspace{1cm} (4.16)
$$

On the other hand, for all $t > 0$

$$
\| \Psi(t, 0)v \| \leq \| v \| + \int_0^t \| (\omega I + A(\tau)) \Psi(\tau, 0)v \| d\tau + \int_0^t d\tau \int_0^\tau \| e^{\omega(\tau-s)} H(\tau, s)\Psi(s, 0)v \| ds
$$

$$
\leq \| v \| + (\omega + A) \int_0^t \| \Psi(\tau, 0)v \| d\tau + \int_0^\tau d\tau \int_0^\tau \| e^{\omega(\tau-s)} H(\tau, s)\Psi(s, 0)v \| ds
$$

$$
\leq \| v \| + (\omega + A) \int_0^\infty \| \Psi(s, 0)v \| ds + \int_0^\infty \left( \| \Psi(s, 0)v \| \int_s^\infty e^{\beta(\tau-s)} \| H(\tau, s) \| d\tau \right) ds.
$$

From Assumption 4.6, we have

$$
\int_s^\infty e^{\beta(\tau-s)} \| H(\tau, s) \| d\tau = \int_s^{s+1} e^{\beta(\tau-s)} \| H(\tau, s) \| d\tau + \int_{s+1}^\infty e^{\beta(\tau-s)} \| H(\tau, s) \| d\tau
$$

$$
\leq N_1 e^{\beta} + \int_s^\infty e^{\beta(\tau-s)} \| H(\tau, s) \| (t-s)d\tau \leq N_1 e^{\beta} + \overline{H}.
$$

Therefore,

$$
\| \Psi(t, 0)v \| \leq (1 + (\omega + A + N e^{\beta} + \overline{H})K_1) \| v \|.
$$

Let $H_2 = 1 + (\omega + A + N e^{\beta} + \overline{H})K_1$, we get $\| \Psi(t, 0)v \| \leq H_2 \| v \|$, which implies

$$
\| \Psi(t, 0) \| \leq H_2, \quad t \geq 0.
$$

Combining this inequality with (4.9), we get

$$
\| \Phi(t, 0) \| \leq H_2 e^{-\omega t}, \quad t \geq 0.
$$

By a similar argument we see that

$$
\| \Phi(t, s) \| \leq H_2 e^{-\omega(t-s)}, \quad t \geq s \geq 0.
$$

The proof is complete. \hfill \Box

**Corollary 4.8.** Suppose that $A(\cdot)$, $H(\cdot, \cdot)$ are bounded and there is a $\delta > 0$ such that $H(t, s) = 0$ if $t - s > \delta$. Then, $\mathcal{N}$ is a surjective map if and only if the solution of (2.2) is $\omega$-exponentially stable for a certain $\omega > 0$. 

Proof. From the assumptions of Corollary 4.8 we see that
\[
H = \sup_{s>0} \int_s^\infty e^{\delta(t-s)} \|H(t,s)\| (t-s)dt \leq \sup_{s>0} \int_s^{s+\delta} e^{\delta(t-s)} \|H(t,s)\| (t-s)dt
\leq \frac{\delta^2 e^{\delta^2}}{2} \sup_{0 \leq t-s \leq \delta} \|H(t,s)\| < \infty.
\]

Then, the assumption in Theorem 4.7 is satisfied and we have the conclusion of the corollary.

Example 4.9. Consider the linear delay equation
\[
\dot{x}(t) = A(t)x(t) + \int_0^\tau R(t,h)x(t-h)dh. \tag{4.17}
\]
We can rewrite this equation as
\[
\dot{x}(t) = A(t)x(t) + \int_{t-\tau}^t R(t,s)x(s)ds.
\]
In assuming that \(R(t,s) = 0\) when \(0 \leq t < s < \tau\) and \(A(t)\) is bounded we have the equation
\[
\dot{x}(t) = A(t)x(t) + \int_0^t H(t,h)x(h)dh, \quad t \geq 0,
\]
where \(H(t,s) = 1_{(t-\tau,t)}(s)R(t,t-s)\). It is seen that \(H(t,s) = 0\) if \(t-s \geq \tau\), which says that the assumptions of Corollary 4.8 are satisfied. Thus, the equation (4.17) is exponentially stable if and only if the operator \(N\) given by (4.11) is surjective. This is equivalent to the equation
\[
\dot{x}(t) = A(t)x(t) + \int_0^\tau R(t,h)x(t-h)dh + f(t)
\]
having bounded solution for every \(f \in L_1([0,\infty),X)\).

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References


