REFLECTED BSDES DRIVEN BY INHOMOGENEOUS SIMPLE LÉVY PROCESSES WITH RCLL BARRIER

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ABSTRACT. In this paper, we study the solution of a backward stochastic differential equation driven by an inhomogeneous simple Lévy process with a rcll reflecting barrier. We show the existence and uniqueness of solution by means of the Snell envelope and the fixed point theorem when the coefficient is stochastic Lipschitz. In term of application, we provide the fair price of the American option in Lévy market.

Introduction. The theory of backward stochastic differential equations (BSDEs for short) was developed by Pardoux and Peng [25]. These equations have attracted great interest due to their connections with mathematical finance [10, 8], stochastic control and stochastic games [7, 15, 17, 16]. There have been many studies done on this topic lately, and we can’t talk about those studies without mentioning the Situ’s one [29], which was on BSDEs driven by a Brownian motion and a Poisson point process. In addition to the study of Nualart and Schoutens [24] who have established the existence and uniqueness of solutions for BSDEs driven by a Lévy process. Also, the great study of Bahlali et al. [1] in which they treated the case where the BSDE is driven by a Brownian motion and the martingales of Teugels associated with an independent Lévy process. And last but not least, El Jamali and El Otmani’s [5] in which we have established the existence and uniqueness of solutions for BSDEs driven by an inhomogeneous Lévy processes when the coefficient is stochastic Lipschitz.

In the framework of a Brownian filtration, the notion of reflected BSDE has been introduced by El-Karoui et al. [11]. A solution of such an equation that is associated with a coefficient \( f \), terminal value \( \xi \) and a barrier \( L \), is a triple process \((Y, Z, K)\) Achieving:

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \\
Y_t &\geq L_t \quad \mathbb{P} - \text{a.s. for all } t \leq T.
\end{align*}
\]

The role of the continuous increasing process \( K \) is to push upwards the process \( Y \) in order to keep it above the barrier with minimal energy, that is, \( \int_0^T (Y_t - L_t) dK_t = 0 \). This type of BSDEs is motivated by pricing the American options [9] and studying the mixed game problems [18].

The extension to the cases of reflected BSDE with jumps, which are first, a standard reflected BSDE driven by a Brownian motion and an independent Poisson point process, has been established by Hamadène and Ouknine [19]. Second, Essaky’s [13] studied on the reflected BSDEs with jumps and right continuous left hand limited (rcll for short) obstacle. Third, El Otmani [12] has considered a reflected BSDE driven by a Brownian motion and the martingales of Teugels associated with a pure jump independent Lévy process and rcll obstacle (see e.g. [14, 27, 30]). And last but not least, Lü [23] who treated the case where the reflected BSDE driven by a Brownian motion and the martingales

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of Teugels associated with an independent Lévy process having a stochastic Lipschitz coefficient when the barrier is continuous.

The purpose of the present paper is to consider the reflected BSDEs driven by an inhomogeneous simple Lévy process (i.e. driven by an Itô process with a mix of non-stationary Poisson processes. For the homogeneous case see e.g. [21] p.208) when the barrier is just rcll. On the one hand, the existence and uniqueness of the solution to the reflected BSDE is proved by means of the Snell envelope and the fixed point theorem when the coefficient is stochastic Lipschitz. On the other hand, the use risk-neutral probability measure (which the discounted assets are martingales) and the Girsanov Theorem to apply the result of previous sections to construct an American option in the inhomogeneous case.

This study is a sequence of three sections. The first one includes some notations and assumptions needed in this paper. While the second section is devoted to give the comparison theorem for the solutions of reflected BSDEs, and also to prove the existence and uniqueness of the solution to reflected BSDEs when the coefficient is stochastic Lipschitz. The third and the last section is an attempt at applying the fixed point theorem when the coefficient is stochastic Lipschitz. While the second section is devoted to give the comparison theorem for the solutions of reflected BSDEs, and also to prove the existence and uniqueness of the solution to reflected BSDEs when the coefficient is stochastic Lipschitz. On the other hand, the use risk-neutral probability measure (which the discounted assets are martingales) and the Girsanov Theorem to apply the result of previous sections to construct an American option in the inhomogeneous case.

1. Setting of the problem and assumptions. Let $T > 0$ be a fixed time and $t \in [0, T]$. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ carrying a standard Brownian motion $(B_t)_{t \leq T}$ and an independent martingale measure $(\tilde{N}^k_t)_{t \leq T}$ corresponding to an inhomogeneous Poisson random measure $N^k_t$ for all $k \in [1, d]$ where $d \in \mathbb{N}$, it holds: $\tilde{N}^k_t = N^k_t - \int_0^t \lambda^k_s \, ds$, where the intensity function $\lambda$ is positive. We assume that $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$ where $(X_t)_{t \leq T}$ is an inhomogeneous Lévy process, i.e $(X_t)_{t \leq T}$ has independent increments and is continuous in probability with $X_0 = 0$ such that its canonical representation can be written as

$$X_t = \int_0^t \sqrt{c_s} \, dB_s + \sum_{k=1}^d \int_0^t \alpha^{(k)}_s \, d\tilde{N}^k_s.$$  

Here the volatility coefficients $c_t \in \mathbb{R}_+$ and the processes $(\alpha^{(k)}_t)_{t \leq T}$ satisfying the condition

$$\int_0^T |c_t| \, dt + \sum_{k=1}^d \int_0^T |\alpha^{(k)}_t|^2 \tilde{N}^k_t \, dt < +\infty.$$  

Let $\beta > 0$ and let $(\alpha_t)_{t \leq T}$ be a non-negative $\mathcal{F}_t$-adapted process. Let $(A_t)_{t \leq T}$ be the increasing continuous process defined by $A_t = \int_0^t a^2_s \, ds$, for all $t \leq T$. Let us introduce the following appropriate spaces:

- $\mathcal{K}^2$: the subspace of the $\mathcal{F}_t$-predictable, rcll and non-decreasing processes $(K_t)_{t \leq T}$ such that $K_0 = 0$ and $\mathbb{E}[K_T]^2 < +\infty$.
- $\mathcal{H}^2$: the subspace of the $\mathcal{F}_t$-predictable processes $(Z_t)_{t \leq T}$ such that $\|Z\|_{\mathcal{H}^2}^2 = \mathbb{E} \int_0^T |Z_t|^2 \, dt < +\infty$.
- $\mathcal{T}^\lambda$: $\mathbb{R}^d$-valued and $\mathcal{P}$-measurable mapping $U : \Omega \to \mathbb{R}$ such that

$$\|U\|_{\mathcal{T}^\lambda}^2 = \sum_{k=1}^d |U^{(k)}|^2 \lambda^{(k)} < +\infty.$$
In this paper, we will devote ourselves to the following reflected BSDE:

\[ \begin{align*}
Y_t &= \xi + \int_t^T f(s,Y_s,Z_s,U_s)ds + K_T - K_t - \int_t^T Z_s dB_s - \sum_{k=1}^d \int_t^T U_s^{(k)} d\tilde{N}_s^{(k)}, \\
Y_t &\geq L_t \quad \text{P - a.s. for all } t \leq T, \\
\text{the Skorokhod condition:} \\
\int_0^T (Y_t - L_t)dK_t^c &= 0 \quad \text{and} \quad K_t^d = \sum_{0 \leq s \leq t} (Y_s - L_s-)^{-1}\mathbb{1}_{\{\Delta L_s < 0\}},
\end{align*} \]

where \( K^c \) (resp. \( K^d \)) is the continuous (resp. purely discontinuous) part of \( K \) and the data \( (\xi, f, L) \) satisfies the following assumptions:

(A.1) The terminal value \( \xi \) is in \( \mathcal{L}^2(\beta, A) \).

(A.2) The coefficient \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{T}^\lambda \to \mathbb{R} \) is such that the following conditions hold:
1. For all \((y, z, u) \in \mathbb{R} \times \mathbb{R} \times \mathcal{T}^\lambda\), the function \(f(\cdot, y, z, u)\) is \(\mathcal{F}\)-progressively measurable and
\[
E \int_0^T e^{\beta A_L} \frac{|f(s, 0, 0, 0)|^2}{a^2} \, ds < +\infty.
\]

2. There exist three non-negative \(\mathcal{F}_t\)-adapted processes \((p_t)_{t \leq T}, (q_t)_{t \leq T}\) and \((\rho_t)_{t \leq T}\) such that
   \((i)\) for all \(t \in [0, T]\), \(y, y', z, z' \in \mathbb{R}^d\) and \(u, u' \in \mathcal{T}^\lambda\)
   \[
   |f(t, y, z, u) - f(t, y', z', u')| \leq p_t|y - y'| + q_t|z - z'| + \rho_t\|u - u'\|_\lambda;
   \]
   \((ii)\) there exists \(\epsilon > 0\) such that \(a_t^2 = p_t + q_t^2 + \rho_t^2 \geq \epsilon\), \(\forall t \in [0, T]\).

Remark 1. \(\star\) \(K^d\) is predictable (because it's purely discontinuous). Therefore, \(K^d\) acts only when the process \(Y\) has a predictable jump which occurs at a predictable negative jump point of \(L\). This implies that
\[
\Delta K_t^d = (-\Delta Y_t)^+ = (Y_t^+ - Y_t^-)^+ = (Y_t^+ - L_t^-)^{-}\mathbb{I}_{\{Y_t^- = L_t^-\}}.
\]

   • We can show the Skorokhod condition is equivalent to \(\int_0^T (Y_t^- - L_t^-)dK_t = 0\). Indeed
   \[
   \int_0^T (Y_t^- - L_t^-)dK_t = \int_0^T (Y_t^- - L_t^-)dK_t^c + \sum_{0 < t \leq T} (Y_t^- - L_t^-)\Delta K_t^d
   = \int_0^T (Y_t^- - L_t^-)dK_t^c + \sum_{0 < t \leq T} |(Y_t^- - L_t^-)^-|^2 \mathbb{I}_{\{\Delta L_t < 0\}} = 0.
   \]

   Conversely, \(0 = \int_0^T (Y_t^- - L_t^-)dK_t = \int_0^T (Y_t^- - L_t^-)dK_t^c + \int_0^T (Y_t^- - L_t^-)dK_t^d\); therefore
   \(\int_0^T (Y_t^- - L_t^-)dK_t^c = 0\) and
   \[
   \Delta K_t^d = (Y_t^- - L_t^-)^-\mathbb{I}_{\{Y_t^- = L_t^-\}} = (Y_t^- - L_t^-)^-\mathbb{I}_{\{Y_t^- = L_t^-\} \cap \{\Delta L_t < 0\}} = (Y_t^- - L_t^-)^-\mathbb{I}_{\{\Delta L_t < 0\}}.
   \]

2. Reflected BSDEs: Existence and uniqueness of solution. The case of the reflected BSDE driven by a homogeneous Lévy process with discontinuous barrier has been studied by El Jamali and El Otmani [6] using the penalization method. In this section, we will use the Snell envelope method to prove the existence and uniqueness of solution of reflected BSDE (with rcll barrier) driven by a simple Lévy processes in the non-homogeneous case.

Definition 1. A solution to reflected BSDE associated with data \((\xi, f, L)\) is a quadruple processes \((Y, Z, U, K)\) satisfying (1) such that \((Y, Z, U, K) \in \mathcal{M}^2(\beta, A)\).
Remark 2. If \((Y, Z, U) \in \mathcal{M}(\beta, A)\), then \(\left(\int_0^t e^{\beta A_t} Y_s Z_s dB_s\right)_{t \leq T}\) and \(\left(\int_0^t e^{\beta A_t} Y_s U_s^{(k)} d\tilde{N}_s^{(k)}\right)_{t \leq T}\) are uniformly integrable martingales. Indeed, using Burkholder-Davis-Gundy’s inequality, one can write

\[ E \sup_{0 \leq t \leq T} \left| \int_0^t e^{\beta A_t} Y_s Z_s dB_s \right| \leq c_2 E \left[ \int_0^T e^{\beta A_t} |Y_t|^2 |Z_t|^2 ds \right]^{1/2} \leq c_2 E \left( \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right)^{1/2} \left( \int_0^T e^{\beta A_t} |Z_t|^2 ds \right)^{1/2} \leq \frac{c_2}{2} E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right] + \frac{c_2}{2} E \int_0^T e^{\beta A_t} |Z_t|^2 ds. \]

And by the same arguments we have for all \(k = 1, \ldots, d\)

\[ E \sup_{0 \leq t \leq T} \left| \int_0^t e^{\beta A_t} Y_s U_s^{(k)} d\tilde{N}_s^{(k)} \right| \leq \frac{c_2}{2} E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right] + \frac{c_2}{2} E \int_0^T e^{\beta A_t} |U_s^{(k)}|^2 \lambda_s ds. \]

2.1. Comparison Theorem. The comparison theorem is one of the principal tools in the theories of the BSDEs. But it does not hold in general for solutions of BSDEs driven by a simple Lévy process, reflected or not (see e.g. [2] for a counter-example).

Theorem 1. Let \((Y, Z, U, K)\) and \((Y', Z', U', K')\) be solutions of the reflected BSDE (1) with data \((\xi, f, L)\) and \((\xi', f', L)\) respectively which satisfy assumption (A.2). Furthermore, We suppose

- \(f\) is independent of \(z\) and \(u;\)
- \(\mathbb{P}\)-a.s. for any \(t \leq T\) one has \(f(t, Y'_t) \leq f'(t, Y'_t, Z'_t, U'_t)\) and \(\xi \leq \xi'.\)

Then \(Y_t \leq Y'_t\) \(\mathbb{P}\)-a.s. for all \(t \leq T\).

Proof. Let \((\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{\xi}) := (Y - Y', Z - Z', U - U', K - K', \xi - \xi').\) The Meyer-Itô formula (see e.g. [26], Theorem 66 p.210) with the convex function \(x \rightarrow (x^+)^2\) implies that, for \(t \in [0, T],\)

\[
(\tilde{Y}_t)^+ = (\tilde{Y}_0)^+ - 2 \int_0^t \tilde{Y}_s^+ (f(s, Y_s) - f'(s, Y'_s, Z'_s, U'_s)) dr - 2 \int_0^t (\tilde{Y}_s^-)^+ d\tilde{K}_s
+ 2 \int_0^t \tilde{Y}_s^+ \tilde{Z}_s dB_s + 2 \sum_{k=1}^d \int_0^t \tilde{Y}_s^+ \tilde{U}_s^{(k)} d\tilde{N}_s^{(k)} + \tilde{C}_t
\]

where \((\tilde{C}_t)_{t \leq T}\) is a continuous non-decreasing process. The integration by parts formula implies that

\[
e^{\beta A_t} (\tilde{Y}_t)^+ = e^{\beta A_t} (\tilde{Y}_0)^+ + 2 \int_t^T e^{\beta A_s} (\tilde{Y}_s)^+ [f(s, Y_s) - f'(s, Y'_s, Z'_s, U'_s)] ds + 2 \int_t^T e^{\beta A_s} (\tilde{Y}_s^-)^+ d\tilde{K}_s
- 2 \int_t^T e^{\beta A_s} (\tilde{Y}_s^+)^+ \tilde{Z}_s dB_s - 2 \sum_{k=1}^d \int_t^T e^{\beta A_s} (\tilde{Y}_s^-)^+ \tilde{U}_s^{(k)} d\tilde{N}_s^{(k)}
- \int_t^T e^{\beta A_s} \tilde{C}_s - \beta \int_t^T e^{\beta A_s} (\tilde{Y}_s^+)^2 dB_s.
\]
Using the assumptions of Theorem 1, we can write
\[ e^{\beta A_t}(\bar{Y}_t)^2 + \beta \int_t^T e^{\beta A_s}(\bar{Y}_s)^2 dA_s + \int_t^T e^{\beta A_s} d\mathcal{C}_s \]
\[ \leq 2 \int_t^T e^{\beta A_s} \bar{Y}_s^+ (f(s, Y_s) - f(s, Y'_s)) ds + 2 \int_t^T e^{\beta A_s} \bar{Y}_s^+ d\bar{K}_s \]
\[-2 \int_t^T e^{\beta A_s} \bar{Y}_s^+ d\bar{Z}_s dB_s - 2 \sum_{k=1}^d \int_t^T e^{\beta A_s} \bar{Y}_s^+ \bar{U}_s^k d\tilde{N}_s^k.\]
Taking the expectation on both sides above, we obtain from the stochastic Lipschitz condition on \( f \) that
\[ \mathbb{E}\left[ e^{\beta A_t}(\bar{Y}_t)^2 \right] + (\beta - 2) \int_t^T e^{\beta A_s}(\bar{Y}_s)^2 dA_s + \mathbb{E} \int_t^T e^{\beta A_s} d\mathcal{C}_s \leq 2 \mathbb{E} \int_t^T e^{\beta A_s} \bar{Y}_s^+ \left[ d\bar{K}_s - d\bar{K}'_s \right]. \]
Note that if \( Y > Y' \) then \( Y > L \) which implies that \( d\bar{K}_c = 0 \) and then \( \int_t^T e^{\beta A_s} \bar{Y}_s^+ \bar{Z}_s dB_s = 0 \). Also, when the purely discontinuous \( K^d \) increases at \( s \) we should have \( \bar{Y}_s - \bar{Y}'_s = L_s - L'_s \), which implies that
\[ \sum_{t<s \leq T} \bar{Y}^+_s \Delta K^d_s = \sum_{t<s \leq T} (L_s - L'_s)^+ \Delta K^d_s = 0. \]
In the same way, \( \int_t^T e^{\beta A_s} (\bar{Y}_s^-)^+ dK'_s = 0 \). Choosing \( \beta \geq 2 \), we conclude that \( \mathbb{E}\left[ e^{\beta A_t}(\bar{Y}_t)^2 \right] \leq 0 \) and thus \( Y_t \leq Y'_t \) \( \mathbb{P} \)-a.s. for all \( t \leq T \).

2.2. Uniqueness. In this part, we will show the uniqueness of the solutions to reflected BSDEs. For that, let \((Y, Z, U, K)\) and \((Y', Z', U', K')\) be two solutions of the reflected BSDE (1) with data \((\xi, f, L)\) and \((\xi', f', L')\) respectively. First, we give an a priori estimate of the difference of the above solutions which will be useful in the sequel.

Proposition 1. Under the condition (A.2)(2), there exists a constant \( C_\beta > 0 \) such that
\[ \|\delta Y\|_{\mathcal{S}_2}^2 + \|\delta Y\|_{\mathcal{S}^2+A}^2 + \|\delta Z\|_{\mathcal{H}_2}^2 + \|\delta U\|_{\mathcal{F}_2}^2 \]
\[ \leq C_\beta \left\{ \mathbb{E}\left[ e^{\beta A_T} |\delta \xi| \right]^2 + \mathbb{E} \int_0^T e^{\beta A_s} \left| f(s, Y'_s, U'_s) - f(s, Y'_s, U'_s) \right|^2 ds + \mathbb{E} \int_0^T e^{\beta A_s} \delta L_{s-} d(K_s) \right\} \]
where \( \delta \mathcal{G} = \mathcal{G} - \mathcal{G}' \) for \( \mathcal{G} = Y, Z, U, K, L \) and \( \xi \).
Proof. Applying Itô’s formula (see e.g. [26], Theorem 32 p.78), we can write
\[ e^{\beta A_T} |\delta \xi|^2 = e^{\beta A_0} |\delta Y_0|^2 + \beta \int_t^T e^{\beta A_s} |\delta Y_s|^2 \, dA_s + 2 \int_t^T e^{\beta A_s} \delta Y_s \, d(\delta Y_s) + \int_t^T e^{\beta A_s} (\delta Y_s, \delta Y_s)_s \]
\[ = e^{\beta A_0} |\delta Y_0|^2 + \beta \int_t^T e^{\beta A_s} |\delta Y_s|^2 \, dA_s - 2 \int_t^T e^{\beta A_s} \delta Y_s (f(s, Y_s, Z_s, U_s) - f'(s, Y'_s, Z'_s, U'_s)) \, ds \]
\[ - 2 \int_t^T e^{\beta A_s} \delta Y_s \, d(\delta K_s) + 2 \int_t^T e^{\beta A_s} \delta Y_s \, dZ_s \, dB_s + 2 \sum_{k=1}^d \int_t^T e^{\beta A_s} \delta Y_s \, dU^{(k)}_s \]
\[ + \int_t^T e^{\beta A_s} |\delta Z_s|^2 \, ds + \sum_{k=1}^d \int_t^T e^{\beta A_s} |\delta U^{(k)}_s|^2 \, dN^{(k)}_s. \]
Taking the expectation and using condition (A.2)(2) and the fact \( \int_0^T e^{\beta A_s} \delta Y_s \, dB_s \) and \( \int_0^T e^{\beta A_s} \delta Y_s \, dU^{(k)}_s \) are uniformly integrable martingales, we obtain
\[ \mathbb{E} e^{\beta A_t} |\delta Y_t|^2 + \beta \mathbb{E} \int_t^T e^{\beta A_s} |\delta Y_s|^2 \, dA_s + \mathbb{E} \int_t^T e^{\beta A_s} |\delta Z_s|^2 \, ds + \sum_{k=1}^d \mathbb{E} \int_t^T e^{\beta A_s} |\delta U^{(k)}_s|^2 \, d\lambda^{(k)}_s \, ds \]
\[ \leq \mathbb{E} \left( e^{\beta A_T} |\delta \xi|^2 \right) + 2\mathbb{E} \int_t^T e^{\beta A_s} |\delta Y_s| \left| f(s, Y'_s, Z'_s, U'_s) - f'(s, Y'_s, Z'_s, U'_s) \right| \, ds \]
\[ + 2\mathbb{E} \int_t^T e^{\beta A_s} |\delta Y_s| (p_s |\delta Y_s| + q_s |\delta Z_s| + r_s |\delta U^{(k)}_s|) \, ds + 2\mathbb{E} \int_t^T e^{\beta A_s} \delta Y_s \, d(\delta K_s) \]
\[ \leq \mathbb{E} \left( e^{\beta A_T} |\delta \xi|^2 \right) + \left( \beta - \frac{1}{2} \right) \mathbb{E} \int_t^T e^{\beta A_s} |\delta Y_s|^2 \, ds + \frac{1}{2} \mathbb{E} \int_t^T e^{\beta A_s} |\delta Z_s|^2 \, ds \]
\[ + \frac{1}{\beta} \mathbb{E} \int_t^T e^{\beta A_s} |f(s, Y'_s, Z'_s, U'_s) - f'(s, Y'_s, Z'_s, U'_s)|^2 \, ds. \]

for \( \beta > \frac{5}{2} \). To conclude, using the fact that
\[ \delta Y_{s-} \, d(\delta K_s) = (L_{s-} - Y'_{s-}) \, dK_s - (Y_{s-} - L'_{s-}) \, dK'_s \leq (L_{s-} - L'_{s-}) \, d(\delta K_s) = \delta L_{s-} \, d(\delta K_s), \]
we obtain that
\[ \mathbb{E} \int_0^T e^{\beta A_s} |\delta Y_s|^2 \, dA_s + \mathbb{E} \int_0^T e^{\beta A_s} |\delta Z_s|^2 \, ds + \mathbb{E} \int_0^T e^{\beta A_s} |\delta U_s|^2 \, ds \]
\[ \leq C_\beta \left\{ \mathbb{E} e^{\beta A_T} |\delta \xi|^2 + \mathbb{E} \int_0^T e^{\beta A_s} \left| f(s, Y'_s, Z'_s, U'_s) - f'(s, Y'_s, Z'_s, U'_s) \right|^2 \, ds \right\}. \]

Finally, the result follows from the Burkholder-Davis-Gundy inequality. \( \square \)
Corollary 1. Under the assumption \( (A.2)(2) \), if \((Y, Z, U, K)\) verifies reflected BSDE (1), then we have
\[
E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right] + E \int_0^T e^{\beta A_t} |Y_s|^2 dA_s + E \int_0^T e^{\beta A_t} |Z_s|^2 ds + E \int_0^T e^{\beta A_t} |U_s|^2 ds + E |K_T|^2 \\
\leq C_\beta \left\{ E \left[ e^{\beta A_T} |\xi|^2 \right] + E \int_0^T e^{\beta A_t} \frac{f(s, 0, 0, 0)^2}{a_s^2} ds + E \sup_{0 \leq t \leq T} \left[ e^{2\beta A_t} |L_t^+|^2 \right] \right\}.
\]

Proof. Note that \((Y', Z', U', K') = (0, 0, 0, 0)\) is a solution of the reflected BSDE (1) with data \((\xi', f', L') = (0, 0, 0)\). Then, by proposition 1, we can write
\[
E \sup_{0 \leq t \leq T} \left[ e^{\beta A_t} |Y_t|^2 \right] + E \int_0^T e^{\beta A_t} |Y_s|^2 dA_s + E \int_0^T e^{\beta A_t} |Z_s|^2 ds + E \int_0^T e^{\beta A_t} |U_s|^2 ds \\
\leq C_\beta \left\{ E \left[ e^{\beta A_T} |\xi|^2 \right] + E \int_0^T e^{\beta A_t} \frac{f(s, 0, 0, 0)^2}{a_s^2} ds + E \int_0^T e^{\beta A_t} |L_s - dK_s| \right\} \\
\leq C_\beta \left\{ E \left[ e^{\beta A_T} |\xi|^2 \right] + E \int_0^T e^{\beta A_t} \frac{f(s, 0, 0, 0)^2}{a_s^2} ds + \eta E \sup_{0 \leq t \leq T} e^{2\beta A_t} |L_t^+|^2 + \frac{1}{\eta} E |K_T|^2 \right\}.
\]
(2)

On the other hand, from the equation
\[
K_T = Y_0 - \xi - \int_0^T f(s, Y_s, Z_s, U_s) ds + \int_0^T Z_s dB_s + \sum_{k=1}^d \int_0^T U^{(k)}_s dN^{(k)}_s,
\]
we have
\[
E |K_T|^2 \leq 5 \left\{ E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right] + E \left[ e^{\beta A_T} |\xi|^2 \right] + E \int_0^T e^{\beta A_t} |Z_s|^2 ds \\
+ E \int_0^T e^{\beta A_t} |U_s|^2 ds + \frac{1}{\beta} E \int_0^T e^{\beta A_t} \frac{f(s, Y_s, Z_s, U_s)^2}{a_s^2} ds \right\} \\
\leq 5 \left\{ E \left[ \sup_{0 \leq t \leq T} e^{\beta A_t} |Y_t|^2 \right] + E \left[ e^{\beta A_T} |\xi|^2 \right] + \frac{4}{\beta} E \int_0^T e^{\beta A_t} |Y_s|^2 dA_s \\
+ \frac{4}{\beta} E \int_0^T e^{\beta A_t} \frac{f(s, 0, 0, 0)^2}{a_s^2} ds + \left( 1 + \frac{4}{\beta} \right) E \int_0^T e^{\beta A_t} |Z_s|^2 ds + \left( 1 + \frac{4}{\beta} \right) E \int_0^T e^{\beta A_t} |U_s|^2 ds \right\}.
\]
(3)

Finally, the desired result is obtained by (2) and (3) for \(\eta > 5C_\beta \left( 1 + \frac{4}{\beta} \right)\). \(\square\)

Proposition 2. Assuming that \((A.2)(2)\) and \((A.3)(1)\) hold. Then, there exists at most one quadruple \((Y, Z, U, K)\) solution of the reflected BSDE (1).

Proof. Let \((Y^1, Z^1, U^1, K^1)\) and \((Y^2, Z^2, U^2, K^2)\) be two solutions of (1) associated with data \((\xi, f, L)\). It follows immediately from Proposition 1 that \(Y^1 = Y^2, Z^1 = Z^2\) and \(U^1 = U^2\). And thus \(K^1 = K^2\). \(\square\)
2.3. Existence of solution. In this part, we will prove the existence of a solution to reflected BSDE (1) via the Snell envelope. We must show the more general result (L is inhomogeneous and rcll) than El Karoui et al. [11] (L is only continuous), El Otmani [12] and Lepeltier and Xu [22] (L is only rcll). First, we consider the special case when the coefficient does not depend on \( (y, z, u) \), that is, \( f(t, y, z, u) = g(t) \).

**Proposition 3.** Let \( \mathcal{T}_{[t, T]} \) be a collection of all stopping times \( \tau \) with values between \( t \) and \( T \). Then

\[
Y_t = \operatorname{ess} \sup_{\tau \in \mathcal{T}_{[t, T]}} E \left[ \int_t^\tau g(s)ds + L_r \mathbb{I}_{\{\tau < T\}} + \xi \mathbb{I}_{\{\tau = T\}} / \mathcal{F}_t \right].
\]

**Proof.** Let \( \tau \in \mathcal{T}_{[t, T]} \), we may take the conditional expectation in (1) between times \( t \) and \( \tau \), hence

\[
Y_t = E \left[ \int_t^\tau g(s)ds + Y_\tau + K_\tau - K_t / \mathcal{F}_t \right] \geq E \left[ \int_t^\tau g(s)ds + L_r \mathbb{I}_{\{\tau < T\}} + \xi \mathbb{I}_{\{\tau = T\}} / \mathcal{F}_t \right].
\]

On the other hand, we define the stopping time \( D'_t(\epsilon) = \inf\{ s \geq t; Y_s \leq L_s + \epsilon \} \land T \), for all \( \epsilon > 0 \). Obviously, by definition of \( D'_t(\epsilon) \), we have that

\[
\left\{\begin{array}{ll}
Y_{D'_t(\epsilon)} \leq L_{D'_t(\epsilon)} + \epsilon, & \text{on the set } \{ D'_t < T \};
Y_s > L_s + \epsilon > L_s, & \text{for } t \leq s < D'_t.
\end{array}\right.
\]

Then between \( t \) and \( D'_t(\epsilon) \) included, \( Y_{s-} > L_{s-} \). From the Skorokhod conditions, we obtain \( \int_{t}^{D'_t(\epsilon)} (Y_{s-} - L_{s-})dK_s = 0 \), so we deduce that \( K_{D'_t(\epsilon)} = K_t \) and

\[
Y_t = E \left[ \int_t^{D'_t(\epsilon)} g(s)ds + Y_{D'_t(\epsilon)} / \mathcal{F}_t \right] \leq E \left[ \int_t^{D'_t(\epsilon)} g(s)ds + L_{D'_t(\epsilon)} \mathbb{I}_{\{D'_t < T\}} + \xi \mathbb{I}_{\{D'_t = T\}} / \mathcal{F}_t \right] + \epsilon.
\]

Using (4) and (5), then we find the desired result. \( \square \)

**Theorem 2.** Assume that \( \frac{q}{a} \in \mathcal{H}^2(\beta, A) \), (A.1) and (A.3) hold. Then the reflected BSDE associated with parameters \( (\xi, g, L) \) admits a unique solution.

**Proof.** For all \( t \leq T \), we define the processes \( \eta \) by

\[
\eta_t := L_t \mathbb{1}_{\{t < T\}} + \xi \mathbb{1}_{\{t = T\}} + \int_0^t g(s)ds.
\]

Note that \( \eta \) are rcll and uniformly square integrable processes such that

\[
E \left[ \sup_{0 \leq t \leq T} |\eta_t|^2 \right] \leq 3 \left\{ E \left[ e^{\beta A_T} |\xi|^2 \right] + \frac{1}{\beta} E \int_0^T e^{\beta A_s} \left| \frac{g(s)}{a(s)} \right|^2 ds + E \left[ \sup_{0 \leq t \leq T} e^{2\beta A_t} |L_t|^2 \right] \right\}.
\]

For all \( t \leq T \) and \( \tau \in \mathcal{T}_{[t, T]} \), we define \( S_t(\eta) := \operatorname{ess} \sup_{\tau \in \mathcal{T}_{[t, T]}} E[\eta_\tau / \mathcal{F}_t] \) the Snell envelope of the process \( \eta \). Recall that \( S(\eta) \) is supermartingale (the smallest rcll supermartingale which dominates the process \( \eta \)). Then, by Doob-Meyer decomposition theorem, there exists an \( \mathcal{F}_t \)-adapted rcll increasing processes \( K \).
with $\mathbb{E}[K_T] < +\infty$ and $K_0 = 0$ such that
\[
S_t(\eta) = \mathbb{E} \left[ \xi + \int_0^T g(s)ds + K_T/\mathcal{F}_t \right] - K_t = M_t - K^c_t - K^d_t.
\]

By the martingale representation theorem in the inhomogeneous case (see [5]), there exist two processes $Z \in \mathcal{H}^2$ and $U \in \mathcal{T}^2$ such that
\[
\xi + \int_0^T g(s)ds + K_T = \mathbb{E} \left[ \xi + \int_0^T g(s)ds + K_T/\mathcal{F}_t \right] + \int_t^T Z_sdB_s + \sum_{k=1}^d \int_t^T U_s^{(k)}d\tilde{N}_s^{(k)}.
\]

We conclude by Proposition 3 that
\[
Y_t + \int_0^t g(s)ds + K_t = S_t(\eta) + K_t = \xi + \int_0^T g(s)ds + K_T - \int_t^T Z_sdB_s - \sum_{k=1}^d \int_t^T U_s^{(k)}d\tilde{N}_s^{(k)}.
\]

Since $Y_t + \int_0^t g(s)ds = S_t(\eta)$ and by the assumption (A.3)(1), we have $Y_t \geq L_t$ for all $t \in [0, T]$. Now, we would like to show the Skorokhod conditions. Since the filtration is generated by an inhomogeneous simple Lévy process, the jumping times of $(M_t)_{t \leq T}$ are those of the power-jump processes associated with inhomogeneous simple Lévy process. Therefore, when $K^d$ jumps, the process $S_t(\eta)$ has the same jump. Then $\{\Delta K^d > 0\} \subset \{S_\cdot(\eta) = \eta_\cdot\}$ and
\[
\int_0^T (Y_s - L_s)dK^c_s = \sum_{0 < s \leq T} (Y_s - L_s)\mathbb{1}_{\{\Delta K^d > 0\}} \Delta K^d_s = \sum_{0 < s \leq T} (Y_s - L_s)\mathbb{1}_{\{S_s(\eta) = \eta_s\}} = 0.
\]

On the other hand, by some property of the Snell envelope (see Lemma A.4 in [22]), we get
\[
0 = \int_0^T (S_t(\eta) - \eta_t)dK^c_t = \int_0^T (Y_t - L_t)dK^c_t.
\]

Finally, let’s check that $(Y, Z, U) \in \tilde{\mathcal{M}}^2(\beta, A)$. First, using Proposition 3 yield that
\[
e^{\beta A_t}|Y_t|^2 \leq 3 \text{ess sup}_{\tau \in \mathcal{T}(t, \tau)} \mathbb{E} \left[ \frac{1}{\beta} \int_t^\tau e^{\beta A_s} \left| g(s) \right|^2 ds + e^{2\beta A_t} |L_T|^2 \mathbb{1}_{\tau < T} + e^{\beta A_T} |\xi|^2 \mathbb{1}_{\tau = T} \right]/\mathcal{F}_t.
\]

Since $(\xi, g/a, L^+ \in \mathcal{L}^2(\beta, A) \times \mathcal{H}^2(\beta, A) \times \mathcal{S}^2(\beta, A)$, it follows that $Y \in \mathcal{S}^2(\beta, A)$. Furthermore, by the same way as proof of Proposition 1 we obtain
\[
\beta \mathbb{E} \int_0^T e^{\beta A_t}|Y_t|^2dA_t + \mathbb{E} \int_0^T e^{\beta A_t}|Z_t|^2ds + \sum_{k=1}^d \mathbb{E} \int_0^T e^{\beta A_t}|U_t^{(k)}|^2\lambda_k ds
\]
\[
\leq \mathbb{E} \left[ e^{\beta A_T} |\xi|^2 \right] + (\beta - 1)\mathbb{E} \int_0^T e^{\beta A_t}|Y_t|^2dA_t
\]
\[
+ \frac{1}{\beta - 1} \mathbb{E} \int_0^T e^{\beta A_t} \left| g(s) \right|^2 ds + \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{2\beta A_t} |L_t^+|^2 \right] + \mathbb{E} |K_T|^2.
\]

Which implies that $(Y, Z, U) \in \tilde{\mathcal{M}}^2(\beta, A)$. Theorem 2 is then proved. 
\[\square\]
The main result of this paper is the following:

**Theorem 3.** Assume that (A.1), (A.2) and (A.3) hold. Then the reflected BSDE associated with parameters \((\xi, f, L)\) has a unique solution.

**Proof.** Let \(\Phi\) be the map from \(\mathcal{M}^{2}(\beta, A)\) into itself and let \((P, Q, R)\) and \((P', Q', R')\) be two elements of \(\mathcal{M}^{2}(\beta, A)\). Now, set \((Y, Z, U) = \Phi(P, Q, R)\) and \((Y', Z', U') = \Phi(P', Q', R')\) where \((Y, Z, U, K)\) and \((Y', Z', U', K')\) are the solutions of the reflected BSDE associated with parameters \((\xi, f(t, P_t, Q_t, R_t), L)\) and \((\xi, f(t, P'_t, Q'_t, R'_t), L)\) respectively. Set \(\delta G = G - G'\) for \(G = Y, Z, U, K, P, R, Q\). The Itô formula implies that

\[
\delta E \int_{t}^{T} e^{\beta A_t} |\delta Y_t|^2 dA_t + \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta Z_t|^2 ds + \mathbb{E} \int_{t}^{T} \|\delta U_t\|^2 ds \\
\leq 2 \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta Y_t| |f(s, P_s, Q_s, R_s) - f(s, P'_s, Q'_s, R'_s)| ds + 2 \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta Y_{s-}d(\delta K_t)| \\
\leq (\beta - 1) \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta Y_t|^2 dA_t + \frac{3}{\beta - 1} \left\{ \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta P_t|^2 dA_t + \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta Q_t|^2 ds + \mathbb{E} \int_{t}^{T} e^{\beta A_t} |\delta R_t|^2 ds \right\}.
\]

Here we have used the assumption (A.2) and the fact that \(\delta Y_{s-}d(\delta K_t) \leq 0\). We choose \(\beta > 4\). Then \(\Phi\) is a strict contraction mapping on \(\mathcal{M}^{2}(\beta, A)\). Henceforth, there exists a triplet processes \((Y, Z, U)\) that is a fixed point to \(\Phi\) which, with \(K\), is the unique solution of the reflected BSDE (1). \(\square\)

### 3. Application in finance

Let’s have a look at the pricing problem of an American option. We propose a model aimed at extending the usage of Black-Scholes model with jumps which is given by the following system of Stochastic Differential Equations (SDEs for short)

\[
\begin{align*}
\frac{dS_t^0}{S_t} &= r_t S_t^0 dt; \quad S_0^0 = 1; \\
\frac{dS_t}{S_t} &= \mu_t dt + \sqrt{\sigma_t} dB_t + \sum_{k=1}^{d} \alpha_{i}^{(k)} d\tilde{N}_t^{(k)}, \\
\frac{dP_t^j}{P_t} &= \mu_t dt + \sqrt{c_t} dB_t + \sum_{k=1}^{d} \alpha_{i}^{(k,j)} d\tilde{N}_t^{(k)}, \quad j = 1, \ldots, d.
\end{align*}
\]

where \((S_t^0)_{t \leq T}\) is the risk-free asset and \((S_t)_{t \leq T}\) is the risky asset. \((r_t)_{t \leq T}\) is a positive process representing the interest rate.

Recall that a market is complete if and only if there is a one risk neutral probability measure \(Q\) (see Proposition 9.3 in [4]). The problem of evaluating an option is to find a unique risk-neutral probability measure \(Q\) equivalent to \(\mathbb{P}\) such that the discounted assets \((\tilde{S}_t)_{t \leq T} = (e^{-\int_{0}^{t} r_s ds} S_t)_{t \leq T}\) are martingales with respect to \(Q\). For that and since we have \(d + 1\) processes \((B_t, \tilde{N}_t^{(1)}, \ldots, \tilde{N}_t^{(d)})\), we need \(d\) price processes that verify the following equations (for the homogeneous case, see [21]):
The representation of price processes under $\mathbb{Q}$, described by (6) is ensured by Girsanov’s theorem if there exists $u$ and $A^{(k)}$ with $A^{(k)}_t < A^{(1)}_t$, $k = 1, \ldots, d$ and $t \leq T$ such that

$$
\begin{align}
\mu_t &= r_t + u_t \sqrt{c_t} + \sum_{k=1}^{d} \alpha_t^{(k)} A^{(k)}_t, \\
\mu_t^i &= r_t + u_t \sqrt{c_t^i} + \sum_{k=1}^{d} \alpha_t^{(k)} j A^{(k)}_t, \quad j = 1, \ldots, d.
\end{align}
$$

(7)

Moreover, Girsanov’s theorem explains the relation between the Brownian motion and the intensities of Poisson processes under $\mathbb{P}$ and $\mathbb{Q}$

- $d\tilde{B}^0_t = dB_t + u_t dt$,
- $\tilde{\lambda}^{(k)}_t = \lambda^{(k)}_t - A^{(k)}_t$, for all $k = 1, \ldots, d$.

Note that the hypothesis $A^{(k)} < \lambda^{(k)}$ is introduced to ensure the positivity of the intensities $\tilde{\lambda}^{(k)}$ for all $k = 1, \ldots, d$. The equations given by (7) can be represented under the following matrix form

$$
\begin{pmatrix}
\sqrt{c_t^1} & \alpha_t^{(1)(1)} & \ldots & \alpha_t^{(d)(1)} \\
\sqrt{c_t^1} & \alpha_t^{(1)(1)} & \ldots & \alpha_t^{(d)(1)} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{c_t^d} & \alpha_t^{(1)(d)} & \ldots & \alpha_t^{(d)(d)}
\end{pmatrix}
\begin{pmatrix}
u_t \\
A^{(1)}_t \\
\vdots \\
A^{(d)}_t
\end{pmatrix}
= \begin{pmatrix}
\mu_t - r_t \\
\mu_t^{(1)} - r_t \\
\vdots \\
\mu_t^{(d)} - r_t
\end{pmatrix}.
$$

(8)

A sufficient condition for the system (8) has a unique solution $[u_t, A^{(1)}_t, \ldots, A^{(d)}_t]^T$ that is the square matrix associated with this system is invertible for all $t \in [0, T]$. Hence there exists a unique risk-neutral probability measure $\mathbb{Q}$ such that the discounted prices $\tilde{S}, \tilde{P}^{(1)}, \ldots, \tilde{P}^{(d)}$ are martingales.

Let us consider the valuation problem of an American contingent claim process $\mathbf{J}(S_t)$. The fair price of the American option is given by

$$
V_t = \text{ess sup}_{\tau \in T_t \cap T} \mathbb{E}_{\mathbb{Q}} \left[ e^{-\int_t^\tau r_s \, ds} \mathbf{J}(S_\tau) / F_t \right].
$$

For background on American options under Lévy models and numerical methods, we refer the reader to [3, 20, 28].

In terms of BSDEs, the price of an American option corresponds to the solution of a linear reflected BSDE. Indeed, let $(Y, Z, U, K)$ be a solution of the reflected BSDE

$$
\begin{align}
Y_t = \mathbf{J}(S_T) &- \int_t^T r_s Y_s ds + K_T - K_t - \int_t^T Z_s d\tilde{B}^Q_s - \sum_{k=1}^{d} \int_t^T U^{(k)}_s d\tilde{N}^{(k),Q}_s, \\
Y_t \geq \mathbf{J}(S_t) &\quad \mathbb{Q} \text{- a.s. for all } t \leq T, \\
\int_0^T (Y_t - \mathbf{J}(S_t)) dK^d_t &= 0 \text{ and } K^d_t = \sum_{0 < s < t} (Y_s - \mathbf{J}(S_{s-}))^- \mathbb{I}_{\{Y_{s-} = \mathbf{J}(S_{s-})\}}.
\end{align}
$$
Proposition 4. The value of an American option is given by

\[ Y_t = \text{ess sup}_{\tau \in [t, \tau]} \mathbb{E}_Q\left[ e^{-\int_t^\tau r_s ds} J(S_\tau) / \mathcal{F}_t \right] = V_t. \]

Proof. Let us consider on \( [t, \tau] \) the BSDE

\[ \tilde{Y}_t = J(S_t) - \int_t^\tau r_s \tilde{Y}_s ds - \int_t^\tau \tilde{Z}_s dB_s^Q - \sum_{k=1}^d \int_t^\tau \tilde{U}_s^{(k)} d\tilde{N}_s^{(k),Q}. \]

where \( \tau \) is a stopping time taking values in \( [t, T] \). Using the Meyer-Itô formula, we get

\[ |(\tilde{Y}_t - Y_t)^+|^2 = -2 \int_t^\tau r_s (\tilde{Y}_s - Y_s)^+ (\tilde{Y}_s - Y_s) ds - 2 \int_t^\tau (\tilde{Y}_s - Y_s)^+ dK_s - 2 \int_t^\tau (\tilde{Y}_s - Y_s)^+ (\tilde{Z}_s - Z_s) dB_s^Q \]

\[ -2 \sum_{k=1}^d \int_t^\tau (\tilde{Y}_s - Y_s)^+ (\tilde{U}_s^{(k)} - U_s^{(k)}) d\tilde{N}_s^{(k),Q}(\tau) - (A_\tau - A_t), \]

where \( A \) is a continuous non-decreasing process. This implies that \( \mathbb{E}_Q |(\tilde{Y}_t - Y_t)^+|^2 \leq 0 \). Then, for all \( t \in [0, \tau] \) we have

\[ Y_t \geq \tilde{Y}_t = \mathbb{E}_Q\left[ e^{-\int_t^\tau r_s ds} J(S_\tau) / \mathcal{F}_t \right]. \]

Thus \( Y_t \geq V_t \) for all \( t \in [0, T] \). On the other hand, let \( D_t = \inf\{t \leq s \leq T : Y_s \leq J(S_s)\} \). By the Skorokhod condition \( \int_t^{D_t} (Y_s - J(S_s)) dK^{c}_s = 0 \) and the continuity of \( K^c \) we obtain \( K^{c}_D = K^c_t \). Furthermore, we have also \( K^{d}_D = K^d_t \). Indeed, \( Y_s > J(S_s) \) for all \( s \in [t, D_t] \) and

\[ K^{d}_D - K^d_t = \sum_{t \leq s < D_t} (Y_s - J(S_{s^-}))^- \mathbb{I}_{\{Y_{s^-} = J(S_{s^-})\}} = 0. \]

We conclude that \( Y_t = \mathbb{E}_Q\left[ e^{-\int_t^\tau r_s ds} J(S_\tau) / \mathcal{F}_t \right] \leq V_t \) and the result follows. \( \square \)

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