Some new weakly singular nonlinear integral inequalities and their application

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Abstract. Our purpose of this article is to establish some new weakly singular nonlinear integral inequalities, which generalizes some known integral inequalities. The inequalities given here can be used in the analysis of the qualitative properties of fractional differential equations and integral equations. Applications are also provided to illustrate the usefulness of our results.

Keywords: Integral inequalities; weakly singular; explicit bounds

MSC: 39A12; 26A33

1. Introduction

The integral inequalities not only provide explicit upper bound on unknown functions, but also be an important tool to study the qualitative properties of solutions for differential equations and integral equations. Variants of Gronwall-Bellman integral inequality and their applications have attracted great interests of many mathematicians (such as [1-13] and references therein). With the development of theory for fractional differential equations, integral inequalities with weakly singular kernels have attracted great interests ([14-26]). In 1981, Henry [14] proved global existence and exponential decay results for a parabolic Cauchy problem by using the following singular integral inequality

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s)ds.$$ 

In 1997, Medved [16] discussed the following useful integral inequality

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1} f(s)u(s)ds.$$ 

In 2008, by using a modification of Medved’s method, Ma and Pečarić [20] studied certain class of nonlinear inequalities of Henry-type

$$u_p(t) \leq a(t) + b(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^\gamma f(s)u^q(s)ds, t \in R_+.$$ 

In 2014, Lin [23] established new weakly singular integral inequality

$$u(t) \leq a(t) + \sum_{i=1}^n b_i(t) \int_0^t (t-s)^{\beta_i-1} c_i(s)u^{\gamma_i}(s)ds.$$ 

In 2016, Xu and Meng [25] presented Gronwall-Bellman type integral inequality with weakly nonlinear weakly singular integral kernel of the form

$$u_p(t) \leq a(t) + b(t) \int_0^t (t-s)^{\beta-1} c(s)u^{\gamma}(s)ds + d(t) \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^\gamma f(s)u^q(s)ds, t \in R_+.$$ 

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In this paper, motivated by the works above, we will establish the following weakly singular integral inequalities:

\[ u(t) \leq a(t) + \int_{t_0}^{t} (t^\beta - s^\beta)(\gamma-1)s^{(\xi-1)}f(s) \left[ u^\alpha(s) + \int_{s_0}^{s} (s^\beta - \tau^\beta)(\gamma-1)\tau^{(\xi-1)}g(\tau)u^\alpha(\tau)d\tau \right]^{\rho} ds, \]  

(1)

\[ \phi(u(t)) = a(t) + \int_{t_0}^{t} (t^\beta - s^\beta)(\gamma-1)s^{(\xi-1)}f_1(t,s)w_1(u(s))ds \]

\[ + \int_{t_0}^{t} (t^\beta - s^\beta)(\gamma-1)s^{(\xi-1)}f_2(t,s)w_2(u(s)))ds \]

\[ + \int_{t_0}^{t} (t^\beta - s^\beta)(\gamma-1)s^{(\xi-1)}f(t,s) \left( \int_{s_0}^{s} (s^\beta - \tau^\beta)(\gamma-1)\tau^{(\xi-1)}g(s,\tau)w_3(u(\tau))d\tau \right) ds. \]  

(2)

Finally, two examples are included to illustrate the usefulness of our results.

2. Main results

Throughout this paper, \( R \) denotes the set of real numbers, \( R_+ = [0, +\infty) \) is the given subset of \( R \), and \( C(M, S) \) denotes the class of all continuous functions defined on set \( M \) with range in the set \( S \).

We cite some very useful lemmas in the procedures of our proof in our main results as follows.

**Lemma 1** ([27]) Let \( a \geq 0, p \geq q \geq 0 \) and \( p \neq 0 \), then

\[ a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q}{p}} a + \frac{p - q}{p} K^{\frac{q}{p}}, \quad K > 0. \]

We give two special cases of above result:

(a) If \( K = 1 \), we have

\[ a^{\frac{q}{p}} \leq \frac{q}{p} a + \frac{p - q}{p}, \quad a \geq 0, p \geq q \geq 0, p \neq 0. \]  

(3)

(b) If \( K = 1, p = 1 \), we have

\[ a^q \leq qa + (1-q), \quad a \geq 0, q \geq 0. \]  

(4)

**Lemma 2** ([19]) Let \( \beta, \gamma, \xi \) and \( p \) be positive constants. Then

\[ \int_{0}^{t} (t^\beta - s^\beta)p^{(\gamma-1)}s^{(\xi-1)}ds = \frac{\theta}{\beta} B \left[ \frac{p(\xi-1) + 1}{\beta}, p(\gamma-1) + 1 \right], \quad t \in R_+, \]

where \( B[x, y] = \int_{0}^{1} s^{x-1}(1-s)^{y-1}ds(0 < y < 1) \) is the well-known beta-function and \( \theta = p[\beta(\gamma-1) + \xi - 1] + 1. \)

**Lemma 3** ([19]) Suppose that the positive constants \( \beta, \gamma, \xi, p_1 \) and \( p_2 \) satisfy conditions:

(1) if \( \beta \in (0, 1], \gamma \in \left( \frac{1}{2}, 1 \right) \) and \( \xi \geq \frac{1}{2} - \gamma, p_1 = \frac{1}{2}; \)

(2) if \( \beta \in (0, 1], \gamma \in (0, \frac{1}{2}) \) and \( \xi > \frac{1-2\gamma^2}{1+\gamma^2}, p_2 = \frac{1+4\gamma^2}{1+\gamma^2}, \) then

\[ B \left[ \frac{p_i(\xi-1)+1}{\beta}, p_i(\gamma-1) + 1 \right] \in R_+, \quad \text{and} \quad \theta_i = p_i[\beta(\gamma-1) + \xi - 1] + 1 \geq 0 \]

are valid for \( i = 1, 2. \)

**Lemma 4** Let \( u(t), a(t), b(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+). \) If \( u(t) \) satisfied the following inequality:

\[ u(t) \leq a(t) + b(t) \int_{0}^{t} h(s)u(s)ds, \]  

(5)

then

\[ u(t) \leq a(t) + \frac{b(t)}{c(t)} \int_{0}^{t} h(s)a(s)c(s)ds, \]  

(6)
where
\[ e(t) = \exp \left( - \int_0^t h(s)b(s)ds \right). \]

**Proof.** Define a function \( v(t) \) on \( \mathbb{R}_+ \) by
\[ v(t) = e(t) \int_0^t h(s)u(s)ds, \]
we have \( v(0) = 0 \). Differentiating \( v(t) \) with respect to \( t \), we have
\[ v'(t) = -e(t)h(t)b(t) \int_0^t h(s)u(s)ds + e(t)h(t)u(t) \]
\[ \leq -e(t)h(t)b(t) \int_0^t h(s)u(s)ds + e(t)h(t)a(t) + e(t)h(t)b(t) \int_0^t h(s)u(s)ds \]
\[ = e(t)h(t)a(t). \]

Integrating both sides of the inequality (8) from 0 to \( t \), because \( v(0) = 0 \) we have
\[ v(t) \leq e(t)h(t)a(t). \]

From (7) and (9), we have
\[ \int_0^t h(s)u(s)ds \leq \frac{1}{e(t)} \int_0^t h(s)a(s)e(s)ds. \]

Substituting the inequality (10) into (5), we can get the required estimation (6). This completes the proof.

Consider the inequality (1)

**Theorem 1** Let \( u(t), a(t), f(t), g(t) \in C([t_0, +\infty), \mathbb{R}_+) \), and \( a(t) \) are nondecreasing functions, let \( \beta, \gamma, \xi \) be positive constants. Suppose \( u(t) \) satisfies the inequality (1).

(i) If \( \beta \in (0, 1], \gamma \in (\frac{1}{2}, 1) \) and \( \xi \geq \frac{3}{4} - \gamma \), we have
\[ u(t) \leq \tilde{B}_1(t) + \frac{\tilde{b}_1(t)}{\tilde{e}_1(t)} \int_{t_0}^t \left( \frac{pm}{1 - \gamma} f^{1+\gamma}(s) + \frac{pm}{(1 - \gamma)} f^{1+\gamma}(s) \tilde{b}_1(s) \int_{t_0}^{s} g^{1+\gamma}(\tau)d\tau \right) \tilde{A}_1(s)\tilde{e}_1(s)ds \]
where
\[ \tilde{a}_1(t) = a(t) + \frac{\gamma}{1 - \gamma} \tilde{b}_1(t) \left( \frac{p}{1 - \gamma} \int_{t_0}^t f^{1+\gamma}(s)\tilde{b}_1(s)ds + 1 \right), \]
\[ \tilde{b}_1(t) = (1 - \gamma)(M_1^\theta_1)^\gamma, \quad M_1 = \frac{1}{\beta^\gamma} \mathbb{B}\left[ \frac{\gamma + \xi - 1}{\beta^\gamma}, \frac{2\gamma - 1}{\gamma}, \theta_1 = \frac{1}{\gamma} [\beta(\gamma - 1) + \xi - 1] + 1, \right. \]
\[ \tilde{B}_1(t) = \tilde{a}_1(t) + \tilde{A}_1(t), \]
\[ \tilde{A}_1(t) = \tilde{b}_1(t) \int_{t_0}^t f^{1+\gamma}(s) \left[ 1 - \frac{p}{1 - \gamma} \left( m\tilde{a}_1(s) + (1 - m) \right) \right] ds \]
\[ + \tilde{b}_1(t) \int_{t_0}^t \frac{p}{1 - \gamma} f^{1+\gamma}(s)\tilde{b}_1(s) \int_{t_0}^{s} g^{1+\gamma}(\tau) \left[ \frac{n}{1 - \gamma} \tilde{a}_1(\tau) + \left( 1 - \frac{n}{1 - \gamma} \right) \right] d\tau ds, \]
\[ \tilde{B}_1(t) = \tilde{a}_1(t) + \tilde{A}_1(t), \]
\[ \tilde{e}_1(t) = \exp \left( - \int_{t_0}^t \tilde{b}_1(s) \left[ \frac{pm}{1 - \gamma} f^{1+\gamma}(s) + \frac{pm}{(1 - \gamma)} f^{1+\gamma}(s) \tilde{b}_1(s) \int_{t_0}^{s} g^{1+\gamma}(\tau)d\tau \right] ds \right). \]

(ii) If \( \beta \in (0, 1], \gamma \in (0, \frac{1}{2}) \) and \( \xi > \frac{1 - 2\gamma^2}{1 - \gamma^2} \), we have
\[ u(t) \leq \tilde{B}_2(t) + \frac{\tilde{b}_2(t)}{\tilde{e}_2(t)} \int_{t_0}^t \left( \frac{pm(1 + 4\gamma^2)}{\gamma} f^{1+\gamma}(s) + \frac{pm(1 + 4\gamma^2)}{(1 - \gamma)^2} f^{1+\gamma}(s) \tilde{b}_2(s) \int_{t_0}^{s} g^{1+\gamma}(\tau)d\tau \right) \tilde{A}_2(s)\tilde{e}_2(s)ds \]
where

\[
\tilde{a}_2(t) = a(t) + \frac{1}{\gamma} \tilde{b}_2(t) \left( \frac{p(1 + 4\gamma)}{\gamma} \int_{t_0}^{t} f^{\frac{1+\epsilon}{\gamma}}(s) \tilde{b}_2(s) ds + 1 \right),
\]

\[
\tilde{b}_2(t) = \frac{\gamma}{1 + 4\gamma} (M_2 b_{\beta})^{\frac{1+\epsilon}{\gamma}}, \quad M_2 = \frac{1}{\beta} B[\frac{\xi (1 + 4\gamma) - \gamma}{\beta (1 + 3\gamma)}, \frac{4\gamma^2}{1 + 3\gamma}], \quad \tilde{b}_2 = \frac{1 + 4\gamma}{1 + 3\gamma} [\beta (\gamma - 1) + \xi - 1] + 1,
\]

\[
\tilde{B}_2(t) = \tilde{a}_2(t) + \tilde{A}_2(t),
\]

\[
\tilde{A}_2(t) = \tilde{b}_2(t) \int_{t_0}^{t} f^{\frac{1+\epsilon}{\gamma}}(s) \left[ (1 - \frac{p(1 + 4\gamma)}{\gamma}) + \frac{p(1 + 4\gamma)}{\gamma} (m \tilde{a}_2(s) + (1 - m)) \right] ds
\]

\[
+ \tilde{b}_2(t) \int_{t_0}^{t} \frac{p(1 + 4\gamma)}{\gamma} f^{\frac{1+\epsilon}{\gamma}}(s) \tilde{b}_2(s) \int_{s}^{t} g^{\frac{1+\epsilon}{\gamma}}(\tau) \left[ \frac{n(1 + 4\gamma)}{\gamma} \tilde{a}_2(\tau) + \left( 1 - \frac{n(1 + 4\gamma)}{\gamma} \right) \right] d\tau ds,
\]

\[
\tilde{B}_2(t) = \tilde{a}_2(t) + \tilde{A}_2(t),
\]

\[
\tilde{e}_2(t) = \exp \left( - \int_{t_0}^{t} \tilde{b}_2(s) \left[ \frac{p n (1 + 4\gamma)}{\gamma} f^{\frac{1+\epsilon}{\gamma}}(s) + \frac{p (1 + 4\gamma)}{\gamma} f^{\frac{1+\epsilon}{\gamma}}(s) \tilde{b}_2(s) \int_{s}^{t} g^{\frac{1+\epsilon}{\gamma}}(\tau) d\tau \right] ds \right).
\]

**Proof.** If \( \beta \in (0, 1], \gamma \in (\frac{1}{2}, 1) \) and \( \xi \geq \frac{3}{2} - \gamma \), let

\[
p_1 = \frac{1}{\gamma}, \quad q_1 = \frac{1}{1 - \gamma}.
\]

If \( \beta \in (0, 1], \gamma \in (0, \frac{1}{2}] \) and \( \xi > \frac{1 - 2\gamma}{1 - \gamma} \), let

\[
p_2 = \frac{1 + 4\gamma}{1 + 3\gamma}, \quad q_2 = \frac{1 + 4\gamma}{\gamma},
\]

then

\[
\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2.
\]

Using Hölder's inequality applied to (1), we have

\[
u(t) \leq a(t) + \left[ \int_{t_0}^{t} \left( t^{\beta - s^\beta} p_{\beta}(\gamma - 1) s^{\beta}(\xi - 1) ds \right)^{\frac{1}{p_i}} \right]^{\frac{1}{p_i}}
\]

\[
\leq a(t) + \left[ \int_{t_0}^{t} f^{\frac{n_i}{\gamma}}(s) \left( u^{m}(s) + \int_{s}^{t} (s^{\beta - \tau^\beta})^{\gamma - 1} \tau^{\xi - 1} g(\tau) u^{n_q}(\tau) d\tau \right) \right]^{\frac{1}{p_i}} \int_{t_0}^{t} f^{\frac{n_i}{\gamma}}(s) (u^{m}(s))^{\frac{1}{p_i}} \int_{t_0}^{t} f^{\frac{n_i}{\gamma}}(s) (u^{m}(s))^{\frac{1}{p_i}} ds
\]

\[
+ \left( \int_{t_0}^{t} (s^{\beta - \tau^\beta})^{\gamma - 1} \tau^{\xi - 1} d\tau \right)^{\frac{1}{p_i}} \left( \int_{t_0}^{t} g^{n_q}(\tau) u^{n_q}(\tau) d\tau \right)^{\frac{1}{n_q}} ds \right]^{\frac{1}{p_i}}.
\]

**Lemma 2**, the inequality (13) can be restated as

\[
u(t) \leq a(t) + (M_i t^\beta)^\frac{1}{p_i} \left[ \int_{t_0}^{t} f^{\frac{n_i}{\gamma}}(s) \left( u^{m}(s) + (M_i s^{\beta})^{\frac{1}{p_i}} \int_{t_0}^{t} g^{n_q}(\tau) u^{n_q}(\tau) d\tau \right)^{\frac{1}{p_i}} ds \right]^{\frac{1}{p_i}}.
\]
By (3) and (4), we obtain
\[
    u(t) \leq a(t) + \frac{1}{q_i} (M_i t^{\theta_i})^{\frac{1}{\theta_i}} \left[ \int_{t_0}^{t} f^q(s) \left( u^m(s) + (M_i s^{\theta_i})^{\frac{1}{\theta_i}} \left( \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau \right)^{\frac{1}{\alpha_q}} \right) ds + (q_i - 1) \right] \\
    \leq a(t) + \frac{1}{q_i} (M_i f^{\theta_i})^{\frac{1}{\theta_i}} \left[ \int_{t_0}^{t} pq_i f^q(s) \left( u^m(s) + (M_i s^{\theta_i})^{\frac{1}{\theta_i}} \left( \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau \right)^{\frac{1}{\alpha_q}} \right) + \frac{q_i}{q_0} \right] ds \\
    + (1 - pq_i) f^q(s) ds + (q_i - 1) \\
    \leq a(t) + \frac{1}{q_i} (M_i f^{\theta_i})^{\frac{1}{\theta_i}} \left[ \int_{t_0}^{t} f^q(s) \left( u^m(s) + \frac{1}{q_i} (M_i s^{\theta_i})^{\frac{1}{\theta_i}} \left( \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau \right)^{\frac{1}{\alpha_q}} \right) + \frac{q_i}{q_0} \right] ds \\
    + (1 - pq_i) f^q(s) ds + (q_i - 1) \\
    = \tilde{a}_i(t) + \frac{1}{q_i} (M_i f^{\theta_i})^{\frac{1}{\theta_i}} \int_{t_0}^{t} f^q(s) \left[ pq_i \left( u^m(s) + \frac{1}{q_i} (M_i s^{\theta_i})^{\frac{1}{\theta_i}} \left( \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau \right) \right) \right] ds + (1 - pq_i) \right] ds.
\]

that is,
\[
    u(t) \leq \tilde{a}_i(t) + \tilde{b}_i(t) \int_{t_0}^{t} f^q(s) \left[ pq_i \left( u^m(s) + \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau \right) \right] ds + (1 - pq_i) \right] ds.
\]

Define a function \( w(t) \) by
\[
    w(t) = \tilde{b}_i(t) \int_{t_0}^{t} (1 - pq_i) f^q(s) ds + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^q(s) u^m(s) ds \\
    + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) u^{\alpha_q}(\tau) d\tau ds,
\]
we can conclude that \( w(t) \) is a nondecreasing function, from (14) and (15), we have
\[
    u(t) \leq \tilde{a}_i(t) + w(t).
\]

By (4) and (16), we obtain
\[
    u^m(t) \leq (\tilde{a}_i(t) + w(t))^m \leq m(\tilde{a}_i(t) + w(t)) + (1 - m), \\
    u^{\alpha_q}(t) \leq (\tilde{a}_i(t) + w(t))^{\alpha_q} \leq nq_i(\tilde{a}_i(t) + w(t)) + (1 - nq_i).
\]

Substituting the inequality (17) and (18) into (15), we have
\[
    w(t) \leq \tilde{b}_i(t) \int_{t_0}^{t} (1 - pq_i) f^q(s) ds + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^q(s) [m(\tilde{a}_i(s) + w(s)) + (1 - m)] ds \\
    + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) [nq_i(\tilde{a}_i(\tau) + w(\tau)) + (1 - nq_i)] ds \\
    + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) [nq_i(\tilde{a}_i(\tau) + (1 - nq_i)] ds \\
    + \tilde{b}_i(t) \int_{t_0}^{t} pq_i m f^q(s) w(s) ds + \tilde{b}_i(t) \int_{t_0}^{t} mnq_i^2 f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) w(\tau) d\tau ds \\
    = \tilde{A}_i(t) + \tilde{b}_i(t) \int_{t_0}^{t} pq_i m f^q(s) w(s) ds + \tilde{b}_i(t) \int_{t_0}^{t} mnq_i^2 f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) w(\tau) d\tau ds \\
    \leq \tilde{A}_i(t) + \tilde{b}_i(t) \int_{t_0}^{t} pq_i m f^q(s) w(s) ds + \tilde{b}_i(t) \int_{t_0}^{t} \left( mnq_i^2 f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) d\tau \right) w(s) ds \\
    = \tilde{A}_i(t) + \tilde{b}_i(t) \int_{t_0}^{t} \left( pq_i m f^q(s) + mnq_i^2 f^q(s) \tilde{b}_i(s) \int_{t_0}^{s} g^q(\tau) d\tau \right) w(s) ds,
\]
where
\[ \tilde{A}_i(t) = \tilde{b}_i(t) \int_{t_0}^{t} f^a(s)[(1 - pq_i) + pq_i(m\tilde{a}_i(s) + (1 - m))]ds \]
\[ + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau ds + \tilde{b}_i(t) \int_{t_0}^{t} pq_i f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau ds. \]

By Lemma 4, we get
\[ w(t) \leq \tilde{A}_i(t) + \frac{\tilde{b}_i(t)}{\tilde{c}_i(t)} \int_{t_0}^{t} \left( pq_i m f^a(s) + pq_i^2 f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau \right) \tilde{A}_i(s)\tilde{c}_i(s)ds, \]
where
\[ \tilde{c}_i(t) = \exp \left[ -\int_{t_0}^{t} \tilde{b}_i(s) \left( pq_i m f^a(s) + pq_i^2 f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau \right) ds \right]. \]

From (16) and (19), we have
\[ u(t) \leq \tilde{a}_i(t) + \tilde{A}_i(t) + \frac{\tilde{b}_i(t)}{\tilde{c}_i(t)} \int_{t_0}^{t} \left( \left( pq_i m f^a(s) + pq_i^2 f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau \right) \tilde{A}_i(s)\tilde{c}_i(s)ds \right) \]
\[ = \tilde{B}_i(t) + \frac{\tilde{b}_i(t)}{\tilde{c}_i(t)} \int_{t_0}^{t} \left( \left( pq_i m f^a(s) + pq_i^2 f^a(s)\tilde{b}_i(s) \int_{t_0}^{s} g^a(\tau)d\tau \right) \tilde{A}_i(s)\tilde{c}_i(s)ds \right), \]
where
\[ \tilde{B}_i(t) = \tilde{a}_i(t) + \tilde{A}_i(t). \]

This completes the proof.

Consider the inequality (2)

Suppose that
(H1) \( \phi \) is a strictly increasing continuous function on \([0, \infty)\), \( \phi(u) > 0 \), for all \( u > 0 \),
(H2) all \( w_j(j = 1, 2, 3) \) are nondecreasing continuous functions on \([0, \infty)\) and positive on \([0, \infty)\), and \( \frac{w_{j+1}}{w_j}, (j = 1, 2) \) are nondecreasing functions,
(H3) \( a(t) \) is a continuous function on \([t_0, \infty), a(t_0) \neq 0 \),
(H4) \( f_j(t, s), (j = 1, 2) \) and \( f(t, s), g(t, s) \) are continuous functions on \([t_0, \infty) \times [t_0, \infty)\).

Moreover, we define the following function:
\[ W_j(u) = \int_1^{u} \frac{ds}{w_j(\phi^{-1}(s))}, \quad j = 1, 2, 3, \]
(20)

\( W_j \) are strictly increasing and continuous functions. Letting \( W_j^{-1} \) denote the inverse function of \( W_j \),
then \( W_j^{-1} \) are also continuous and increasing functions.

Theorem 2 Let \( \beta, \gamma, \xi \) be positive constants. \( u(t) \) is a nonnegative continuous function at \( t \geq t_0 \geq 0 \),
satisfying the integral inequality (2). If (H1) – (H4) hold, then we have the following results.
(i) If \( \beta \in (0, 1], \gamma \in (\frac{1}{2}, 1) \), \( \xi \geq \frac{3}{2} - \gamma \) and
\[ b_1(t) \sum_{j=1}^{3} \tilde{F}_{1,j}(t, s)w_j(u(s)) > 0, (j = 1, 2, 3), \]
(21)

we have
\[ u(t) \leq \phi^{-1}(W_3^{-1}(\tilde{c}_{1,4}(t))). \]
where

\[ b_1(t) = (M_1 t^{\theta_1} \gamma), \quad M_1 = \frac{1}{\beta} \frac{\beta + \xi - 1 \gamma}{\beta \gamma}, \quad \theta_1 = \frac{1}{\gamma} [\beta(1) + \xi - 1] + 1, \]

\[ F_{11}(t, s) = f_1^{\gamma}(t, s), \quad F_{12}(t, s) = f_2^{\gamma}(t, s), \]

\[ F_1(t, s) = b^{\tau}(t, s) f \gamma^{\gamma}(t, s), \quad G_1(t, s) = g^{\gamma}(t, s), \]

\[ \tilde{F}_{13}(t, s) = \int_s \max_{t_0 \leq \tau \leq t} F_1(t, \tau) G_1(\tau, s) \, d\tau, \]

\[ \tilde{F}_{1,j}(t, s) = \max_{t_0 \leq \tau \leq t} F_1(t, \tau), \quad j = 1, 2, \]

\[ A_1(t) = a(t) + 3 \gamma b_1(t) - \gamma b_1(t) \int_{t_0}^t F_{11}(t, s) \, ds - \gamma b_1(t) \int_{t_0}^t F_{12}(t, s) \, ds \]

\[ = \gamma b_1(t) \left( \int_{t_0}^t F_1(t, \tau) \left( \int_{t_0}^\tau G_1(s, \tau) \, d\tau \right) \right) \, ds, \]

\[ e_{1,1}(t) = \max_{t_0 \leq \tau \leq t} |A_1(\tau)|, \quad e_{1,2}(t) = W_1(e_{1,1}(t)) + b_1(T) \int_{t_0}^T \tilde{F}_{1,1}(T, \tau) \, d\tau, \quad T \in [t_0, \infty), \]

\[ e_{1,j+1}(t) = W_j(W_{j-1}^{-1}(e_{1,j}(t))) + b_1(T) \int_{t_0}^T \tilde{F}_{1,j}(T, \tau) \, d\tau, \quad j = 2, 3, \quad T \in [t_0, \infty), \]

\[ \tilde{e}_{1,1}(t) = e_{1,1}(t), \quad \tilde{e}_{1,2}(t) = W_1(\tilde{e}_{1,1}(t)) + b_1(t) \int_{t_0}^t \tilde{F}_{1,1}(t, \tau) \, d\tau, \]

\[ \tilde{e}_{1,j+1}(t) = W_j(W_{j-1}^{-1}(\tilde{e}_{1,j}(t))) + b_1(t) \int_{t_0}^t \tilde{F}_{1,j}(t, \tau) \, d\tau, \quad j = 2, 3, \]

\[ (ii) \text{ If } \beta \in (0, 1], \gamma \in (0, \frac{1}{2}], \xi > \frac{1-2\gamma^2}{1-2\gamma} \text{ and } \]

\[ b_2(t) \sum_{j=1}^3 \tilde{F}_{2,j}(t, s) w_j(u(s)) > 0, (j = 1, 2, 3), \quad (22) \]

we have

\[ u(t) \leq \phi^{-1}(W_3^{-1}(\tilde{e}_{2,4}(t))) \]
where
\[
\begin{align*}
 b_2(t) &= (M_2\beta_2)^{\frac{1+3\gamma}{1+4\gamma}}, \quad M_2 = \frac{1}{\beta} B\left(\frac{4\gamma^2}{\beta(1+3\gamma)}, \frac{4\gamma^2}{1+3\gamma}\right), \quad \beta_2 = \frac{1+4\gamma}{1+3\gamma} \beta(\gamma-1) + \xi - 1] + 1, \\
 F_{21}(t, s) &= F_{22}(t, s) = f_2^{1+4\gamma}(t, s), \\
 F_2(t, s) &= b^{1+4\gamma}(s)f_2^{1+4\gamma}(t, s), \\
 F_{2,3}(s, j) &= \int_s^t \max_{t_0 \leq \tau \leq t} F_2(t, \tau)G_2(\tau, s)\,d\tau, \\
 \hat{F}_{2,j}(t, s) &= \max_{t_0 \leq \tau \leq t} F_{2,j}(\tau, s), \quad j = 1, 2, \\
 A_2(t) &= a(t) + 3(1+3\gamma) \beta_2 - \frac{1+3\gamma}{1+4\gamma} b_2(t) \int_t^s F_{21}(t, s)ds - \frac{1+3\gamma}{1+4\gamma} b_2(t) \int_t^s F_{22}(t, s)ds \\
 e_{2,1}(t) &= \max_{t_0 \leq \tau \leq t} |A_2(\tau)|, \\
 e_{2,2}(t) &= W_1(e_{2,1}(t)) + b_1(T) \int_t^s \hat{F}_{2,1}(T, s)ds, \quad T \in [t_0, \infty), \\
 e_{2,j+1}(t) &= W_j(W_{j-1}(e_{2,j}(t))) + b_1(T) \int_t^s \hat{F}_{2,j}(T, s)ds, \quad j = 2, 3, \quad T \in [t_0, \infty), \\
 \hat{e}_{2,1}(t) &= e_{2,1}(t), \\
 \hat{e}_{2,2}(t) &= W_1(\hat{e}_{2,1}(t)) + b_2(t) \int_t^s \hat{F}_{2,1}(t, s)ds, \\
 \hat{e}_{2,j+1}(t) &= W_j(W_{j-1}(\hat{e}_{2,j}(t))) + b_2(t) \int_t^s \hat{F}_{2,j}(t, s)ds, \quad j = 2, 3,
\end{align*}
\]

Proof. If \(\beta \in (0, 1], \gamma \in (\frac{1}{2}, 1)\) and \(\xi \geq \frac{3}{2} - \gamma\), let
\[
\begin{align*}
p_1 &= \frac{1}{2}, \\
q_1 &= \frac{1}{\gamma - 1}.
\end{align*}
\]

If \(\beta \in (0, 1], \gamma \in (0, \frac{1}{2})\) and \(\xi > \frac{1-2\gamma^2}{1-3\gamma}\), let
\[
\begin{align*}
p_2 &= \frac{1+4\gamma}{1+3\gamma}, \\
q_2 &= \frac{1+4\gamma}{\gamma},
\end{align*}
\]

then
\[
\frac{1}{p_i} + \frac{1}{q_i} = 1, \quad i = 1, 2.
\]

Using Holder’s inequality applied to (2), we have
\[
\phi(u(t)) \leq a(t) + \left[ \int_{t_0}^t (t^\beta - \sigma^\beta) p_1(\gamma-1) s^{p_1}(\xi-1) ds \right]^{\frac{1}{p_1}} \left[ \int_{t_0}^t f_{1,1}^1(t, s) u^{q_1}(u(s)) ds \right]^{\frac{1}{q_1}} + \left[ \int_{t_0}^t (t^\beta - \sigma^\beta) p_1(\gamma-1) s^{p_1}(\xi-1) ds \right]^{\frac{1}{p_1}} \left[ \int_{t_0}^t f_{1,1}^2(t, s) u^{q_1}(u(s)) ds \right]^{\frac{1}{q_1}} + \left[ \int_{t_0}^t (t^\beta - \sigma^\beta) p_1(\gamma-1) s^{p_1}(\xi-1) ds \right]^{\frac{1}{p_1}} \left[ \int_{t_0}^t f_{1,1}^3(t, s) \left( \int_{t_0}^s (t^\beta - \sigma^\beta) p_1(\gamma-1) s^{p_1}(\xi-1) d\sigma \right) \left( \int_{t_0}^s g^{q_1}(s, \tau) u^{q_1}(u(\tau)) d\tau \right) u^{q_1}(u(s)) ds \right]^{\frac{1}{q_1}}. 
\]
(23)
Using Lemma 2, the inequality (23) can be restated as

\[
\phi(u(t)) \leq a(t) + (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^0 (t, s) w_1^0 (u(s)) ds \right] \frac{1}{q_1} + (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_2^0 (t, s) w_2^0 (u(s)) ds \right] \frac{1}{q_1} \\
+ (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^2 (t, s) \cdot (M_{s}^{\alpha})(s) \right] \left( \int_{t_0}^{s} g^0 (s, \tau) w_2^0 (u(\tau)) d\tau \right) ds \frac{1}{q_1}, \quad i = 1, 2.
\]

By (3) and (4), we obtain

\[
\phi(u(t)) \leq a(t) + \frac{1}{q_1} (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^0 (t, s) w_1^0 (u(s)) ds \right] + \frac{1}{q_1} (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_2^0 (t, s) w_2^0 (u(s)) ds \right] \\
+ \frac{1}{q_1} (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^2 (t, s) \cdot (M_{s}^{\alpha})(s) \right] \left( \int_{t_0}^{s} g^0 (s, \tau) w_2^0 (u(\tau)) d\tau \right) ds + \frac{3(q_i - 1)}{q_i} (M_{t}^{\alpha})(s) \\
\leq a(t) + (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^0 (t, s) w_1 (u(s)) ds \right] + \frac{1}{q_1} (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_2^0 (t, s) ds \right] \\
+ (M_{t}^{\alpha})(s) \left[ \int_{t_0}^{t} f_1^2 (t, s) \cdot (M_{s}^{\alpha})(s) \right] \left( \int_{t_0}^{s} g^0 (s, \tau) w_3 (u(\tau)) d\tau \right) ds \frac{1}{q_1} \\
+ \frac{3(q_i - 1)}{q_i} (M_{t}^{\alpha})(s) \\
= A_i(t) + b_i(t) \int_{t_0}^{t} F_{1i}(t, s) w_1 (u(s)) ds + b_i(t) \int_{t_0}^{t} F_{2i}(t, s) w_2 (u(s)) ds \\
+ b_i(t) \int_{t_0}^{t} F_{1i}(t, s) \left( \int_{t_0}^{s} G_i (s, \tau) w_3 (u(\tau)) d\tau \right) ds, \quad i = 1, 2,
\]

that is,

\[
\phi(u(t)) \leq A_i(t) + b_i(t) \int_{t_0}^{t} F_{1i}(t, s) w_1 (u(s)) ds + b_i(t) \int_{t_0}^{t} F_{2i}(t, s) w_2 (u(s)) ds \\
+ b_i(t) \int_{t_0}^{t} F_{1i}(t, s) \left( \int_{t_0}^{s} G_i (s, \tau) w_3 (u(\tau)) d\tau \right) ds, \quad i = 1, 2,
\]

where

\[
b_i(t) = (M_{t}^{\alpha})(s), \\
F_{1i}(t, s) = f_1^i (t, s), \quad F_{2i}(t, s) = f_2^i (t, s), \\
F_i(t, s) = b_i (s) f_1 (t, s), \quad G_i (t, s) = g^0 (t, s), \\
A_i(t) = a(t) + \frac{3(q_i - 1)}{q_i} b_i(t) + \frac{1}{q_i} b_i(t) \int_{t_0}^{t} F_{1i}(t, s) ds + \frac{1}{q_i} b_i(t) \int_{t_0}^{t} F_{2i}(t, s) ds \\
+ \frac{1}{q_i} b_i(t) \int_{t_0}^{t} F_{1i}(t, s) \left( \int_{t_0}^{s} G_i (s, \tau) d\tau \right) ds, \quad i = 1, 2,
\]

by (H_3) and (H_4), \(A_i(t)\) is a continuous function on \([t_0, \infty)\), \(A_i(t_0) \neq 0\), \(F_{ij}(t, s)(j = 1, 2)\) and \(F_i(t, s), G_i (t, s)\) are continuous functions on \([t_0, \infty) \times [t_0, \infty)(i=1,2)\).

Let

\[
\tilde{F}_{i,3}(t, s) = \int_{t_0}^{t} \max_{s \leq \tau \leq t} F_i(t, \tau) G_i (\tau, s) d\tau, \quad i = 1, 2,
\]
From (29) and (30), we obtain

\[
\int_{t_0}^{t} F_i(t, s) \int_{t_0}^{s} G_i(s, \tau) w_3(u(\tau)) d\tau ds = \int_{t_0}^{t} w_3(u(\tau)) \int_{\tau}^{t} F_i(t, s) G_i(s, \tau) ds d\tau \\
= \int_{t_0}^{t} w_3(u(s)) \int_{s}^{t} F_i(t, \tau) G_i(\tau, s) d\tau ds \\
\leq \int_{t_0}^{t} \tilde{F}_{i,3}(t, s) w_3(u(s)) ds, \quad i = 1, 2, 
\]

(26)

from (25), (26), then (24) can be written

\[
\phi(u(t)) \leq e_{i,1}(t) + b_i(t) \sum_{j=1}^{3} \int_{t_0}^{t} \tilde{F}_{i,j}(t, s) w_j(u(s)) ds, 
\]

(27)

where

\[
\tilde{F}_{i,j}(t, s) = \max_{t_0 \leq \tau \leq t} F_{ij}(\tau, s), \quad j = 1, 2, \quad i = 1, 2,
\]

then \( \tilde{F}_{i,j}(t, s), (j = 1, 2, 3) \) are nondecreasing in \( t \) for each fixed \( s \), and satisfy \( \tilde{F}_{i,j}(t, s) \geq F_{ij}(t, s) \). For any fixed \( T \in [t_0, \infty) \) and for arbitrary \( t \in [t_0, T] \), from (31), we obtain

\[
\phi(u(t)) \leq e_{i,1}(t) + b_i(T) \sum_{j=1}^{3} \int_{t_0}^{t} \tilde{F}_{i,j}(T, s) w_j(u(s)) ds.
\]

(27)

Set

\[
v(t) = e_{i,1}(t) + b_i(T) \sum_{j=1}^{3} \int_{t_0}^{t} \tilde{F}_{i,j}(T, s) w_j(u(s)) ds, \quad i = 1, 2,
\]

(28)

then \( v(t) \) is a nonnegative and nondecreasing continuous function, and

\[
\phi(u(t)) \leq v(t), \quad u(t) \leq \phi^{-1}(v(t)), \quad v(t_0) = e_{i,1}(t_0), \quad v(t) > e_{i,1}(t),
\]

by (21) and (22), taking the derivative with respect to \( t \) in (28), we have

\[
v'(t) = e'_{i,1}(t) + b_i(T) \sum_{j=1}^{3} \tilde{F}_{i,j}(T, t) w_j(u(t)), \quad i = 1, 2,
\]

(29)

let

\[
\psi_j(t) = \frac{w_j(t)}{w_1(t)}, \quad j = 1, 2, 3,
\]

(30)

by \((H_2)\), we see that each \( \psi_j(j = 1, 2, 3) \) is a nondecreasing function. Then

\[
\frac{\psi_{j+1}(t)}{\psi_j(t)} = \frac{w_{j+1}(t)}{w_j(t)}, \quad j = 1, 2,
\]

(31)

which are also continuous, nondecreasing and positive on \([t_0, T]\).

From (29) and (30), we obtain

\[
\frac{v'(t)}{w_1(\phi^{-1}(v(t)))} = \frac{e'_{i,1}(t) + b_i(T) \sum_{j=1}^{3} \tilde{F}_{i,j}(T, t) w_j(u(t))}{w_1(\phi^{-1}(v(t)))} \\
\leq \frac{e'_{i,1}(t) + b_i(T) \sum_{j=1}^{3} \tilde{F}_{i,j}(T, t) w_j(\phi^{-1}(v(t)))}{w_1(\phi^{-1}(v(t)))} \\
= \frac{e'_{i,1}(t)}{w_1(\phi^{-1}(v(t)))} + b_i(T) \tilde{F}_{i,1}(T, t) + \frac{b_i(T) \sum_{j=2}^{3} \tilde{F}_{i,j}(T, t) \psi_j(\phi^{-1}(v(t)))}{w_1(\phi^{-1}(v(t)))} \\
\leq \frac{e'_{i,1}(t)}{w_1(\phi^{-1}(e_{i,1}(t)))} + b_i(T) \tilde{F}_{i,1}(T, t) + b_i(T) \sum_{j=1}^{2} \tilde{F}_{i,j+1}(T, t) \psi_{j+1}(\phi^{-1}(v(t))),
\]

(32)
integrating both sides of (32) from $t_0$ to $t$, and using the definition of $W_j(u)$ in (20), we have

$$W_1(v(t)) \leq W_1(e_{i,1}(t)) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,j+1}(T,s)ds + b_i(T) \int_{t_0}^{t} \sum_{j=1}^{2} \tilde{F}_{i,j+1}(T,s)\psi_{j+1}(\phi^{-1}(v(s)))ds, \ i = 1, 2. \tag{33}$$

Let

$$\eta_1(t) = W_1(v(t)), \tag{34}$$

$$e_{i,2}(t) = W_1(e_{i,1}(t)) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,j+1}(T,s)ds, \ i = 1, 2, \tag{35}$$

from (34) and (35), then (33) can be restated as

$$\eta_1(t) \leq e_{i,2}(t) + b_i(T) \sum_{j=1}^{2} \int_{t_0}^{t} \tilde{F}_{i,j+1}(T,s)\psi_{j+1}(\phi^{-1}(v(s)))ds$$

$$= e_{i,2}(t) + b_i(T) \sum_{j=1}^{2} \int_{t_0}^{t} \tilde{F}_{i,j+1}(T,s)\psi_{j+1}(\phi^{-1}(W_1^{-1}(\eta_1(s))))ds.$$ 

Set

$$v_1(t) = e_{i,2}(t) + b_i(T) \sum_{j=1}^{2} \int_{t_0}^{t} \tilde{F}_{i,j+1}(T,s)\psi_{j+1}(\phi^{-1}(W_1^{-1}(\eta_1(s))))ds, \ j = 1, 2, 3, \ i = 1, 2,$$

then $v_1(t)$ is a nondecreasing continuous function on $[t_0, t]$, and $\eta_1(t) \leq v_1(t), e_{i,2}(t) \leq v_1(t), v_1(t_0) = e_{i,2}(t_0)$. Defining a function

$$\Phi_{j+1}(u) := \int_{0}^{u} \frac{\psi_j(\phi^{-1}(W_1^{-1}(s)))}{\psi_{j+1}(\phi^{-1}(W_1^{-1}(s)))} ds = \int_{0}^{u} \frac{w_j(\phi^{-1}(W_1^{-1}(s)))}{w_{j+1}(\phi^{-1}(W_1^{-1}(s)))} ds, \ j = 1, 2, \tag{36}$$

then

$$\frac{v_1'(t)}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))} = \frac{e_{i,2}'(t) + b_i(T) \sum_{j=1}^{2} \tilde{F}_{i,j+1}(T,t)\psi_{j+1}(\phi^{-1}(W_1^{-1}(\eta_1(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))}$$

$$\leq \frac{e_{i,2}'(t) + b_i(T) \sum_{j=1}^{2} \tilde{F}_{i,j+1}(T,t)\psi_{j+1}(\phi^{-1}(W_1^{-1}(v_1(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(t))))}, \tag{37}$$

taking the integral on both sides of (37) from $t_0$ to $t$, we have

$$\int_{t_0}^{t} \frac{w_1(\phi^{-1}(W_1^{-1}(v_1(s))))v_1'(s)}{w_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds = \int_{t_0}^{t} \frac{v_1'(s)}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds$$

$$\leq \int_{t_0}^{t} \frac{e_{i,2}'(s) + b_i(T) \sum_{j=1}^{2} \tilde{F}_{i,j+1}(T,s)\psi_{j+1}(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds$$

$$\leq \int_{t_0}^{t} \frac{e_{i,2}'(s)}{\psi_2(\phi^{-1}(W_1^{-1}(e_{i,2}(s))))} ds + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s)ds$$

$$+ b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T,s)\psi_3(\phi^{-1}(W_1^{-1}(v_1(s)))) ds$$

$$= \int_{t_0}^{t} \frac{w_1(\phi^{-1}(W_1^{-1}(e_{i,2}(s))))e_{i,2}'(s)}{w_2(\phi^{-1}(W_1^{-1}(e_{i,2}(s))))} ds + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s)ds$$

$$+ b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T,s)\psi_3(\phi^{-1}(W_1^{-1}(v_1(s)))) ds, \tag{38}$$
from (36) and (38), we obtain

\[
\Phi_2(v_1(t)) - \Phi_2(v_1(t_0)) \leq \Phi_2(e_{i,2}(t)) - \Phi_2(e_{i,2}(t_0)) + b_i(t) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s) ds \\
+ b_i(T) \int_{t_0}^{t} \frac{\tilde{F}_{i,3}(T,s) \psi_3(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds,
\]

(39)

from (39) and using the formula \( v_1(t_0) = e_{i,2}(t_0) \), we have

\[
\Phi_2(v_1(t)) \leq \Phi_2(e_{i,2}(t)) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s) ds + b_i(T) \int_{t_0}^{t} \frac{\tilde{F}_{i,3}(T,s) \psi_3(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds.
\]

(40)

From (20) and (36), we can conclude that

\[
\Phi_{j+1}(u) := \int_{0}^{u} \frac{\psi_j(\phi^{-1}(W_j^{-1}(s)))}{\psi_{j+1}(\phi^{-1}(W_j^{-1}(s)))} ds = \int_{0}^{u} \frac{w_j(\phi^{-1}(W_j^{-1}(s)))}{w_{j+1}(\phi^{-1}(W_j^{-1}(s)))} ds \\
= \int_{1}^{W_j^{-1}(u)} \frac{ds}{w_{j+1}(\phi^{-1}(s))} = W_{j+1}(W_j^{-1}(u)), \quad j = 1, 2.
\]

(41)

By (41), (40) can be written as

\[
W_2(W_1^{-1}(v_1(t))) \leq W_2(W_1^{-1}(e_{i,2}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s) ds \\
+ b_i(T) \int_{t_0}^{t} \frac{\tilde{F}_{i,3}(T,s) \psi_3(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds.
\]

(42)

Let

\[
\eta_2(t) = W_2(W_1^{-1}(v_1(t))),
\]

(43)

\[
e_{i,3}(t) = W_2(W_1^{-1}(e_{i,2}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,2}(T,s) ds, i = 1, 2,
\]

(44)

then \( W_1^{-1}(v_1(t)) = W_2^{-1}(\eta_2(t)) \), from (47) and (48), then (46) can be written

\[
\eta_2(t) \leq e_{i,3}(t) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T,s) ds \\
\quad + b_i(T) \int_{t_0}^{t} \frac{\tilde{F}_{i,3}(T,s) \psi_3(\phi^{-1}(W_1^{-1}(v_1(s))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_1(s))))} ds.
\]

Let

\[
v_2(t) = e_{i,3}(t) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T,s) \psi_3(\phi^{-1}(W_1^{-1}(v_2(s)))) ds, i = 1, 2,
\]

then \( v_2(t) \) is a nondecreasing function on \([t_0, t)\), and \( \eta_2(t) \leq v_2(t), v_2(t_0) = e_{i,3}(t_0) \). Taking the derivative of \( v_2(t) \) with respect to \( t \), we have

\[
v'_2(t) = e'_{i,3}(t) + b_i(T) \frac{\tilde{F}_{i,3}(T,t) \psi_3(\phi^{-1}(W_1^{-1}(\eta_2(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(\eta_2(t))))} \\
\quad \leq e'_{i,3}(t) + b_i(T) \frac{\tilde{F}_{i,3}(T,t) \psi_3(\phi^{-1}(W_1^{-1}(v_2(t))))}{\psi_2(\phi^{-1}(W_1^{-1}(v_2(t))))},
\]

(45)

from (45), we have

\[
\frac{\psi_2(\phi^{-1}(W_2^{-1}(v_2(t))))}{\psi_3(\phi^{-1}(W_1^{-1}(v_2(t))))} v'_2(t) \leq \frac{\psi_2(\phi^{-1}(W_2^{-1}(v_2(t))))}{\psi_3(\phi^{-1}(W_1^{-1}(v_2(t))))} e'_{i,3}(t) + b_i(T) \tilde{F}_{i,3}(T,t),
\]

(46)
taking the integral on both sides of (46) from $t_0$ to $t$, by using $(H_2)$ and (31), we have

$$
\Phi_3(v_2(t)) - \Phi_3(v_2(t_0)) \leq \int_{t_0}^{t} \frac{\psi_2(\phi^{-1}(W_2^{-1}(v_2(s))))}{\psi_3(\phi^{-1}(W_1^{-1}(v_2(s))))} e_{i,3}(s) ds + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds
$$

$$
\leq \int_{t_0}^{t} \frac{\psi_2(\phi^{-1}(W_2^{-1}(e_{i,3}(s))))}{\psi_3(\phi^{-1}(W_1^{-1}(e_{i,3}(s))))} e_{i,3}(s) ds + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds
$$

$$
\Phi_3(e_{i,3}(t)) - \Phi_3(e_{i,3}(t_0)) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds,
$$

that is

$$
\Phi_3(v_2(t)) \leq \Phi_3(e_{i,3}(t)) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds, \quad (47)
$$

from (41) and (47), we have

$$
v_2(t) \leq W_2 \left( W_3^{-1} \left( W_3(W_2^{-1}(e_{i,3}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds \right) \right), \quad (48)
$$

from (34), (43), (48), and using $\eta_1(t) \leq v_1(t), \eta_2(t) \leq v_2(t)$, we obtain

$$
v(t) = W_3^{-1}(\eta_1(t)) \leq W_3^{-1}(v_1(t)) = W_2^{-1}(\eta_2(t)) \leq W_2^{-1}(v_2(t))
$$

$$
\leq W_3^{-1} \left( W_3(W_2^{-1}(e_{i,3}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds \right), \quad i = 1, 2,
$$

then, we have

$$
u(t) \leq \phi^{-1}(v(t))
$$

$$
\leq \phi^{-1} \left( W_3^{-1} \left( W_3(W_2^{-1}(e_{i,3}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds \right) \right)
$$

$$
\leq \phi^{-1}(W_3^{-1}(e_{i,4}(t))), \quad t \in [t_0, T], \quad i = 1, 2,
$$

where $e_{i,4}(t) = W_3^{-1} \left( W_3(W_2^{-1}(e_{i,3}(t))) + b_i(T) \int_{t_0}^{t} \tilde{F}_{i,3}(T, s) ds \right)$, by the arbitrarily of $T$, we have

$$
u(t) \leq \phi^{-1}(W_3^{-1}(e_{i,4}(t))), \quad t \in [t_0, \infty], \quad i = 1, 2.
$$

This completes the proof.

3. Applications

[1] In this section, we apply our result to study the boundedness of the solution of integral equations. Consider the following Volterra type retarded weakly singular integral equations:

$$
x(t) - \int_{t_0}^{t} \int_{tn}^{s} (t^\beta - s^\beta)^{\gamma-1} s^{\xi-1} f(s) \left[ x(s) + \int_{t_0}^{s} (s^\beta - \tau^\beta)^{\gamma-1} \tau^{\xi-1} g(\tau) d\tau \right] ds = h(t), \quad (49)
$$

which arises very often in various problems, especial describing physical processes with aftereffects.

**Example 1** Let $x(t), f(t), g(t)$ and $h(t)$ be continuous functions on $[0, +\infty)$. Let $p, \beta, \gamma, \xi$ be positive constants. Suppose that $x(t)$ satisfies equation (49), then we have the estimate for $x(t)$.

(i) If $\beta \in (0, 1], \gamma \in (\frac{1}{2}, 1)$ and $\xi \geq \frac{1}{2} - \gamma$, we have

$$
|x(t)| \leq \hat{B}_1(t) + \hat{b}_1(t) \int_{t_0}^{t} \int_{tn}^{s} \left( \frac{p}{1 - \gamma} f^{\frac{1}{1-\gamma}}(s) + \frac{p}{(1 - \gamma)^2} f^{\frac{1}{1-2\gamma}}(s) \right) \tilde{b}_1(s) ds, \quad (50)
$$
Applying Theorem 1 (with Proof) we have

\[ u(t) \leq \tilde{B}_2(t) + \tilde{\tilde{b}}_2(t) \int_{t_0}^{t} \left( \frac{p(1 + 4\gamma)}{\gamma} \right)^{\frac{1 + 4\gamma}{\gamma}} s \left( 1 - \frac{p(1 + 4\gamma)}{\gamma} \right) + \frac{p(1 + 4\gamma)}{\gamma} \frac{1 + 4\gamma}{\gamma} ds, \]  

where

\[ \tilde{a}_2(t) = |h(t)| + \frac{1 + 3\gamma}{\gamma} \tilde{b}_2(t) \left( \frac{p(1 + 4\gamma)}{\gamma} \right)^{\frac{1 + 4\gamma}{\gamma}} s \left( 1 - \frac{p(1 + 4\gamma)}{\gamma} \right) + \frac{p(1 + 4\gamma)}{\gamma} \frac{1 + 4\gamma}{\gamma} ds + 1, \]

\[ \tilde{b}_2(t) = \frac{1}{\gamma + 4\gamma} M_2 \tilde{a}_2(t), \]

\[ \tilde{A}_2(t) = \tilde{b}_2(t) \int_{t_0}^{t} \left( 1 - \frac{p(1 + 4\gamma)}{\gamma} \right) + \frac{p(1 + 4\gamma)}{\gamma} \tilde{\tilde{a}}_2(t) ds + \tilde{\tilde{b}}_2(t) \int_{t_0}^{t} \left( 1 - \frac{p(1 + 4\gamma)}{\gamma} \right) + \frac{p(1 + 4\gamma)}{\gamma} \tilde{\tilde{a}}_2(t) ds, \]

\[ \tilde{\tilde{B}}_2(t) = \tilde{\tilde{a}}_2(t) + \tilde{\tilde{A}}_2(t), \]

\[ \tilde{\tilde{e}}_2(t) = \exp \left( - \int_{t_0}^{t} \tilde{b}_2(s) \left( 1 - \frac{p(1 + 4\gamma)}{\gamma} \right) + \frac{p(1 + 4\gamma)}{\gamma} \tilde{\tilde{a}}_2(t) ds \right). \]

Proof. From (49), we have

\[ |x(t)| \leq |h(t)| + \int_{t_0}^{t} (t^\beta - s^\beta)^{\gamma - 1} s^{\gamma - 1} |f(s)| \left[ |x(s)| + \int_{t_0}^{s} (s^\beta - \tau^\beta)^{\gamma - 1} \tau^{\gamma - 1} g(\tau) |x(\tau)| d\tau \right] \]  

Applying Theorem 1 (with \( m = n = 1, a(t) = |h(t)|, u(t) = |x(t)| \)) to (52), we get the desired estimations (50) and (51).

[2] Consider the following differential system:

\[ \frac{dx(t)}{dt} = F(t, x, t), \quad t \in [0, \infty), \]

\[ x(t_0) = x_0, \]

where \( x_0 \) is a constant, \( F(t, x, t) \) is continuous with respect to \( t \) and \( x \) on \([0, \infty) \times [0, \infty) \times (-\infty, +\infty)\).

Example 2 Suppose \( F(t, x, t) \) satisfies

\[ |F(t, s, x)| \leq (t^\beta - s^\beta)^{\gamma - 1} s^{\gamma - 1} (t^{\gamma - 1} |x| + c|x| + d|x|ln|x|), \]  

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and $c, d$ are constants, $\beta \in (0, 1], \gamma \in (\frac{1}{2}, 1), \xi \geq \frac{3}{2} - \gamma$, then we have the estimate for the solution $x(t)$ of (53).

$$|x(t)| \leq \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t)t^{1/\gamma} \right) \right)^{\frac{1}{2}} + c e^{t^{1/\gamma} b(t)t^{1/\gamma}},$$

where

$$b(t) = (M_1 t^\beta)^\gamma, \ M_1 = \frac{1}{\beta} B \left[ \gamma + \xi - 1, \frac{2\gamma - 1}{\gamma} \right], \ \theta_1 = \frac{1}{\gamma} \left[ \beta(\gamma - 1) + \xi - 1 \right] + 1,$$

$$\tilde{F}_1(t,s) = t^{1/\gamma}, \ \tilde{F}_2(t,s) = e^{1/\gamma}, \ \tilde{F}_3(t,s) = d^{1/\gamma},$$

$$A_1(t) = x_0 + 3\gamma b(t) + (1 - \gamma)b(t)t^{1/\gamma} - \gamma e^{1/\gamma} t \cdot b(t) - \gamma d^{1/\gamma} t \cdot b(t),$$

$$w_1(u) = \sqrt{u}, \ w_2(u) = |u|, \ w_3(u) = |u|\ln|u|,$$

$$W_1(u) = \int_0^u \frac{ds}{\sqrt{s}} = \frac{3}{2} \left( u^2 - u_0^2 \right), \ W_1^{-1}(u) = \left( \frac{2}{3} u + u_0^2 \right)^{1/2}, \ u_0 > 0,$$

$$W_2(u) = \ln \frac{u}{u_1}, \ W_2^{-1}(u) = u_1 e^u, \ u_1 > 0,$$

$$W_3(u) = \ln(\ln u), \ W_3^{-1}(u) = e^u,$$

$$\tilde{e}_1(t) = x_0, \ \tilde{e}_2(t) = \frac{3}{2} \left( x_0^2 - u_0^2 \right) + \frac{\gamma - 1}{\gamma} b(t)t^{1/\gamma},$$

$$\tilde{e}_3(t) = \ln \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t)t^{1/\gamma} \right) \right)^{1/2} + c e^{t^{1/\gamma} b(t)t^{1/\gamma}},$$

$$\tilde{e}_4(t) = \ln \left( \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t)t^{1/\gamma} \right) \right)^{1/2} + c e^{t^{1/\gamma} b(t)t^{1/\gamma}} \right) + d^{1/\gamma} b(t)t^{1/\gamma}.$$

**Proof.** Differential system (53) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t F(s, t, x)ds. \quad (55)$$

Using the condition (54), from (55), we have

$$|x(t)| \leq x_0 + \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} (t^{1/\gamma}\sqrt{|x(s)|})ds + c \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} |x(s)|ds$$

$$+ d \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} |x(s)|\ln|x(s)|ds,$$

let $u(t) = |x(t)|$, we get

$$u(t) \leq x_0 + \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} (t^{1/\gamma} u(s))ds + c \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} u(s)ds$$

$$+ d \int_0^t (t^\beta - s^\beta)^{\gamma - 1} s^{\xi - 1} u(s)\ln u(s)ds. \quad (56)$$

(56) is a special form of (2), where $f_1(t,s) = t$, $f_2(t,s) = c$, and the functions of (56) satisfy the conditions
of Theorem 2, using the result of Theorem 2, we obtain
\[
\tilde{e}_1(t) = x_0, \\
\tilde{e}_2(t) = W_1(\tilde{e}_1(t)) + b(t) \int_0^t \frac{s}{t + r} ds = \frac{3}{2} \left( x_0^2 - u_0^2 \right) + \frac{\gamma - 1}{\gamma} b(t) t^{\frac{\gamma}{\gamma - 1}}, \\
\tilde{e}_3(t) = W_2(W_1^{-1}(\tilde{e}_2(t))) + b(t) \int_0^t \frac{s}{t + r} ds \\
= W_2 \left( \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t) t^{\frac{\gamma}{\gamma - 1}} \right) \right) \right) + c \frac{1}{r} b(t) t, \\
\tilde{e}_4(t) = W_3(W_2^{-1}(\tilde{e}_3(t))) + b(t) \int_0^t \frac{s}{t + r} ds \\
= W_3 \left( \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t) t^{\frac{\gamma}{\gamma - 1}} \right) \right) \right) + c \frac{1}{r} b(t) t, \\
then
\[
\begin{align*}
\quad u(t) & \leq W_3^{-1}(\tilde{e}_4(t)) \\
& = W_3^{-1} \left( \ln \left( \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t) t^{\frac{\gamma}{\gamma - 1}} \right) \right) \right) + c \frac{1}{r} b(t) t \right) \\
& = \left( \left( x_0^2 + \frac{2}{3} \left( \frac{\gamma - 1}{\gamma} b(t) t^{\frac{\gamma}{\gamma - 1}} \right) \right) \right) e^{c \frac{1}{r} b(t) t}. \\
\end{align*}
\]
This implies that \( u(t) \) is bounded for \( t \in [0, \infty) \). The proof is completed.

3. Conclusion

In this paper, by using analytical methods and known results in the lemmas, we study some new weakly singular nonlinear integral inequalities. By the Examples, we see that the inequalities given here can be used to analyze the boundedness of fractional differential equations and integral equations.

In addition, the following inequalities are the general form of inequalities (1) and (2),
\[
\begin{align*}
\quad u(t) & \leq a(t) + \int_{t_0}^t \left( b^2 - s b_1 \right) g(s) ds, \\
\quad \phi(u(t)) & \leq a(t) + \int_{t_0}^t \left( b^2 - s b_1 \right) g(s) ds + \int_{t_0}^t f(s) w_1(u(s)) ds \\
& + \int_{t_0}^t f(s) w_2(u(s)) ds \\
& + \int_{t_0}^t \left( b^2 - s b_1 \right) g(s) w_3(u(s)) ds, \\
\end{align*}
\]
then, by using a similar procedure for \( \beta_i \in (0, 1], \gamma_i \in (0, \frac{1}{2}], \xi_i > \frac{1 - 2\xi_i^2}{1 - \gamma_i}, i = 1, 2, 3, 4, \) we can get corresponding results for the estimations on \( u(t) \).

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.
Authors contributions
YL carried out the main results and completed the corresponding proof. RX participated in Section 3 - Applications, and also check and revise the whole text. All authors read and approved the final manuscript.

Data availability statement
We declare that the data in the paper can be used publicly.

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