THE REDUCED DIVISOR CLASS GROUP AND THE TORSION NUMBER

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Abstract. The reduced divisor class group of a normal Cohen–Macaulay graded domain together with its torsion number is introduced. They are studied in detail especially for normal affine semigroup rings.

Introduction

Let P be a finite partially ordered set and R the normal affine semigroup ring introduced in [8]. Nowadays authors call R the *Hibi ring*, but in the present paper we call R the *join-meet ring* arising from P, because its relations are given by the joins and meets of the distributive lattice defined by P. It is shown [5] that the divisor class group $\mathrm{Cl}(R)$ of R is free of rank $p+q+e-d-1$, where p is the number of minimal elements of P, q is the number of maximal elements of P, e is the number of edges of the Hasse diagram of P and $d = |P|$. On the other hand, in [8], by studying the generators of the canonical module ω_R of R, it is proved that R is Gorenstein if and only if R is pure, i.e., every maximal chain of P has the same cardinality. In general, it is known that R is Gorenstein if and only if the canonical class $[\omega_R]$ of R is equal to 0 in $\mathrm{Cl}(R)$. In other words, $[\omega_R] = 0$ in $\mathrm{Cl}(R)$ if and only if P is pure. It is reasonable to ask how to compute $[\omega_R]$ in terms of combinatorics of P. This natural question is what motivated the authors to write this paper in the first place. Its satisfied solution will be given in Section 2.

Let R be a Noetherian local ring or a finitely generated graded K-algebra for which R is a normal Cohen–Macaulay domain with a canonical module ω_R. In the first half of Section 1, the new concepts, the *reduced divisor class group* of R and the *torsion number* of R, are introduced. The reduced divisor class group of R is $\overline{\mathrm{Cl}}(R) = \mathrm{Cl}(R)/\mathbb{Z}[\omega_R]$ and the torsion number of R is the nonnegative integer $d(R)$ defined as follows: let $\mathrm{Fitt}_i(G)$ denote the ith Fitting ideal of a finite Abelian group, and let $r = \mathrm{rank}\,\overline{\mathrm{Cl}}(R)$. If $\mathrm{Fitt}_r(\mathrm{Cl}(R)) = \mathrm{Fitt}_r(\overline{\mathrm{Cl}}(R))$, then we set $d(R) = 0$. Otherwise, $d(R)$ is given by the identity $\mathrm{Fitt}_r(\overline{\mathrm{Cl}}(R)) = (d(R))$. One has $d(R) = 0$ if and only if R is Gorenstein (Lemma 1.1). When $\mathrm{Cl}(R)$ is free of rank r, the torsion number $d(R)$ has a concrete interpretation. In fact, one has $\overline{\mathrm{Cl}}(R) \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d(R))$ and $[\omega_R]$ is part of a basis of $\mathrm{Cl}(R)$ if and only if $d(R) = 1$ (Lemma 1.2). When $S \subset \mathbb{Z}^n$ is a normal affine semigroup, the divisor class group of the associated normal

2010 *Mathematics Subject Classification.* Primary 13H10; Secondary 13D02, 05E40.

Jürgen Herzog, the first author, passed away on 23 April 2024 at the age of 82. His passing marked the end to our unbroken research collaboration which began on 13 February 1995. The present paper is the last published in our 29 years of collaboration.

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semigroup ring $R = K[S]$ is well understood. In the latter half of Section 1, the basic facts related to the divisor class group $\mathrm{Cl}(K[S])$ of $R = K[S]$, especially the result by Chouinard [2] on a set of generating relations of $\mathrm{Cl}(R)$ are summarized in short.

Section 2 will be devoted to the study of the divisor class groups of the join-meet ring of a finite partially ordered set. As was discussed in [5], the information of the facets of the cone coming from P (Stanley [10]) yields the relation matrix of $\mathrm{Cl}(R)$ and it gives the explicit expression of $[\omega_R]$ in terms of the basis of $\mathrm{Cl}(R)$, which is the satisfied solution of the original question as well as which directly explains why $[\omega_R] = 0$ in $\mathrm{Cl}(R)$ if and only if P is pure (Theorem 2.2).

On the other hand, the detailed study of torsion numbers is achieved in Section 3. In the join-meet ring R, the torsion number can be an arbitrary nonnegative integer (Example 3.1). Furthermore, if a join-meet ring R is nearly Gorenstein but not Gorenstein, then one has $d(R) = 1$ (Corollary 3.3). However, in general, even though a normal affine semigroup ring is nearly Gorenstein but not Gorenstein, it happens that $d(R) > 1$ (Example 3.5).

1. The canonical class and the torsion number

Let R be a Noetherian local ring or a finitely generated graded K-algebra. We furthermore assume that R is a normal Cohen-Macaulay domain with a canonical module ω_R. The canonical module can be identified with a divisorial ideal. Let $\mathrm{Cl}(R)$ be the divisor class group of R. The class of a divisorial ideal I of R will be denoted by $[I]$. We choose of system a of generators $g_1, \dots, g_m$ of $\mathrm{Cl}(R)$. Then $[\omega]$ can be written as a linear combination of these generators, say, $[\omega_R] = \sum_{i=1} a_i g_i$. The integer coefficients of this presentation depend of course on the choice of the generators. Of special interest is the case that $[\omega_R] = 0$, because this is the case if and only if R is Gorenstein. However the above linear combination does not tell us immediately, whether of not $[\omega_R] = 0$. Thus we are looking for a more intrinsic invariant of the canonical class. To this end, we consider the group $\overline{\mathrm{Cl}}(R) = \mathrm{Cl}(R)/\mathbb{Z}[\omega_R]$, and a certain Fitting ideal of it. We call $\overline{\mathrm{Cl}}(R)$ the *reduced divisor class group* of R.

Let us briefly recall the concept of Fitting ideals and their basic properties. Let M be a finitely generated module over a commutative ring R with generators $u_1, \dots, u_n$ and with a relation matrix $A = [a_{ij}]_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$. In other words, $\sum_{i=1,\dots,n} a_{ij} m_i = 0$ for all j, and these are the generating relations of M with respect to these generators. Given these data, the ith Fitting ideal $\mathrm{Fitt}_i(M)$ of M is the ideal $I_{n-i}(A)$ of $(n-i)$-minors of A. The Fitting ideals are invariants of the module, that is, they do not depend on the choice of the system of generators and the relation matrix. One has $\mathrm{Fitt}_0(M) \subseteq \mathrm{Fitt}_1(M) \subseteq \cdots \subseteq \mathrm{Fitt}_n(M) = R$. If R is a domain, then $\operatorname{rank} M = \min\{i\colon \mathrm{Fitt}_i(M) \neq 0\}$. Moreover, M is free of rank r if and only if $\mathrm{Fitt}_i(M) = 0$ for $i < r$ and $\mathrm{Fitt}_r(M) = R$.

We may view any finitely generated Abelian group G as a $\mathbb{Z}$-module, and hence the Fitting ideals of G are defined. Suppose G has n generators and the relation matrix A has rank m. Then there exists an exact sequence $0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to G \to 0$,

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which implies that $\operatorname{rank} G = n - m$. Thus, if $r = \operatorname{rank} G$, then r is the smallest integer for which $\operatorname{Fitt}_r(G) \neq 0$.

Now we are ready to define the *torsion number* $d(R)$ of R. Let $r = \operatorname{rank} \overline{\operatorname{Cl}}(R)$. If $\operatorname{Fitt}_r(\operatorname{Cl}(R)) = \operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R))$, then we set $d(R) = 0$. Otherwise, $d(R)$ is given by the identity

$$\operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) = (d(R)).$$

We have

Lemma 1.1. *R is Gorenstein if and only if $d(R) = 0$.*

Proof. Suppose that R is Gorenstein. Then $\overline{\operatorname{Cl}}(R) = \operatorname{Cl}(R)$, and so $\operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) = \operatorname{Fitt}_r(\operatorname{Cl}(R))$.

Conversely, suppose that $\operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) = \operatorname{Fitt}_r(\operatorname{Cl}(R))$. Let $s = \operatorname{rank} \operatorname{Cl}(R)$. Then $s \geq r \geq s-1$. Suppose $r = s-1$. Then $\operatorname{Fitt}_{s-1}(\operatorname{Cl}(R)) = \operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) \neq 0$, a contradiction. Hence $\operatorname{rank} \operatorname{Cl}(R) = \operatorname{rank} \overline{\operatorname{Cl}}(R)$, and $\operatorname{Cl}(R) \cong \mathbb{Z}^r \oplus H$,where H is a finite group. Since $\operatorname{rank} \operatorname{Cl}(R) = \operatorname{rank} \overline{\operatorname{Cl}}(R)$, it follows that $[\omega_R] \in H$. Therefore, $\overline{\operatorname{Cl}}(R) \cong \mathbb{Z}^r \oplus \overline{H}$, where $\overline{H} = H/\mathbb{Z}[\omega_R]$. It follows that $(|H|) = \operatorname{Fitt}_r(\operatorname{Cl}(R)) = \operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) = (|\overline{H}|)$. Therefore, $H = \overline{H}$. This implies that $[\omega_R] = 0$. □

When the divisor class group is free, then $d(R)$ has a concrete interpretation.

Lemma 1.2. *Suppose* $\operatorname{Cl}(R)$ *is free of rank r. Then* $\operatorname{Cl}(R) \cong \mathbb{Z}^r$. *Under this isomorphism, let $[\omega_R] = (a_1, \ldots, a_r)$ with $a_i \in \mathbb{Z}$. Then $d(R) = \gcd(a_1, \ldots, a_r)$. In particular, $\overline{\operatorname{Cl}}(R) \cong \mathbb{Z}^{r-1} \oplus \mathbb{Z}/(d(R))$ and $[\omega_R]$ is part of a basis of* $\operatorname{Cl}(R)$ *if and only if $d(R) = 1$,*

Proof. With respect to the basis of $\operatorname{Cl}(R)$ corresponding to the isomorphism $\operatorname{Cl}(R) \cong \mathbb{Z}^r$, the relation matrix of $\overline{\operatorname{Cl}}(R)$ is given by $[a_1, \ldots, a_r]$. We have $[\omega_R] = 0$, if and only if all $a_i = 0$, and this is the case if and only if $\operatorname{rank} \overline{\operatorname{Cl}}(R) = r$. In this case, $\operatorname{Fitt}_r(\overline{\operatorname{Cl}}(R)) = \operatorname{Fitt}_r(\operatorname{Cl}(R))(= \mathbb{Z})$, and hence $d(R) = 0$ according to our definition. On the other hand, if $a_i \neq 0$ for some i, then $\operatorname{rank} \overline{\operatorname{Cl}}(R) = r-1$ and $\operatorname{Fitt}_{r-1}(\overline{\operatorname{Cl}}(R)) = (\gcd(a_1, \ldots, a_r))$. This yields the statements of the lemma. □

Let K be a field. For a normal affine semigroup $S \subset \mathbb{Z}^n$ the divisor class group of the associated semigroup ring $R = K[S]$ is well understood. We use the notation introduced in [1] and denote by $\mathbb{Z}S$ the smallest subgroup of $\mathbb{Z}^n$ containing S and by $\mathbb{R}_+S \subset \mathbb{R}^n$ the smallest cone containing S. Since R is normal, Gordon's lemma [1, Proposition 6.1.2] guarantees that $S = \mathbb{Z}^n \cap \mathbb{R}_+S$. After a suitable change of coordinates, one may always assume that $\mathbb{Z}S = \mathbb{Z}^n$. Notice that $\mathbb{R}_+S \subset \mathbb{R}^n$ is a positive rational cone. Given any such cone C, one has that $\mathbb{Z}^n \cap C$ is a normal affine semigroup. Let $H_1, \ldots, H_r$ be the supporting hyperplanes of C. Since for each i, the hyperplane H_i is spanned by lattice points, a linear form $f_i = \sum_{i=1}^n a_{ij} x_j$ defining H_i has rational coefficients. By clearing denominators we may assume that all a_{ij} are integers, and then dividing f_i by the greatest common divisor of the a_{ij}, we may furthermore assume that $\gcd(a_{i1}, \ldots, a_{in}) = 1$. Up to sign, this linear form f_i is uniquely determined by H_i. Let p be a lattice point in the relative interior of

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C. By replacing f_i by $-f_i$, if necessary, we may assume that $f_i(p) > 0$ for all i. We call this normalized uniquely determined linear form f_i the *support form* of H_i.

We recall the following facts:

(i) Let $P_i \subset R$ be the K subvector space of $K[S]$ spanned by all monomials $\mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} \in C \setminus H_i$. Then P_i is a monomial prime ideal of height 1, and we have $\{P_1, \ldots, P_r\}$ is the set of all monomial prime ideals of height 1 in R.

(ii) $\mathrm{Cl}(R)$ is generated by the classes $[P_1], \ldots, [P_r]$.

(iii) (Chouinard [2]) $\sum_{i=1}^{r} a_{ij}[P_i] = 0$ for $j = 1, \ldots, n$, and this is a set of generating relations of $\mathrm{Cl}(R)$. In other words, the $r \times n$-matrix $A_R = [a_{ij}]_{\substack{i=1,\ldots,r \\ j=1,\ldots,n}}$ is a relation matrix of $\mathrm{Cl}(R)$. and we have an exact sequence of abelian groups

$$0 \longrightarrow \mathbb{Z}^n \xrightarrow{A_R} \mathbb{Z}^r \longrightarrow \mathrm{Cl}(R) \longrightarrow 0.$$

(iv) $\mathrm{Cl}(R)$ is free of rank s if and only if $\mathrm{Fitt}_i(\mathrm{Cl}(R)) = 0$ for $i < s$ and $\mathrm{Fitt}_s(\mathrm{Cl}(R)) = \mathbb{Z}$, equivalently, if $I_{n-s}(A_R) = \mathbb{Z}$ and $\mathrm{rank}\, A_R = n - s$.

By a theorem of Danilov and Stanley (see [1, Theorem 6.3.5]), ω_R is generated by the monomials $\mathbf{x}^{\mathbf{a}}$ for which $\mathbf{a}$ belongs to the relative interior of C. This implies that $\omega_R = \bigcap_{i=1}^{r} P_i$, and hence $[\omega_R] = \sum_{i=1}^{r}[P_i]$. Consequently, $\overline{\mathrm{Cl}}(R)$ has the relation matrix $\overline{A}_R$, where $\overline{A}_R$ is obtained from A_R by adding a column whose entries are all one.

If $\mathrm{Cl}(R)$ is free of rank r, then $\mathrm{rank}\, \overline{\mathrm{Cl}}(R) = r-1$, and hence $d(R)$ is the generator of the principal ideal $\mathrm{Fitt}_{r-1}(\overline{\mathrm{Cl}}(R)) = I_{n-r+1}(\overline{A}_R)$.

2. The Divisor class group of a join-meet ring

The present section will be devoted to the discussion of the divisor class group of the normal semigroup ring, introduced in [8], arising from a finite partially ordered set. Let $P = \{x_1, \ldots, x_n\}$ be a finite partially ordered set and suppose that i is smaller than j whenever $x_i < x_j$ in P. Let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$, where $\hat{0} < x_i < \hat{1}$ for $1 \leq i \leq n$. Let $E(\hat{P})$ denote the set of edges of the Hasse diagram of $\hat{P}$. Thus $(x, y) \in \hat{P} \times \hat{P}$ belongs to $E(\hat{P})$ if $x < y$ in $\hat{P}$ and $x < z < y$ for no $z \in \hat{P}$. Following [10, p. 10], one associates each $e \in E(\hat{P})$ with the linear form f_e by setting

$$f_e = \begin{cases} x_i & \text{if} \quad e = (x_i, \hat{1}); \\ x_i - x_j & \text{if} \quad e = (x_i, x_j) \in P \times P; \\ x_0 - x_j & \text{if} \quad e = (\hat{0}, x_j). \end{cases}$$

Let $C \subset \mathbb{R}_+^{n+1}$ denote the cone whose supporting hyperplanes are those H_e defined by f_e with $e \in E(\hat{P})$. Let K be a field and $R = K[C \cap \mathbb{Z}^{n+1}]$ the affine semigroup ring, called the *join-meet ring* arising from P. It is known [8] that the the join-meet ring $R = K[C \cap \mathbb{Z}^{n+1}]$ is normal. In particular, $R = K[C \cap \mathbb{Z}^{n+1}]$ is Cohen–Macaulay. The divisor class group $\mathrm{Cl}(R)$ of $R = K[C \cap \mathbb{Z}^{n+1}]$ is generated by the classes $[P_e]$ with $e \in E(\hat{P})$, where P_e is the monomial prime ideal of height 1 arising from H_e. It is shown [5] that $\mathrm{Cl}(R)$ is free of rank $|E(\hat{P})| - (n + 1)$.

Following [5] one fixes a spanning tree $T = \{e_0, \ldots, e_n\}$ of $E(\hat{P})$, where $e_i = (x_i, x_{i'})$ with $x_0 = \hat{0}$. Let $E(\hat{P}) = \{e_0, \ldots, e_n, e_{n+1}, \ldots, e_r\}$. Let $A_R = [a_{ij}]_{\substack{i=0,\ldots,r \\ j=0,\ldots,n}}$

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denote the relation matrix of $\mathrm{Cl}(R)$, where a_{ij} is the coefficient of x_j in f_{e_i}. The choice of the tree T says that the submatrix of A_R consisting of the first $n+1$ rows is an upper triangle matrix with each diagonal entry 1. It then follows that $[P_{n+1}], \ldots, [P_r]$ is a basis of the free abelian group $\mathrm{Cl}(R)$, where $P_i = P_{e_i}$. In the divisor class group $\mathrm{Cl}(R)$, for each $0 \leq i \leq n$ one writes

$$[P_i] = \sum_{j=n+1}^{r} c_j^{(i)}[P_j], \quad c_j^{(i)} \in \mathbb{Q}. \tag{1}$$

Each $c_j^{(i)} \in \mathbb{Q}$ can be computed as follow: For each edge $e_j = (x, y)$ with $n+1 \leq j \leq r$, the subgraph G_j consisting of the edges $e_0, \ldots, e_n, e_j$ possesses a unique cycle C_j. One fixes the orientation of C_j with $x \to y$. If $e_i = (x_i, x_{i'})$ with $0 \leq i \leq n$ appears in C_j whose orientation is $x_i \to x_{i'}$, then one has $c_j^{(i)} = 1$. If $e_i = (x_i, x_{i'})$ with $0 \leq i \leq n$ appears in C_j whose orientation is $x_{i'} \to x_i$, then one has $c_j^{(i)} = -1$. If e_i with $0 \leq i \leq n$ does not appear in C_j, then one has $c_j^{(i)} = 0$.

One claims the validity of the above computation of $c_j^{(i)}$. In other words, $[P_0], \ldots, [P_n]$ with the expression (1) together with $[P_{n+1}], \ldots, [P_r]$ satisfy the relations of the columns of A_R. Let $x_i \in P \cup \{\hat{0}\}$ with $\hat{0} = x_0$. Let $\mathcal{A}$ denote the set of edges of $\hat{P}$ of the form $(x_i, x_{i'})$ and $\mathcal{B}$ that of the form $(x_{i''}, x_i)$. If the cycle C_j, where $n+1 \leq j \leq r$ intersects $\mathcal{A} \cup \mathcal{B}$, then one of the followings occurs:

(i) $|C_j \cap \mathcal{A}| = |C_j \cap \mathcal{B}| = 1$;
(ii) $|C_j \cap \mathcal{A}| = 2$ and $C_j \cap \mathcal{B} = \emptyset$;
(iii) $C_j \cap \mathcal{A} = \emptyset$ and $|C_j \cap \mathcal{B}| = 2$.

In each of the above (i), (ii) and (iii), the total sum of $[P_j]$ appearing in $[P_e]$'s with $e \in \mathcal{A}$ is equal to that of $[P_j]$ appearing in $[P_e]$'s with $e \in \mathcal{B}$. Hence $[P_0], \ldots, [P_n]$ with the expression (1) together with $[P_{n+1}], \ldots, [P_r]$ satisfy the relations of the ith column of A_R, as desired.

Example 2.1. Let $P = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the finite partially ordered set of Figure 1. The tree $T = \{e_0, \ldots, e_6\}$ of Figure 2 satisfies the above condition. The cycle C_7 consists of the edges $e_7, e_1, e_4, e_6, e_5, e_2, e_0$ (Figure 3). Fix the orientation of C_7 with

$$x_0 \to x_1 \to x_4 \to \hat{1} \to x_6 \to x_5 \to x_2 \to x_0.$$

Thus the coefficient of $[P_7]$ in each of $[P_1], [P_4]$ is 1, the coefficient of $[P_7]$ in each of $[P_0], [P_2], [P_5], [P_6]$ is -1 and the coefficient of $[P_7]$ in $[P_3]$ is 0. One has

$$[P_0] = -[P_7] - [P_8], \quad [P_1] = [P_7], \quad [P_2] = -[P_7] - [P_8] - [P_9],$$
$$[P_3] = [P_8] - [P_{10}], \quad [P_4] = [P_7] + [P_9] + [P_{10}],$$
$$[P_5] = -[P_7] - [P_9] - [P_{10}], \quad [P_6] = -[P_7] - [P_9] - [P_{10}].$$

Thus in particular

$$[\omega_R] = -[P_7] - [P_9] - [P_{10}].$$

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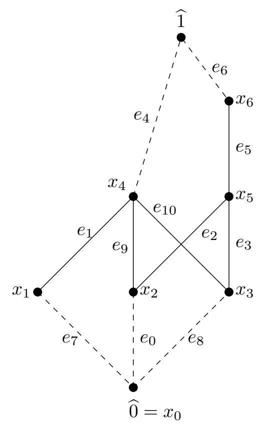

FIGURE 1. poset P

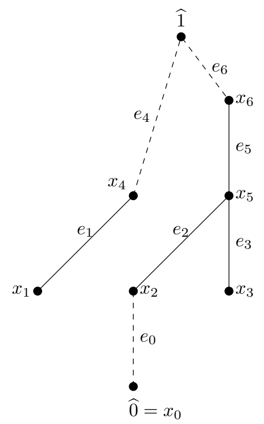

FIGURE 2. tree T

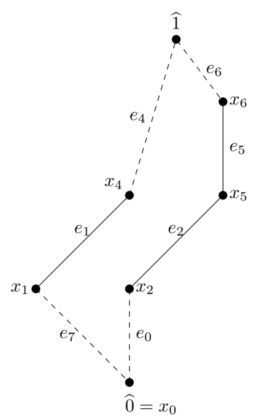

FIGURE 3. cycle C_7

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Now, it is of interest to know when $[\omega_R] = \sum_{e \in E(\hat{P})}[P_e] = 0$ in $\mathrm{Cl}(R)$, because this is the case if and only if R is Gorenstein. The explicit computation done in Example 2.1 easily enables us to prove the following

Theorem 2.2. *In* $\mathrm{Cl}(R)$, *one has* $[\omega_R] = 0$ *if and only if* P *is pure.*

Proof. One employes the notation as above.

("if") Suppose that P is pure. Then, clearly, in each cycle C_j, the number of $e_i = (x_i, x_{i'})$ with $0 \leq i \leq n$ appearing in C_j whose orientation is $x_i \to x_{i'}$ is exactly one less than that of $e_i = (x_i, x_{i'})$ with $0 \leq i \leq n$ appearing in C_j whose orientation is $x_{i'} \to x_i$. Hence each coefficient q_j of $[\omega_R] = \sum_{j=n+1}^{r} q_i[P_j]$ is equal to 0.

("only if") Suppose that P is *not* pure and

$$C : x < x_{i_1} < \cdots < x_{i_s} < y, \qquad C' : x < x_{i'_1} < \cdots < x_{i'_{s'}} < y$$

are maximal chains of the interval $[x, y]$ of $\hat{P}$ with $s < s'$ for which $i_j \neq i'_{j'}$ for each j and j'. One can choose a tree T which contains all edges except for $(x, x_{i'_1})$ appearing in the chains C and C'. Let $e_j = (x, x_{i'_1})$. It then follows that the coefficient q_j of $[\omega_R] = \sum_{j=n+1}^{r} q_j[P_j]$ is equal to $s' - s \neq 0$. Hence $[\omega_R] \neq 0$, as desired. □

Theorem 2.2 gives an alternative proof to the old results that the join-meet ring $R = K[C \cap \mathbb{Z}^{n+1}]$ is Gorenstein if and only if P is pure ([8, p. 105]).

3. Computation of the torsion number

Let R be a normal Cohen–Macaulay domain with free divisor class group of rank r, and let $b_1, \ldots, b_r$ be a basis of $\mathrm{Cl}(R)$. Then $[\omega_R] = \sum_{i=1}^{r} c_i b_i$ with $c_i \in \mathbb{Z}$ for all i. Of course, a basis of $\mathrm{Cl}(R)$ is not uniquely determined. In Section 2 we recalled that for given poset P each spanning tree of $E(\hat{P})$ yields a basis of the divisor class group of the associated join-meet ring. For different bases the coefficients c_i in the presentation of $[\omega_R]$ differ. However $\gcd(c_1, \ldots, c_r)$ is independent of the choice of the basis, because it is just the torsion number $d(R)$ of R, defined in Section 1.

Example 3.1. P be the poset with components P_1 and P_2 where P_1 and P_2 are chains of length a and b, say, $P_1 : x_0 < \cdots < x_a$ and $P_2 : y_0 < \cdots < y_b$. Fix the tree T in $\hat{P}$ consisting of the edges belonging to $E(\hat{P}) \setminus (x_0, x_a)$, where $x_0 = \hat{0}$. Then $[P_e]$ with $e = (x_0, x_a)$ is a basis of $\mathrm{Cl}(R)$. The computation in Section 2 yields $[P_{e'}] = [P_e]$ if $e' \in E(P_1) \cup \{(x_a, \hat{1})\}$ and $[P_{e''}] = -[P_e]$ if $e'' \in E(P_2) \cup \{(\hat{0}, y_1), (y_b, \hat{1})\}$. Hence $[\omega_R] = (a - b)[P_e]$ and $d(R) = a - b$.

The Example 3.1 shows that $d(R)$ can be any number. However, for any join-meet ring, the torsion number can be bounded as follow.

Proposition 3.2. *Let* P *be a finite poset. Let* $L_1 : x_0 < \cdots < x_a$ *and* $L_2 : y_0 < \cdots < y_b$ *be maximal chains of* P *for which* $x_i \neq y_j$ *for each* i *and* j. *Then* $d(R)$ *divides* $a - b$.

Proof. Fix a tree T in $\hat{P}$ whose edges contains all edges belonging to

$$E = E(L_1) \cup E(L_2) \cup \{(x_a, \hat{1}), (\hat{0}, y_0), (y_b, \hat{1})\}.$$

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Then $e = (\hat{0}, x_0) \notin E(T)$. The unique cycle in $T \cup \{e\}$ consists of the edges belonging to $E \cup \{e\}$. Hence, as was done in Example 3.1, the coefficient of $[P_e]$ of $[\omega_R]$ is equal to $a - b$. Thus in particular $d(R)$ divides $a - b$, as desired. □

When, in general, R is a Cohen–Macaulay graded K-algebra over a field K with canonical module ω_R, it is called nearly Gorenstein [6] if the canonical trace ideal $\mathrm{tr}(\omega_R)$ contains the maximal graded ideal of R. Here $\mathrm{tr}(\omega_R)$ is the ideal generated by the image of ω_R through all homomorphism of R-modules into R. As $\mathrm{tr}(\omega_R)$ describes the non-Gorenstein locus of R, one has $\mathrm{tr}(\omega_R) = R$ if and only if R is a Gorenstein ring.

If the join-meet ring R is nearly Gorenstein but not Gorenstein, then one has $a - b = 1$ ([6, Theorem 5.4]). In particular, one has $d(R) = 1$.

Corollary 3.3. *If the join-meet ring R is nearly Gorenstein but not Gorenstein, then $d(R) = 1$.*

Here is another example of a nearly Gorenstein ring which is not Gorenstein and whose torsion number is 1.

Proposition 3.4. *Let K be a field, let X be an $m \times n$-matrix of indeterminates with $m \leq n$, and let $R = K[X]/I_{r+1}(X)$. Then $\mathrm{Cl}(R)$ is free of rank 1, and if R is nearly Gorenstein but not Gorenstein then $d(R) = 1$.*

Proof. The divisor class group of R is isomorphic to $[P]\mathbb{Z} \cong \mathbb{Z}$, where P is the prime ideal in R generated by the r-minors of the first r rows X modulo $I_{r+1}(X)$, see [1, Theorem 7.3.5]. Furthermore, $\omega_R = P^{(n-m)}$, see [1, Theorem7.3.6].

In [7, Theorem 1.1] it is shown that $\mathrm{tr}(\omega_R) = I_r(X)^{n-m}R$. From this fact it follows that R is nearly Gorenstein but not Gorenstein if and only if $r = 1$ and $n - m = 1$, and that in this case $[\omega_R] = [P]$. This implies that $d(R) = 1$. □

One would expect that torsion number, if defined, is always 1 for rings which are nearly Gorenstein but not Gorenstein. However, the following family of examples show that this is not the case.

Example 3.5. Let $R_m = K[x_1, \ldots, x_m]$ denote the polynomial ring in m variables over a field K and $S_n = K[y_1, \ldots, y_n]$ that in n variables over K. Let $R_m^{(p)}$, where $1 \leq p \in \mathbb{Z}$, be the pth Veronese subring of R_m. It is known that $R_m^{(p)}$ is normal and Cohen–Macaulay ([4, p. 193]). Furthermore, $R_m^{(p)}$ is Gorenstein if and only if p divides m ([9])). Fix positive integers m, n, p and q and write $R = R_m^{(p)} \# S_n^{(q)}$ for the Segre product of $R_m^{(p)}$ and $S_n^{(q)}$.

Let $\mathcal{P} \subset \mathbb{R}^{m+n}$ denote the convex polytope consisting of those

$$(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathbb{R}^{m+n}$$

for which

(i) $a_i \geq 0$ for $1 \leq i \leq m$;
(ii) $b_j \geq 0$ for $1 \leq j \leq n$;
(iii) $\sum_{i=1}^{m} a_i = p$;
(iv) $\sum_{j=1}^{n} b_j = q$.

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As is discussed in [3], the convex polytope $\mathcal{P}$ is a lattice polytope of dimension $m+n-2$. (A convex polytope is called a *lattice polytope* if each of the vertices has integer coordinates.) The Segre product $R = R_m^{(p)} \# S_n^{(q)}$ is the toric ring of $\mathcal{P}$. In other words, R is generated by those monomials

$$\left(\prod_{i=1}^{m} x_i^{a_i}\right)\left(\prod_{j=1}^{n} y_i^{a_i}\right)$$

with $(a_1, \ldots, a_m, b_1, \ldots, b_n) \in \mathcal{P} \cap \mathbb{Z}^{n+m}$. Furthermore, R is normal and Cohen–Macaulay ([4, p. 198]). Now, one introduces the lattice polytope $\mathcal{Q} \subset \mathbb{R}^{m+n-2}$ of dimension $m+n-2$ consisting of those

$$(a_1, \ldots, a_{m-1}, b_1, \ldots, b_{n-1}) \in \mathbb{R}^{m+n-2}$$

for which
(i) $a_i \geq 0$ for $1 \leq i \leq m-1$;
(ii) $b_j \geq 0$ for $1 \leq j \leq n-1$;
(iii) $\sum_{i=1}^{m-1} a_i \leq p$;
(iv) $\sum_{j=1}^{n-1} b_j \leq q$.
The facets of $\mathcal{Q}$ are
(i) $x_i = 0$ for $1 \leq i \leq m-1$;
(ii) $y_j = 0$ for $1 \leq j \leq n-1$;
(iii) $\sum_{i=1}^{m-1} x_i = p$;
(iv) $\sum_{j=1}^{n-1} y_j = q$.
One can regard the Segre product R to be the toric ring of $\mathcal{Q}$. Let $C \subset \mathbb{R}_+^{m+n+1}$ denote the cone whose supporting hyperplanes are
(i) $H_i : x_i = 0$ for $1 \leq i \leq m-1$;
(ii) $H'_j : y_j = 0$ for $1 \leq j \leq n-1$;
(iii) $H : -\sum_{i=1}^{m-1} x_i + pt = 0$;
(iv) $H' : -\sum_{j=1}^{n-1} y_j + qt = 0$.
Let P_i denote the monomial prime ideal of height 1 arising from H_i and Q_j that arising from H'_j. Let P denote the monomial prime ideal of height 1 arising from H and Q that arising from H'. The divisor class group $\mathrm{Cl}(R)$ is generated by

$$[P_1], \ldots, [P_{m-1}], [P], [Q_1], \ldots, [Q_{n-1}], [Q]$$

whose relations are

$$[P_1] = \cdots = [P_{m-1}] = [P],\ [Q_1] = \cdots = [Q_{n-1}] = [Q],\ p[P] + q[Q] = 0.$$

Hence

$$\mathrm{Cl}(R) = (\mathbb{Z}[P] \bigoplus \mathbb{Z}[Q])/(p[P] + q[Q]).$$

In particular one has $\mathrm{Cl}(R) = \mathbb{Z}$ if and only if p and q are relatively prime. Since the canonical class is $[\omega_R] = m[P] + n[Q]$, it follows that R is Gorenstein if and only if $(m,n) = c(p,q)$ for some integer $c > 1$. In particular if R is Gorenstein, then each of $R_m^{(p)}$ and $R_n^{(q)}$ is Gorenstein. (See also [4, chapter 4].) Furthermore, the Segre product R is nearly Gorenstein, but not Gorenstein if and only if p divides m, q

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divides n and $|m/p - n/q| = 1$ ([6]). If p and q are relatively prime and if p' and q' are integers with $p'p + q'q = 1$, then $\mathrm{Cl}(R)$ is free of rank 1 which is generated by $-q'[P] + p'[Q]$.

For example, $R = R_4^{(2)} \# S_9^{(3)}$ is nearly Gorenstein, but not Gorenstein and $\mathrm{Cl}(R)$ is free of rank 1 which is generated by $[P] + 2[Q]$. Since

$$[\omega_R] = 4[P] + 9[Q] = -(2[P] + 3[Q]) + 6([P] + 2[Q]),$$

one has $d(R) = 6$.

Finally, we add an example of the computation of the torsion number of R when $\mathrm{Cl}(R)$ is not free.

Example 3.6. Let K be a filed, and let $R = K[x_1, \ldots, x_n]^{(r)}$ be the rth Veronese subring of the polynomial ring $K[x_1, \ldots, x_n]$. Then the support forms of the hyperplanes describing the cone of the natural embedding of the semigroup describing R are $x_i \geq 0$ for $i = 1, \ldots, n-1$ and $-(x_1 + \cdots + x_{n-1}) + rt \geq 0$. Therefore, we have

$$\overline{A}_R = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 1 \\ -1 & -1 & \cdots & -1 & r & 1 \end{bmatrix}$$

The torsion number of R is then given by $I_n(\overline{A}_R) = (r, n)$. Therefore, $d(R) = \gcd(r, n)$.

It is shown in [6, Corollary 4.8] that any Veronese subring of the polynomial is nearly Gorenstein, but again $d(R)$ can be any number.

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