DISTRIBUTIVE MODULES OVER THE RINGS OF FORMAL POWER SERIES AND POLYNOMIALS

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Abstract. A module over a ring is called distributive if for every submodules A, B and C, the equality $A \cap (B + C) = A \cap B + A \cap C$ holds true. In this paper, we give all distributive modules, up to isomorphism, over the rings of formal power series and polynomials, both with coefficients in a field.

1. Introduction

A module over a ring is called *distributive* if for every submodules A, B and C, the equality $A \cap (B + C) = A \cap B + A \cap C$ holds true. The study of distributive modules was motivated by Stephenson [16], where he obtained some comprehensive general results about these modules. After that, many authors have studied and developed the theory and many interesting results about distributive modules over commutative and noncommutative rings have been published (see, for example, [2, 4, 9, 10, 12, 16, 17, 18] and the references there in). We refer the reader to two of the most recent articles [7, 8] on the subject.

The first key result of this paper is the following theorem which gives us all distributive modules, up to isomorphism, over the ring of formal power series.

Theorem A. *Let k be a field and $R = k[[x_1, \dots, x_n]]$ be the ring of formal power series in n variables over the field k. Then every nonzero distributive R-module is isomorphic to one of the following*

(1) *R-module $R/(x_1 - S_1(x_i), \dots, x_n - S_n(x_i))$,*
(2) *R-module $R/(x_1 - S_1(x_i), \dots, x_n - S_n(x_i), x_i^m)$,*
(3) *R-module $k[[x_i]]_{(S_1(x_i),\dots,S_n(x_i))}$,*
(4) *R-module $k((x_i))_{(S_1(x_i),\dots,S_n(x_i))}$,*

where $1 \leq i \leq n$, $S_1(x_i), \dots, S_n(x_i) \in k[[x_i]]$ with $S_1(0) = \cdots = S_n(0) = 0$ and $S_i(x_i) = x_i$, and $m \geq 1$.

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Our second key result, which is a byproduct of the first one, is the following theorem which gives us all distributive modules, up to isomorphism, over the ring of polynomials.

Theorem B. *Let k be an algebraically closed field and $R = k[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over the field k. Then every nonzero distributive (uniserial) R-module is isomorphic to one of the following*

(1) *R-module $(k[(x_i - a_i)^{-1}])_{(S_1(x_i-a_i),\ldots,S_n(x_i-a_i))}$,*

(2) *R-module $(k[(x_i - a_i)^{-1}]/((x_i - a_i)^m))_{(S_1(x_i-a_i),\ldots,S_n(x_i-a_i))}$,*

where $1 \le i \le n$, $(a_1, \ldots, a_n) \in k^n$, $S_1(x_i), \ldots, S_n(x_i) \in k[[x_i]]$ with $S_1(0) = \cdots = S_n(0) = 0$ and $S_i(x_i) = x_i$, and $m \ge 1$.

Although, the above two theorems are our key results, two other major results arise when we give the proofs (see Theorems 3.2 and 4.2). The rest of the paper is organized as follows. In Section 2, we provide some background material. In Section 3, we characterize all ideals which their residue rings are discrete valuation rings. In Section 4, we show that when the modules appear in the preceding section are isomorphic. Finally, in Sections 5 and 6, we give the proofs of Theorems A and B.

2. Background materials

In this section, we provide some necessary background material. Let us restate the three major results of [8] in the following propositions. These are needed for later use. Indeed, the skeleton of our proofs are based on these propositions. Note that in the following propositions, $\mathrm{E}_R(-)$ refers to the *injective envelope* of an R-module $-$, which is the smallest injective R-module containing $-$. By a theorem of Eckmann–Schopf in 1953 (see [1, Theorem 18.10] as a more accessible reference), every module over commutative Noetherian rings has an injective envelope and all such envelopes are isomorphic.

Proposition 2.1 (Theorem 3.5, [8])**.** *Let R be a commutative Noetherian ring and let M be an R-module. Then M is distributive if and only if there exist a family $(\mathfrak{p}_i)_{i\in I}$ of pairwise comaximal prime ideals of R and a family $(M_i \mid M_i \subseteq \mathrm{E}_R(R/\mathfrak{p}_i))_{i\in I}$ of distributive R-modules with $M = \bigoplus_{i\in I} M_i$*

The following proposition gives us a characterization of the distributive modules over commutative Noetherian complete local rings. Its third part is crucial in our work, where D refers to the Matlis dual. We recall that M is said to be *uniserial* if its submodules are totally ordered by inclusion (see [5, 6] for some interesting results about uniserial modules).

Proposition 2.2 (Proposition 4.3, [8])**.** *Let R be a commutative Noetherian complete local ring with maximal ideal $\mathfrak{m}$ and let $k = R/\mathfrak{m}$. Then for a submodule M of $\mathrm{E}_R(k)$ the following statements are equivalent:*

(1) *M is distributive.*

(2) *M is uniserial.*

(3) *$M \cong D(R/I)$, where $I \subseteq \mathfrak{m}$ is an ideal of R such that R/I is a discrete valuation ring.*

(4) *$M \cong \mathrm{E}_{R/I}(k)$, where I is as in (3).*

(5) *$\dim_k (M \cap E_n)/(M \cap E_{n-1}) \leq 1$ for every $n \geq 1$.*

Furthermore, in cases (3) and (4), R/I is either an Artinian discrete valuation ring or a complete discrete valuation domain. Also, in case (5), if for some $m \geq 1$, $\dim_k(M \cap E_m)/(M \cap E_{m-1}) = 1$ holds true, then for every $1 \leq n \leq m$, we have $\dim_k(M \cap E_n)/(M \cap E_{n-1}) = 1$.

We now combine the above two propositions with [8, Theorem 4.5] to obtain the following characterization of the distributive modules over commutative Noetherian rings.

Proposition 2.3 (Main Theorem, [8])**.** *Let R be a commutative Noetherian ring and M be an R-module. Then M is distributive if and only if there exist a family $(\mathfrak{p}_i)_{i \in I}$ of pairwise comaximal prime ideals of R and a family $(M_i \mid M_i \subseteq \mathrm{E}_R(R/\mathfrak{p}_i))_{i \in I}$ of R-modules with $M = \bigoplus_{i \in I} M_i$ and satisfying the following conditions:*

(a) *when $\mathfrak{p}_i$ is maximal, then M_i meets one of the following equivalent statements:*

  (1) *M_i is distributive.*

  (2) *M_i is uniserial.*

  (3) *$M_i \cong D_{\widehat{R_{\mathfrak{p}_i}}}(\widehat{R_{\mathfrak{p}_i}}/I)$, where I is an ideal of $\widehat{R_{\mathfrak{p}_i}}$, which is contained in its maximal ideal, and $\widehat{R_{\mathfrak{p}_i}}/I$ is a discrete valuation ring.*

  (4) *$M_i \cong \mathrm{E}_{\widehat{R}/I}(R/\mathfrak{p}_i)$, where I is as in (3).*

  (5) *$\dim_{R/\mathfrak{p}_i}(M_i \cap E_n)/(M_i \cap E_{n-1}) \leq 1$ for every $n \geq 1$.*

(b) *when $\mathfrak{p}_i$ is not maximal, then $M_i \subseteq K$, where K is the field of fractions of $R/\mathfrak{p}_i$, and $R/\mathfrak{p}_i$ is a Dedekind domain.*

2.1. **The idea behind the main results.** Let k be a field and $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series in n variables over the field k. It is well known that R is a complete local ring. Thus, by using Propositions 2.3 and 2.2, we see that to find all the distributive R-modules, we need to find all the ideals I of R such that R/I is a discrete valuation ring and all prime ideals $\mathfrak{p}$ of R such that $R/\mathfrak{p}$ is a Dedekind domain. But R is local, and thus, any such $R/\mathfrak{p}$ is local and therefore if it is Dedekind, then it is also a discrete valuation domain. Thus, our problem then is to find all the ideals I of R such that R/I is a discrete valuation ring and then identify those such I that are also prime ideals. In this direction, let R/I be a discrete valuation ring with maximal ideal (π). If $\pi^m = 0$ for some $m \geq 1$, then R/I is not a domain (and is, in fact, Artinian). If $\pi^m \neq 0$ for every $m \geq 1$, then by Serre [15, Proposition 2], R/I is an integral domain, and so, it is a discrete valuation domain. Thus, in this case, I is a prime ideal. If $\mathfrak{m}/I = (\pi)$, by Serre [15, §2], R/I is an integral domain if and only if $\pi^m \neq 0$ for every $m \geq 1$. We note that if $\pi^m = 0$ for some $m \geq 1$, then R/I has finite length. We also note that finding all such I as above is equivalent to the problem of finding all finitely generated uniserial modules.

Remark 2.4. At this point, we note that our terminology differs from that of Serre's. What Serre calls a discrete valuation ring is an integral domain (so our discrete valuation domain). This change allows us to avoid the somewhat cumbersome terminology "local principal ideal ring".

Based on the observations described just before the above remark, the following result is in order.

Proposition 2.5. *Let k be a field and $R = k[[x_1, \dots, x_n]]$ be the ring of formal power series in n variables over the field k. Then an R-module M is a finitely generated uniserial module if and only if $M \cong R/I$, where $I \subseteq R$ is an ideal such that R/I is a discrete valuation ring.*

Proof. ($\Rightarrow$): Let M be a finitely generated uniserial R-module. Then with $\mathfrak{m} = (x_1, \dots, x_n)$, $M/\mathfrak{m}M$ is also uniserial over R and over $k \cong R/\mathfrak{m}$. Hence, its dimension over k is at most one. Thus, $M/\mathfrak{m}M$ is a cyclic R-module. Therefore, by Nakayama's lemma, M is cyclic. Let $M \cong R/I$. Then we argue that $(\mathfrak{m}/I)/(\mathfrak{m}/I)^2$ has dimension at most one over k, and so, $\mathfrak{m}/I$ is cyclic, i.e., is a principal ideal. Hence, R/I is a discrete valuation ring.

($\Leftarrow$): This implication is trivial. $\square$

The following observation is useful in classifying the finitely generated uniserial modules.

Lemma 2.6. *Let k be a field and $R = k[[x_1, \dots, x_n]]$ be the ring of formal power series in n variables over the field k. Also, let M and N be two finitely generated uniserial R-modules. Then $M \cong N$ if and only if $\operatorname{ann}_R(M) = \operatorname{ann}_R(N)$.*

Proof. ($\Rightarrow$): This implication is trivial.

($\Leftarrow$): Given M, N, let $M \cong R/I$ and $N \cong R/J$. Since $\operatorname{ann}_R(M) = I$ and $\operatorname{ann}_R(N) = J$, the assumption implies that $I = J$. Thus, $M \cong R/I$ and $N \cong R/I$. This means that $M \cong N$, as required. $\square$

As usual, let k be a field and $R = k[[x_1, \dots, x_n]]$ be the ring of formal power series in n variables over the field k. For every i with $1 \leq i \leq n$, we can think of R as $k[[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]][[x_i]]$, i.e., as the ring of formal power series in x_i with coefficients in $k[[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]]$. This observation is useful when we make use of the algebraic Weierstrass preparation theorem. Let us now recall the statement of this theorem.

2.2. **The algebraic Weierstrass preparation theorem.** Let R be a complete local ring with maximal ideal $\mathfrak{m}$. A polynomial $p(x) \in R[x]$ of degree $n \geq 0$ is called *distinguished* if $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, where $a_i \in \mathfrak{m}$ for $0 \leq i \leq n-1$. Then we have the following so-called algebraic Weierstrass preparation theorem (see [11] for a short and elegant proof of this theorem based on the Banach contraction mapping theorem).

Theorem 2.7 (Weierstrass). *Let R be a complete local ring with maximal ideal $\mathfrak{m}$. Let $f(x) = \sum_{i=0}^{\infty} b_i x_i \in R[[x]]$ be such that $b_i \notin \mathfrak{m}$ for at least one i, and let n be the least such i. Then there exists a unique unit $u \in R[[x]]$ and a unique distinguished polynomial $p(x) \in R[x]$ of degree n such that $uf(x) = p(x)$.*

Note that, in the above theorem, for such an $f(x)$ and $p(x)$, we have $(f(x)) = (p(x))$ for these two principal ideals of $R[[x]]$. The n in the theorem is called the *Weierstrass degree* of $f(x)$ in x. Thus, for example, consider $k[[x,y]]$. We have $k[[x,y]] = k[[x]][[y]] = k[[y]][[x]]$. Hence, for $f \in k[[x,y]]$, we can consider the (possible) Weierstrass degrees of f in x or in y.

3. Ideals with discrete valuation residue rings

Let k be a field and $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series in n variables over the field k. Our aim is to find the ideals I of R such that R/I is a discrete valuation ring and then identify those I such that I is a prime ideal. Here, and in several other pages of this paper, we let $n = 3$ and write $R = k[[x,y,z]]$. Each time we do this, it will be clear how to generalize our results to $k[[x_1, \ldots, x_n]]$ for arbitrary $n \geq 1$.

Thus, let $R = k[[x,y,z]]$, where k is a field, and let I be an ideal of R such that R/I is a discrete valuation ring. We are interested in the case that $R/I \neq 0$. This means that $I \subseteq \mathfrak{m} = (x,y,z)$. Note that we cannot have $I \subseteq \mathfrak{m}^2$; because otherwise R/I has $R/\mathfrak{m}^2$ as a quotient, but $R/\mathfrak{m}^2$ is not a discrete valuation ring while R/I is assumed to be such. Thus, we can find $f \in I \subseteq \mathfrak{m}$ with $f \notin \mathfrak{m}^2$. Now, $f \in \mathfrak{m}$ implies that $f(0,0,0) = 0$, and since $f \notin \mathfrak{m}^2$, we have

$$f = \alpha x + \beta y + \gamma z + \text{ terms of degree} \geq 2,$$

where at least one of $\alpha, \beta, \gamma \in k$ is nonzero. Assume that $\gamma \neq 0$. Then $f \in k[[x,y]][[z]]$ has Weierstrass degree one in z. Therefore, by Theorem 2.7, we have a unit $u \in R$ and a distinguished polynomial $p \in k[[x,y]][[z]]$ of degree one, where

$$uf = z - T(x,y)$$

with $T(x,y) \in k[[x,y]]$ and $T(0,0) = 0$. (Note that the minus in the above equality is just for convenience.) Therefore, the equalities $(f) = (uf) = (z - T(x,y))$ holds true for these principal ideals. Hence, if we can compute $R/(z - T(x,y))$, it will help us to compute R/I, since this quotient ring is a quotient ring of $R/(z - T(x,y)) = R/(f)$. But the quotient ring $R/(z - T(x,y))$ is isomorphic to $k[[x,y]]$. In order to see this, consider the continuous surjective k-homomorphism $\varphi : R \to k[[x,y]]$, where

$\varphi(x) = x$, $\varphi(y) = y$, and $\varphi(z) = T(x, y)$. Thus, by using this isomorphism, we see that $k[[x, y]]$ can be considered as an R-module, where the scalar multiplication by x and y is as expected, and the multiplication by z agrees with the multiplication by $T(x, y)$, i.e., for $g \in k[[x, y]]$, we have $zg = T(x, y)g$. In this situation and in other similar situations below, we have the obvious action by $\alpha \in k$.

Thus, now we have $k[[x, y]]$ as a ring and as an R-module. In $R/(z - T(x, y))$ we have the ideal $I/(z - T(x, y))$. We call this ideal J. Using the isomorphism above we can think of I as an ideal of $k[[x, y]]$. Since $k[[x, y]]$ is not a principal ideal ring and also it is nonzero, we can find $g(x, y) \in J$ with $g(0, 0) = 0$, i.e., $g \in (x, y)$ while $g \notin (x, y)^2$. Thus, as with f, we can argue that g has Weierstrass degree one in x or in y. For convenience, suppose it is in y. Then, again by Theorem 2.7, there exists a unit $v \in k[[x, y]]$ such that we have $vg = y - S(x)$, where $S(x) \in k[[x]]$. Hence, we see that $k[[x, y]]/(g) \cong k[[x]]$, where $k[[x]]$ is a $k[[x, y]]$-module with y acting like $S(x)$. But $k[[x]]$ as such is a quotient module of the R-module $k[[x, y]]$ with z action being multiplication by $T(x, y)$. Since the y action on $k[[x]]$ is multiplication by $S(x)$, we see that the z action on $k[[x]]$ is multiplication by $T(x, S(x))$. Therefore, we have the ideal

$$(y - S(x), z - T(x, y)) = (y - S(x), z - T(x, S(x))) \subseteq I$$

such that

$$R/(y - S(x), z - T(x, S(x))) \cong k[[x]]$$

as a ring. But $k[[x]]$ is a discrete valuation domain with ideals (x^m), $m \geq 0$, and zero.

Thus, we have found I if $I = (y - S(x), z - T(x, S(x)))$. If this is not the case, we need to add one more generator to $(y - S(x), z - T(x, S(x)))$ to get I. Thus, again identifying this just means adding some x^m, $m \geq 1$. Then we get the quotient ring $k[[x]]/(x^m)$. If we can describe $k[[x]]$ as R-module, then we can easily find the module structure on $k[[x]]/(x^m)$.

We will now concentrate on describing such a $k[[x]]$ as an R-module. To this end, we only need to describe the scalar action by y and z on $k[[x]]$. But y acts like $S(x)$, i.e., $yf = S(x)f$ and z like $T(x, S(x))$, i.e., $zf = T(x, S(x))f$, for $f \in k[[x]]$. Also, note that x acts like x on $k[[x]]$. With the proper choices, we see that $S(x)$ and $T(x, S(x))$ can be arbitrary elements of $k[[x]]$ contained in (x). This means that they have zero as their constant terms. In what follows we can think of $T(x, S(x))$ as just some such element of $k[[x]]$. Thus, we simplify and call it $T(x)$.

Notations and Remarks 3.1. Given $R = k[[x, y, z]]$, where k is a field, if we regard $k[[x]]$ as an R-module, where y acts like $S(x)$ and z acts like $T(x)$, $S(x), T(x) \in k[[x]]$ with $S(0) = T(0) = 0$, we let $k[[x]]_{(x,S(x),T(x))}$ denote this module. Also, its quotient by (x^m) for some $m \geq 1$ is denoted by $(k[[x]]/(x^m))_{(x,S(x),T(x))}$. Note that the x in $(x, S(x), T(x))$ may seem superfluous, but in the future results it will be useful.

Note that the submodules of $k[[x]]_{(x,S(x),T(x))}$ as an R-module are precisely the submodules of $k[[x]]$ as a $k[[x]]$-module. In a similar manner, we get the uniserial R-module $k[[y]]$, where x acts like $U(y)$ and z like $V(y)$, where $U(0) = V(0) = 0$.

Thus, we have $k[[y]]_{(U(y),y,V(y))}$. Likewise we get the R-modules $k[[z]]_{(S(z),T(z),z)}$, and then, also the uniserial $k[[x]]/(x^m)$, $k[[y]]/(y^m)$, and $k[[z]]/(z^m)$ for $m \geq 1$ with the appropriate R-module structure.

By generalizing the above arguments to the ring of formal power series in n variables for arbitrary $n \geq 1$, we obtain one of our main results.

Theorem 3.2. *Let k be a field and $R = k[[x_1, \ldots, x_n]]$ be the ring of formal power series in n variables over the field k. Also, let I be an ideal of R such that R/I is a nonzero discrete valuation ring. Then there exists $1 \leq i \leq n$ such that for some $S_1(x_i), \ldots, S_n(x_i) \in k[[x_i]]$ with $S_1(0) = \cdots = S_n(0) = 0$ and $S_i(x_i) = x_i$, one of the following cases occur:*

(1) $I = (x_1 - S_1(x_i), \ldots, x_n - S_n(x_i))$,

(2) $I = (x_1 - S_1(x_i), \ldots, x_n - S_n(x_i), x_i^m)$ *for some* $m \geq 1$.

Moreover, R/I when I is as (1) *or* (2)*, respectively is isomorphic to either*

$$k[[x_i]]_{(S_1(x_i),\ldots,S_n(x_i))} \ \textit{or} \ \left(\frac{k[[x_i]]}{(x_i^m)}\right)_{(S_1(x_i),\ldots,S_n(x_i))}$$

as an R-module. Note that, the ideal I in (1) *is a prime ideal while in* (2) *is not, unless $m = 1$.*

4. Isomorphic distributive modules

Clearly it is of interest to classify the distributive modules. Thus, we would like to know when two of these distributive (here, in fact, uniserial) modules are isomorphic. We first note that no $k[[x_i]]$ (with some given action) is isomorphic to a $k[[x_j]]/(x_j^m)$, $m \geq 1$ (again with an action), since the first of these modules is of infinite length and the second is of length m. Hence, we first consider the problem of when for some i, j and for some $(S_1(x_i), \ldots, S_n(x_i))$ and $(T_1(x_j), \ldots, T_n(x_j))$ as above we have

$$k[[x_i]]_{(S_1(x_i),\ldots,S_n(x_i))} \cong k[[x_j]]_{(T_1(x_j),\ldots,T_n(x_j))}.$$

To simplify the notation, we write S for $(S_1(x_i), \ldots, S_n(x_i))$ and likewise for T. We first consider the case where $i = j$.

Proposition 4.1. *Keep the above notations and let k be a field. Then $k[[x_i]]_S \cong k[[x_i]]_T$ if and only if $S_j = T_j$ for every $1 \leq j \leq n$.*

Proof. If these conditions are satisfied, then we have the identity isomorphism. For the converse, we again consider the case $n = 3$ and write $R = k[[x, y, z]]$. Thus, our question is when do we have an isomorphism $k[[x]]_S \to k[[x]]_T$.

To this end, we recall that $S_1(x) = T_1(x) = x$. Thus, $S_1 = T_1$. We see that the annihilator of $k[[x]]_S$ is the ideal $I = (y - S_2(x), z - S_3(x))$. This follows from the isomorphism $k[[x]] \cong R/I$. Then the annihilator of $k[[x]]_T$ is $(y - T_2(x), z - T_3(x))$. This gives that these two ideals are equal, and so, $y - S_2(x) \in (y - T_2(x), z - T_3(x))$. Let

$$y - S_2(x) = f(x, y)(y - T_2(x)) + g(x, y)(z - T_3(x)).$$

24 May 2024 09:33:33 PDT
231124-Pournaki Version 2 - Submitted to J. Comm. Alg.

Substituting $T_3(x)$ for z on both sides we get

$$y - S_2(x) = f(x,y)(y - T_2(x)).$$

Let $x = 0$ on both sides. Then $y - 0 = f(0,y)(y-0)$. Hence, $f(0,y) = 1$, and thus, $f(0,0) = 1$. Therefore, $f(x,y)$ is a unit. Note that $y - S_2(x)$ and $y - T_2(x)$ are distinguished polynomials for $k[[x]][y]$. Thus, by Theorem 2.7, we see that $f(x,y) = 1$ and that $y - S_2(x) = y - T_2(x)$. Thus, $S_2(x) = T_2(x)$. Similarly, we get $S_3(x) = T_3(x)$. □

Before we consider the next kind of isomorphisms, we recall some facts about power series $S, T \in R[[x]]$, where R is any commutative ring. If $S = a_0 + a_1x + \cdots$, we say S has *order* n if $a_0 = 0, \ldots, a_{n-1} = 0$, but $a_n \neq 0$. We then write $\omega(S) = n$. If there is no such n, then $S = 0$ and we write $\omega(S) = \infty$. An S has a multiplicative inverse in $R[[x]]$ if and only if $\omega(S) = 0$ and $S(0) = a_0$ has an inverse in R, i.e., a_0 is a unit of R. If $S, T \in R[[x]]$ and if $\omega(T) \geq 1$, we can define $S \circ T$ in the obvious fashion. If $T(x) = x$, then $S \circ T = S$. We say S has an inverse for this operation if there exists a T with $\omega(T) \geq 1$ such that $S \circ T(x) = T \circ S(x) = x$. We then write $T = S^{(-1)}$. For a given S with $\omega(S) \geq 1$, $S^{(-1)}$ exists if and only if $S = a_1x + a_2x^2 + \cdots$, where a_1 is a unit of R. In this case, T is unique and is denoted by $S^{(-1)}$. The usual argument then gives that $S^{(-1)}$ has an inverse and that its inverse is T. We also note that when $\omega(S), \omega(T) \geq 1$ and when $S(T(x)) = x$, then $T(S(x)) = x$ and $T = S^{(-1)}$.

Hence, we now try to answer the question of when $k[[x]]_S$ and $k[[y]]_T$ are isomorphic. The annihilator of $k[[x]]_S$ is $(y - S_2(x), z - S_3(x))$ and that of $k[[y]]_T$ is $(x - T_1(y), z - T_3(y))$. Thus, if these two modules are isomorphic, then we have

$$(y - S_2(x), z - S_3(x)) = (x - T_1(y), z - T_3(y)).$$

Thus, then this means we can write

$$y - S_2(x) = f(x,y,z)(x - T_1(y)) + g(x,y,z)(z - T_3(y)).$$

We substitute $T_3(y)$ for z in the above equation and get that

$$y - S_2(x) = f(x,y,T_3(y))(x - T_1(y)).$$

Considering the orders of both sides of this equation, we see that $f(x,y,T_3(y))$ has order zero, and so, it is a unit of $k[[x,y,z]]$. Now, we substitute $T_1(y)$ for x and get $y - S_2(T_1(y)) = 0$. Thus, $y = S_2(T_1(y))$. This gives that $S_2^{(-1)}(y) = T_1(y)$, and so, also that $T_1^{(-1)}(y) = S_2(y)$.

We again use the equality of the ideals

$$(y - S_2(x), z - S_3(x)) = (x - T_1(y), z - T_3(y))$$

and write

$$z - S_3(x) = f \cdot (x - T_1(y)) + g \cdot (z - T_3(y)).$$

Substituting $T_1(y)$ for x we get

$$z - S_3(T_1(y)) = g(T_1(y), y, z)(z - T_3(y)),$$

24 May 2024 09:33:33 PDT
231124-Pournaki Version 2 - Submitted to J. Comm. Alg.

where $g(T_1(y), y, z)$ is a unit. Since the two polynomials $z - S_3(T_1(y))$ and $z - T_3(y)$ are distinguished polynomials in $k[[x, y]][[z]]$, the uniqueness part of Theorem 2.7 gives that $z - S_3(T_1(y)) = z - T_3(y)$. Thus, we have that if $k[[x]]_S \cong k[[y]]_T$, then $S_1(x)$ has an inverse $S^{(-1)}(x)$ and, in fact, $S_1^{(-1)}(y) = T_1(y)$. Then also we have $S_3(T_1(y)) = T_3(y)$. Hence, all together, we see that if

$$k[[x]]_{(S_1,S_2,S_3)} \cong k[[y]]_{(T_1,T_2,T_3)}$$

with

$$(S_1(x), S_2(x), S_3(x)) = (x, S_2(x), S_3(x))$$

and

$$(T_1(y), T_2(y), T_3(y)) = (T_1(y), y, T_3(y)),$$

then $S_1(x)$ has an inverse $S^{(-1)}(x)$ and $S_1^{(-1)}(x) = T_2(x)$. Thus, we get $(T_1(y), y, T_3(y))$ from $(x, S_2(x), S_3(x))$ by substituting $T_1(y)$ for x in $(x, S_2(x), S_3(x))$, or equivalently,

$$(T_1(y), T_2(y), T_3(y)) = (S_1(T_1(y)), S_2(T_1(y)), S_3(T_1(y))).$$

Now, conversely, we want to argue that when we have these equalities (where T_2 and S_1 have inverses and $S^{(-1)}(x) = T(x)$), the modules $k[[x]]_{(S_1,S_2,S_3)}$ and $k[[y]]_{(T_1,T_2,T_3)}$ are isomorphic, i.e.,

$$k[[x]]_{(S_1,S_2,S_3)} \cong k[[y]]_{(T_1,T_2,T_3)}.$$

The equality above gives us a hint that our isomorphism $\varphi : k[[x]] \to k[[y]]$ might be such that $\varphi(f(x)) = f(T_1(y))$. We do have a k-homomorphism φ which is continuous and such that $\varphi(x) = T_1(y)$. Clearly, φ is a k-linear. We want it to be $k[[x, y, z]]$-linear. For $f \in k[[x]]$, we consider $\varphi(xf)$ and $x\varphi(f)$. We have

$$\varphi(xf) = T_1(y)f(T_1(y)) = T_1(y)\varphi(f).$$

But $k[[y]] = k[[y]]_{(T_1(y),T_2(y),T_3(y))}$, and so, $T_1(y)\varphi(f) = x\varphi(f)$. The arguments that $\varphi(yf) = y\varphi(f)$ and $\varphi(zf) = z\varphi(f)$ are given below. Since $T^{(-1)}(y) = S_2(y)$, we have

$$\varphi(yf) = \varphi(S_2(x)f(x)) = S_2(T_1(y))f(T_1(y)) = y\varphi(f).$$

Hence, $\varphi(yf) = y\varphi(f)$. We have

$$\varphi(zf) = \varphi(S_3(x)f(x)) = S_3(T_1(y))\varphi(f).$$

But $S_3(T_1(y)) = T_3(y)$ and $T_3(y)\varphi(f) = z\varphi(f)$. Thus,

$$\varphi : k[[x]]_{(S_1,S_2,S_3)} \to k[[y]]_{(T_1,T_2,T_3)}$$

is $k[[x, y, z]]$-linear. It is an isomorphism and $\varphi^{-1} : k[[y]] \to k[[x]]$ is as φ but where $\varphi^{-1}(y) = S_2(x) = T_1^{(-1)}(x)$. Hence, generalizing to $k[[x_1, \ldots, x_n]]$ and using our customary notations we have the following result:

Theorem 4.2. *Keep the above notations and let k be a field. Then the isomorphism*

$$k[[x_i]]_{(S_1(x_i),\ldots,S_n(x_i))} \cong k[[x_j]]_{(T_1(x_j),\ldots,T_n(x_j))}$$

holds true if and only if one of the following cases occur:

(a) *$i = j$ and $S_1 = T_1, \ldots, S_n = T_n$,*

(b) *$i \neq j$ and $S_j(x_i)$ and $T_i(x_j)$ have inverses such that $S_j^{(-1)}(x_i) = T_i(x_j)$, and we can get $(T_1(x_j), \ldots, T_n(x_j))$ from $(S_1(x_i), \ldots, S_n(x_i))$ by substituting $T_i(x_j)$ for x_i, i.e.,*

$$S_1(T_i(x_j)) = T_1(x_j),$$
$$\vdots$$
$$S_n(T_i(x_j)) = T_n(x_j).$$

There is a version of the above theorem for the modules $(k[[x_i]]/(x_i^m))_S$, where $m \geq 1$. First, note that this module is of length m. Therefore, if $(k[[x_i]]/(x_i^m))_S \cong (k[[x_j]]/(x_j^{m'}))_T$, then $m = m'$ by comparing the lengths of both sides. Thus, our question will be when do we have isomorphism

$$(k[[x_i]]/(x_i^m))_S \cong (k[[x_j]]/(x_j^m))_T?$$

To have these modules isomorphic, we would have conditions analogous to (a) and (b) where, for example, (a) would be replaced by "$i = j$ and $S_1 \cong T_1 \pmod{m}, \ldots$". Here, $S \cong T \pmod{m}$ for formal power series means that S and T have the same terms of degree i for $0 \leq i \leq m-1$.

5. The proof of Theorem A

We recall that, by Lemma 2.6, if M and N are two finitely generated uniserial R-modules with $R = k[[x_1, \ldots, x_n]]$, then $M \cong N$ if and only if $\mathrm{ann}_R(M) = \mathrm{ann}_R(N)$. Thus, for example, if $M = k[[x_i]]_{(S_1,\ldots,S_n)}$, then we have

$$k[[x_i]]_{(S_1,\ldots,S_n)} = R/(x_1 - S_1(x_i), \ldots, x_n - S_n(x_i)).$$

Thus, $\mathrm{ann}_R(M) = (x_1 - S_1(x_i), \ldots, x_n - S_n(x_i))$. We have analogous description of $\mathrm{ann}_R(M)$ for the other finitely generated uniserial modules.

We now claim that we know all the distributive (i.e., uniserial since our rings are local) modules. Note that since $R = k[[x_1, \ldots, x_n]]$ is local, there are no pairwise comaximal prime ideals of R. Therefore, the sum $M = \bigoplus_{i \in I} M_i$ in Proposition 2.1 only has (at most) one summand. Then, by this proposition and also Proposition 2.2, the distributive R-modules are the $D(R/I)$, where R/I is a discrete valuation ring and the submodules $M \subseteq K$, where K is the field of fractions of R/I when R/I is also a domain.

When R/I is a domain, we found that as a module it is $k[[x_i]]_S$ for some $1 \leq i \leq n$ and for our usual $S = (S_1, \ldots, S_n)$. Thus, as a domain, it is $k[[x_i]]$. Its field of fractions is $k((x_i))$, i.e., the field of Laurent series in x_i with coefficients in k. As an R-module, $k[[x_i]]$ is annihilated by each $x_j - S_j(x_i)$, and so, these annihilate $k((x_i))$ as an R-module. This just means that multiplication by x_j is the same as multiplication by $S_j(x_i)$. Thus, we call this module $k((x_i))_S$. Note that its submodules as an R- or as

24 May 2024 09:33:33 PDT
231124-Pournaki Version 2 - Submitted to J. Comm. Alg.

an $k[[x_i]]$-module are (x_i^m), $m \in \mathbb{Z}$, $k((x_i))$, and zero. Therefore, up to isomorphism, we only have two nonzero $M \subseteq k((x_i))_S$, i.e., $k[[x_i]]_S$ and $k((x_i))_S$.

Now, what remains is to describe the Matlis dual $D(R/I)$, where R/I is a discrete valuation ring. In order to do this, we recall Northcott's description of $\mathrm{E}_R(k)$ with $k = k[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)$ (see [[14]]). We consider the set $k[x_1^{-1}, \ldots, x_n^{-1}]$ of inverse polynomials in $x_1^{-1}, \ldots, x_n^{-1}$ with coefficients in k. This set with the obvious addition is made into a $k[[x_1, \ldots, x_n]]$-module using the following rules:

(a) $x_j x_i^{-m} = 0$ for all $i \neq j$ and all $m \geq 0$ (note that $m = 0$ gives $x_j 1 = 0$),

(b) $x_i x_i^{-m-1} = x_i^{-m}$ for all i and all $m \geq 1$.

Theorem 5.1 (Northcott)**.** *Keep the above notations. Then* $\mathrm{E}_R(k) \cong k[x_1^{-1}, \ldots, x_n^{-1}]$.

This description of $\mathrm{E}_R(k)$ helps us to describe our dual $D(R/I)$. It is at this point that our description of R/I as an R-module will be useful. We give one example. This example suggests what the dual is in general. Then using Matlis duality we will argue that this suggestion is correct.

Example 5.2. Let $n = 2$ and $R = k[[x, y]]$, where k is a field. We want to find the Matlis dual of the R-module $k[[x]]_{(x,x^2)}$. Thus, $I = (y - x^2) \subseteq k[[x, y]]$ in this situation. Recall that for any $I \subseteq R$,

$$D(R/I) = \mathrm{Hom}_R(R/I, E) \cong E',$$

where $E' = \{z \in E \mid Iz = 0\}$. Therefore, in our example, we want to find all $z = f(x^{-1}, y^{-1}) \in k[x^{-1}, y^{-1}]$ with $(y - x^2)f(x^{-1}, y^{-1}) = 0$. With a little work, we see that the set of such f's is generated as a vector space over k by

$$1,\ x^{-1},\ x^{-2} + y^{-1},\ x^{-3} + x^{-1}y^{-1},\ x^{-4} + x^{-2}y^{-1} + y^{-2},\ \ldots\ .$$

This gives an awkward presentation of the dual. As a vector space, we have an isomorphism between this dual and $k[x^{-1}]$ with

$$1 \to 1,\ x^{-1} \to x^{-1},\ x^{-2} \to x^{-2} + y^{-1},\ x^{-3} \to x^{-3} + x^{-1}y^{-1},\ \ldots\ ,$$

where the isomorphism is $k[x^{-1}] \to D(R/I)$. If we translate the $k[[x, y]]$-module structure to $k[x^{-1}]$, we see that x acts as x and y as x^2. Thus, we get $k[x^{-1}]_{(x,x^2)}$, where we are using of the obvious notion.

At this point a caution is necessary. We have the module $k[x^{-1}]_{(x,x^2)}$ and the submodule $k[x^{-1}] \subseteq k[x^{-1}, y^{-1}]$. Note that y acts like x^2 in the first module and like zero in the second module, and so, there is no possible isomorphism between these two modules.

Since $R = k[[x_1, \ldots, x_n]]$ is a complete local ring, we can use the full force of Matlis duality. Hence, this means the correspondence $M \to D(M)$ gives us a bijective correspondence between isomorphism classes of finitely generated distributive (so uniserial) modules and Artinian distributive modules. Thus, with $R = k[[x_1, \ldots, x_n]]$, our list of all the finitely generated uniserial (i.e., distributive) modules is the R/I, where

$I = (x_1 - S_1(x_i), \ldots, x_n - S_n(x_i))$ or

$$I = (x_1 - S_1(x_i), \dots, x_n - S_n(x_i), x_i^m),$$

$1 \leq i \leq n$, $S_1, \dots, S_n$ as usual, and $m \geq 1$. For two such, e.g., R/I and R/I', we have $R/I \cong R/I'$ as R-modules if and only if $I = I'$. By Matlis duality, then the Artinian uniserial (i.e., distributive) R-modules will be the Matlis duals $D(R/I)$ for such I.

Note that since any such R/I is reflexive then $\mathrm{ann}_R(R/I) = \mathrm{ann}_R(D(R/I))$. Also, $D(R/I) \cong D(R/I')$ if and only if $R/I \cong R/I'$ (as modules). Thus, the Artinian uniserial modules A are classified by their annihilators $\mathrm{ann}_R(A)$ and every such $\mathrm{ann}_R(A)$ is one of our I's. This means that if for each I as above we can find an Artinian uniserial R-module A with $\mathrm{ann}_R(A) = I$, then we will have found all the Artinian uniserial R-modules. Also, by Matlis duality, we get a criterion for two such to be isomorphic. Hence, we revert to our $k[x_i^{-1}]_{(S_1,\dots,S_n)}$. Note that, $k[x_i^{-1}]$ is Artinian as an $k[[x_i]]$-module, and so, as an $R = k[[x_1, \dots, x_n]]$-module (i.e., R-module $k[x_i^{-1}]_{(S_1,\dots,S_n)}$) is Artinian. With a little work, we see that

$$\mathrm{ann}_R\big(k[x_i^{-1}]_{(S_1,\dots,S_n)}\big) = (x_1 - S_1(x_i), \dots, x_n - S_n(x_i)).$$

Now we consider the Artinian R-module $k[x_1^{-1}]_{(S_1,\dots,S_n)}$. The set of $k + kx_i^{-1} + \cdots + kx_i^{-m+1}$'s for $m \geq 1$ is a submodule. A convenient way to denote this module is as $k[x_i^{-1}]/(x_i^{-m})$ (here, thinking of $k[x_i^{-1}]$ as a ring). Thus, we can think

$$k[x_i^{-1}]/(x_i^{-m}) = k + kx_i^{-1} + \cdots + kx_i^{-m+1},$$

and then we get the $R = k[[x_1, \dots, x_n]]$-module which we will denote as

$$\Big(k[x_i^{-1}]/\big(x_i^{-(m+1)}\big)\Big)_{(S_1,\dots,S_n)}.$$

Note that, its annihilator is the ideal $I = (x_1 - S_1(x_i), \dots, x_n - S_n(x_i), x^m)$, and so, we have a concrete description of all our $D(R/I)$ (see Proposition 2.2). This completes the proof of Theorem A.

Remark 5.3. Let $R = k[[x_1, \dots, x_n]]$, where k is a field. Using Proposition 2.3, we have that for the complete local ring R, the uniserial R-modules, up to isomorphism, are $D(R/I)$'s, where R/I is a discrete valuation ring, R/I, and its field of fractions K when R/I is a discrete valuation domain. We note that when R/I is a discrete valuation ring, it is also uniserial. Therefore, R/I, up to isomorphism, should occur in our list. If R/I is a domain, then it does. If R/I is not a domain, then we must have $R/I \cong D(R/J)$ for some J, where R/J is a principal ideal ring. But recalling that $\mathrm{ann}_R(R/I) = I$ and that $\mathrm{ann}_R(D(R/J)) = J$, we see $I = J$, i.e., that $R/I \cong D(R/I)$. Thus, R/I is not only isomorphic to its bidual, but also it is isomorphic to its dual. Thus, we might call such an R/I *self-reflexive*. These R/I's are precisely the uniserial modules A of finite length.

Besides the $D(R/I)$ (R/I a principal ideal ring), our list of uniserial modules includes the submodules of K, where K is the field of fractions of R/I, where R/I is a discrete valuation domain. Up to isomorphism, the only submodules are zero, R/I and K itself (if $\mathfrak{m}/I = (\pi)$, then 0, K and the (π^m)'s, $m \in \mathbb{Z}$, are the submodules).

24 May 2024 09:33:33 PDT
231124-Pournaki Version 2 - Submitted to J. Comm. Alg.

Clearly, 0 and R/I are reflexive, but so is K. For according to Bourbaki ([3, exercises, §8, 2]) if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over a complete local ring with M' finitely generated and M'' Artinian, then M is Matlis reflexive. We have such a sequence $0 \to R/I \to K \to K/(R/I) \to 0$ with R/I and K as above. Recall that $R/I \cong k[x_i]_S$ for some S, and so, $K \cong k((x_i))_S$. Our short exact sequence is then

$$0 \to k[[x_i]] \to k((x_i)) \to k[x_i^{-1}] \to 0$$

and we see that $k((x_i))_S$ is reflexive. Then its dual is also uniserial and must occur in our list of uniserial modules. The only possibility is $k((x_i))_S$ itself. Therefore, $D(k((x_i))_S) \cong k((x_i))_S$. This can also be proved by taking the dual of the exact sequence above.

6. The proof of Theorem B

In this section, we use the results of the previous sections to describe the distributive and uniserial modules over $R = k[x_1, \dots, x_n]$, where k is an algebraically closed field. We then also give these modules over R/I, where I is an ideal of R. We first appeal to Proposition 2.3. From that proposition, we see that we need to find all uniserial R-modules M, where $M \subseteq \mathrm{E}_R(R/\mathfrak{m})$ and $\mathfrak{m} \subseteq R$ is a maximal ideal.

We begin with $\mathfrak{m} = (x_1, \dots, x_n)$. In this case, we have $R/\mathfrak{m} \cong k$ (with $x_i k = 0$ for each i) and

$$\mathrm{E}_R(k) = \mathrm{E}_{R_\mathfrak{m}}(k) = \mathrm{E}_{\widehat{R_\mathfrak{m}}}(k),$$

where the notion of a submodule of this module is the same for all three rings. Here, we have $\widehat{R_\mathfrak{m}} = k[[x_1, \dots, x_n]]$. Note that, we know all uniserial $\widehat{R_\mathfrak{m}}$-modules M, where $M \subseteq \mathrm{E}_{\widehat{R_\mathfrak{m}}}(k)$. These are described in Theorem A. Indeed, they are the $k[x_i^{-1}]_{(S_1,\dots,S_n)}$'s and the $\big(k[x_i^{-1}]/(x_i^m)\big)_{(S_1,\dots S_n)}$'s with $m \geq 1$. These modules are uniserial, and so, distributive as $k[[x_1, \dots, x_n]]$-modules, and thus, as $k[x_1, \dots, x_n]$-modules. Note that we do not need to require that the $S_1(x_i), \dots, S_n(x_i)$ be polynomials in the R situation. Thus, we know, up to isomorphism, all the uniserial $M \subseteq \mathrm{E}_R(R/\mathfrak{m})$ with $\mathfrak{m} = (x_1, \dots, x_n)$.

To find the all uniserial R-modules $M \subseteq \mathrm{E}_R(R/\mathfrak{n})$ for other maximal ideals $\mathfrak{n}$, we note that since k is algebraically closed, every such ideal is of the form $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$ for a unique $(a_1, \dots, a_n) \in k^n$. Note that we have the equality

$$k[x_1, \dots, x_n] = k[x_1 - a_1, \dots, x_n - a_n].$$

Thus, we can let $y_i = x_i - a_i$, $1 \leq i \leq n$, which implies that $y_1, \dots, y_n$ are algebraically independent over k and $\mathfrak{n} = (y_1, \dots, y_n)$. Thus, we are (essentially) in the previous situation. Therefore, we use the obvious notations of $k[[x_1 - a_1, \dots, x_n - a_n]]$ and $k[(x_1 - a_1)^{-1}, \dots, (x_n - a_n)^{-1}]$ and get the uniserial $k[(x_i - a_i)^{-1}]_{(S_1(x_i - a_i), \dots, S_n(x_i - a_i))}$ and then these "modulo $(x_i - a_i)^{-m}$" for $m \geq 1$. We note that the annihilators of any such module is $\mathfrak{n} = (x_1 - a_1, \dots, x_n - a_n)$.

24 May 2024 09:33:33 PDT
231124-Pournaki Version 2 - Submitted to J. Comm. Alg.

Using the mentioned fact and Theorem 4.2 we can give necessary and sufficient conditions for two such uniserial modules to be isomorphic. Using Proposition 2.3 we now see that to find all the distributive R-modules, we also need to find all the prime ideals $\mathfrak{p} \subseteq R$ such that $R/\mathfrak{p}$ is a Dedekind domain. Thus, we need to find all nonsingular algebraic curves in k^n. Then the M's that we are considering are the submodules of the R-module K, where K is the field of fractions of D. If moreover, for such an $M \neq 0$, we want M to be uniserial, then $R/\mathfrak{p}$ (as isomorphic to a submodule of M) would be uniserial, and so, a discrete valuation domain. But this would correspond to a curve as above with only one point. There is no such curve and consequently no such prime ideal $\mathfrak{p} \subseteq R$.

We note that, if $\mathfrak{n} \subseteq R$ is a maximal ideal, then $\mathfrak{n}$ and $\mathfrak{p}$ as above are comaximal if and only if the point corresponding to $\mathfrak{n}$ is not on the curve corresponding to $\mathfrak{p}$. Two such $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are comaximal if and only if the corresponding curves share no points. Thus, with all this information we can use Proposition 2.3 to describe all our distributive and uniserial R-modules and even up to isomorphism. We note that over $R = k[x_1, \ldots, x_n]$, the uniserial modules are precisely the Artinian distributive modules. This completes the proof of Theorem B.

Furthermore, given an ideal $I \subseteq R$, it is easy to describe the distributive and uniserial modules M over R/I. Note that any such M is a distributive (or uniserial) module over R. As an R-module, we have $IM = 0$. Conversely, if $IM = 0$ for a distributive (uniserial) R-module, then M is a distributive (uniserial) R/I-module. It is easy to pick out such distributive (uniserial) R-modules.

Finally, we close this paper by making some remarks.

Remarks 6.1. (1) Perhaps the logical next step in considering distributive modules is to define and study distributive sheaves of modules on a scheme Σ. A starting point might be the scheme $\mathbb{P}^n(k)$, where k is an algebraically closed field and $n \geq 1$. Thus, for example, in $\mathbb{P}^2(k)$, we have Bézout's theorem (see [13, Corollary 7.8, page 54]), and so, there avoid the situation of having nonintersecting curves.

(2) Another interesting problem is that of finding the distributive (so uniserial) modules over $\mathbb{Z}_p[[\alpha]]$, where $\mathbb{Z}_p$ is the ring of p-adic integer. This problem seems to have connections with the theory of local fields and with number theory.

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