BOUNDS ON HILBERT FUNCTIONS OF SUBALGEBRAS IN DEGREE TWO

JULIAN VILL

ABSTRACT. Given a subspace $U \subseteq \mathbb{C}[x_1, \ldots, x_n]_d$ we consider the closure of the image of the rational map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{\dim U - 1}$ given by U. Its coordinate ring is isomorphic to $\bigoplus_{i \geq 0} U^i$ where U^i is the degree i component. We consider the Hilbert function of this algebra in the case where U contains a regular sequence, equivalently the map is a morphism, and find lower bounds for the dimension of the degree 2 component.

1. Introduction

A well-studied property of ideals in standard-graded polynomial rings is their Hilbert function. A natural question to ask therefore is which sequences of integers can appear as Hilbert functions of such ideals. Macaulay answered this question and showed that whenever there exists an ideal having a certain Hilbert function, there also exists a lex-ideal with the same Hilbert function. In particular, this implies that it suffices to consider monomial ideals.

Later, Eisenbud, Green and Harris [7] asked a more refined question. Given an ideal I in the polynomial ring over $\mathbb{C}$ containing a regular sequence of degrees $d_1, \ldots, d_s$. Can we find a monomial ideal containing $x_i^{d_i}$ $(i = 1, \ldots, s)$ with the same Hilbert function. This question turned out to be very difficult and currently only several special cases have been proven.

We are interested in a similar refinement in a different situation. Let $n, d \in \mathbb{N}$, $n, d \geq 2$. We denote by $A := \mathbb{C}[x_1, \ldots, x_n]$ the polynomial ring over $\mathbb{C}$. We also write $A(n)$ to emphasize the number of variables but usually omit the n in the notation. For the polynomial ring over the reals we write $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$, $\underline{x} = (x_1, \ldots, x_n)$.

Given a positive integer $r \leq \dim A_d$ and a subspace $U \subseteq A_d$ of dimension r, one may consider the rational map $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{r-1}$ defined by U. The coordinate ring of the closure of the image of this map is then given as the subalgebra $\mathbb{C}[U]$ defined by U. It carries a natural $\mathbb{Z}_+$-grading with the degree i component being U^i, the subspace spanned by all homogeneous polynomials $\prod_{j=1}^{i} f_j$ with $f_j \in U$.

In [3] the authors ask the following question: For given r, i, n, d what is the minimum of $\{\dim U^i \colon U \subseteq A_d, \dim U = r\}$. They show that it is attained by a strongly stable subspace. These are monomial subspaces, hence the minimum can be computed in small cases as one only has to check finitely many cases.

We are interested in the analogue of the EGH conjecture in this situation. Namely, determine the minimum if we additionally require the subspace U to be base-point-free, i.e. there is no point $\xi \in \mathbb{P}^{n-1}$ such that for every $f \in U \colon f(\xi) = 0$. This is also equivalent to requiring the rational map defined by U to be a morphism rather than a rational map.

2020 *Mathematics Subject Classification.* 13D40, 13A02.

Key words and phrases. Hilbert function, syzygy, Macaulay inverse, apolarity.

The author thanks the editors and the anonymous referees for the careful reading of the manuscript and their insightful comments and suggestions.

26 Aug 2024 04:30:05 PDT
231122-Vill Version 3 - Submitted to J. Comm. Alg.

As can be expected from the EGH conjecture this makes the problem even harder. Especially, it is no longer clear if it suffices to consider monomial subspaces.

In this paper, we are exclusively interested in the degree 2 component U^2. It is easy to see that the maximal dimension is attained by a generic subspace, i.e. being generic in the Grassmannian of r-planes in affine $(\dim A_d)$-space. One might expect that for a generic subspace the multiplication map $\mathcal{S}_2 U \to U^2$ from the second symmetric power is either injective or surjective. However, as shown in [3, Prop 2.8.] this is not true in general.

Regarding the minimal possible dimension, if we do not require the subspace to be base-point-free, we know we may take a strongly stable subspace. However, every strongly stable subspace of codimension at least 1 has a base-point which makes it useless for our purpose.

We note that in general, even in small cases it is a hard task to determine this value. We do for example not know the precise value already for subspaces of codimension 2 in $\mathbb{R}[x,y]_5$.

It turns out to be more convenient to consider the codimension of such subspaces instead of the dimension. Therefore, we are now interested in the maximum of $\{\operatorname{codim}_{A_{2d}} U^2 \colon U \subseteq A_d,\ \operatorname{codim} U = k,\ U \text{ base-point-free}\}$. Our main result is the following bound for subspaces of small codimension.

Theorem (Theorem 8.12)**.** *Let $k \leq d-1$. Then for every $n \geq 2$ and every base-point-free subspace $U \subseteq \mathbb{C}[x_1, \ldots, x_n]_d$ of codimension k we have*

$$\operatorname{codim}_{\mathbb{C}[\underline{x}]_{2d}} U^2 \leq k^2 + \binom{k+2}{3} = \tfrac{1}{6}(k^3 + 9k^2 + 2k).$$

We want to emphasize that this bound is independent of the number of variables n which heavily contrasts the general case where we allow the subspace to have base-points. If for example U has codimension 1, then the unique strongly stable subspace of codimension 1 satisfies $\operatorname{codim} U^2 = n$.

Next we explain the structure of the paper as well as the methods used in the proofs. In Section 2 we recall some results about strongly stable subspaces and the bounds for $\operatorname{codim} U^2$ found by Boij and Conca [3]. At the end we introduce well-known theorems by Macaulay, Gotzmann and Green concerning the growth of Hilbert functions. Section 3 contains a first easy upper bound whereas in Section 4 we are concerned with subspaces of codimension 1 and 2. Especially the methods used in the codimension 2 case are used again later on. After this we start proving the general bound which is split into Section 5 to Section 8. In the first of the sections we study the dependence of $m(n,d,k) = \max\{\operatorname{codim} U^2 \colon U \subseteq A_d \text{ subspace}, \operatorname{codim} U = k\}$ on the degree d. After that, we analyse how to change the number of variables and end by determining the number $m(k,k,k)$.

2. Preliminaries

In this first section, we recall upper bounds for $\operatorname{codim}_{A_{2d}} U^2$ proven by Boij and Conca [3]. The subspaces realizing the bounds are monomial subspaces and may even be taken to be strongly stable. Strongly stable subspaces and generic initial ideals form one of our main tools for the rest of the paper.

For most results, the monomial order does not play a role. For simplicity, if not stated otherwise, we always work with the lex-ordering. Whenever we consider the orthogonal complement of a subspace, we always work with the apolarity pairing.

Definition 2.1. Let $U \subseteq A_d$ be a monomial subspace. U is called *strongly stable* if for every $1 \leq i \leq n$ the following holds: For every monomial $M \in U$ such that $x_i | M$ and every $i < j \leq n$, the monomial $x_j \frac{M}{x_i}$ is contained in U.

26 Aug 2024 04:30:05 PDT
231122-Vill Version 3 - Submitted to J. Comm. Alg.

For every monomial ordering $\succeq$ where $x_1 < \cdots < x_n$ and every monomial M such that $x_i | M$ we have $x_j \frac{M}{x_i} \succeq M$ for every $j > i$.

Remark 2.2. Another way of thinking about strongly stable subspaces is via their complements. If $U \subseteq A_d$ is strongly stable and $W = U^\perp$, then for every $1 \le i \le n$ the following holds: For every monomial $M \in W$ such that $x_i | M$ and every $1 \le j < i$, the monomial $x_j \frac{M}{x_i}$ is contained in W, i.e. the inequality sign is reversed.

The following statements are all immediate to check from the definition.

Lemma 2.3. *Let $U \subseteq A(n)_d$ be a strongly stable subspace of codimension k.*

(i) The subspace $V := x_1 U \oplus \mathbb{C}[x_2, \ldots, x_n]_{d+1}$ is also strongly stable.
(ii) If $\operatorname{codim}_{A(n)_{2d}} U \le d$, every monomial in $U^\perp$ is divisible by x_1^s with $s := d - k + 1$.
(iii) If $\operatorname{codim}_{A(n)_{2d}} U \le n$ then every monomial in $U^\perp$ is contained in $A(k)_d = \mathbb{C}[x_1, \ldots, x_k]_d$.

The main reason why strongly stable subspaces are particularly useful are the following propositions.

Proposition 2.4 ([6, Theorem 15.18])**.** *Let $I \subseteq A$ be a homogeneous ideal. There exists a Zariski-open subset $V \subseteq \mathrm{GL}_n(\mathbb{C})$ such that for every $G_1, G_2 \in V$ the initial ideals satisfy $\mathrm{in}(G_1 I) = \mathrm{in}(G_2 I)$ where $G_i I := \{p(G_i^{-1} x) \colon p \in I\}$ is the ideal I is mapped to by the coordinate change G_i for $i = 1, 2$.*

Definition 2.5. Let $I \subseteq A$ be a homogeneous ideal and $G \in V$ as in Proposition 2.4 then

$$\mathrm{gin}(I) := \mathrm{in}(GI)$$

is called the *generic initial ideal* of I.

Generic initial ideals have already been used by Hartshorne in 1966 ([10]) to show the connectedness of Hilbert schemes, and later on to get hold of invariants of projective varieties. The first systematic study of generic initial ideals in characteristic 0 was done by Galligo in [8].

Proposition 2.6 ([6, Theorem 15.20, 15.23])**.** *Let $I \subseteq A$ be a homogeneous ideal, then for every $s \ge 0$ the vector space $\mathrm{gin}(I)_s$ is strongly stable.*

The main idea is the following easy observation.

Lemma 2.7. *Let $I \subseteq A$ be a homogeneous ideal then $\mathrm{in}(I)^2 \subseteq \mathrm{in}(I^2)$.*

Remark 2.8. In general we have $\mathrm{in}(U)^2 \subsetneq \mathrm{in}(U^2)$. Consider $U = \operatorname{span}(x_1^2 + x_2^2)^\perp \subseteq A(n)_2$ with $n \ge 3$. Then U is spanned by all monomials except for x_1^2 and x_2^2 and by the binomial $x_1^2 - x_2^2$. The initial ideal $\mathrm{in}(U)_2$ is spanned by all monomials except for x_1^2, i.e. $\mathrm{in}(U)_2 = \operatorname{span}(x_1^2)^\perp$. Now one easily checks that $\operatorname{codim}_{A(n)_{2d}} \mathrm{in}(U^2)_4 = \operatorname{codim}_{A(n)_{2d}} U^2 = 2$ and $\operatorname{codim}_{A(n)_{2d}} (\mathrm{in}(U)_2)^2 = n$.

We want to compare the following two values.

Definition 2.9. For $n \ge 2$ and $d, k \ge 1$ let

$$m(n,d,k) = \max\{\operatorname{codim} U^2 \colon U \subseteq A_d \text{ subspace, } \operatorname{codim} U = k\},$$
$$m^0(n,d,k) = \max\{\operatorname{codim} U^2 \colon U \subseteq A_d \text{ base-point-free subspace, } \operatorname{codim} U = k\}.$$

We say that a subspace $U \subseteq A_d$ of codimension k *realizes* $m(n,d,k)$ (resp. $m^0(n,d,k)$) if $\operatorname{codim}_{A_{2d}} U^2 = m(n,d,k)$ (resp. $= m^0(n,d,k)$).

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As the following proposition shows, the number $m(n,d,k)$ can be computed combinatorially.

Proposition 2.10 ([3, Proposition 2.2]). *For all positive integers $n \geq 2$, $d \geq 2$ and k, there exists a strongly stable subspace $U \subseteq A_d$ of codimension k such that*

$$m(n,d,k) = \operatorname{codim}_{A_{2d}} U^2.$$

Remark 2.11. In fact, any strongly stable subspaces $U \subseteq A_d$ of codimension $k \leq d$ is the space of all forms of degree d vanishing at some k points (counted with multiplicity). Or equivalently, the Hilbert function $t \mapsto \dim (A/\langle U \rangle)_t$ is equal to k for any $t \geq d$. However, it is not clear in general which configuration of k points realizes $m(n,d,k)$.

Remark 2.12. For small n,d,k this is a list of $m(n,d,k)$ for $n = 2,3,4,5,6$. This has been calculated using `SAGE` [13] by first finding all strongly stable subspaces of some fixed codimension and then finding the maximum of all $\operatorname{codim} U^2$.

We have $m(n,d,1) = n$ and $m(n,d,2) = 2n$ in every case shown in the table. This can also easily be checked to hold in general.

Moreover, we see that in these examples $m(n,d,k) = m(n,k,k)$ for any $d \geq k$, i.e. values are constant on the right side of the diagonal. This is always true as we show later (see Corollary 5.5).

Remark 2.13. We now discuss the asymptotic behavior of $m(n,d,k)$. As we have mentioned, for fixed n and k, the number $m(n,d,k)$ stabilizes for large d. To be more precise, we have $m(n,d,k) = m(n,k,k)$ for every $d \geq k$.

Determining exactly the growth of $m(n,d,k)$ for increasing n or k seems to be a rather difficult combinatorial problem. However, it is clear that increasing n or k while fixing the other value and the degree, results in larger values for $m(n,d,k)$. More precisely, we do get a lower bound for the growth using the Alexander-Hirschowitz Theorem.

Let $X \subseteq \mathbb{P}^{n-1}$ be a set of k general points, in the sense of the Alexander-Hirschowitz Theorem. Assume that d is large enough, then the subspace U of forms in A_d vanishing on X has codimension k. By the Alexander-Hirschowitz Theorem, the space V of all forms of degree $2d$ vanishing to order at least 2 at every point of X has codimension kn. Since $U^2 \subseteq V$, it follows that $\operatorname{codim}_{A_{2d}} U^2 \geq kn$, especially $m(n,d,k) \geq kn$. And therefore also $m(n,d',k) \geq kn$ for every $d' \geq k$.

Remark 2.14. Let $U \subseteq A_d$ be a strongly stable subspace such that $U \neq A_d$. If $x_1^d \in U$, we see from the definition that $U = A_d$. Therefore, $x_1^d \in U^\perp$ which reveals a base-point of U. This shows that if $f \in U^2$, then f is singular at the point $(1 : 0 : \cdots : 0)$.

For later reference, we now consider base-point-free monomial subspaces U and find bounds for $\operatorname{codim}_{A_{2d}} U^2$. This will be needed later on as we will reduce to the monomial case.

Lemma 2.15. *Let $d \geq 2$ and let $U \subseteq A_d$ be a base-point-free, monomial subspace of codimension 1. Then the following hold:*

(i) If $d = 2$ then $\operatorname{codim}_{A_{2d}} U^2 = 2$,
(ii) if $d \geq 3$ then $\operatorname{codim}_{A_{2d}} U^2 \in \{0,1\}$.

Proof. Let $U^\perp = \operatorname{span}(M)$ for some monomial $M \in A_d$. Up to permutation of the variables there are only two monomials in A_{2d} that have only one decomposition into a product of monomials of degree d, those are x_1^{2d} and $x_1^{2d-1}x_2$.

	$n=3$								$n=4$							
$d=$	2	3	4	5	6	7	8	9	2	3	4	5	6	7	8	9
codim $U=$																
1	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	4
2	6	6	6	6	6	6	6	6	8	8	8	8	8	8	8	8
3	10	10	10	10	10	10	10	10	13	13	13	13	13	13	13	13
4	12	13	13	13	13	13	13	13	20	20	20	20	20	20	20	20
5	14	16	17	16	16	16	16	16	23	24	25	24	24	24	24	24
6	–	21	21	21	21	21	21	21	26	29	29	31	28	28	28	28
7	–	23	24	24	25	24	24	24	30	35	35	35	37	35	35	35
8	–	25	27	27	28	29	27	27	32	39	40	41	41	43	40	40
9	–	27	30	31	31	32	33	31	34	45	45	45	47	47	49	45

	$n=5$								$n=6$							
$d=$	2	3	4	5	6	7	8	9	2	3	4	5	6	7	8	9
codim $U=$																
1	5	5	5	5	5	5	5	5	6	6	6	6	6	6	6	6
2	10	10	10	10	10	10	10	10	12	12	12	12	12	12	12	12
3	17	16	16	16	16	16	16	16	21	19	19	19	19	19	19	19
4	24	25	24	24	24	24	24	24	28	31	28	28	28	28	28	28
5	35	35	35	35	35	35	35	35	40	40	41	40	40	40	40	40
6	39	40	40	41	40	40	40	40	56	56	56	56	56	56	56	56
7	43	47	45	46	49	45	45	45	61	62	62	62	62	62	62	62
8	48	54	55	54	54	57	54	54	66	71	68	68	68	71	68	68
9	55	60	60	63	61	62	65	59	73	79	81	79	79	79	81	79

Let $T \in A_{2d}$ be any monomial that is not x_1^{2d} or $x_1^{2d-1}x_2$ (after permutation of the variables). Then there are two decompositions into a product of two monomials of degree d. Especially, one of the decompositions does not use the monomial M, hence $T \in U^2$.

The decompositions of the two monomials above are $x_1^{2d} = (x_1^d)(x_1^d)$ and $x_1^{2d-1}x_2 = x_1^d(x_1^{d-1}x_2)$. Therefore, both are not contained in U^2 if and only if $M = x_1^d$, and only $x_1^{2d-1}x_2$ is not contained in U^2 if and only if $M = x_1^{d-1}x_2$. In the first case, U has a base-point, in the second case the only monomial not contained in U^2 is $x_1^{2d-1}x_2$ if $d \geq 3$ and thus $\operatorname{codim}_{A_{2d}} U^2 = 1$.

If $d = 2$ and U is base-point-free, $U^\perp$ is spanned by x_1x_2 (after permutation of the variables). We easily check that U^2 contains every monomial of degree 4 except for $x_1^3x_2$ and $x_1x_2^3$. Hence $\operatorname{codim}_{A_{2d}} U^2 = 2$. □

Lemma 2.16. *Let $U \subseteq A_d$ be a base-point-free, monomial subspace of codimension 2. Then the following hold:*

(i) $\operatorname{codim}_{A_{2d}} U^2 \leq 6$ *if* $d = 2$,
(ii) $\operatorname{codim}_{A_{2d}} U^2 \leq 4$ *if* $d \in \{3, 4\}$,
(iii) $\operatorname{codim}_{A_{2d}} U^2 \leq 2$ *if* $d \geq 5$.

Moreover the bound is tight for $d \leq 4$.

Proof. The proof works the same as for Lemma 2.15 by considering all monomials with at most two distinct decompositions. □

For the following sections, we need some knowledge about the Hilbert functions of ideals generated by subspaces. We introduce theorems of Macaulay and Gotzmann concerning Hilbert functions and Green's Hyperplane Restriction Theorem for later reference.

Definition 2.17. Let $a, d \in \mathbb{N}$, then a can be uniquely written in the form

$$a = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1},$$

where $k(d) > k(d-1) > \cdots > k(1) \geq 0$, called the *d-th Macaulay representation* of a (see [4, Lemma 4.2.6.]). For any integers $s, t \in \mathbb{Z}$ define

$$a_{(d)}|_t^s := \binom{k(d)+s}{d+t} + \binom{k(d-1)+s}{d-1+t} + \cdots + \binom{k(1)+s}{1+t}.$$

Furthermore for $a < b$ we define $\binom{a}{b} = 0$.

Theorem 2.18 (Macaulay's Theorem, [11, Corollary C.7.], [4, Theorem 4.2.10])**.** *Let $I \subseteq A$ be a homogeneous ideal and let $H = (h_i)_{i \geq 0}$ be the Hilbert function of I. Then*

(i) $h_{i+1} \leq (h_i)_{(i)}|_1^1$ *for every* $i \geq 0$, *and*
(ii) if there exists $j \in \mathbb{N}$ *such that* $j \geq h_j$, *then* $h_i \geq h_{i+1}$ *for every* $i \geq j$.

In fact, Macaulay showed in 1927 [12] that whenever we have a sequence $H = (h_i)_{i \geq 0}$ which satisfies property (i) in Theorem 2.18, there exists $n \geq 2$ and a homogeneous ideal $I \subseteq A$ such that the Hilbert function of I is exactly H. This ideal I can even be chosen to be monomial.

Theorem 2.19 (Gotzmann's Persistence Theorem, [11, Corollary C.17.], [1, Theorem 2.6])**.** *Let $d \geq 0$ be an integer and let I be a homogeneous ideal that is generated in degrees at most d ($I = \langle I_{\leq d} \rangle$). Denote by $H = (h_i)_{i \geq 0}$ the Hilbert function of I. If $h_{d+1} = (h_d)_{(d)}|_1^1$, then $h_{d+l} = (h_d)_{(d)}|_l^l$ for all $l \geq 1$.*

By Macaulay's Theorem $h_{d+1} \leq (h_d)_{(d)}|_1^1$. Therefore, Gotzmann's Theorem determines the complete Hilbert function whenever we have maximal growth from some degree d to the next degree $d+1$. Namely, the growth is maximal for all following degrees as well. For ideals generated by subspaces, this has the following meaning.

Corollary 2.20. *Let $U \subseteq A_d$ be a subspace of codimension $k \leq d$ and let $H = (h_i)$, $h_i := h_{\langle U \rangle}(i)$ be the Hilbert function of $\langle U \rangle$. Then*

(i) $h_{d+1} = \operatorname{codim}_{A_{d+1}} A_1 U \leq k$ *and*
(ii) if $h_{d+1} = k$, *then* $h_{d+i} = \operatorname{codim}_{A_{d+i}} A_i U = k$ *for all* $i \geq 1$.

In case (ii) $\mathcal{V}(U) \neq \varnothing$ *is finite.*

Proof. (i): We have $\operatorname{codim}_{A_d} U = h_d = k \leq d$ and therefore $\operatorname{codim}_{A_{d+1}} A_1 U = h_{d+1} \leq h_d = k$ by Theorem 2.18 (ii).

(ii): We first note that the d-th Macaulay representation of h_d is given by

$$h_d = \binom{d}{d} + \cdots + \binom{d-k+1}{d-k+1} = \sum_{i=0}^{k-1} \binom{d-i}{d-i}$$

and therefore $(h_d)_{(d)}|_1^1 = h_d = k$. Hence, the assumption $h_{d+1} = k = (h_d)_{(d)}|_1^1$ allows us to use Theorem 2.19 from which we get

$$h_{d+i} = (h_d)_{(d)}|_i^i = \binom{d+i}{d+i} + \cdots + \binom{d-h_d+1+i}{d-h_d+1+i} = k$$

for every $i \geq 1$.

In (ii) the Hilbert polynomial is the constant polynomial k, hence $\mathcal{V}(U)$ is non-empty and finite. □

Corollary 2.21. *Let $U \subseteq A_d$ be a base-point-free subspace with $\operatorname{codim}_{A_d} U = k \leq d$. Then $h_{\langle U \rangle}(2d-1) \leq 1$. If $k < d$ then $h_{\langle U \rangle}(2d-1) = 0$.*

Proof. The Hilbert function of $\langle U \rangle$ has to be smaller than $(\dots, k, k-1, k-2, \dots, 1, 0)$ (dimension dropping by at least 1 in every degree): indeed, if we had equality in any two consecutive degrees s and $s+1$ with $s \geq d$ such that $h_s \neq 0$, it follows from Corollary 2.20 (ii) that $\mathcal{V}(U) \neq \varnothing$. Therefore, we get the inequality on the degree $2d-1$ component of $A/\langle U \rangle$. □

For the degree $2d$ component, there is a stronger result due to Blekherman using Cayley-Bacharach duality.

Theorem 2.22 ([2, Theorem 2.5.])**.** *Let $n \geq 3$, $d \geq 3$ and let $U \subseteq A_d$ be a base-point-free subspace. If $\operatorname{codim}_{A_d} U < 3d-2$, then $UA_d = A_{2d}$. If $n \geq 4$, $d = 2$ and $\operatorname{codim}_{A_2} U < 5$, then $UA_2 = A_4$.*

Remark 2.23. If we would use the same argument as in Corollary 2.21, we only get $h_{\langle U \rangle}(2d) = 0$ if $\operatorname{codim}_{A_d} U \leq d$, instead of whenever $\operatorname{codim}_{A_d} U < 3d-2$ (< 5 if $d = 2$).

This also shows that the bound $\operatorname{codim}_{A_d} U \leq d$ in Corollary 2.21 is far off from being necessary to obtain $h_{\langle U \rangle}(2d-1) = 0$ in general.

However, Theorem 2.22 only tells us something about UA_d and not about UA_{d-1} and the proof does not easily generalize to other degrees but is very specific to the degree $2d$ component UA_d.

Definition 2.24. Let $I \subseteq A$ be a homogeneous ideal and $p \in A_s$ for some $s \geq 1$. We define the *ideal quotient*

$$(I : p) := \bigoplus_{\nu \geq 0} (I : p)_\nu$$

where

$$(I : p)_\nu := \{q \in A_\nu : pq \in I\} \subseteq A_\nu$$

for every $\nu \geq 0$. If $U \subseteq A_d$ is a subspace, we write $(U : p) := (\langle U \rangle : p)_{d-s} \subseteq A_{d-s}$.

2.25. We consider the following setup. Let $I \subseteq A$ be a homogeneous ideal and $l \in A_1$ a linear form. We have the graded exact sequence

$$0 \to A/(I : l)(-1) \xrightarrow{\cdot l} A/I \to A/\langle I, l \rangle \to 0.$$

Let $h_i = \dim(A/I)_i$ and $c_i = \dim(A/(I,l))_i$.

In this situation, we have the following theorem due to Green.

Theorem 2.26 (Green's Hyperplane Restriction Theorem, [9, Theorem 1])**.** *For any $d \geq 0$ and a generic linear form $l \in A_1$ we have*

$$c_d \leq (h_d)_{(d)}|_0^{-1}.$$

This can either be seen as a lower bound for $\dim\langle I, l\rangle_d$ or equivalently as an upper bound for $\dim(I : l)_{d-1}$ which tells us how many elements in I are divisible by l.

Notation-wise this means that if $h_d = \binom{k(d)}{d} + \binom{k(d-1)}{d-1} + \cdots + \binom{k(1)}{1}$, then

$$c_d \leq \binom{k(d)-1}{d} + \binom{k(d-1)-1}{d-1} + \cdots + \binom{k(1)-1}{1}.$$

Example 2.27. Let $U \subseteq A_d$ be a subspace of codimension 1 and let $l \in A_1$ be a generic linear form. This means $h_d = A_d/U = 1 = \binom{d}{d}$. Therefore, Theorem 2.26 shows

$$c_d \leq \binom{d-1}{d} = 0.$$

On the one hand, this means $\langle U, l\rangle_d = A_d$, and on the other hand

$$\operatorname{codim}_{A_{d-1}}(U : l)_{d-1} = \operatorname{codim}_{A_d} U - \dim\langle I, l\rangle_d = 1.$$

I.e. the subspace $(U : l)_{d-1}$ also has codimension 1.

3. A first upper bound for base-point-free subspaces

One way to get a good upper bound for base-point-free subspaces of small dimension is the following.

Proposition 3.1. *Let $U \subseteq A_d$ be a base-point-free subspace of dimension r. Then*

$$\dim U^2 \geq nr - \binom{n}{2}.$$

Proof. Since U is base-point-free it follows that $\dim U \geq n$ and U contains a regular sequence $p_1, \ldots, p_n$. Consider the map

$$\prod_{i=1}^{n} U \to A_{2d}, \quad (q_1, \ldots, q_n) \mapsto \sum_{i=1}^{n} p_i q_i.$$

Since $p_1, \ldots, p_n$ is a regular sequence, the syzygies are the ones coming from the Koszul complex of the sequence. Namely, the kernel is spanned by the vectors $(0, \ldots, 0, p_j, 0, \ldots, 0, -p_i, 0, \ldots, 0)$ with $i < j$ and p_j at position i in the vector and $-p_i$ at position j. There are exactly $\binom{n}{2}$ of those vectors, hence the image $\operatorname{span}(p_1, \ldots, p_r)U$ has dimension $n \cdot \dim U - \binom{n}{2}$ and the image is contained in U^2. □

Corollary 3.2. *For any $n, d \geq 1$, $1 \leq k \leq \dim A_d$ we have*

$$m^0(n, d, k) \leq \binom{n-1+2d}{n-1} + \binom{n}{2} + nk - n\binom{n-1+d}{n-1}$$

Example 3.3. The easiest example where this bound is tight is $U = \operatorname{span}(x_1^d, \ldots, x_n^d)$. Since U is spanned by a regular sequence the condition $\operatorname{span}(p_1, \ldots, p_n)U = U^2$ is certainly true.

In general, one should expect this bound to be good whenever the dimension of the subspace is close to n and rather bad whenever $\operatorname{codim} U$ is small.

However, the bound can also be tight or almost tight even for larger subspaces. Indeed, in any of the following cases, there exists a base-point-free subspace $U \subseteq A_d$ of dimension r such that the bound in Proposition 3.1 is tight.

(i) $r = n$,
(ii) d is even, $n \geq 3$ and $r = n + 3$,
(iii) $s := r - n + 1 \in \mathbb{N}$ and s divides d.

In the next two cases, there exists a base-point-free subspace $U \subseteq A_d$ of dimension r such that the bound is 1 off, i.e. $\dim U^2 = nr - \binom{n}{2} + 1$.

(iv) d is even, $n \geq 4$ and $r = n + 6$,
(v) $3|d$, $n \geq 3$ and $r = n + 7$.

The proof is not hard but we do not include a proof here as this is not needed later. The main reason is that the second Veronese of $\mathbb{P}^2$ and the s-th Veronese of $\mathbb{P}^1$ ($s \in \mathbb{N}$) are both varieties of minimal degree and the second (resp. third) Veronese of $\mathbb{P}^3$ (resp. $\mathbb{P}^2$) is an arithmetically Cohen-Macaulay variety of almost minimal degree.

The third case is especially interesting since it shows that if the degree is large enough, there exist subspaces of any dimension in any number of variables such that the bound is tight.

The main downside of this bound is that for large subspaces, i.e. small codimension, this bound depends on n which is not necessary as shown in Theorem 8.12. From now on we consider only the case where the codimension of U is small.

4. Subspaces of codimension 1 and 2

We start by determining bounds for $\operatorname{codim}_{A_{2d}} U^2$ in the cases $\operatorname{codim}_{A_d} U = 1, 2$. We show that there is a uniform bound for $\operatorname{codim}_{A_{2d}} U^2$ not depending on n or d. This is also our main motivation for the next sections where we generalize this result to higher codimensions. Furthermore, we show Theorem 4.5 which is our main tool in the next sections to reduce the number of variables.

Lemma 4.1. *Let $U \subseteq A_d$ be a base-point-free subspace and $W := U^\perp$. If $\mathcal{V}(W)$ is not contained in any linear variety of codimension 2, then there exists a change of coordinates such that $\operatorname{in}(U)_d$ is base-point-free.*

In the case $n = 2$, this should be understood as $\mathcal{V}(W) \neq \varnothing$, i.e. $\dim \mathcal{V}(W) \in \{0, 1\}$.

Proof. By assumption there exist linearly independent linear forms $l_1, \dots, l_{n-1} \in A_1$ such that $l_1^d, \dots, l_{n-1}^d \in U$. After a change of coordinates, we can assume that $x_1^d, \dots, x_{n-1}^d \in U$. Since $x_1 < x_2 < \cdots < x_n$, it holds that $x_n^d \geq x^\alpha$ for any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n = \{(a_1, \dots, a_n) \in \mathbb{Z}^n : a_i \geq 0, \forall\, i = 1, \dots, n\}$, $|\alpha| := \sum_{i=1}^n \alpha_i = d$. Since U is base-point-free, there exists a form in U such that x_n^d occurs in it. Hence, $\operatorname{in}(U)_d$ contains $x_1^d, \dots, x_n^d$ which shows that the subspace $\operatorname{in}(U)_d$ is base-point-free. □

With this lemma in place, we look at subspaces of codimension 1. Firstly, we consider the simple case where our subspace does have a base-point.

Lemma 4.2. *If $U \subseteq A_d$ is a subspace of codimension 1 and U has a base-point, then $\operatorname{codim}_{A_{2d}} U^2 = n$.*

Proof. We can apply a change of coordinates such that $U^\perp = \operatorname{span}(x_1^d)$. Then U is the subspace spanned by all monomials except x_1^d. Now we see that for every $1 \leq i \leq n$ the monomial $x_1^{2d-1} x_i$ is not contained in U^2 and thus $\operatorname{codim}_{A_{2d}} U^2 = n$. □

Proposition 4.3. *Let $d \geq 2$ and $U \subseteq A_d$ be a base-point-free subspace of codimension 1. Then the following hold:*

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(i) If $d \geq 3$*, then* $\operatorname{codim}_{A_{2d}} U^2 \leq 1$*,*
(ii) if $d = 2$*, then* $\operatorname{codim}_{A_{2d}} U^2 \leq 2$*.*

Proof. Write $W := U^\perp = \operatorname{span}(q)$ for some $q \in A_d$. No hypersurface is contained in a linear variety of codimension 2, hence by Lemma 4.1 we can apply a change of coordinates and assume that the subspace $\operatorname{in}(U)_d$ is base-point-free. Then $\dim U^2 = \dim \operatorname{in}(U^2)_{2d} \geq \dim (\operatorname{in}(U)^2)_{2d}$ by Lemma 2.7. It therefore suffices to check the claim for monomial subspaces which is done in Lemma 2.15. □

Remark 4.4. In the case $d = 2$ it is even true that $\operatorname{codim}_{A_4} U^2 \in \{0, 2\}$, i.e. the case $\operatorname{codim}_{A_4} U^2 = 1$ does not occur. This can be shown by considering the orthogonal complement of U. It is spanned by a quadratic form $q \in A_2$ of rank at least 2. As quadratic forms are diagonalizable, one can reduce to a combinatorial situation. If the rank is equal to 2, the codimension of U^2 is 2, in any other case $U^2 = A_4$.

Now we turn to the codimension 2 case. We find a bound for $\operatorname{codim}_{A_{2d}} U^2$ by reducing either to monomial subspaces or subspaces of binary forms.

First, we show how to reduce the number of variables. The idea of the proof is the following: if $U \subseteq A[x_{n+1}]_d = \mathbb{C}[x_1, \ldots, x_{n+1}]_d$ is a subspace of the form $U = x_{n+1}A[x_{n+1}]_{d-1} \oplus U'$ with $U' \subseteq A_d$, then $U^2 = x_{n+1}^2 A[x_{n+1}]_{2d-2} \oplus x_{n+1}A_{d-1}U' \oplus (U')^2$. This shows

$$\operatorname{codim}_{A(n+1)_{2d}} U^2 = \operatorname{codim}_{A(n)_{2d}} (U')^2 + \operatorname{codim}_{A(n)_{2d-1}} A_{d-1}U'.$$

If U does not have this nice form, we have to argue slightly more carefully using the same idea.

Theorem 4.5. *Let* $U \subseteq A_d$ *be a subspace of codimension* k*. If there exists* $2 \leq m \leq n$ $(R := A(m))$ *such that* $U' := U \cap R_d$ *satisfies* $\operatorname{codim}_{R_d} U' = k$*, then*

$$\operatorname{codim}_{A_{2d}} U^2 \leq (n - m) \operatorname{codim}_{R_{2d-1}} U'R_{d-1} + \operatorname{codim}_{R_{2d}} (U')^2.$$

Proof. Let $\mathfrak{m} = \langle x_{m+1}, \ldots, x_n \rangle \subseteq A_d$, then $\mathfrak{m}_d = \sum_{i=m+1}^n x_i A_{d-1}$. We write

$$U = U' \oplus V \oplus W$$

with $U' \subseteq R_d$, $V \subseteq \mathfrak{m}_d$ and $W = \operatorname{span}(p_i + q_i \colon i = 1, \ldots, s)$ where $p_i \in R_d$ and $0 \neq q_i \in \mathfrak{m}_d$ for $i = 1, \ldots, s$. By assumption $\operatorname{codim}_{R_d} U' = k$ which means $U + R_d = A_d$ and thus

$$V \oplus \operatorname{span}(q_1, \ldots, q_s) = \mathfrak{m}_d. \tag{1}$$

Calculating U^2 we get

$$U^2 = (U')^2 + (V + W)^2 + U'(V + W).$$

Since we are working with the lex-ordering (and $x_1 < \cdots < x_n$), any monomial of degree d containing any x_i, $i \geq m + 1$ is bigger than any monomial in R_d.

Firstly, fix any monomial x^α such that $\alpha \in \mathbb{Z}_+^n, |\alpha| = 2d$ and $\sum_{j \geq m+1} \alpha_j \geq 2$, then there exist $\beta, \gamma \in \mathbb{Z}_+^n, |\beta| = |\gamma| = d$ and x_i, x_j, $i, j \geq m + 1$ such that $x_i | x^\beta, x_j | x^\gamma$ and $x^\alpha = x^\beta x^\gamma$. Then we have $x^\beta + p_\beta$, $x^\gamma + p_\gamma \in V + W$ for some $p_\beta, p_\gamma \in R_d$. Hence

$$x^\alpha = \operatorname{in}((x^\beta + p_\beta)(x^\gamma + p_\gamma)) \in \operatorname{in}((V + W)^2)_{2d} \subseteq \operatorname{in}(U^2)_{2d}.$$

Secondly, we have

$$\operatorname{in}(U'(V + W)) \overset{Eq.\ (1)}{\supseteq} \operatorname{in}(U'\mathfrak{m}_d) = \operatorname{in}\left(\bigoplus_{i=m+1}^n x_i(U'A_{d-1}) \right) \supseteq \operatorname{in}\left(\bigoplus_{i=m+1}^n x_i(U'R_{d-1}) \right).$$

This shows that for every $i = m+1, \dots, n$ we have

$$\mathfrak{m}^2 A_{2d-2},\ \mathrm{in}((U')^2)_{2d},\ \mathrm{in}(x_i U' R_{d-1})_{2d} \subseteq \mathrm{in}(U^2)_{2d}.$$

Counting dimensions, we get

$$\mathrm{codim}_{A_{2d}} \mathrm{in}(U^2)_{2d} \le (n-m)\,\mathrm{codim}_{R_{2d-1}} U' R_{d-1} + \mathrm{codim}_{R_{2d}} (U')^2.$$

□

Remark 4.6. The bound is sharp whenever $U = (x_{m+1}, \dots, x_n)A_{d-1} \oplus U'$ as can be seen from the comment above Theorem 4.5.

Corollary 4.7. *If the subspaces U, U' in Theorem 4.5 are base-point-free and $k \le d-1$, then*

$$\mathrm{codim}_{A_{2d}} U^2 \le \mathrm{codim}_{R_{2d}} (U')^2.$$

Proof. By Corollary 2.21 the degree $2d-1$ component of $R/\langle U' \rangle$ has dimension 0. Therefore, the result follows from Theorem 4.5. □

Theorem 4.8. *Let $U \subseteq A_d$ be a base-point-free subspace of codimension 2. Then the following hold:*

(i) If $d = 2$, then $\mathrm{codim}_{A_4} U^2 \le 6$,
(ii) if $d \ge 3$, then $\mathrm{codim}_{A_{2d}} U^2 \le 4$.

For $d \le 4$ the bounds are tight.

Proof. Let $W = U^\perp$. If $\mathcal{V}(W) \ne \mathcal{V}(l, l')$ for any two linear forms $l, l' \in A_1$, the claim follows from Lemma 4.1 and Lemma 2.16 with the same arguments as in the codimension 1 case (Proposition 4.3) as we can reduce to base-point-free monomial subspaces.

Otherwise we can assume after a change of coordinates that $\mathcal{V}(W) = \mathcal{V}(x_1, x_2)$ and thus $x_3^d, \dots, x_n^d \in U,\ x_1^d, x_2^d \notin U$. Hence, we can write

$$U = \mathrm{span}(x^\alpha + \nu_\alpha x_1^d + \lambda_\alpha x_2^d \colon \alpha \in \mathbb{Z}_+^n,\ |\alpha| = d,\ \exists\, i \ge 3 \colon x_i | x^\alpha) \oplus U'$$

where $U' \subseteq \mathbb{C}[x_1, x_2]_d$ is a subspace of codimension 2. We distinguish two cases. Either (a) for all α we have $\nu_\alpha = \lambda_\alpha = 0$, or (b) there exists α such that $(\nu_\alpha, \lambda_\alpha) \ne (0,0)$.

(a): Here U has the form

$$U = \mathrm{span}(x_3, \dots, x_n)A_{d-1} \oplus U'.$$

If $d = 2$, this case cannot occur since $\dim U' = 1$ and thus U has a base-point. Hence, we can assume that $d \ge 3$. Since U is base-point-free it follows that U' is base-point-free as a subspace of $\mathbb{C}[x_1, x_2]_d$. Then $\mathrm{codim}_{A_{2d}} U^2 \le \mathrm{codim}_{A(2)_{2d}} (U')^2 \le 4$ by Corollary 4.7.

(b): Fix $\alpha \in \mathbb{Z}_+^n$, $|\alpha| = d$ such that $(\nu_\alpha, \lambda_\alpha) \ne (0,0)$. Consider the subspace

$$V := U' \oplus \mathrm{span}(\nu_\alpha x_1^d + \lambda_\alpha x_2^d) \subseteq \mathbb{C}[x_1, x_2]_d.$$

This subspace has codimension 1 and thus $V^\perp = \mathrm{span}(h)$ for some $h \in \mathbb{C}[x_1, x_2]_d$. Especially, there exists $l \in \mathbb{C}[x_1, x_2]_1$ such that $l^d \in V$, namely the one evaluating h in one of its zeroes. Therefore, there exist $a \in \mathbb{C}$ and $\beta \in \mathbb{Z}_+^n$, $|\beta| = d$, $x^\beta \notin \mathbb{C}[x_1, x_2]$ such that $ax^\beta + l^d \in U$. Write $x^\beta = x_1^{\beta_1} x_2^{\beta_2} M$ with $M \in \mathbb{C}[x_3, \dots, x_n]$. Let ϕ be a change of coordinates on $\mathbb{C}[x_1, x_2]$ that maps l to x_2. Then $aMg + x_2^d \in \phi(U)$ with $g \in \mathbb{C}[x_1, x_2]$ the image of $x_1^{\alpha_1} x_2^{\alpha_2}$ under ϕ. Now take any monomial ordering such that $x_1 > x_2 > \dots > x_n$ and such that x_2^d is greater than any monomial in Mg, for example, the ordering given in Remark 4.9. With respect to this ordering $\mathrm{in}(\phi(U))_d$ contains $x_1^d, \dots, x_n^d$ and is therefore base-point-free: the monomials

$x_3^d, \dots, x_n^d$ are contained in U and therefore in $\phi(U)$ by assumption. The monomial x_2^d is the initial monomial of $aMg + x_2^d$ and x_1^d appears in some form in $\phi(U)$ since it is base-point-free and by the choice of the monomial ordering x_1^d is the initial monomial of that form. Now we finish as earlier, $\operatorname{codim}_{A_{2d}} U^2 = \operatorname{codim}_{A_{2d}} \phi(U)^2 \leq \operatorname{codim}_{A_{2d}} (\operatorname{in}(\phi(U))_d)^2$ and using Lemma 2.16 we get the bounds we wanted.

The bounds are tight for $d \leq 4$ by Lemma 2.16. □

Remark 4.9. We want to define a monomial ordering such that $x_1 > x_2 > \cdots > x_n$ and such that x_2^d is greater than any monomial of degree d that is divisible by x_i for any $i \in \{3, \dots, n\}$.

We consider a block ordering $\succeq_b$ on the sets $\{x_1, x_2\}$ and $\{x_3, \dots, x_n\}$ and on each set the graded-lexicographic-ordering (grlex). Let α, $\beta \in \mathbb{Z}_+^n$, then the grlex ordering is defined as

$$x^\alpha \succeq_{grlex} x^\beta : |\alpha| > |\beta| \text{ or } |\alpha| = |\beta| \text{ and } x^\alpha \succeq_{lex} x^\beta$$

where $\succeq_{lex}$ is the usual lex-ordering and the variables are ordered as $x_1 > x_2 > \cdots > x_n$. Then the block-ordering is defined as follows. Let α, $\beta \in \mathbb{Z}_+^n$, then

$$x^\alpha \succeq_b x^\beta :\Leftrightarrow x_1^{\alpha_1} x_2^{\alpha_2} \succ_{grlex} x_1^{\beta_1} x_2^{\beta_2} \text{ or } x_1^{\alpha_1} x_2^{\alpha_2} = x_1^{\beta_1} x_2^{\beta_2} \text{ and } \frac{x^\alpha}{x_1^{\alpha_1} x_2^{\alpha_2}} \succeq_{grlex} \frac{x^\beta}{x_1^{\beta_1} x_2^{\beta_2}}.$$

Now let $x^\alpha \in A_d$ be any monomial such that $\alpha_i > 0$ for some $i \in \{3, \dots, n\}$. Then $\alpha_1 + \alpha_2 < d$ and therefore $x_2^d \succ_b x^\alpha$.

Remark 4.10. The idea to choose a monomial ordering such that the degree d component of the initial ideal is base-point-free is unlikely to work as easily for higher codimensions.

The lex-plus-powers conjecture (or EGH conjecture) due to Eisenbud, Green, and Harris [7] predicts that for any homogeneous ideal $I \subseteq A$ containing a regular sequence $p_1, \dots, p_n$ with $d_i := \deg p_i$, there is also a monomial ideal containing $x_i^{a_i}$ for $i = 1, \dots, n$ with the same Hilbert function as I. The conjecture has only been proven in some special cases, see for example [5].

This is certainly not exactly what we are after. On the one hand, we are only interested in the case where all p_i are of the same degree and the ideal is generated in that degree. On the other hand, it is not enough that the monomial ideal has the same Hilbert function, since the Hilbert function of I does not determine the Hilbert function of I^2.

This however shows that we should not expect a reduction to monomial ideals to easily work in more general cases.

We have seen that in the codimension 1 case the codimension of U^2 can be bounded by 1 if $d \geq 3$ (resp. 2 if $d = 2$) and in the case $\operatorname{codim}_{A_d} U = 2$, the codimension can be bounded by 4 if $d \geq 3$ (resp. 6 if $d = 2$). We would like to generalize this to higher codimensions. It seems that the correct way to do this is to show that there is a bound for $\operatorname{codim}_{A_{2d}} U^2$ that is independent of n or d, as long as d is large enough.

We have already seen how we can reduce the number of variables and therefore make our bounds independent of n using Theorem 4.5.

In the next section, we show how to find bounds that are independent of the degree d.

5. Reducing the degree

In this section, we show that for $d \geq k$ the function $d \mapsto m(n, d, k)$ is constant for every fixed n, k. By definition, this is equivalent to showing that for certain subspaces

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$U \subseteq A_d$ of codimension k there exists a subspace $V \subseteq A_{d-1}$ of codimension k such that $\operatorname{codim}_{A_{2d}} U^2 = \operatorname{codim}_{A_{2d-2}} V^2$ whenever $d > k$.

For the next proofs, let us recall that for any subspace $U \subseteq A_d$ and any $l \in A_1$ we have the exact sequence in 2.25:

$$0 \to A_{d-1}/(U : l) \to A_d/U \to A_d/\langle U, l\rangle_d \to 0.$$

Especially, if $\langle U, l\rangle_d = A_d$ it follows that $\operatorname{codim}_{A_{d-1}}(U : l) = \operatorname{codim}_{A_d} U$.

Lemma 5.1. *Let $U \subseteq A_d$ be a subspace of codimension k and $k \leq d$. Then for a generic linear form $l \in A_1$ we have $\langle U, l\rangle_d = A_d$ and $\operatorname{codim}_{A_{d-1}}(U : l) = \operatorname{codim}_{A_d} U$.*

Proof. With the notation from 2.25 and with $I = \langle U\rangle$ we have

$$h_I(d) = h_d = k = \sum_{i=0}^{k-1} \binom{d-i}{d-i}$$

since $k \leq d$. Hence, by Green's Theorem

$$\dim A_d/\langle U, l\rangle_d = c_d \leq (h_d)_{(d)}|_0^{-1} = \sum_{i=0}^{k-1} \binom{d-i-1}{d-i} = 0$$

which means $A_d/\langle U, l\rangle_d = 0$, and therefore the first claim follows. The second one is immediate from the exact sequence above. □

Theorem 5.2. *Let $U \subseteq A_d$ be a subspace of codimension $k \leq d$ and let $l \in A_1$ be a generic linear form. With $V := (U : l) \subseteq A_{d-1}$ the following inequality holds*

$$\operatorname{codim}_{A_{2d}} U^2 \leq \operatorname{codim}_{A_{2d-1}} UV.$$

If furthermore $k \leq d-1$, then

$$\operatorname{codim}_{A_{2d}} U^2 \leq \operatorname{codim}_{A_{2d-2}} V^2.$$

Proof. Since l is generic and $k \leq d$ it follows from Lemma 5.1 that $\operatorname{codim}_{A_{d-1}} V = \operatorname{codim}_{A_d} U$ and $\langle U, l\rangle_d = A_d$. Furthermore, we have

$$A_{2d} = (\langle U, l\rangle_d)^2 \subseteq \langle U^2, l\rangle_{2d},$$

hence $\operatorname{codim}_{A_{2d-1}}(U^2 : l) = \operatorname{codim}_{A_{2d}} U^2$ by the exact sequence in 2.25. Since $UV \subseteq (U^2 : l)$ we have

$$\operatorname{codim}_{A_{2d}} U^2 = \operatorname{codim}_{A_{2d-1}} (U^2 : l) \leq \operatorname{codim}_{A_{2d-1}} UV.$$

Now we do the same for UV if $k \leq d-1$. If we show that $\langle V, l\rangle_{d-1} = A_{d-1}$, then

$$A_{2d-1} = \langle U, l\rangle_d \langle V, l\rangle_{d-1} \subseteq \langle UV, l\rangle_{2d-1}.$$

Thus $\operatorname{codim}_{A_{2d-2}}(UV : l) = \operatorname{codim}_{A_{2d-1}} UV$ and $V^2 \subseteq (UV : l)$ which means $\operatorname{codim}_{A_{2d-1}} UV \leq \operatorname{codim}_{A_{2d-2}} V^2$.

It is left to show that $\langle V, l\rangle_{d-1} = A_{d-1}$. This is equivalent to showing that $\operatorname{codim}_{A_{d-2}}(V : l) = \operatorname{codim}_{A_{d-1}} V$. Since $((U : l) : l) = (U : l^2)$ this again is equivalent to showing that $\operatorname{codim}_{A_{d-2}}(U : l^2) = \operatorname{codim}_{A_{d-1}} V = \operatorname{codim}_{A_d} U$ or $\langle U, l^2\rangle_d = A_d$. Since $l \in A_1$ is generic, we can also apply a generic change of coordinates to U, hence assume that $\operatorname{in}(U) = \operatorname{gin}(U)$ and $l = x_1$. Then

$$\dim \langle U, x_1^2\rangle_d = \dim \operatorname{in}(\langle U, x_1^2\rangle)_d \geq \dim \langle \operatorname{in}(U), x_1^2\rangle_d.$$

Here the first equality follows from the fact that any ideal and its initial ideal have the same Hilbert function, the second one is immediate since $\text{in}(\langle U, x_1^2\rangle) \supseteq \langle \text{in}(U), x_1^2\rangle$.

It is therefore enough to show that $\langle \text{gin}(U), x_1^2\rangle_d = A_d$. Since $k \leq d-1$ every monomial of degree d not contained in $\text{gin}(U)_d$ is divisible by x_1^2. But this means exactly that $\text{gin}(U)_d + x_1^2 A_{d-2} = A_d$. □

Remark 5.3. (i) The reason we pass to initial ideals in the second part of the proof is that we need to show $\langle UV, l\rangle_{2d-1} = A_{2d-1}$. As we have seen $\langle U, l\rangle_d = A_d$ and if we take another generic linear form l' we also have $\langle V, l'\rangle_{d-1} = A_{d-1}$. However, since $V = (U : l)$ we do not know that l behaves generically for V.

(ii) It is not true in general that $(U : l)$ is base-point-free if U is. Let $n = 3$ and let $U = \text{span}(x^2y, x^2z, xy^2)^\perp \subseteq \mathbb{C}[x, y, z]_3$. Then U contains $A_1 \,\text{span}(yz, z^2)$, and thus for a generic linear form $l \in A_1$ we have $(U : l) = \text{span}(yz, z^2) \oplus \text{span}(p)$ for some $p \in A_2$. Hence, the space $(U : l)$ has a base-point, namely $\mathcal{V}(z, p)$.

One can show however that $(U : l)$ is base-point-free whenever the degree is large enough.

Now we show the reverse inequality from Theorem 5.2 in the case of strongly stable subspaces.

Proposition 5.4. *Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k \leq d-1$ and let $V := (U : x_1)$. Then $\text{codim}_{A_{2d-2}} V^2 \leq \text{codim}_{A_{2d}} U^2$*

Proof. Let $M \in A_{2d-2} \setminus V^2$. We show that if $x_1^2 M \notin U^2$, then $x_1^2((V^2)^\perp) \subseteq (U^2)^\perp$, and the claimed inequality follows.

Assume $x_1^2 M \in U^2$. Then either

(i) there exist monomials $S, T \in A_{d-1}$ such that $x_1^2 M = (x_1 S)(x_1 T)$ and $x_1 S$, $x_1 T \in U$, or

(ii) there exist $S \in A_{d-2}$ and $T \in A_d$ such that $x_1^2 = (x_1^2 S)T$ and $x_1^2 S$, $T \in U$.

In both cases $M = ST$. In case (i) we have $S, T \in V$ since $V = (U : x_1)$ and hence $M = ST \in V^2$, a contradiction.

In case (ii) we see $x_1 S \in V$. If $x_1 | T$, then $\frac{T}{x_1} \in V$ and again we have $M = (x_1 S)\frac{T}{x_1} \in V^2$. Thus we can assume that x_1 does not divide T. Since $k \leq d-1$ every monomial of degree $d-1$ not contained in V is divisible by x_1. Hence, for every $i \in \{2, \dots, n\}$ such that $x_i | T$, the monomial $\frac{T}{x_i}$ is contained in V. Fix any such $i \in \{2, \dots, n\}$. Since V is strongly stable and $x_1 S \in V$, the monomial $x_i S$ is also contained in V. Combined this gives

$$M = ST = (x_i S)\frac{T}{x_i} \in V^2,$$

which is again a contradiction. □

Combining the two inequalities of Theorem 5.2 and Proposition 5.4, we get the following result.

Corollary 5.5. *If $k \leq d$ then*

$$m(n, d, k) = m(n, k, k).$$

Proof. If $d = k$ there is nothing to show, we can thus assume that $k \leq d-1$. Let $U \subseteq A_d$ be a subspace of codimension k such that $\text{codim}\, U^2 = m(n, d, k)$ and $k < d$. By Theorem 5.2 we have $\text{codim}_{A_{2d}} U^2 \leq \text{codim}_{A_{2d-2}} V^2$ with $V = (U : l)$ for a generic linear form $l \in A_1$. By definition $\text{codim}_{A_{2d-2}} V^2 \leq m(n, d-1, k)$, thus $m(n, d, k) \leq m(n, d-1, k)$. On the other hand, let $V \subseteq A_{d-1}$ be a strongly stable subspace of codimension k such that $\operatorname{codim}_{A_{2d-2}} V^2 = m(n, d-1, k)$. Let $U := x_1 V \oplus \mathbb{C}[x_2, \dots, x_n]_d$, then $V = (U : x_1)$ and U is strongly stable. By Proposition 5.4 it follows that $m(n, d-1, k) = \operatorname{codim}_{A_{2d-2}} V^2 \leq \operatorname{codim}_{A_{2d}} U^2 \leq m(n, d, k)$. Combined this gives $m(n, d, k) = m(n, d-1, k)$ and we are done by induction. □

Remark 5.6. The proofs also show that if $V \subseteq A_d$ is a strongly stable subspace of codimension k such that $\operatorname{codim}_{A_{2d}} V^2 = m(n, d, k)$ and $k \leq d$, the subspace $U := x_1 V \oplus \mathbb{C}[x_2, \dots, x_n]_{d+1}$ satisfies $\operatorname{codim}_{A_{2d+2}} U^2 = m(n, d+1, k)$.

6. Lifting subspaces

In Theorem 4.5 we showed how to reduce the number of variables, now we also want to increase that number while preserving $\operatorname{codim} U^2$.

Definition 6.1. Let $U \subseteq A_d$ be a subspace of codimension k. Define

$$U^{(1)} := x_{n+1} A(n+1)_{d-1} \oplus U \subseteq A(n+1)_d$$

and for any $l \geq 2$

$$U^{(l)} := (U^{(l-1)})^{(1)} \subseteq A(n+l)_d$$

$(U^{(0)} := U)$.

For any $l \geq 1$ the subspace $U^{(l)}$ also has codimension k in $A(n+l)_d$. And in fact, we know the whole Hilbert function of $U^{(l)}$.

Proposition 6.2. *Let $U \subseteq A_d$ be a subspace of codimension k. Let $H = (h_i)_{i \geq 0}$ be the Hilbert function of $\langle U \rangle$. Then for every $l \geq 0$ the following hold:*

(i) The Hilbert function $K = (k_i)_{i \geq 0}$ of the ideal generated by $U^{(l)}$ in $A(n+l)$ satisfies
- *$k_i = \dim A(n+l)_i$ for $0 \leq i \leq d-1$ and*
- *$k_i = h_i$ for $i \geq d$.*

Furthermore, we have

(ii) $\operatorname{codim}_{A(n+l)_{2d}} (U^{(l)})^2 = \operatorname{codim}_{A_{2d}} U^2 + l \cdot h_{2d-1}$.

Proof. It is enough to show this for $l = 1$ since the rest follows by induction. Write $A' = A[y]$ with a new indeterminate y, then

$$V := U^{(1)} = y A'_{d-1} \oplus U \subseteq A'_d.$$

For any $s \geq 0$, we have

$$\begin{aligned} V A'_s = y A'_{d-1} A'_s + U A'_s &= \left(\bigoplus_{i=1}^{d+s} y^i A_{d+s-i} \right) + A_s U + y A_{s-1} U + \cdots + y^s U \\ &= \bigoplus_{i=1}^{d+s} y^i A_{d+s-i} \oplus U A_s \end{aligned}$$

which shows (i) since $A'_{d+s} = \bigoplus_{i=0}^{d+s} y^i A_{d+s-i}$.

For (ii) we calculate V^2 and with the same argument as above we get

$$V^2 = y^2 A'_{2d-2} + y A'_{d-1} U + U^2 = \bigoplus_{i=2}^{2d} y^i A_{2d-i} \oplus y(A_{d-1} U) \oplus U^2$$

and
$$\operatorname{codim}_{A'_{2d}} V^2 = \operatorname{codim}_{A_{2d}} U^2 + h_{2d-1}.$$
□

This enables us to determine the Hilbert function of codimension 2 subspaces of A_2 as an easy application.

For generic U the Hilbert function of $\langle U\rangle$ is as small as possible. In the codimension 2 case this means that the Hilbert function is $(1, n, 2)$ generically. We show that this holds whenever U is base-point-free.

Proposition 6.3. *Let $U \subseteq A_2$ be a base-point-free subspace of codimension 2. Then the Hilbert function of $\langle U\rangle$ is $(1, n, 2)$.*

Proof. By Theorem 2.18 the Hilbert function is smaller or equal to $(1, n, 2, 2, \dots)$. So assume $h_{\langle U\rangle}(3) > 0$. Then by Proposition 6.2, the subspace $U^{(l)} \subseteq A(n+l)_2$ has codimension 2 and for $l \geq 7$ we have $\operatorname{codim}_{A(n+l)_4}(U^{(l)})^2 \geq 7$ which is not possible by Theorem 4.8. □

Corollary 6.4. *Let $U \subseteq A_d$ be a base-point-free subspace of codimension $k \in \{1, 2\}$ and $\operatorname{codim}_{A_{2d}} U^2 = s$. Then for every $N \geq n$, there exists a base-point-free subspace $V \subseteq A(N)_d$ of codimension k such that $\operatorname{codim}_{A(N)_{2d}} V^2 = s$.*

Proof. By Proposition 6.3 and Corollary 2.21 the degree $2d-1$ component of $A/\langle U\rangle$ has dimension 0. Hence
$$\operatorname{codim}_{A(n)_{2d}} U^2 = \operatorname{codim}_{A(n+l)_{2d}} (U^{(l)})^2.$$
by Proposition 6.2 (ii). □

7. Arbitrary codimension

We show bounds for $m^0(n, d, k)$ that are independent of n and d, if d is large enough. The most important step is to also consider the orthogonal complement alongside our starting space. This is made precise in Lemma 7.2.

The main idea is the following: if $U \subseteq A_d$ is a base-point-free subspace of codimension k, we consider $U' := U \cap A(m)_d$ for some $2 \leq m \leq n$. To use Theorem 4.5 we need to make sure that $\operatorname{codim}_{A_{2d}} U = \operatorname{codim}_{A(m)_{2d}} U'$ and to get bounds that are independent of n we want U' to be base-point-free as well (and $k \leq d-1$), we then use Corollary 4.7 to conclude $\operatorname{codim}_{A_{2d}} U^2 \leq \operatorname{codim}_{A(m)_{2d}} (U')^2$. To get a bound that is independent of d, we use the results of Section 5.

We still always assume that n, $d \geq 2$, and $k \in \mathbb{N}$.

Remark 7.1 (The dual problem)**.** Let $U \subseteq A_d$ be a base-point-free subspace of codimension k. Instead of asking if $U' := U \cap A(m)_d$ satisfies

(i) $\operatorname{codim}_{A(m)_d} U' = k$ and
(ii) $\mathcal{V}(U') = \varnothing$ with $\mathcal{V}(U') \subseteq \mathbb{P}^{m-1}$,

as in Theorem 4.5 and Corollary 4.7, we can also look at the dual problem:

Let $W = U^\perp$. Does $W' := W(x_1, \dots, x_m, 0, \dots, 0)$ have the same dimension as W and does W' not contain the d-th power of a linear form.

The next lemma is an easy statement from linear algebra.

Lemma 7.2. *Let $U \subseteq A_d$ be a subspace and $W := U^\perp$. Let $l_1, \dots, l_s \in A_1$ be linearly independent linear forms and $V := \mathbb{C}[l_1, \dots, l_s]_d \subseteq A_d$. Then*

$$(U \cap V)^\perp \cong (W + V^\perp)/V^\perp,$$

and $V^\perp = \operatorname{span}(\lambda_1, \dots, \lambda_{n-s})A_{d-1}$ where $\operatorname{span}(l_1, \dots, l_s)^\perp = \operatorname{span}(\lambda_1, \dots, \lambda_{n-s}) \subseteq A_1$.

Write $\overline{W}$ for $(W + V^\perp)/V^\perp$, then we especially have $\operatorname{codim}_{A_d} U \cap V = \dim \overline{W}$ and $U \cap V$ is base-point-free if and only if $\overline{W}$ contains no d-th power of a linear form.

Now we want to work on condition (i) in Remark 7.1 to ensure that $\dim W = \dim \overline{W}$. Since W will play the role of $U^\perp$, k will usually denote the dimension of W, and not the codimension.

Proposition 7.3. *Let $k < n$ and let $W \subseteq A_d$ be a subspace of dimension k. Let $l \in A_1$ be generic. Then $\dim \overline{W} = \dim W$ where $\overline{W} = W + \langle l \rangle_d / \langle l \rangle_d$.*

Proof. After a change of coordinates we may assume that W is in general coordinates. Then

$$\operatorname{gin}(W + \langle l \rangle)_d = \operatorname{in}(W + \langle l \rangle)_d = \langle x_n, x_{n-1}^d, x_{n-1}^{d-1}x_{n-2}, \dots, x_{n-1}^{d-1}x_{n-k} \rangle_d$$

as the initial term of l is x_n and the other monomials are the largest k monomials appearing in the generators of W after removing all terms divisible by x_n. Furthermore, there cannot be any other monomials as

$$\dim \operatorname{in}(W + \langle l \rangle)_d = \dim(W + \langle l \rangle_d) \leq \dim W + \dim \langle l \rangle_d = k + \dim A_{d-1}$$

and

$$\dim \langle x_n, x_{n-1}^d, x_{n-1}^{d-1}x_{n-2}, \dots, x_{n-1}^{d-1}x_{n-k} \rangle_d = k + \dim A_{d-1}.$$

□

Remark 7.4. The condition of Proposition 7.3 on the dimension, namely $k < n$ is necessary and tight in the following sense. Let $0 \neq F \in A_{d-1}$ and let $W = FA_1$. Then $\dim W = n$ and $\dim \overline{W} = n - 1 < \dim W$ for a generic linear form $l \in A_1$.

In fact, it follows from [1, Theorem 3.2] that every subspace W of dimension n such that $\dim \overline{W} < \dim W$ for a generic linear form $l \in A_1$, has the form FA_1 for some $F \in A_{d-1}$.

Corollary 7.5. *Let $U \subseteq A_d$ be a subspace of codimension k with $k \leq n$. Then*

$$\operatorname{codim}_{\mathbb{C}[l_1,\dots,l_k]_d}(U \cap \mathbb{C}[l_1, \dots, l_k]_d) = k$$

for generic linear forms $l_1, \dots, l_k \in A_1$.

Especially, after applying a generic change of coordinates to U, we have

$$\operatorname{codim}_{A(k)_d}(U \cap A(k)_d) = k.$$

Proof. Let $W = U^\perp$. If $k = n$ there is nothing to show, hence assume $k < n$. By Lemma 7.2 it is enough to consider the dimension of $\overline{W} \subseteq A_d/\langle l_{k+1}, \dots, l_n \rangle_d$ for any basis $l_{k+1}, \dots, l_n$ of the orthogonal complement of $\operatorname{span}(l_1, \dots, l_k)$. Since $k < n$, it follows from Proposition 7.3 that $\dim W = \dim \overline{W}$. □

Corollary 7.6. *Let $W \subseteq A_d$ be a subspace of dimension $k < n$ and let $l \in A_1$ a generic linear form. Then $\dim (W : l) = 0$.*

Proof. This follows from Proposition 7.3 and the exact sequence

$$0 \to A_{d-1}/(W:l) \to A_d/W \to A_d/\langle W, l\rangle_d \to 0.$$

in 2.25: the space in the middle has dimension $\dim A_d - k$ and the space on the right has dimension $\dim A(n-1) - k$ by Proposition 7.3. Hence, the one on the left-hand side has dimension $\dim A_d - k - \dim A(n-1) + k = \dim A_{d-1}$ and thus $\dim(W:l) = 0$. □

This shows that condition (i) in Remark 7.1 is satisfied whenever $k < n$ and we go down by one variable. And it is in general not satisfied if $n \leq k$ since we can take $W = FA_1$ for some $0 \neq F \in A_{d-1}$.

Now we want to look at condition (ii) in Remark 7.1. By Lemma 7.2 asking whether $U' := U \cap A(n-1)$ is base-point-free is the same as asking if the orthogonal complement contains no d-th power of a linear form. Thus assume that U is base-point-free and W contains no d-th powers.

Is it true in general that $\overline{W} \subseteq (A/\langle l\rangle)_d$ contains no d-th powers for generic $l \in A_1$ whenever $\dim W = \dim \overline{W}$? Sadly this is not the case as the next example shows.

Example 7.7. Let $W := x_n^{d-1}A(n-1)_1 \subseteq A_d$ and let $l \in A_1$ be a generic linear form. After scaling l, we can write $l = x_n + l'$ for some $l' \in A(n-1)_1$, hence $\overline{W} \subseteq (A/\langle l\rangle)_d$ is isomorphic to $(l')^{d-1}A(n-1)_1$. Then $(l')^d \in \overline{W}$ and $\dim W = \dim \overline{W}$.

However, it is true whenever the number of variables is large as the next theorem shows. For convenience, we use the following notation. For a subspace $W \subseteq A_d$ we say $l \in A_1$ is W-generic if l is generic in the sense of Green's Theorem 2.26.

Theorem 7.8. *Let $W \subseteq A_d$ be a subspace of dimension k and let $n \geq 2k+1$. If W contains no d-th power of a linear form, then the same holds for $\overline{W} = W + \langle l\rangle_d/\langle l\rangle_d \subseteq (A/\langle l\rangle)_d$ where $l \in A_1$ is a generic linear form.*

Proof. Assume this is wrong. Let $X = \mathcal{V}(W) \subseteq \mathbb{P}^{n-1}$. Let $l \in A_1$ be generic, then $\mathcal{V}(W, l) = \mathcal{V}(W, l, L^d)$ for some $L \in A_1$ since $L^d + pl \in W$ for some $p \in A_{d-1}$. Hence a generic hyperplane section of X is degenerate. We claim that X is degenerate. If any irreducible component of X is non-degenerate, then so is a generic hyperplane section of this component and hence X. Therefore all irreducible components are degenerate. Assume X is non-degenerate. Then there exist components $X_1, \ldots, X_s$ each contained in a linear variety T_i but the union of all T_i is non-degenerate. Note that all T_i have codimension at most k and $n \geq 2k+1$, thus no T_i has dimension 0. Intersecting X with $\mathcal{V}(l)$, each X_i is a non-degenerate variety inside $T_i \cap \mathcal{V}(l)$ and the union $\cup_{i=1}^s (T_i \cap \mathcal{V}(l))$ is non-degenerate inside $\mathcal{V}(l)$. But then $X \cap \mathcal{V}(l)$ is non-degenerate in $\mathcal{V}(l)$, a contradiction.

Hence there exists $H_1, \ldots, H_r \in A_1$ linearly independent such that $X \subseteq \mathcal{V}(H_1, \ldots, H_r)$, $r \leq k$ and X is non-degenerate in $\mathcal{V}(H_1, \ldots, H_r)$. Let $l_1, \ldots, l_{k+1} \in A_1$ generic, then there exist $L_1, \ldots, L_{k+1} \in \operatorname{span}(H_1, \ldots, H_r)$ such that $L_i^d + l_i g_i \in W$ for some $g_i \in A_{d-1}$ and $i = 1, \ldots, k+1$. Since $\dim W = k$ these $k+1$ forms are linearly dependent and there exist $\lambda_1, \ldots, \lambda_{k+1} \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{k+1} \lambda_i (L_i^d + l_i g_i) = 0.$$

Since $l_1, \ldots, l_{k+1}$ are generic and $r+k+1 \leq 2k+1$, the linear forms $H_1, \ldots, H_r, l_1, \ldots, l_{k+1}$ are linearly independent. After a change of coordinates we may therefore assume that $H_i = x_i$

$(i = 1, \dots, r)$ and $l_i = x_{r+i}$ $(i = 1, \dots, k+1)$. Since

$$\sum_{i=1}^{k+1} \lambda_i L_i^d + \sum_{i=1}^{k+1} \lambda_i l_i g_i = 0.$$

but no monomial in the second sum is contained in $\mathbb{C}[x_1, \dots, x_r]$ we see that both sums individually have to be 0. With $j := \max(i \colon \lambda_i \neq 0)$ we have

$$l_j g_j \in \operatorname{span}(l_1 g_1, \dots, l_{j-1} g_{j-1}).$$

Since $\dim \operatorname{span}(l_1 g_1, \dots, l_{j-1} g_{j-1}) \leq k$ it follows from Green's theorem that

$$(\operatorname{span}(l_1 g_1, \dots, l_{j-1} g_{j-1}) \colon l) = \{0\}$$

for generic $l \in A_1$. Especially, this is true for l_j, a contradiction. □

Theorem 7.9. *Let $k \leq d-1$. Then for every $n \geq 2$ and every base-point-free subspace $U \subseteq A_d$ of codimension k we have*

$$\operatorname{codim}_{A_{2d}} U^2 \leq m(2k, k, k).$$

Especially, the number $m(2k, k, k)$ is independent of n and d.

Proof. If $n \leq 2k$ and $U \subseteq A_d$ is a base-point-free subspace of codimension k, then by Proposition 6.2 we have $\operatorname{codim}_{A_{2d}} U^2 = \operatorname{codim}_{A(2k)_{2d}} (U^{(2k-n+1)})^2$ and $\operatorname{codim}_{A_d} U = \operatorname{codim}_{A(2k)_d} U^{(2k-n+1)}$ with $U^{(2k-n+1)} \subseteq A(2k+1)_d$. It is therefore enough to only consider the case $n > 2k$.

We apply a generic change of coordinates to U. By Corollary 7.5 it follows that $V := U \cap A(2k)_d$ has codimension k in $A(2k)_d$ and by Theorem 7.8 the subspace V is still base-point-free.

Using Corollary 4.7 we get $\operatorname{codim}_{A_{2d}} U^2 \leq \operatorname{codim}_{A(2k)_{2d}} V^2 \leq m(2k, d, k)$. By Corollary 5.5 we finally have $m(2k, d, k) = m(2k, k, k)$ which concludes the proof. □

Remark 7.10. Assuming that d is large enough is essential. Consider the following example: Let $R = A(4), \mathfrak{m} = \langle x_5, \dots, x_n \rangle \subseteq A(n), n \geq 5$ and

$$U = \operatorname{span}(x_1^3, x_2^3, x_3^3, x_4^3, x_1^2 x_2 + x_3^2 x_4) \subseteq A(4)_3.$$

This subspace has codimension 15 in $A(4)_3$. We check with `SAGE` [13] that the Hilbert function of $\langle U \rangle$ is given by $(1, 4, 10, 15, 15, 7, 1)$. Define the subspace

$$V := U^{(n-4)} = \bigoplus_{i=1}^{3} \mathfrak{m}^i R_{3-i} \oplus U \subseteq A(n)_3.$$

This subspace also has codimension 15 in $A(n)$ by Proposition 6.2. Again by Proposition 6.2 we know that $\operatorname{codim}_{A(n)_6} V^2 = \operatorname{codim}_{A(4)_6} U^2 + 7 \cdot (n-4)$.

This shows that we cannot have a uniform bound for this combination of codimension and degree not depending on n.

It seems likely that one cannot only reduce to $2k$ variables in Theorem 7.8 but actually to $k+1$ variables. This is at least the only counterexample we know of (for $k \leq n-1$). We also checked this in small cases for all monomial subspaces on a computer.

Conjecture 7.11. *Let $k \leq d-1, n-1$, and $n \geq 3$. Let $W \subseteq A_d$ be a subspace of dimension k and suppose that W contains no d-th power of a linear form. Then for a generic linear form $l \in A_1$ it holds that either*

(i) $\overline{W}$ *contains no d-th power of a linear form, or*
(ii) $n = k+1$ *and* $W = L_1^{d-1}\mathbb{C}[L_2,\dots,L_{k+1}]_1$ *for some basis* $L_1,\dots,L_{k+1}$ *of* $A(k+1)_1$.

This would allow us to show $\operatorname{codim}_{A_{2d}} U^2 \le m(k,k,k)$ in Theorem 7.9 with an additional argument.

Remark 7.12. (i) The conjecture is certainly false if $n = k$ is allowed: for $n = k$ let $W = x_1^{d-1}\mathbb{C}[x_2,\dots,x_n]_1 \oplus \operatorname{span}(p)$ for some generic $p \in A_d$. Since p is generic, W contains no d-th powers, but $\overline{W} \cong x_1^{d-1}\mathbb{C}[x_1,\dots,x_{n-1}]_1 \oplus \operatorname{span}(\overline{p})$ does contain one where $\overline{W} \subseteq A_d/\langle l\rangle_d$ for a generic linear form $l \in A_1$.

(ii) The conjecture is true for $n \ge 2k+1$ by Theorem 7.8. For $k = 1$ it also follows from a simple geometric observation: if $W = \operatorname{span}(p)$ and p is not a power of a linear form, then $\mathcal{V}(p)$ is non-degenerate. Hence, the same holds for a generic hyperplane section which therefore cannot be defined by the power of a linear form.

8. Determining the upper bound

In this section, we determine the value $m(2k,k,k)$. Moreover, we find the strongly stable subspace that realizes $m(k,k,k)$, and show that is it unique. We rely on Proposition 8.8 and Proposition 8.11 to do most of the combinatorial work for the main results, which are Theorem 8.3 and Theorem 8.12.

We always assume $k \in \mathbb{N}$, $k \ge 2$.

Lemma 8.1. *For any* $k \ge 2$ *we have* $m(2k,k,k) \le k^2 + m(k,k,k)$.

Proof. Let $U \subseteq A(2k)_k$ be some strongly stable subspace of codimension k that realizes $m(2k,k,k)$. Since $\operatorname{codim}_{A(2k)_k} U = k$, every monomial not contained in U is contained in $A(k)_k$. Especially, the codimension of $V := U \cap A(k)_k$ in $A(k)_k$ is equal to k. Using Theorem 4.5 we see

$$\begin{aligned}\operatorname{codim}_{A(2k)_{2k}} U^2 &\le \operatorname{codim}_{A(k)_{2k}} V^2 + (2k-k)\dim(A(k)/\langle V\rangle)_{2k-1}\\ &\le m(k,k,k) + k\dim(A(k)/\langle V\rangle)_{2k-1}.\end{aligned}$$

Since $U \subseteq A(k)_k$ has codimension k, we see from Corollary 2.20 (i) that

$$\dim(A(k)/\langle V\rangle)_{2k-1} \le k$$

which finishes the proof. □

In fact, it follows from Remark 2.11 and Remark 4.6 that we have equality in the last lemma.

To determine $m(2k,k,k)$ it now suffices to determine a bound for $m(k,k,k)$. We show that the only strongly stable subspace realizing the bound $m(k,k,k)$ for $k \ge 2$ is $U = \operatorname{span}(x_1^k, x_1^{k-1}x_2,\dots,x_1^{k-1}x_k)^\perp$ and $m(k,k,k) = \binom{k+2}{3}$.

First, we show that the subspace U above does realize the bound $m(k,k,k)$.

Lemma 8.2. *Let* $U \subseteq A(k)_k$ *be the subspace of codimension* k *spanned by all monomials of degree* k *except for* $x_1^d, x_1^{d-1}x_2,\dots,x_1^{d-1}x_k$*. Then* $\operatorname{codim}_{A(k)_{2k}} U^2 = \binom{k+2}{3}$.

Proof. We easily see that the only monomials not contained in U^2 are

- $x_1^{2k-1}x_i$ for $i = 1,\dots,k$,
- $x_1^{2k-2}x_ix_j$ for $i,j = 2,\dots,k$,
- $x_1^{2k-3}x_ix_jx_l$ for $i,j,l = 2,\dots,k$

or equivalently $x_1^{2k-3}x_ix_jx_l$ for $i,j,l = 1,\dots,n$. These are a total of $\frac{1}{6}(k^3+3k^2+2k) = \binom{k+2}{3}$ monomials. $\square$

Theorem 8.3. *For any $k \geq 2$ we have $m(k,k,k) = \binom{k+2}{3}$.*

Proof. We prove this by induction on k. For $k = 2$ this is immediately checked. Assume the claim holds for some $k \geq 2$. Let $U \subseteq A(k+1)_{k+1}$ be a strongly stable subspace realizing the bound $m(k+1,k+1,k+1)$. We write $d = k+1$ for the degree. If $x_1^{d-1}x_{k+1}$ is not contained in U, then $U = \operatorname{span}(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_{k+1})^\perp$ by Lemma 8.7. By Lemma 8.2 we then have $\operatorname{codim}_{A(k+1)_{2d}} U^2 = \binom{k+3}{3} = \binom{(k+1)+2}{3}$. We show that this is the only strongly stable subspace realizing the upper bound. For the sake of contradiction assume that $x_1^{d-1}x_{k+1}$ is contained in U, then no monomial in $U^\perp$ is divisible by x_{k+1}. Indeed, if there was a monomial $M \in U^\perp$ divisible by x_{k+1}, we can write $M = Tx_{k+1}$ with $T \in A(k+1)_{d-1}$, then $M = T\frac{x_1^{d-1}x_{k+1}}{x_1^{d-1}} \in U$ as $x_1^{d-1}x_{k+1} \in U$ and U is strongly stable. Therefore, we can write

$$U = x_{k+1}A(k+1)_{d-1} \oplus V$$

where $V \subseteq A(k)_d$ is a strongly stable subspace of codimension $k+1$. By Theorem 4.5 we now have

$$m(k+1,d,k+1) = \operatorname{codim}_{A(k+1)_{2d}} U^2 \leq \operatorname{codim}_{A(k)_{2d}} V^2 + k + 1 \leq m(k,d,k+1) + k + 1.$$

In Proposition 8.11 we show that $m(k,d,k+1) < m(k,k,k) + \binom{k+1}{2}$, then we have

$$m(k+1,k+1,k+1) = \operatorname{codim}_{A(k+1)_{2d}} U^2 < \underbrace{m(k,k,k)}_{=\binom{k+2}{3}} + \binom{k+1}{2} + k + 1 = \binom{k+3}{3}.$$

However, the subspace $W := \operatorname{span}(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_{k+1})^\perp$ satisfies $\operatorname{codim}_{A(k+1)_{2d}} W^2 = \binom{k+3}{3}$. This yields the following contradiction

$$\binom{k+3}{3} = \operatorname{codim}_{A(k+1)_{2d}} W^2 \leq m(k+1,k+1,k+1) < \binom{k+3}{3}.$$

$\square$

It is left to prove Proposition 8.11 which we used in the last proof. For this we first need some more preparation.

Definition 8.4. For any monomial $M \in A_d$ denote by $\mathrm{p}(M)$ the smallest integer $j > 1$ such that $x_j | M$. If $M = x_1^d$, we define $\mathrm{p}(M) := 1$. By M^- we denote the *reduction of* M, which is the monomial $x_1\frac{M}{x_{\mathrm{p}(M)}}$. On the other hand, we write M^+ for the set of all monomials $T \in A_d$ such that M is the reduction of T.

Example 8.5. Consider the monomial $M = x_1^2x_3x_4 \in A(4)_4$, then $\mathrm{p}(M) = 3$ and $M^- = x_1^3x_4$. The set M^+ consists of the monomials $x_1x_2x_3x_4$ and $x_1x_3^2x_4$. The monomial $x_1x_3x_4^2$ is not contained in M^+ since its reduction is $x_1^2x_4^2$.

If $M = x_1^d$, then by definition $M^- = x_1^d$ and $M^+ = \{x_1^{d-1}x_i \colon 1 \leq i \leq n\}$.

Lemma 8.6. *Let $M \in A_d$ be a monomial divisible by x_1 with $\mathrm{p}(M) > 1$. Then $|M^+| = \mathrm{p}(M) - 1$.*

Proof. M^+ consists of the elements $x_j\frac{M}{x_1}$ for $j = 2, \dots, \mathrm{p}(M)$: The variables $x_2, \dots, x_{\mathrm{p}(M)-1}$ do not appear in M by definition of $\mathrm{p}(M)$. Hence, for $2 \le j \le \mathrm{p}(M)$ we see $\mathrm{p}\left(x_j\frac{M}{x_1}\right) = j$, and therefore $\left(x_j\frac{M}{x_1}\right)^- = \frac{x_1}{x_j}\left(x_j\frac{M}{x_1}\right) = M$. □

Lemma 8.7. *Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k \le n$. If there exists a monomial $M \in U^\perp$ such that $\mathrm{p}(M) = k$, then $U = \mathrm{span}(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_k)^\perp$.*

Proof. Since $M \notin U$ and U is strongly stable, the monomials $x_i\frac{M}{x_k}$ are also not contained in U for $i = 1, \dots, k-1$. Hence, $U^\perp$ is spanned by the monomials $x_i\frac{M}{x_k}$ $(i = 1, \dots, k-1)$ and M. Especially, if $M = x_1^{d-1}x_k$, then U has the asserted form.

Assume $M \ne x_1^{d-1}x_k$, hence $M = x_1^a x_k T$ for some $a \in \mathbb{N}$ and some monomial $1 \ne T \in \mathbb{C}[x_k, \dots, x_n]$ of degree $d-a-1$. Just as above, the monomials $x_1^a x_i T$ are not contained in U for $i = 1, \dots, k$. However, since $T \ne 1$, there exists $k < i \le n$ such that $x_i | T$, hence $x_1^a x_k^2 \frac{T}{x_i}$ is also not contained in U. This is a contradiction since $k+1$ different monomials are not contained in U. □

Proposition 8.8. *Let $U \subseteq A_d$ be a strongly stable subspace of codimension k with $2 \le k \le n$. Let S be the set of all monomials of degree d not contained in U. Then $|\bigcup_{M \in S} M^+| \le \binom{k}{2} + n$ with equality if and only if $U = \mathrm{span}(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_k)^\perp$. Furthermore, every monomial in S is contained in $\bigcup_{M \in S} M^+$.*

Remark 8.9. We give a brief idea why Proposition 8.8 holds. Assume we know that equality in Proposition 8.8 holds whenever $U = \mathrm{span}(x_1^d, x_1^{d-1}x_2, \dots, x_1^{d-1}x_k)^\perp$. We swap one monomial in $U^\perp$ with some other monomial in A_d. For U to stay strongly stable, we need to remove $x_1^{d-1}x_k$ and add $x_1^{d-2}x_2^2$. By Lemma 8.6 we know that $|(x_1^{d-2}x_2^2)^+| = 1$ and $|(x_1^{d-1}x_k)^+| = k-1$, hence the inequality is now strict.

Continuing doing this, we have to remove $x_1^{d-1}x_j$ for some large j and add a monomial M in fewer variables, especially $\mathrm{p}(M) < \mathrm{p}(x_1^{d-1}x_j) = j$ and by Lemma 8.6 the set $\bigcup_{M \in S} M^+$ contains even fewer elements now.

Proof of Proposition 8.8. Let $T \in S$ such that $\mathrm{p}(T) = \max\{\mathrm{p}(M) : M \in S\}$. Since U is strongly stable, the elements $x_j\frac{T}{x_{\mathrm{p}(T)}}$ for $j = 2, \dots, \mathrm{p}(T) - 1$ are also not contained in U and therefore lie in S. Moreover, they satisfy $\mathrm{p}(x_j\frac{T}{x_{\mathrm{p}(T)}}) = j$ for all $j = 2, \dots, \mathrm{p}(T) - 1$. Thus we have $\mathrm{p}(T) - 1$ elements in S for which we know the value $\mathrm{p}(\cdot)$, namely T, $x_j\frac{T}{x_{\mathrm{p}(T)}}$ $(j = 2, \dots, \mathrm{p}(T) - 1)$.

Next, we look at the element $x_1^d \in S$. We have $(x_1^d)^+ = \{x_1^d, \dots, x_1^{d-1}x_n\}$, and therefore $|(x_1^d)^+| = n$. In total, we identified $\mathrm{p}(T)$ elements in S for which we know the value $\mathrm{p}(\cdot)$ and all other $k - \mathrm{p}(T)$ elements M in S satisfy $\mathrm{p}(M) \le \mathrm{p}(T)$ by the choice of T. We write

$$S' = S \setminus \{x_1^d, x_2\frac{T}{x_{\mathrm{p}(T)}}, \dots, x_{\mathrm{p}(T)-1}\frac{T}{x_{\mathrm{p}(T)}}, T\}$$

for the set where we removed all monomials from S of which we determined the value $p(\cdot)$. We calculate

$$\begin{aligned}\left|\bigcup_{M\in S} M^+\right| &\leq \sum_{i=2}^{\mathrm{p}(T)}\left|\left(x_i\frac{T}{x_{\mathrm{p}(T)}}\right)^+\right| + |(x_1^d)^+| + \left|\bigcup_{M\in S'} M^+\right| \\ \text{(Lemma 8.6)} &\leq \sum_{i=2}^{\mathrm{p}(T)}(i-1) + n + \underbrace{(|S|-\mathrm{p}(T))}_{=|S'|}(\mathrm{p}(T)-1) \\ &= \sum_{i=1}^{\mathrm{p}(T)-1} i + (k-\mathrm{p}(T))(\mathrm{p}(T)-1) + n \\ &= \sum_{i=1}^{\mathrm{p}(T)-1} i + \sum_{j=\mathrm{p}(T)}^{k-1}(\mathrm{p}(T)-1) + n \leq \sum_{i=1}^{k-1} i + n = \binom{k}{2} + n.\end{aligned}$$

Next, we show that equality holds if and only if $U = \mathrm{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)^\perp$. First, we note that $\mathrm{p}(T) = k$ implies $S = \{x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k\}$ by Lemma 8.7.

Assume U is not of this form, then $\mathrm{p}(T) < k$ and therefore the sum $\sum_{j=\mathrm{p}(T)}^{k-1}(\mathrm{p}(T)-1)$ is non-zero. Moreover, we have a strict inequality $\sum_{j=\mathrm{p}(T)}^{k-1}(\mathrm{p}(T)-1) < \sum_{j=\mathrm{p}(T)}^{k-1} j$ which means the second to last inequality above is also strict in this case.

Therefore, equality can only hold if $U = \mathrm{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)^\perp$. Assume we are in this case, then $\mathrm{p}(T) = k$. First, we note that this means that $S' = \varnothing$, hence we get

$$\left|\bigcup_{M\in S} M^+\right| \leq \sum_{i=2}^{\mathrm{p}(T)}\left|\left(x_i\frac{T}{x_{\mathrm{p}(T)}}\right)^+\right| + |(x_1^d)^+| \overset{\text{(Lemma 8.6)}}{=} \sum_{i=2}^{\mathrm{p}(T)}(i-1) + n = \sum_{i=1}^{k-1} i + n = \binom{k}{2} + n.$$

We thus need to show that the inequality is an equality or equivalently that the sets M^+ with $M \in S$ are disjoint.

Let $x_1^{d-1}x_i \in S$, $i \geq 2$, then $(x_1^{d-1}x_i)^+ = \{x_1^{d-2}x_ix_l \colon 2 \leq l \leq i\}$. We see that if $(x_1^{d-1}x_i)^+$ and $(x_1^{d-1}x_j)^+$, $i,j \geq 2$ contain a common element, it has the form $x_1^{d-2}x_ix_j$ with $i \leq j$ and $j \leq i$, which means $i = j$. For x_1^d we have $(x_1^d)^+ = \{x_1^d, \ldots, x_1^{d-1}x_n\}$ and every element has degree at least $d-1$ in x_1, hence it is not contained in $(x_1^{d-1}x_i)^+$ for any $i \geq 2$. This finishes the first part.

The last claim we need to prove is $S \subseteq \bigcup_{M\in S} M^+$. Let $T \in S$. Since U is strongly stable, the element $x_1\frac{T}{x_{\mathrm{p}(T)}} = T^-$ is also contained in S. But by definition this means

$$T \in \left(x_1\frac{T}{x_{\mathrm{p}(T)}}\right)^+ \subseteq \bigcup_{M\in S} M^+$$

which finishes the proof. □

Lemma 8.10. *Let $U \subseteq A_d$ be a strongly stable subspace of codimension $k+1$, then there exists a strongly stable subspace $V \subseteq A_d$ of codimension k such that $U \subseteq V$.*

Proof. If $U = \{0\}$, take $V = \mathrm{span}(x_n^d)$. Now assume $U \neq \{0\}$. Let M be any monomial not contained in U, and let $I := \{i \in \mathbb{N} \colon x_i | M\} \smallsetminus \{x_n\}$. Then either $x_{i+1}\frac{M}{x_i} \in U$ for every $i \in I$ or there exists $j \in I$ such that $x_{j+1}\frac{M}{x_j} \notin U$. If all of them are contained in U, we may add

M to U and the subspace $V := U \oplus \operatorname{span}(M)$ is still strongly stable. Indeed, let $i \in I$ and $i < l \leq n$, then

$$x_l \frac{M}{x_i} = \frac{x_l}{x_{i+1}} \underbrace{\left(x_{i+1} \frac{M}{x_i} \right)}_{\in U} \in U.$$

Any other monomial in V except for M is already contained in U and U is strongly stable.

If $x_{j+1} \frac{M}{x_j}$ is not contained in U for some $j \in I$, then for any monomial ordering $\succeq$ we have $x_{j+1} \frac{M}{x_j} \succeq M$. Now we continue with $x_{j+1} \frac{M}{x_j}$ instead of M. After finitely many steps we find M such that $x_{i+1} \frac{M}{x_i} \in U$ for every $i \in I$, since we reach $x_{n-1} x_n^{d-1}$ and this monomial is only divisible by x_{n-1} and x_n and $x_n \frac{x_{n-1} x_n^{d-1}}{x_{n-1}} = x_n^d \in U \neq \{0\}$. □

Proposition 8.11. *For every $k \geq 2$ we have $m(k, k+1, k+1) < m(k, k, k) + \binom{k+1}{2}$.*

Proof. It is enough to show $m(k, k+1, k+1) < m(k, k+1, k) + \binom{k+1}{2}$ since by Corollary 5.5 we have $m(k, k+1, k) = m(k, k, k)$. Again we denote the degree by $d := k+1$. Let $U \subseteq A(k)_d$ be a strongly stable subspace of codimension $k+1$ realizing $m(k, d, k+1)$ and let $V \subseteq A(k)_d$ be any strongly stable subspace of codimension k containing U. Such a subspace V always exists by Lemma 8.10.

Now we compare U^2 and V^2. We can write $V = U \oplus \operatorname{span}(M)$ for some monomial $M \in A(k)_d$ and hence $V^2 = U^2 + \operatorname{span}(M)V$.

Let $MT \in \operatorname{span}(M)V$ for some monomial $T \in V$. We claim that

$$MT = \left(x_{\mathrm{p}(T)} \frac{M}{x_1} \right) T^-$$

is contained in U^2 except for at most $\binom{k+1}{2} - 1$ choices of T. We note that the monomial $x_{\mathrm{p}(T)} \frac{M}{x_1}$ is always contained in U since U is strongly stable. After we show this, we are done as follows:

$$\dim V^2 \leq \dim U^2 + \binom{k+1}{2} - 1, \quad \text{or equivalently} \quad \operatorname{codim}_{A(k)_{2d}} V^2 \geq \operatorname{codim}_{A(k)_{2d}} U^2 - \binom{k+1}{2} + 1$$

hence

$$m(k, k+1, k+1) = \operatorname{codim}_{A(k)_{2d}} U^2 \leq \operatorname{codim}_{A(k)_{2d}} V^2 + \binom{k+1}{2} - 1 \leq m(k, k+1, k) + \binom{k+1}{2} - 1.$$

Whenever T^- is not contained in U this means that T is contained in M^+ for some monomial $M \in A(k)_d \setminus U$. The subspace U has codimension $k+1$ and is contained in $A(k)_d$. Therefore, it cannot be the orthogonal complement of $\operatorname{span}(x_1^d, x_1^{d-1} x_2, \ldots, x_1^{d-1} x_{k+1})$ in $A(k+1)_d$ as the variable x_{k+1} appears.

Let S be the set of all monomials in $A(k)_d$ that are not contained in U. It follows from Proposition 8.8 that $|\bigcup_{M \in S} M^+| < \binom{k+1}{2} + k$. This means that there are at most $\binom{k+1}{2} + k - 1$ monomials $M \in A(k)_d$ such that $M^- \notin U$.

Since $U \subseteq V$ and V is strongly stable, all monomials not contained in V are also not contained U, hence are in S and therefore also in $\bigcup_{M \in S} M^+$ by Proposition 8.8.

We can now finish the proof: Let $T \in V$ such that $T^- \notin U$. This means

$$T \in \left(\bigcup_{M \in S} M^+ \right) \cap V = \left(\bigcup_{M \in S} M^+ \right) \setminus V^\perp$$

and we just showed that the set on the right-hand side has cardinality at most $(\binom{k+1}{2}+k-1)-k=\binom{k+1}{2}-1$. ☐

Theorem 8.12. *Let $k \le d-1$. Then for every $n \ge 2$ and every base-point-free subspace $U \subseteq A_d$ of codimension k we have*

$$\operatorname{codim}_{A_{2d}} U^2 \le k^2 + \binom{k+2}{3} = \frac{1}{6}(k^3+9k^2+2k).$$

Proof. From Theorem 7.9 we know $\operatorname{codim}_{A_{2d}} U^2 \le m(2k,k,k)$. From Lemma 8.1 we see $m(2k,k,k) \le k^2 + m(k,k,k)$ and Theorem 8.3 shows $m(k,k,k) = \binom{k+2}{3}$. ☐

Theorem 8.13. *Let $k \ge 1$ and let $n, d \ge k$. Then $m(n,d,k) = \binom{k+2}{3} + (n-k)k$. Furthermore the only strongly stable subspace $U \subseteq A_d$ realizing $m(n,d,k)$ is given by*

$$U = \operatorname{span}(x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k)^\perp \subseteq A_d.$$

Proof. For $k = 1$, this follows from Lemma 4.2 and the fact that there only exists a single strongly stable subspace of codimension 1. For $k \ge 2$ we calculate

$$\begin{aligned} m(n,d,k) &\overset{(4.5)}{=} m(k,d,k) + (n-k)k \overset{(5.5)}{=} m(k,k,k) + (n-k)k \\ &\overset{(8.3)}{=} \binom{k+2}{3} + (n-k)k = \frac{1}{6}(k^3-3k^2+2k) + nk. \end{aligned}$$

For the first equality, we also use Remark 4.6 and the fact that the Hilbert function of the ideal generated by the subspace $U = \operatorname{span}(x_1^k, x_1^{k-1}x_2, \ldots, x_1^{k-1}x_k)^\perp \subseteq A(k)_k$ is k for any degree at least k: from Corollary 2.20 we see that the Hilbert function is at most k in those degrees, but we immediately check that $x_1^d, x_1^{d-1}x_2, \ldots, x_1^{d-1}x_k$ are not contained in UA_{d-k}.

Let $U \subseteq A_d$ be a strongly stable subspace of codimension k realizing $m(n,d,k)$. Every monomial not contained in U is contained in $A(k)_d$, hence $\operatorname{codim}_{A(k)_d} V = k$ where $V := U \cap A(k)_d$. Now we have

$$\begin{aligned} m(k,k,k) + (n-k)k \overset{(8.3)}{=} m(n,d,k) = \operatorname{codim}_{A_{2d}} U^2 &\overset{(4.5)}{\le} \operatorname{codim}_{A(k)_{2d}} V^2 + (n-k)k \\ &\le \ m(k,d,k) + (n-k)k \overset{(5.5)}{=} m(k,k,k) + (n-k)k, \end{aligned}$$

and therefore both inequalities are equalities and we have $\operatorname{codim}_{A(k)_{2d}} V^2 = m(k,k,k)$. We note that $V \subseteq A(k)_d$, and d might still be larger than k. By the last part of the proof of Theorem 5.2 we have $m(k,k,k) = \operatorname{codim}_{A(k)_{2d}} V^2 \le \operatorname{codim}_{A(k)_{2k}} (V : x_1^{d-k})^2 \le m(k,k,k)$. Especially, $(V : x_1^{d-k}) \subseteq A(k)_k$ is a subspace of codimension k realizing $m(k,k,k)$ and thus it is equal to $\operatorname{span}(x_1^k, \ldots, x_1^{k-1}x_k)^\perp \subseteq A(k)_k$. This shows $V^\perp = x_1^{d-k}\operatorname{span}(x_1^k, \ldots, x_1^{k-1}x_k) \subseteq A(k)_d$ and thus also $U = \operatorname{span}(x_1^d, \ldots, x_1^{d-1}x_k)^\perp \subseteq A_d$. ☐

References

[1] Jeaman Ahn, Juan C. Migliore, and Yong-Su Shin. Green's theorem and Gorenstein sequences. *J. Pure Appl. Algebra*, 222(2):387–413, 2018. 6, 17

[2] Grigoriy Blekherman. Positive Gorenstein ideals. *Proc. Am. Math. Soc.*, 143(1):69–86, 2015. 7

[3] Mats Boij and Aldo Conca. On Fröberg-Macaulay conjectures for algebras. *Rend. Ist. Mat. Univ. Trieste*, 50:139–147, 2018. 1, 2, 4

[4] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 1998. 6

[5] Giulio Caviglia and Diane Maclagan. Some cases of the Eisenbud-Green-Harris conjecture. *Math. Res. Lett.*, 15(2-3):427–433, 2008. 12

[6] David Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. Springer New York, 1995. 3

[7] David Eisenbud, Mark Green, and Joe Harris. Higher Castelnuovo theory. In *Journées de géométrie algébrique d'Orsay, France, juillet 20-26, 1992*, pages 187–202. Paris: Société Mathématique de France, 1993. 1, 12

[8] Andre Galligo. A propos du théorème de preparation de Weierstrass. Fonctions de plusieurs Variables complexes, Sem. Francois Norguet, Oct. 1970 - Dec. 1973, Lect. Notes Math. 409, 543-579, 1974. 3

[9] Mark Green. Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann. Algebraic curves and projective geometry, Proc. Conf., Trento/Italy 1988, Lect. Notes Math. 1389, 76-86, 1989. 8

[10] Robin Hartshorne. Connectedness of the Hilbert scheme. *Publications Mathématiques de l'IHÉS*, 29:5–48, 1966. 3

[11] Anthony Iarrobino and Steve L. Kleiman. The Gotzmann theorems and the Hilbert scheme. In *Power Sums, Gorenstein Algebras, and Determinantal Loci*, pages 289–312. Springer Berlin Heidelberg, 1999. 6

[12] F. S. Macaulay. Some properties of enumeration in the theory of modular systems. *Proc. Lond. Math. Soc. (2)*, 26:531–555, 1927. 6

[13] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 8.8)*, 2019. `https://www.sagemath.org`. 4, 19

OvGU Magdeburg, FMA-IAG, D-39106 Magdeburg, Germany
Email address: `julian.vill@ovgu.de`

26 Aug 2024 04:30:05 PDT
231122-Vill Version 3 - Submitted to J. Comm. Alg.