LYUBEZNIK TABLE OF S_r AND CM_r RINGS

JOSEP ÀLVAREZ MONTANER AND SIAMAK YASSEMI

Abstract. We describe the shape of the Lyubeznik table of either rings in positive characteristic or Stanley-Reisner rings in any characteristic when they satisfy Serre's condition S_r or they are Cohen-Macaulay in a given codimension, condition denoted by CM_r. Moreover we show that these results are sharp.

1. Introduction

Let $(R, \mathfrak{m})$ be a regular local ring containing a field $\mathbb{K}$ and set $A = R/I$, where I is an ideal of R. It is known that some vanishing results on local cohomology modules behave similarly in either the case where $\mathbb{K}$ is a field of positive characteristic or I is a squarefree monomial ideal in a polynomial ring in any characteristic. For example, as a consequence of work of Peskine and Szpiro [PS73] in positive characteristic and Lyubeznik [Lyu84] for monomial ideals, there is only one local cohomology different from zero when A is Cohen-Macaulay. The main reason behind this similar behaviour is that the Frobenius morphism in positive characteristic is flat by Kunz theorem [Kun69] and, applying it to our ideal I recursively gives us a cofinal system of ideals with respect to the system given by the usual powers which describe these local cohomology modules. For squarefree monomial ideals we have a similar flat morphism, raising any variable of the polynomial ring to its second power, that plays the same role. This point of view has already been successfully used by Singh and Walther [SW07] and Àlvarez Montaner [AM15].

The vanishing of local cohomology modules implies the vanishing of some Lyubeznik numbers of A introduced in [Lyu93]. Indeed, using an spectral sequence argument one may check that the Lyubeznik table of a Cohen-Macaulay ring A is trivial [AM00, Remark 4.2]. This still holds true replacing the Cohen-Macaulay property for sequentially Cohen-Macaulay [AM15]. We point out that these results are no longer true when A is Cohen-Macaulay containing a field of characteristic zero. For example, consider the ideal generated by the 2×2 minors of a generic 2×3 matrix [AML06].

In this note we continue the study of Lyubeznik numbers of A in either the case where $\mathbb{K}$ is a field of positive characteristic or I is a squarefree monomial ideal in any characteristic. The main results are Theorems 3.4 and 3.5 where we describe the shape of the Lyubeznik table of A when we relax the Cohen-Macaulay condition on A to Serre's condition S_r or being Cohen-Macaulay in codimension r, condition denoted by CM_r.

A priori, there is no reason for thinking that the results we obtain are sharp but this is indeed the case as we will show in Section 4. Finally we highlight that, using results obtained by Conca and Varbaro [CV20], one may compute some apparently complicated Lyubeznik tables in positive characteristic in the event that the ring A has a squarefree Gröbner deformation.

JAM is partially supported by grant PID2019-103849GB-I00 (MCIN/AEI/10.13039/501100011033), AGAUR grant 2021 SGR 00603 and Spanish State Research Agency, through the Severo Ochoa and Maria de Maeztu Program for Centers and Units of Excellence in R&D (project CEX2020-001084-M).

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.

2. Lyubeznik numbers

Let $(R, \mathfrak{m})$ be a regular local ring containing a field $\mathbb{K}$ and I an ideal of R. Some finiteness properties of local cohomology modules $H^r_I(R)$ were proved by Huneke and Sharp [HS93] when the field $\mathbb{K}$ has positive characteristic and Lyubeznik [Lyu93] in the characteristic zero case. In particular, they proved that the Bass numbers of these local cohomology modules are finite. Relying on this fact, Lyubeznik [Lyu93] introduced a set of numerical invariants of local rings containing a field as follows:

Theorem/Definition 2.1. *Let A be a local ring containing a field $\mathbb{K}$, so that its completion $\widehat{A}$ admits a surjective ring homomorphism $R \xrightarrow{\pi} \widehat{A}$ from a regular local ring $(R, \mathfrak{m})$ of dimension n and set $I := \ker(\pi)$. Then, the Bass numbers*

$$\lambda_{p,i}(A) := \mu_p(\mathfrak{m}, H^{n-i}_I(R)) = \mu_0(\mathfrak{m}, H^p_{\mathfrak{m}}(H^{n-i}_I(R)))$$

depend only on A, i and p, but neither on R nor on π.

We refer to these invariants as *Lyubeznik numbers* and they are known to satisfy the following properties: $\lambda_{p,i}(A) = 0$ if $i > d$, $\lambda_{p,i}(A) = 0$ if $p > i$ and $\lambda_{d,d}(A) \neq 0$, where $d = \dim A$. Therefore we can collect them in the so-called *Lyubeznik table*:

$$\Lambda(A) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}.$$

We say that the Lyubeznik table is *trivial* if $\lambda_{d,d}(A) = 1$ and $\lambda_{p,i}(A) = 0$ for p and i different from d. The *highest* Lyubeznik number $\lambda_{d,d}(A)$ has an interpretation in terms of the dual graph $\Gamma_1(A)$, also known as Hochster-Huneke graph, associated to *Spec*(A).

Definition 2.1. Let A be a ring of dimension d and let t be an integer such that $0 \leq t \leq d$. We define the graph $\Gamma_t(A)$ as a simple graph whose vertices are the minimal primes of A and there is an edge between $\mathfrak{p}$ and $\mathfrak{q}$ distinct minimal primes if and only if $ht(\mathfrak{p} + \mathfrak{q}) \leq t$.

Zhang gave the following characterization.

Theorem 2.2. [Zha07, Main Theorem] *Let A be a complete local ring with separably closed residue field. Then:*

$$\lambda_{d,d}(A) = \#\Gamma_1(A).$$

Remark 2.3. More generally $\lambda_{d,d}(A) = \#\Gamma_1(B)$ where $B = \widehat{\widehat{A}^{\mathrm{sh}}}$ is the completion of the strict henselianization of the completion of A.

We point out that Kawasaki already proved in [Kaw02, Theorem 2] that the highest Lyubeznik number $\lambda_{d,d}$ of a Cohen-Macaulay ring (or even S_2) is always one. Other Lyubeznik numbers can be described from the graphs $\Gamma_t(A)$ as shown by Walther [Wal01] and Núñez-Betancourt, Spiroff and Witt [NnBSW19]. Moreover, Walther describe the possible Lyubeznik tables for $d \leq 2$ (see also [RWZ22] for other small dimensional cases).

Proposition 2.4. *Let A be an equidimensional complete local ring of dimension ≥ 3 with separably closed residue field. Then*

(i) [Wal01, Proposition 2.2] $\lambda_{0,1}(A) = \#\Gamma_{d-1}(A) - 1$.
(ii) [NnBSW19, Theorem 5.4 (1)] $\lambda_{1,2}(A) = \#\Gamma_{d-2}(A) - \#\Gamma_{d-1}(A)$.
(iii) [NnBSW19, Theorem 5.4 (2)] $\lambda_{i,i+1}(A) \geq \#\Gamma_{d-i-1}(A) - \#\Gamma_{d-i}(A)$ *for* $1 \leq i \leq d-2$.

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.

3. Lyubeznik tables of S_r and CM_r rings

Throughout this section we will always assume that $(R, \mathfrak{m})$ is a regular local ring and A is a complete local ring containing a field that admits a presentation $A = R/I$ where $I \subseteq R$ is an ideal. We will study the Lyubeznik table when we relax the Cohen-Macaulay condition on the ring A. The classical way of doing so is by means of Serre's conditions. Another way is by asking for being Cohen-Macaulay up to some codimension. This notion has been considered by Miller, Novik and Schwarz [MNS11] and it was further developed in [HYZN12, HYZN12, PPTY22] for the case that A is equidimensional and defined by a squarefree monomial ideal.

Definition 3.1. We say:

(i) A satisfies Serre's condition S_r if

$$\text{depth } A_{\mathfrak{p}} \geq \min\{r, \dim A_{\mathfrak{p}}\},$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

(ii) A satisfies the condition CM_r if it is Cohen-Macaulay in codimension r, that is $A_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\operatorname{ht} \mathfrak{p} \leq d - r$.

Remark 3.2. Schenzel [Sch79] proved that if A satisfies S_r with $r \geq 2$ then it is equidimensional. However, we may have non-equidimensional CM_r rings (see Example 4.3).

Both the S_r and CM_r conditions can be characterized in terms of the *deficiency modules*

$$K_A^i := \operatorname{Ext}_R^{n-i}(A, R).$$

The following result can be found in the the work of Schenzel [Sch82, Lemma 3.2.1] (see also [CV20, Remark 2.9]). For the squarefree monomial ideals case one may consult [PPTY22].

Proposition 3.3. *We have:*

(i) *A is S_r, $r \geq 2$, if and only if $\dim K_A^i \leq i - r$ for all $1 \leq i \leq d$.*
(ii) *A is CM_r if and only if $\dim K_A^i \leq r$ for all $1 \leq i \leq d$.*

Next we present the main results of the paper where the shape of the Lyubeznik tables is given in terms of the S_r and the CM_r conditions.

Theorem 3.4. *Assume that $r \geq 2$ and either that:*

- *A is S_r and contains a field of positive characteristic, or*
- *A is S_r and I is a squarefree monomial ideal.*

Then, the Lyubeznik table of A satisfies $\lambda_{i,i} = \lambda_{i,i+1} = \cdots = \lambda_{i,i+(r-1)} = 0$, for $i \in \{0, \ldots, d-1\}$.

Proof. If A contains a field of positive characteristic, then Huneke and Sharp [HS93, Corollary 2.3] proved that $\operatorname{Ass}(H_I^{n-i}(R)) \subseteq \operatorname{Ass}(K_A^i)$, and thus $\dim(H_I^{n-i}(R)) \leq \dim(K_A^i)$. In the squarefree monomial ideal case, Yanagawa [Yan01, Theorem 2.11] proved that the straight module $H_I^{n-i}(R)$ is equivalent to the squarefree module K_A^i. In particular this gives the equality $\dim(H_I^{n-i}(R)) = \dim(K_A^i)$ [Yan01, Lemma 2.8].

Now assume in both cases that A is S_r and thus we have $\dim(K_A^i) \leq i - r$ and consequently $\dim(H_I^{n-i}(R)) \leq i - r$ for all $1 \leq i \leq d$. Then the result follows from the inequality

$$\operatorname{id}_R(H_I^{n-i}(R)) \leq \dim(H_I^{n-i}(R))$$

proved in [HS93, Corollary 3.9] and [Lyu93, Theorem 3.4]. Namely, the Lyubeznik numbers are the Bass numbers $\lambda_{p,i}(A) = \mu_p(\mathfrak{m}, H_I^{n-i}(R))$ and thus the possible non-zero $\lambda_{p,i}(A)$ are in the range $0 \leq p \leq i - r$. □

Theorem 3.5. *Assume either that:*

- *A is CM_r and contains a field of positive characteristic, or*
- *A is CM_r and I is a squarefree monomial ideal.*

Then the Lyubeznik table of A satisfies $\lambda_{p,i} = 0,\ \forall p \geq r$ and $i \in \{0, \ldots, d-1\}$.

Proof. The proof is analogous to the proof of Theorem 3.4 but in the present case we have $\dim(K_A^i) \leq r$ and thus $\dim(H_I^{n-i}(R)) \leq r$ for all $1 \leq i \leq d$. ☐

Remark 3.6. Under the hypothesis of Theorem 3.5, assume that A is CM_1 and thus the only possible non-zero row of the Lyubeznik table is the 0-th row. Then, the Lyubeznik numbers of A satisfy $\lambda_{d,d} = \lambda_{0,1} + 1$ and $\lambda_{p,d} = \lambda_{0,d-p+1}$ for all $p \in \{2, \ldots, d-1\}$ (see [GLS98, BB05]).

Using Grothendieck's spectral sequence

$$E_2^{p,n-i} = H_{\mathfrak{m}}^p(H_I^{n-i}(R)) \Longrightarrow H_{\mathfrak{m}}^{p+n-i}(R)$$

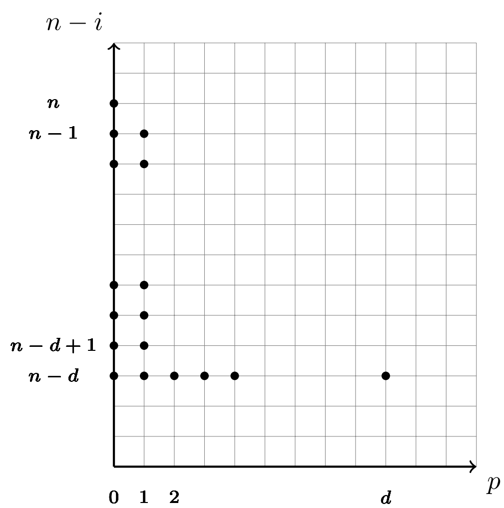
we can give a similar result for the CM_2 case.

Corollary 3.7. *Assume either that :*

- *A is CM_2 and contains a field of positive characteristic, or*
- *A is CM_2 and I is a squarefree monomial ideal.*

Then the Lyubeznik numbers of A satisfy $\lambda_{d,d} = \lambda_{0,1} + \lambda_{1,2} + 1$, $\lambda_{2,d} = \lambda_{0,d-1}$ and $\lambda_{p,d} = \lambda_{0,d-p+1} + \lambda_{1,d-p+2}$ for all $p \in \{3, \ldots, d-1\}$.

Proof. Under the CM_2 condition, the only possibly non-zero terms of Grothendieck spectral sequence are placed at the dot spots in the following diagram:



We have $\lambda_{0,0} = 0$ by Grothendieck's vanishing theorem (see [BS98, Theorem 6.1.2]). We also notice that $\lambda_{0,d} = \lambda_{1,d} = 0$.

The only possible non-zero differentials at each E_j-page, $j \geq 2$, of the spectral sequence are:

$$d_j : E_j^{0,n-j+1} \longrightarrow E_j^{j,n-d} \text{ and } d_j : E_j^{1,n-j+1} \longrightarrow E_j^{j+1,n-d}.$$

By the general theory of spectral sequences, there exist filtrations $0 \subseteq F_n^r \subseteq \cdots \subseteq F_0^r \subseteq H_{\mathfrak{m}}^r(R)$ for all r, such that the consecutive quotients are $F_i^r / F_{i+1}^r = E_\infty^{i,r-i}$. Then, taking into account that $H_{\mathfrak{m}}^r(R) = 0$ for all $r \neq n$, we have first:

- $0 = E_\infty^{0,n-d+1} = E_3^{0,n-d+1} = \ker\left(d_2 : E_2^{0,n-d+1} \longrightarrow E_2^{2,n-d}\right)$,
- $0 = E_\infty^{2,n-d} = E_3^{2,n-d} = E_2^{2,n-d}/\operatorname{Im}\left(d_2 : E_2^{0,n-d+1} \longrightarrow E_2^{2,n-d}\right)$,

and thus $\lambda_{2,d} = \lambda_{0,d-1}$. For the next subdiagonal in the diagram we have, in the third page:

- $E_3^{0,n-d+2} = E_2^{0,n-d+2}$,
- $0 = E_\infty^{1,n-d+1} = E_3^{0,n-d+1} = \ker\left(d_2 : E_2^{1,n-d+1} \longrightarrow E_2^{3,n-d}\right)$,
- $E_3^{3,n-d} = E_2^{3,n-d}/\operatorname{Im}\left(d_2 : E_2^{1,n-d+1} \longrightarrow E_2^{3,n-d}\right)$,

and in the fourth page:

- $0 = E_\infty^{0,n-d+2} = E_4^{0,n-d+2} = \ker\left(d_3 : E_3^{0,n-d+2} \longrightarrow E_3^{3,n-d}\right)$,
- $0 = E_\infty^{2,n-d} = E_4^{2,n-d} = E_3^{2,n-d}/\operatorname{Im}\left(d_3 : E_3^{0,n-d+2} \longrightarrow E_3^{3,n-d}\right)$.

Therefore $\lambda_{3,d} = \lambda_{0,d-2} + \lambda_{1,d-1}$ and analogously we get $\lambda_{p,d} = \lambda_{0,d-p+1} + \lambda_{1,d-p+2}$ for all $p \in \{4, \ldots, d-1\}$. For the last case we only have to put into the picture the fact that $H_{\mathfrak{m}}^n(R)$ is isomorphic to the injective hull of the residue field which accounts for the $+1$ in the formula $\lambda_{d,d} = \lambda_{0,1} + \lambda_{1,2} + 1$.

□

4. Squarefree monomial ideals

The aim of this section is to prove that the results given in Theorems 3.4 and 3.5 are sharp. To make explicit computations we will use the approach given by Àlvarez Montaner and Vahidi [AMV14] (see also [AMY18]) where one can interpret Lyubeznik numbers for the case of squarefree monomial ideals is in terms of the linear strands of the free resolution of the Alexander dual of the ideal. Throughout this section we let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with coefficients in a field $\mathbb{K}$. Bass numbers behave well with respect to localization and completion so there is no inconvenience in working in this setting. We start briefly recalling the results of [AMV14].

Let $I^\vee$ be the Alexander dual of a squarefree monomial ideal $I \subseteq R$. Its minimal $\mathbb{Z}$-*graded free resolution* is an exact sequence of free $\mathbb{Z}$-graded R-modules:

$$\mathbb{L}_\bullet(I^\vee): \quad 0 \longrightarrow L_m \xrightarrow{d_m} \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \longrightarrow I^\vee \longrightarrow 0,$$

where the j-th term is of the form

$$L_j = \bigoplus_{\ell \in \mathbb{Z}} R(-\ell)^{\beta_{j,\ell}(I^\vee)},$$

and the matrices of the morphisms $d_j : L_j \longrightarrow L_{j-1}$ do not contain invertible elements. The $\mathbb{Z}$-graded *Betti numbers* of $I^\vee$ are the invariants $\beta_{j,\ell}(I^\vee)$. Given an integer r, the r-*linear strand* of $\mathbb{L}_\bullet(I^\vee)$ is the complex:

$$\mathbb{L}_\bullet^{<r>}(I^\vee): \quad 0 \longrightarrow L_{n-r}^{<r>} \xrightarrow{d_{n-r}^{<r>}} \cdots \longrightarrow L_1^{<r>} \xrightarrow{d_1^{<r>}} L_0^{<r>} \longrightarrow 0,$$

where

$$L_j^{<r>} = R(-j-r)^{\beta_{j,j+r}(I^\vee)},$$

and the differentials $d_j^{<r>} : L_j^{<r>} \longrightarrow L_{j-1}^{<r>}$ are the corresponding components of d_j.

We point out that these differentials can be described using the so-called *monomial matrices* introduced by Miller [Mil00]. These are matrices with scalar entries that keep track of the degrees

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.

of the generators of the summands in the source and the target. Now we construct a complex of $\mathbb{K}$-vector spaces

$$\mathbb{F}_\bullet^{<r>}(I^\vee)^*: \quad 0 \longleftarrow \underbrace{\mathbb{K}^{\beta_{n-r,n}(I^\vee)}}_{\deg 0} \longleftarrow \cdots \longleftarrow \underbrace{\mathbb{K}^{\beta_{1,1+r}(I^\vee)}}_{\deg n-r-1} \longleftarrow \underbrace{\mathbb{K}^{\beta_{0,r}(I^\vee)}}_{\deg n-r} \longleftarrow 0\,,$$

where the morphisms are given by the transpose of the corresponding monomial matrices and thus we reverse the indices of the complex. Then, the Lyubeznik numbers are described by means of the homology groups of these complexes.

Theorem 4.1. [AMV14, Cor. 4.2] *Let $I^\vee$ be the Alexander dual of a squarefree monomial ideal $I \subseteq R$. Then*

$$\lambda_{p,n-r}(R/I) = \dim_{\mathbb{K}} H_p(\mathbb{F}_\bullet^{<r>}(I^\vee)^*).$$

It has been shown in [HSFYZN18], [VZN19], [PPTY22] that the S_r and CM_r properties on the ring R/I provide conditions on the vanishing of Betti numbers of the Alexander dual ideals $I^\vee$ and consequently the shape of the corresponding *Betti table*. In particular, it describes the linear strands of the free resolution. To compute Lyubeznik numbers we have to take a step further and consider the homology of these linear strands so, a priori, it may seem that the results in Theorems 3.4 and 3.5 are not sharp. The following examples show that indeed the results are sharp.

Example 4.2. Let $I = (x_1, x_2, x_3, x_4) \cap (x_1, x_2, x_4, x_6) \cap (x_1, x_2, x_5, x_6) \cap (x_1, x_2, x_5, x_7) \cap (x_1, x_2, x_7, x_8) \cap (x_1, x_3, x_4, x_6) \cap (x_1, x_3, x_5, x_6) \cap (x_1, x_3, x_5, x_7) \cap (x_1, x_3, x_6, x_8) \cap (x_1, x_6, x_7, x_8) \cap (x_2, x_4, x_5, x_7) \cap (x_2, x_4, x_6, x_8) \cap (x_2, x_4, x_7, x_8) \cap (x_3, x_4, x_5, x_6) \cap (x_3, x_4, x_6, x_8) \cap (x_3, x_4, x_7, x_8) \cap (x_4, x_5, x_6, x_7) \cap (x_5, x_6, x_7, x_8)$ be an ideal in $R = \mathbb{K}[x_1, \ldots, x_8]$. The minimal free resolution of its Alexander dual ideal is

$$\mathbb{L}_\bullet(I^\vee): \quad 0 \longrightarrow R(-8)^5 \longrightarrow R(-7)^{12} \oplus R(-6)^4 \longrightarrow R(-5)^{28} \longrightarrow R(-4)^{18} \longrightarrow I^\vee \longrightarrow 0\,,$$

and thus $I^\vee$ has two linear strands. The Lyubeznik table is:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 7 & 0 \\ & & 0 & 0 & 0 \\ & & & 0 & 7 \\ & & & & 1 \end{pmatrix}$$

The ring R/I is S_2 (see [Hol19, Example 5.6]) but it is not Cohen-Macaulay because it has two local cohomology modules different from zero. We point out that R/I is not CM_1 but it is CM_2 so it satisfies the properties shown in Corollary 3.7.

Example 4.3. Let $I = (x_1, x_2, x_3, x_4, x_5) \cap (x_1, x_2, x_3, y_4, y_5) \cap (y_1, y_2, y_3, y_4, y_5)$ be an ideal in $R = \mathbb{K}[x_1, \ldots, x_5, y_1, \ldots, y_5]$. The minimal free resolution of its Alexander dual ideal is

$$\mathbb{L}_\bullet(I^\vee): \quad 0 \longrightarrow R(-7) \oplus R(-8) \longrightarrow R(-5)^3 \longrightarrow I^\vee \longrightarrow 0\,,$$

and thus $I^\vee$ has three linear strands. The Lyubeznik table is:

$$\Lambda(R/I) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & & 0 & 0 \\ & & & & & 3 \end{pmatrix}$$

The ideal I can be interpreted as the edge ideal of a graph $G(3,2)$ obtained from a Cohen-Macaulay bipartite graph G. Then, the ring R/I is CM_4 by using [HSFYZN18, Theorem 4.5]. The ring R/J with $J = I \cap (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ is not equidimensional but it is still CM_4. Despite the fact that the minimal free resolution of its Alexander dual ideal is

$$\mathbb{L}_\bullet(J^\vee): \quad 0 \longrightarrow R(-10) \longrightarrow R(-7) \oplus R(-8) \oplus R(-9)^2 \longrightarrow R(-5)^3 \oplus R(-8) \longrightarrow J^\vee \longrightarrow 0\,,$$

we have $\Lambda(R/I) = \Lambda(R/J)$.

5. Squarefree initial ideals

Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with coefficients in a field $\mathbb{K}$. Assume that R is equipped with a $\mathbb{Z}^m$-graded structure such that $\deg(x_i) \in \mathbb{Z}^m_{\geq 0}$. We have that some homological invariants behave well with respect to Gröbner deformations. For instance, Conca and Varbaro [CV20, Theorem 1.3] proved that, for a $\mathbb{Z}^m$-graded ideal $I \subseteq R$ such that the initial ideal $\mathrm{in}(I)$ with respect to some term order is squarefree, we have:

$$\dim_{\mathbb{K}} H^i_{\mathfrak{m}}(R/I)_\alpha = \dim_{\mathbb{K}} H^i_{\mathfrak{m}}(R/\mathrm{in}(I))_\alpha$$

for all $i \in \mathbb{Z}_{\geq 0}$ and all $\alpha \in \mathbb{Z}^m$. Therefore, extremal Betti numbers, depth and Castelnuovo-Mumford regularity of R/I and $R/\mathrm{in}(I)$ coincide. Classes of ideals satisfying this condition are ASL ideals, Cartwright-Sturmfels ideals and Knutson ideals (see [CV20] for details).

For our purposes we point out the following result:

Proposition 5.1. [CV20, Corollary 2.11] *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field. Let $I \subseteq R$ be an homogeneous ideal of codimension ≥ 2 such that the initial ideal $\mathrm{in}(I)$ with respect to some term order is squarefree. Then:*

(i) *R/I is S_r, $r \geq 2$, if and only if $R/\mathrm{in}(I)$ is S_r.*
(ii) *R/I is CM_r if and only if $R/\mathrm{in}(I)$ is CM_r.*

It has been proved in [ALNnBRM22] that the graphs $\Gamma_t(R/I)$ of equidimensional rings, and consequently some Lyubeznik numbers, also behave well with respect to Gröbner deformations.

Theorem 5.2. [ALNnBRM22, Theorem 3.4] *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field. Let $I \subseteq R$ be an equidimensional homogeneous ideal of codimension ≥ 2 such that the initial ideal $\mathrm{in}(I)$ with respect to some term order is squarefree. Then,*

$$\#\Gamma_t(R/I) = \#\Gamma_t(R/\mathrm{in}(I)).$$

Corollary 5.3. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field. Let $I \subseteq R$ be an equidimensional homogeneous ideal of codimension ≥ 2 such that the initial ideal $\mathrm{in}(I)$ with respect to some term order is squarefree. Then,*

$$\lambda_{d,d}(R/I) = \lambda_{d,d}(R/\mathrm{in}(I)),\ \ \lambda_{0,1}(R/I) = \lambda_{0,1}(R/\mathrm{in}(I)) \ \text{ and } \ \lambda_{1,2}(R/I) = \lambda_{1,2}(R/\mathrm{in}(I)).$$

In positive characteristic, Nadi and Varbaro [NV20] proved the following inequality between the Lyubeznik numbers of R/I and those of $R/\mathrm{in}(I)$.

Proposition 5.4. [NV20, Corollary 2.5] *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field of positive characteristic. Let $I \subseteq R$ be an homogeneous ideal such that the initial ideal $\mathrm{in}(I)$ with respect to some term order is a squarefree monomial ideal. Then $\lambda_{p,i}(R/I) \leq \lambda_{p,i}(R/\mathrm{in}(I))$.*

Combining this result with Theorems 3.4 and 3.5 we obtain the following:

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.

Corollary 5.5. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field of positive characteristic. Let $I \subseteq R$ be an homogeneous ideal such that the initial ideal* $\mathrm{in}(I)$ *with respect to some term order is a squarefree monomial ideal. Then:*

- *If $R/\mathrm{in}(I)$ is S_r with $r \geq 2$ then the Lyubeznik table of R/I satisfies*
$$\lambda_{i,i}(R/I) = \lambda_{i,i+1}(R/I) = \cdots = \lambda_{i,i+(r-1)}(R/I) = 0, \text{ for } i \in \{0, \ldots, d-1\}.$$
- *If $R/\mathrm{in}(I)$ is CM_r then the Lyubeznik table of R/I satisfies*
$$\lambda_{p,i}(R/I) = 0, \ \ \forall p \geq r \text{ and } i \in \{0, \ldots, d-1\}.$$

It is quite common that the Lyubeznik table of a monomial ideal is trivial and thus the following easy consequence becomes relevant.

Corollary 5.6. *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field of positive characteristic. Let $I \subseteq R$ be an homogeneous ideal such that the initial ideal* $\mathrm{in}(I)$ *with respect to some term order is a squarefree monomial ideal. If the Lyubeznik table of $R/\mathrm{in}(I)$ is trivial then the Lyubeznik table of R/I is trivial as well.*

Using these results, we can compute the Lyubeznik table of the following examples:

Example 5.7. [CV20, Example 3.2] Let $R = \mathbb{K}[x_1, \ldots, x_5]$ be a polynomial ring over a field of positive characteristic. Let I be the homogeneous ideal given by the 2×2-minors of the matrix
$$\begin{pmatrix} x_4^2 + x_5^a & x_3 & x_2 \\ x_1 & x_4^2 & x_3^b - x_2 \end{pmatrix}$$
with $\deg(x_4) = a, \deg(x_1) = \deg(x_3) = 1, \deg(x_2) = b$ and $\deg(x_5) = 2$. On the other hand,
$$\mathrm{in}(I) = (x_1x_3, x_1x_2, x_2x_3)$$
where we consider the lex term order and thus the Lyubeznik table of R/I is trivial in any characteristic.

Binomial edge ideals satisfy that their generic initial ideals are squarefree [CDNG18, Theorem 2.1].

Example 5.8. Let $R = \mathbb{K}[x_1, \ldots, x_6, y_1, \ldots, y_6]$ be a polynomial ring over a field of positive characteristic. Let $J_{C_6} \subseteq R$ be the binomial edge ideal associated to the 6-cycle C_6 and $\mathrm{gin}(J_{C_6})$ its generic initial ideal. Namely, we have:
$$\begin{aligned} J_{C_6} &= (x_1y_2 - x_2y_1, x_1y_6 - x_6y_1, x_2y_3 - x_3y_2, -x_3y_4 + x_4y_3, x_4y_5 - x_5y_4, x_5y_6 - x_6y_5) \\ \mathrm{gin}(J_{C_6}) &= (x_5x_6, x_4x_5, x_3x_4, x_2x_3, x_1x_6, x_1x_2, x_4x_6y_5, x_3x_5y_4, x_2x_6y_1, x_2x_4y_3, x_1x_5y_6, x_1x_3y_2, \\ &\quad x_3x_6y_4y_5, x_3x_6y_1y_2, x_2x_5y_3y_4, x_2x_5y_1y_6, x_1x_4y_5y_6, x_1x_4y_2y_3, x_4x_6y_1y_2y_3, x_3x_5y_1y_2y_6, \\ &\quad x_2x_6y_3y_4y_5, x_2x_4y_1y_5y_6, x_1x_5y_2y_3y_4, x_1x_3y_4y_5y_6) \end{aligned}$$

The Lyubeznik table of $R/\mathrm{gin}(J_{C_6})$ is trivial in any characteristic and thus the Lyubeznik table of R/J_{C_6} is trivial as well.

References

[ALNnBRM22] Lilia Alanís-López, Luis Núñez Betancourt, and Pedro Ramírez-Moreno. Connectedness of square-free Groebner deformations. *Proc. Amer. Math. Soc.*, 150(4):1405–1419, 2022. 7

[AM00] Josep Àlvarez Montaner. Characteristic cycles of local cohomology modules of monomial ideals. *J. Pure Appl. Algebra*, 150(1):1–25, 2000. 1

[AM15] Josep Àlvarez Montaner. Lyubeznik table of sequentially Cohen-Macaulay rings. *Comm. Algebra*, 43(9):3695–3704, 2015. 1

[AML06] Josep Àlvarez Montaner and Anton Leykin. Computing the support of local cohomology modules. *J. Symbolic Comput.*, 41(12):1328–1344, 2006. 1

[AMV14] Josep Àlvarez Montaner and Alireza Vahidi. Lyubeznik numbers of monomial ideals. *Trans. Amer. Math. Soc.*, 366(4):1829–1855, 2014. 5, 6

[AMY18] Josep Àlvarez Montaner and Kohji Yanagawa. Lyubeznik numbers of local rings and linear strands of graded ideals. *Nagoya Math. J.*, 231:23–54, 2018. 5

[BB05] Manuel Blickle and Raphael Bondu. Local cohomology multiplicities in terms of étale cohomology. *Ann. Inst. Fourier (Grenoble)*, 55(7):2239–2256, 2005. 4

[BS98] Markus P. Brodmann and Rodney Y. Sharp. *Local cohomology: an algebraic introduction with geometric applications*, volume 60 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998. 4

[CDNG18] Aldo Conca, Emanuela De Negri, and Elisa Gorla. Cartwright-Sturmfels ideals associated to graphs and linear spaces. *J. Comb. Algebra*, 2(3):231–257, 2018. 8

[CV20] Aldo Conca and Matteo Varbaro. Square-free Gröbner degenerations. *Invent. Math.*, 221(3):713–730, 2020. 1, 3, 7, 8

[GLS98] Ricardo García López and Claude Sabbah. Topological computation of local cohomology multiplicities. volume 49, pages 317–324. 1998. Dedicated to the memory of Fernando Serrano. 4

[Hol19] Brent Holmes. A generalized Serre's condition. *Comm. Algebra*, 47(7):2689–2701, 2019. 6

[HS93] Craig L. Huneke and Rodney Y. Sharp. Bass numbers of local cohomology modules. *Trans. Amer. Math. Soc.*, 339(2):765–779, 1993. 2, 3

[HSFYZN18] Hassan Haghighi, Seyed Amin Seyed Fakhari, Siamak Yassemi, and Rahim Zaare-Nahandi. A generalization of Eagon-Reiner's theorem and a characterization of bi-CM_t bipartite and chordal graphs. *Comm. Algebra*, 46(9):3889–3898, 2018. 6, 7

[HYZN12] Hassan Haghighi, Siamak Yassemi, and Rahim Zaare-Nahandi. A generalization of k-Cohen–Macaulay simplicial complexes. *Ark. Mat.*, 50(2):279–290, 2012. 3

[Kaw02] Ken-ichiroh Kawasaki. On the highest Lyubeznik number. *Math. Proc. Cambridge Philos. Soc.*, 132(3):409–417, 2002. 2

[Kun69] Ernst Kunz. Characterizations of regular local rings of characteristic p. *Amer. J. Math.*, 91:772–784, 1969. 1

[Lyu84] Gennady Lyubeznik. On the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for ideals $\mathfrak{a}$ generated by monomials in an R-sequence. In *Complete intersections (Acireale, 1983)*, volume 1092 of *Lecture Notes in Math.*, pages 214–220. Springer, Berlin, 1984. 1

[Lyu93] Gennady Lyubeznik. Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra). *Invent. Math.*, 113(1):41–55, 1993. 1, 2, 3

[Mil00] Ezra Miller. The Alexander duality functors and local duality with monomial support. *J. Algebra*, 231(1):180–234, 2000. 5

[MNS11] Ezra Miller, Isabella Novik, and Ed Swartz. Face rings of simplicial complexes with singularities. *Math. Ann.*, 351(4):857–875, 2011. 3

[NnBSW19] Luis Núñez Betancourt, Sandra Spiroff, and Emily E. Witt. Connectedness and Lyubeznik numbers. *Int. Math. Res. Not. IMRN*, (13):4233–4259, 2019. 2

[NV20] Parvaneh Nadi and Matteo Varbaro. Canonical Cohen-Macaulay property and Lyubeznik numbers under Gröbner deformations. *Rend. Istit. Mat. Univ. Trieste*, 52:579–589, 2020. 7

[PPTY22] Mohammad R. Pournaki, Milad Poursoltani, Naoki Terai, and Siamak Yassemi. Simplicial complexes satisfying Serre's condition versus the ones which are Cohen-Macaulay in a fixed codimension. *SIAM J. Discrete Math.*, 36(4):2506–2522, 2022. 3, 6

[PS73] Christian Peskine and Lucien Szpiro. Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck. *Inst. Hautes Études Sci. Publ. Math.*, (42):47–119, 1973. 1

[RWZ22] Thomas Reichelt, Uli Walther, and Wenliang Zhang. On Lyubeznik type invariants. *Topology Appl.*, 313:Paper No. 107983, 31, 2022. 2

[Sch79] Peter Schenzel. Zur lokalen Kohomologie des kanonischen Moduls. *Math. Z.*, 165(3):223–230, 1979. 3

[Sch82] Peter Schenzel. *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum-Ringe*, volume 907 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982. With an English summary. 3

[SW07] Anurag K. Singh and Uli Walther. Local cohomology and pure morphisms. *Illinois J. Math.*, 51(1):287–298 (electronic), 2007. 1

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.

[VZN19] Matteo Varbaro and Rahim Zaare-Nahandi. Simplicial complexes of small codimension. *Proc. Amer. Math. Soc.*, 147(8):3347–3355, 2019. 6

[Wal01] Uli Walther. On the Lyubeznik numbers of a local ring. *Proc. Amer. Math. Soc.*, 129(6):1631–1634, 2001. 2

[Yan01] Kohji Yanagawa. Bass numbers of local cohomology modules with supports in monomial ideals. *Math. Proc. Cambridge Philos. Soc.*, 131(1):45–60, 2001. 3

[Zha07] Wenliang Zhang. On the highest Lyubeznik number of a local ring. *Compos. Math.*, 143(1):82–88, 2007. 2

Departament de Matemàtiques and Institut de Matemàtiques de la UPC-BarcelonaTech (IMTech), Universitat Politècnica de Catalunya, Av. Diagonal 647, Barcelona 08028; and Centre de Recerca Matemàtica (CRM), 08193 Bellaterra, Barcelona.

E-mail address: josep.alvarez@upc.edu

Department of Mathematics, Purdue University, West Lafayette, IN 47907

E-mail address: syassemi@purdue.edu

19 Mar 2024 10:49:53 PDT
230330-Yassemi Version 2 - Submitted to J. Comm. Alg.