A CHARACTERISATION OF LOCAL $N$-RINGS AND AN APPLICATION TO ABSTRACT HARMONIC ANALYSIS

BETTINA WILKENS

ABSTRACT. A commutative ring with unity is called an $N$-ring if each of its ideals is contracted from an Noetherian extension ring. The chief result of the paper is a characterisation of local $N$-rings by their subdirectly irreducible quotients. The results are applied to characterise spectral synthesis on $G$-invariant subspaces of the space of complex valued functions on an Abelian group $G$.

1. Introduction

Any ring appearing in this paper is commutative with unity. Elementary properties of localisation - it commutes with homomorphisms, every ideal of a localisation is extended from the ground ring (see [2, Corollary 3.4, Proposition 3.11]) - will be used freely. The paper would not exist without the work of R. Gilmer, W. Heinzer, and D. Lantz, the instigators of the investigation of $N$-rings, introducing the concept in [5] and adding many further results in the subsequent papers [6] and [7].

A ring $R$ is an $N$-ring ([5]) if, for each ideal $I$ of $R$, there exists a Noetherian extension ring $T(I)$ of $R$ such that $I$ is the intersection of $R$ and an ideal of $T(I)$.

A colon ideal is a relative annihilator $(I : S)$, where $I$ is an ideal of a ring and $S$ is a subset. Our investigations are built on the below result from [6]:

**Theorem 1.1.** [6, Theorem 2.3 and p.116, second paragraph] The following conditions on the ring $R$ are equivalent:

a) $R$ is an $N$-ring.

b) $R$ satisfies ACC on colon ideals.

c) $R$ satisfies DCC on colon ideals.

Recall that a ring is subdirectly irreducible if the intersection of its nonzero ideals is nonzero. The unique nonzero minimal ideal of a subdirectly irreducible ring $T$ is commonly denoted $\mu(T)$.

Given an ideal $I$ of a ring $R$ and an element $a$ of $R \setminus I$, Zorn’s Lemma yields an ideal $J$ of $R$ that is maximal with respect to containing $I$, but not $a$. Every nonzero ideal of $R/J$ containing $a + J$, the quotient $R/J$ is subdirectly irreducible. In other word, $R$ is a subdirect product of subdirectly irreducible rings.

The main result of the paper is Theorem 3.6; a local $N$-ring $R$ is characterised by the fact that its subdirectly irreducible quotients have finite length as $R$-modules. Theorem 3.10 and Lemma 3.11 explore the relationship between the $N$-ring property and the structure of subdirectly irreducible rings.

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quotients in a not necessarily local ring.

The incentive for research comes from abstract harmonic analysis, more precisely the study of function spaces on discrete Abelian groups. We consider the space of complex-valued functions on an Abelian group $G$ with the topology of pointwise convergence, the group acting by translations. Spectral synthesis is said to hold on a closed $G$-submodule $V$ of this space if the finite-dimensional submodules of any closed submodule $W$ span a dense subspace of $W$. The final section of the paper applies previous results to obtain a ring-theoretic characterisation of this property. The paper concludes with a description of the module structure of $V$ assuming spectral synthesis to hold on it.

A note on notation: Sequences of ring elements frequently appearing alongside the ideals they generate, we use triangular parentheses for ideals. Given a vector space $X$ and a subspace $U$ of $X$, we denote $\dim(X/U)$ as $\text{codim}_X(U)$.

The remainder of this introductory section sets up the framework for harmonic analysis on discrete Abelian groups. We follow [14] and [15] with regard to definitions and notation.

Let $G$ be an Abelian group. The set of functions $G \to \mathbb{C}$ is denoted $\mathcal{C}G$. We regard $\mathcal{C}G$ as endowed with the product topology induced by the Euclidean topology on $\mathbb{C}$, the previously mentioned topology of pointwise convergence. With respect to this topology, $\mathcal{C}G$ is a locally convex topological space. Indeed, the topology is generated by the family of seminorms $(p_x, x \in G)$ where $p_x(f) = |f(x)|$ ($f \in \mathcal{C}G$).

Consider the bilinear coupling $\mathcal{C}G \times \mathcal{C}G \to \mathbb{C}$ given by

$$\langle \sum_{i=1}^{n} \lambda_ix_i, f \rangle = \sum_{i=1}^{n} \lambda_if(x_i).$$

For $a \in \mathbb{C}G$, let $\nu_a \colon \mathcal{C}G \to \mathbb{C}$ be the function defined by $\nu_a(f) = \langle a, f \rangle$. The continuity of the functions $\nu_x, x \in G$, hence their linear combinations $\nu_a, a \in \mathcal{C}G$, is built into the definition of the product topology.

There are no other continuous linear functionals on $\mathcal{C}G$. To see this, let $\alpha$ be such a functional. Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. The preimage $\alpha^{-1}(D)$ is an open subset of $\mathcal{C}G$ containing $0$, so that there exist a finite subset $\{x_1, \ldots, x_k\}$ of $G$ and positive real numbers $\varepsilon_1, \ldots, \varepsilon_k$ satisfying

$$\alpha(f) \in D \text{ if } |f(x_i)| < \varepsilon_i \ (1 \leq i \leq k).$$

In particular, $\alpha(f) \in D$ whenever $f(x_1) = 0 = \ldots = f(x_k)$. The set $U = \{f \in \mathcal{C}(G) \mid f(x_1) = 0 = \ldots = f(x_k)\}$ is a subspace of $\mathcal{C}(G)$ and $\alpha(U)$ is a subspace of $\mathbb{C}$ contained in $D$. This is possible only if $\alpha(U) = \{0\}$ and $\alpha$ is a linear combination of the maps $\nu_{x_1}, \ldots, \nu_{x_k}$.

The map $a \mapsto \nu_a$ ($a \in \mathbb{C}G$) has turned out to be an isomorphism from $\mathcal{C}G$ to the topological dual of $\mathcal{C}G$. Reversing viewpoints, an element of $\mathcal{C}G$ extends linearly to a unique element of the algebraic dual $\mathcal{C}G^*$.

The group $G$ acts on $\mathcal{C}G$ by translations, i.e. via $(xf)(g) = f(xg)$. A closed $G$-invariant subspace of $\mathcal{C}G$ is called a variety or variety on $G$. Varieties on $G$ typically arise as solution sets of systems of functional equations on $G$. Spectral synthesis on $V$ is a guarantee that the solution of a functional equation in $V$ is the limit of a sequence of tractable functions called exponential polynomials.

For subsets $S$ of $\mathcal{C}G$ and $U$ of $\mathcal{C}G$, orthogonal complements are defined as usual:

$$S^\perp = \{a \in \mathcal{C}G \mid \langle a, f \rangle = 0 \text{ for all } f \in S\},$$

$$U^\perp = \{f \in \mathcal{C}G \mid \langle u, f \rangle = 0 \text{ for all } u \in U\}.$$
Assume that there are finite-codimension subspaces \( W \) whose kernel is \( (A) \) easily extends to \( E \).

Lemma 1.2 (see [11], [13]). Let \( V \) be a variety on \( G \) and \( I \) an ideal of \( \mathcal{C}G \). Then \( V^\perp = V \) and \( I^{\perp\perp} = I \).

Lemma 1.2 means that the map given by \( V \mapsto V^\perp \) from the set of varieties on \( G \) to the set of ideals of \( \mathcal{C}G \) is bijective. A convenient equivalent formulation (see [15]) is:

Lemma 1.3 (see [15]). Let \( V \) be a variety on \( G \). As a \( \mathcal{C}G \)-module, \( V \) is isomorphic to \( (\mathcal{C}G/V^\perp)^* \).

Lemmas 1.2 and 1.3 establish a strict connection between properties of the variety \( V \) and “dual” properties of the ring \( \mathcal{C}G/V^\perp \), an invitation to look at the module structure of \( \mathcal{C}G \) through the lens of commutative algebra.

2. Preliminaries

2.1. Some linear algebra. Let \( X, Y \) and \( V \) be vector spaces over a field \( F \) and let \( \varphi: X \times Y \to V \) be an \( F \)-bilinear map. We abbreviate \( \varphi(x, y) \) to \( xy \) \( (x \in X, y \in Y) \), extending this notation to subsets in the usual way. For subspaces \( W \) of \( V \), define

\[
D_W = \{ x \in X \mid x Y \leq W \}, \quad E_W = \{ y \in Y \mid X y \leq W \}.
\]

Lemma 2.1. The following are equivalent:

a) For every hyperplane \( H \) of \( V \), \( \text{cod}_X D_H < \infty \).

b) There are finite-codimension subspaces \( X_1 \) of \( X \) and \( Y_1 \) of \( Y \) satisfying \( X_1 Y_1 = \{0\} \).

Proof. Let \( H \) be a hyperplane of \( V \). For \( y \in Y \), the map \( \psi_y \), defined by \( \psi_y(x + D_H) = xy + H \) is an element of \( \text{Hom}(X/D_H, V/H) \) and the map given by \( y \mapsto \psi_y \) is a homomorphism \( Y \to \text{Hom}(X/D_H, V/H) \) whose kernel is \( E_H \). If \( \text{cod}_X D_H < \infty \), then \( \dim \text{Hom}(X/D_H, V/H) = \text{cod}_X D_H \geq \dim Y / E_H \). Letting \( X \) and \( Y \) change places in this argument, we obtain that \( \dim X / D_H \) is finite if and only if \( \dim Y / E_H \) is finite (and equal to \( \dim X / D_H \)). In particular:

\[
(A) \text{ Condition a) holds if and only if } \text{cod}_Y E_H \text{ is finite for all hyperplanes } H \text{ of } V.
\]

A subspace \( W \) of \( V \) of finite codimension being the intersection of finitely many hyperplanes, Remark (A) easily extends to

\[
(B) \text{ condition a) holds if and only if both } \text{cod}_X D_W \text{ and } \text{cod}_Y E_W \text{ are finite whenever } \text{cod}_W W \text{ is finite.}
\]

Assume that there are finite-codimension subspaces \( X_1 \) of \( X \) and \( Y_1 \) of \( Y \) satisfying \( X_1 Y_1 = \{0\} \). Let \( H \) be a hyperplane in \( V \). For \( x \in X_1 \), let \( \phi_x \) be the homomorphism \( Y/Y_1 \to V/H \) given by \( \phi_x(y + Y_1) = xy + H \). The map defined via \( x \mapsto \phi_x \) is a homomorphism \( X_1 \to \text{Hom}(Y/Y_1, V/H) \) with kernel \( D_H \cap X_1 \). In particular, \( \text{cod}_X D_H \) is finite, verifying that b) implies a).

Now assume that a) holds.
(C) There is a finite-codimension subspace $U$ of $X$ with the property that $uY$ is finite-dimensional for $u$ in $U$.

Proof of (C): Assume otherwise. We recursively construct a hyperplane that violates condition a).

Pick $a_1 \in X$ and $b_1 \in Y$ such that $a_1b_1 \neq 0$, let $A_0 = \{0_X\}$ and $B_0 = \{0_Y\}$.

Let $n \in \mathbb{N}$. Assume that there are chains of subspaces $A_0 < \ldots < A_n \leq X$ and $B_0 < \ldots < B_n \leq Y$ satisfying the following:

For $k = 1, \ldots, n$, $A_k = A_{k-1} + \langle a_k \rangle$ and $B_k = B_{k-1} + \langle b_k \rangle$ with

\[(D) \quad a_kY \cap \langle A_{k-1}B_{k-1} \rangle = \{0\} \text{ and } a_kb_k \notin a_kB_k - 1.\]

Let $W$ be a complement of $\langle A_nB_n \rangle$ in $V$. As $\text{cod} W \leq n^2$ and (C) is assumed false, there is $a$ in $D_W$ such that $aY$ has infinite dimension, in particular, $aY \neq bB_n$. Let $a_{n+1} = a$ and pick $b_{n+1} \in Y$ satisfying $a_{n+1}b_{n+1} \notin a_{n+1}B_n$.

Put $A_{n+1} = A_n + \langle a_{n+1} \rangle$ and $B_{n+1} = B_n + \langle b_{n+1} \rangle$ to extend the two chains by another term.

Let $A = \bigcup_{n \in \mathbb{N}} A_n$, $B = \bigcup_{n \in \mathbb{N}} B_n$, $S = \langle a_ib_i | i \in \mathbb{N} \rangle$, and $T = \langle a_jb_j | i, j \in \mathbb{N}, j > i \rangle$.

Consider an equation $0 = \sum_{j=1}^{n} a_jc_j$ where $c_j \in B_j$ for $j = 1, \ldots, n$. Since $a_nc_n \in \langle A_{n-1}B_{n-1} \rangle$, property (D) forces $a_nc_n = 0$. Assuming that $a_kc_\ell = 0$ for $\ell = n-i+1, \ldots, n$, we obtain that $a_{n-i}b_n-i \in a_{n-i}Y \cap \langle A_{n-i-1}B_{n-i-1} \rangle = \{0\}$. Thus

\[(E) \quad \sum_{j=1}^{n} a_jc_j = 0 \text{ if and only if } a_jc_j = 0 \text{ for } j = 1, \ldots, n.\]

Statement (E) immediately implies that the vectors $a_ib_i$, nonzero by construction, form a basis of $S$. If $\sum_{i=1}^{k} \lambda_ia_k = 0$ ($\lambda_1, \ldots, \lambda_k \in F$), then $\sum_{i=1}^{k} \lambda_ia_kb_j = 0$ holds for $j = 1, \ldots, k$. By remark (E), $\lambda_ja_kb_j = 0 = \lambda_j$. Thus the vectors $a_i, i \in \mathbb{N}$, are linearly independent. If $S \cap T \neq \{0\}$, then we have an equality $0 = \sum_{j=1}^{n} a_jc_j$ with $c_j \in B_j$ whenever $1 \leq j \leq n$ and $c_k \notin B_{k-1}$ for at least one $k$. However, (E) implies that $a_kc_k = 0$ and $a_kB_k \subseteq a_k(\langle c_k \rangle + B_{k-1}) \subseteq a_kB_{k-1}$, a contradiction.

Having established that $S \cap T = \{0\}$, we pick a complement $W$ of $S$ in $V$ containing $T$. Let $H = W + \langle a_1b_1 - a_i b_i | 1 < i \in \mathbb{N} \rangle$ - note that $H$ is a hyperplane in $V$. Let $0 \neq z = \sum_{i=1}^{m} \lambda_ai$.

For the minimal $j$ with $\lambda_j \neq 0$, we have $z b_j \equiv \lambda_ja_jb_j \neq 0 \pmod{H}$. It follows that $A \cap D_H = \{0\}$. Since the vectors $a_i, i \in \mathbb{N}$, are linearly independent, $\text{cod}XH$ cannot be finite, establishing (C).

Now assume the lemma fails. Remark (C) provides a finite-codimension subspace $U$ of $X$ satisfying $\dim uY < \infty$ whenever $u \in U$. Since $\text{cod} U < \infty$, any finite-codimension subspace of $U$ is one of $X$.

Pick $c_1 \in U$ and $d_1 \in Y$ to satisfy $c_1d_1 \neq 0$; set $C_0 = \{0_X\}$ and $D_0 = \{0_Y\}$.

Let $n \in \mathbb{N}$. Assume that we have found chains $C_0 < \ldots < C_n \leq U$ and $D_0 < \ldots < D_n \leq Y$ such that, for $k = 1, \ldots, n$, we have $C_k = C_{k-1} + \langle c_k \rangle$, $D_k = D_{k-1} + \langle d_k \rangle$ with

\[(F) \quad C_{k-1}d_k = \{0\} \text{ and } c_kd_k \notin \langle C_k \rangle Y.\]
Let $K = \{ d \in Y \mid C_d = \{0\} \}$. Since $\dim C_n$ is finite and $C_n \leq U$, cod $K$ is finite. Choose a complement $W$ of $\langle C_nY \rangle$ in $V$. The lemma being assumed false, we must have $(D_W \cap U)K \neq \{0\}$. Pick elements $c_{n+1}$ of $D_W \cap U$ and $d_{n+1}$ of $K$ satisfying $d_{k+1}b_{k+1} \neq 0$, put $C_{n+1} = C_n + \langle c_{n+1} \rangle$ and $D_{n+1} = D_n + \langle d_{n+1} \rangle$.

The linear independence of the vectors $c_id_i$, $i \in \mathbb{N}$, is immediate from the second condition in (F). If $c = \sum_{i=1}^{n} \mu_ic_i$ with $\mu_i \neq 0$, then $cd_n = \mu_ncnd_n \neq 0$. Thus the vectors $c_i$, $i \in \mathbb{N}$, are linearly independent.

Now the proof proceeds on familiar lines: Let $W$ a complement of $\langle c_id_i \mid i \in \mathbb{N} \rangle$ in $V$ and $H = W + \langle c_1d_1 - c_id_i \mid 2 \leq i \in \mathbb{N} \rangle$. Let $w = \sum_{k=1}^{\ell} \lambda_kc_k$ with $\lambda_k \neq 0$. Then $wd_l = \lambda_kc_id_k \neq 0$ (mod $H$), so that $\langle c_k \mid k \in \mathbb{N} \rangle$ is an infinite-dimensional subspace of $U$ intersecting $D_H$ in $\{0\}$, a contradiction completing the proof.

Retaining the notation of Lemma 2.1, let cod $X_1 = n$ and cod $Y_1 = m$. Let $H$ be a hyperplane in $V$. Then
\[ \dim(X_1/(X_1 \cap D_H)) \leq m, \]
so that cod $D_H \leq m + n$. We note:

**Corollary 2.2.** The codimension of $D_H$ is finite for all hyperplanes $H$ if and only if there is $N > 0$ such that $\text{cod}_X(D_H) \leq N$ holds whenever $H$ is a hyperplane in $V$.

**2.2. Lemmas on local rings.** Let $(R, M)$ be a local ring. The first Lemma in this section is an adaptation of one of various well-known proofs of Krull’s intersection theorem.

**Lemma 2.3.** The following are equivalent:

a) For every ideal $I$ of $R$, $\bigcap_{n \in \mathbb{N}} (M^n + I) = I$.

b) Let $(a_n)$ be a sequence in $R$ satisfying $a_n \in M^n$ for $n \in \mathbb{N}$. The ideal $\langle a_i \mid i \in \mathbb{N} \rangle$ is generated by finitely many terms of the sequence.

c) For every ideal $I$ of $R$, there is $n$ in $\mathbb{N}$ such that $I \cap M^n$ is contained in a finitely generated subideal of $I$.

d) Let $(I_k)_{k \in \mathbb{N}}$ be an ascending chain of ideals in $R$. There is a power $M^n$ such that the chain $(I_k \cap M^n)_{k \in \mathbb{N}}$ stabilises.

**Proof.**

We begin by establishing the equivalence of conditions b), c), and d). Let $I$ be an ideal of $R$ with the property that, for all $n$, $I \cap M^n$ is not contained in a finitely generated subideal. Let $a_0 = 0$ and $0 \neq a_1 \in I \cap M$. Given $i > 1$ and elements $a_1, \ldots, a_{i-1}$ of $I$ such that, for $1 \leq j \leq i - 1$, $a_j \in M^i \setminus \langle a_0, \ldots, a_{j-1} \rangle$, there is $a_i \in (M^i \cap I) \setminus \langle a_1, \ldots, a_{i-1} \rangle$. The sequence $(a_n)_{n \in \mathbb{N}}$, violates condition b).

Suppose that c) holds. Let $(I_k)_{k \in \mathbb{N}}$ be an ascending chain of ideals and let $I = \bigcup_{k \in \mathbb{N}} I_k$. Choose $n$ sufficiently large for $I \cap M^n$ to be contained in a finitely generated subideal $J$ of $I$. Being finitely generated, $J \subseteq I_m$ for some $m$ and $I \cap M^n = I_m \cap M^n = I_m \cap M^n$ whenever $\ell \geq m$. Hence c) implies d).

Finally, assume that d) holds. Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of elements of $R$ satisfying $a_n \in M^n$ for all $n$. For $k \in \mathbb{N}$, let $I_k = \langle a_1, \ldots, a_k \rangle$ and let $I = \bigcup_{k \in \mathbb{N}} I_k$. If $I$ is not generated by finitely many $a_n$, then since $I_{n+t} = I_n + (M^{n+1} \cap I_{n+t})$ holds for all $n$ and $t$, the sequence $(I_k \cap M^n)_{k \in \mathbb{N}}$ cannot stabilise for any $n$. 

Thus d) implies b).

Now assume that a) holds, while b) does not. Let \((a_n)_{n \in \mathbb{N}}\) be a sequence in \(R\) satisfying \(a_n \in M^n\) for all \(n\). Define ideals \(I_k\) and \(I\) as above. Suppose that \(I\) is not generated by finitely many \(a_n\). Given \(n_1 < \ldots < n_i\), there are integers \(m\) and \(n\) with \(n_i < m < n\) and \(a_m \notin I_{n_i} + M^n\). This yields the existence of a strictly ascending integer sequence \((n_i)_{i \in \mathbb{N}}\) such that

\[
\text{for } i \in \mathbb{N}, a_{n_i} \in M^{n_i} \setminus \langle a_{n_1}, \ldots, a_{n_{i-1}} \rangle + M^{n_{i+1}}.
\]

Let \(K = \langle a_{n_i} - a_{n_j} | 2 \leq i \in \mathbb{N} \rangle\). We have \(a_{n_i} \in \bigcap_{n \in \mathbb{N}} (K + M^n) = K\). This makes \(a_{n_i}\) an \(R\)-linear combination

\[
a_{n_i} = \sum_{i=2}^{n} \alpha_i (a_{n_i} - a_{n_j}).
\]

Let \(\alpha = \sum \alpha_i\). Since \((\alpha - 1)a_{n_i} = \sum \alpha_i a_{n_j} \in M^{n_2}\), \(\alpha - 1\) is not a unit in \(R\), i.e. \(\alpha \notin M\). Hence \(\alpha_i \notin M\) for some \(i \geq 2\). Since \(a_{n_j} \in M^{n_{i+1}}\) whenever \(j > i\), we have \(\alpha_j a_{n_j} \equiv (\alpha - 1)a_{n_i} - \sum_{j=2}^{i} \alpha_j a_{n_j}\) (mod \(M^{n_{i+1}}\)). Yet \(\alpha_i\) is a unit in \(R\), so that \(a_{n_j} \in \langle a_{n_1}, \ldots, a_{n_{i-1}} \rangle + M^{n_{i+1}}\) after all, a contradiction. So a) implies b).

Finally, we set to proving that c) implies a). Every proper ideal \(J\) of \(R\) is contained in \(M\) and the quotient ring \(R/J\) is local with maximal ideal \(M/J\). Let \(I/J\) be an ideal in \(R/J\). There is a finitely generated subideal \(I_1\) of \(I\) containing \(M^n \cap I\) for some \(n\). Since \(J \subseteq I\), we have \((J + M^n) \cap I = J + (I \cap M^n)\); in other words,

\[
(I/J) \cap (M/J)^n = (I \cap (M^n + J))/J = ((I \cap M^n) + J)/J \subseteq (I_1 + J)/J.
\]

Thus property c) carries over to homomorphic images and it is sufficient to verify that in a local ring in which c) holds the powers of the maximal ideal intersect in \(\{0\}\).

Let \((S, U)\) be a local ring with property c) and let \(L = \bigcap_{n \in \mathbb{N}} U^n\). Applied to \(I = U\), c) yields that some power of \(U\) is contained in a finitely generated subideal \(J\) of \(S\) and \(L = \bigcap_{n \in \mathbb{N}} J^n\). Since \(L = L \cap U^n\) for all \(n\), condition c) implies that \(L\) must be finitely generated. Once \(L = JL\) is established, Nakayama’s Lemma will yield that \(L = \{0\}\).

Let \(\mathcal{S}\) be the set of ideals \(K\) of \(S\) satisfying \(K \subseteq J\) and \(K \cap L = JL\). Since \(JL \in \mathcal{S}\) and \(\mathcal{S}\) is an inductive set, Zorn’s lemma provides a maximal element \(Q\) of \(\mathcal{S}\).

Let \(a\) in \(J\). For \(i \in \mathbb{N}\), let \(P_i = (Q: a^i)\). Let \(P = \bigcup_{i \in \mathbb{N}} P_i\). Since \(P_i \subseteq P_{i+1}\) for \(i \in \mathbb{N}\), \(P\) is an ideal. Let \(n\) be sufficiently large for \(P \cap J^n \subseteq J^n\) to be contained in a finitely generated subideal \(I\) of \(P\). Being finitely generated, \(I \subseteq P_k\) for some \(k\), so that \(P_k \cap J^m = P_m \cap J^m\) whenever \(m \geq k\). Let \(x \in L \cap (a^k J^n + Q)\). Write \(x = a^k y + h\) with \(h \in Q\) and \(y \in J^n\). Then \(a^{k+1} y + ah = ax \in JL \subseteq Q\), so that \(a^{k+1} y \in Q\) and \(y \in J^n P_{k+1}\). But \(J^n \cap P_{k+1} = J^n P_k\), i.e. \(y \in P_k\). Consequently, \(a^k y \in Q\) and \(x \in Q \cap L = JL\). The maximality of \(Q\) implies that \(a^k J^n + Q = Q\). Let \(\{a_1, \ldots, a_s\}\) be a (finite) generating set for \(J\). For \(t = 1, \ldots, s\), we have established the existence of integers \(k_t\) and \(l_t\) satisfying \(a_t^{k_t} J^{l_t} + Q = Q\). With \(p = k_1 + \ldots + k_s\) and \(q = \max \{l_1, \ldots, l_s\}\), we have \(J^{p+q} \subseteq Q\). Setting \(r = p + q\), we get \(L \subseteq J^r \subseteq Q\), i.e. \(L \subseteq L \cap Q = JL\).

As noted above, this suffices to establish b).
Lemma 2.4. Suppose that $R$ possesses the properties from Lemma 2.3. Let $I$ and $J$ be ideals of $R$ with $I \subseteq J$ and let $Q = J/I$. If $Q$ is a nil ideal, then $Q$ is nilpotent.

Proof. Let $S = R/I$ and $N = M/I$. There is an integer $n$ such that $N^n \cap Q$ is contained in some finitely generated subideal $K$ of $Q$. Being generated by finitely many nilpotent elements, $K$ is nilpotent. This makes $Q^n$, and therefore $Q$, nilpotent.

3. Results

3.1. Local $N$-rings. Throughout this section, $(R, M)$ or simply $R$ denotes a local ring. Up to isomorphism, $F = R/M$ is the only simple $R$-module and a subquotient of $R$ annihilated by $M$ carries the structure of an $F$-vector space. This will be tacitly assumed whenever it applies. A ring will be said to have finite length if it has finite length as a module over itself.

Lemma 3.1. A subdirectly irreducible local ring $(S, U)$ is an $N$-ring if and only if it has finite length.

Proof. An $S$-module of finite length is Noetherian and a fortiori an $N$-ring.

Conversely, assume that $S$ is an $N$-ring. Let $A = \mu(S)$, $F = S/U$. Being a simple $S$-module, $A$ is $S$-module-isomorphic to $F$. Unless $U = \{0\}$ and $\ell(S) = 1, A \subseteq U$. Given $0 \neq b \in \text{Ann}(U)$, the module $Sb$ is simple and $A \subseteq Sb$, so that $Sb = A = \text{Ann}(U)$. Let $y \in U \setminus A$. We have seen that $yU \neq \{0\}$, implying that $A \subseteq yU$ and $(A : y) \subset \text{Ann}_S(y)$. If $y$ is not nilpotent, then every power of $y$ is in $U \setminus A$, so that $\text{Ann}_S(y^n) \subset (A : y^n) \subset \text{Ann}_S(y^{n+1})$ holds for all $n$. The ideals $\text{Ann}_S(y^n)$, $n \in \mathbb{N}$, form a non-stabilising strictly ascending chain, so that $S$ is not an $N$-ring by Theorem 1.1. Accordingly, $U$ is the nilradical of $S$. By the definition of an $N$-ring, $S$ is a subring of a Noetherian ring $T$. The ideal $U$ is contained in the nilradical of $T$, which is nilpotent due to $T$ being Noetherian. So $U$ is nilpotent.

Let $J$ be an ideal of $S$ of finite length $k$ and let $J = (I : U)$. We apply induction on $k$ to show that $J$ has finite length. If $I = \{0\}$, we have $J = \text{Ann}_S(U) = A$. If $k > 0$, then there is an ideal $I_1$ contained in $I$ with $I/I_1 \cong S$. Let $J_1 = (I_1 : U)$, $U_1 = (I_1 : J)$, $X = U/U_1$, and $Y = J/J_1$. Since $JU^2 \subseteq IU \subseteq I_1$ and $JU \subseteq I \subseteq J_1$, both subquotients $X$ and $Y$ are annihilated by $U$.

Consider the $F$-bilinear map $\varphi: X \times Y \rightarrow I/I_1$ given by

$$(x + U_1, y + J_1) \mapsto xy + I_1$$

For $V \subseteq X$ and $W \subseteq Y$, let $V^\perp = \{y \in Y \mid \varphi(v, y) = 0 \text{ for } v \in V\}$ and $W^\perp = \{x \in X \mid \varphi(w, x) = 0 \text{ for } w \in W\}$.

Suppose that $Y$ has infinite dimension. Consider an ascending chain $(W_i)_{i \in \mathbb{N}}$ of subspaces of $Y$ satisfying $\dim W_i = i$ for $i \in \mathbb{N}$. The definitions of $X$ and $Y$ imply that $X^\perp = \{0\} = Y^\perp$. This yields that, for all $i$, $\text{codim}_X(W_i^\perp) = i$ and $W_i = W_i^\perp$. Since, for each $i$, the preimage in $S$ of $W_i^\perp$ is the annihilator modulo $I_1$ of the preimage of $W_i$, the ring $S$ does not satisfy $\text{ACC}$ (or $\text{DCC}$) on colon ideals, a contradiction. So $\dim J/J_1 < \infty$. The length of the S-module $I_1$ being $k - 1$, induction yields that $J_1$ has finite length, so we are done.

We know that $U$ is a nilpotent ideal, so we have a minimal $n$ with $U^n = \{0\}$. Since $U$ annihilates $U^{n-1}$, $U^{n-1} = A$, a simple module. Assume that $U^{n-\ell}$ has finite length. Since $U_{n-\ell-1} \subseteq (U^{n-\ell}, U)$, the previous paragraph’s result says that $U_{n-\ell-1}$ has finite length as well.
By Theorem 1.1 (alternatively, see [5, Lemma 2.3]), homomorphic images of an $N$-ring are $N$-rings. So it follows from Lemma 3.1 that the subdirectly irreducible quotients of a local $N$-ring are of finite length. We prove the converse in the remainder of this section.

**Definition 3.2.** Let $H, I$ and $J$ be ideals of $R$. We call $(H, I, J)$ an unfolding triple of ideals if

$$H \subseteq I, I/H \cong_R F$$

and

$$(J: J)/(H: J) \text{ has infinite length as an } R\text{-module.}$$

The next lemma is an amplified echo of Corollary 2.2.

**Lemma 3.3.** Assume that there are no unfolding triples of ideals in $R$. Let $J/I$ be a subquotient of $R$ annihilated by a power $M^n$. Then the following hold:

a) There are ideals $K$ and $L$ of $R$ such that $I \subseteq K \subseteq J$, $KL \subseteq I$ and both $J/K$ and $R/L$ have finite length.

b) Let $\ell = \ell(M/L)$ and $m = \ell(J/K)$. There is a function $f : \mathbb{N}^3 \to \mathbb{N}$ such that, for each $a \in J \setminus I$, there is an ideal $V$ satisfying $I \subseteq V \subseteq J$, $a \notin V$ and $\ell(J/V) \leq f(\ell, m, n)$.

**Proof.** Let $F = R/M$. Subquotients of $R$ annihilated by $M$ will be regarded as $F$ vector-spaces.

Without loss, $n$ is minimal to satisfy $M^nJ \subseteq I$. We apply induction on $n$. If $n = 0$, then $J = I$ and if $n = 1$, we may take $K = J$ and $L = M$, while every $a$ in $J \setminus I$ avoids the preimage of some hyperplane in $J/I$, so the constant function $f \equiv 1$ will satisfy the Lemma’s requirements. We assume that $n > 1$ from now on. For $k = 1,\ldots,n$, let $I_k = M^{n-k}I + J$. Via induction, there exist ideals $K_1$ and $L_1$ satisfying $I \subseteq K_1 \subseteq J$ and $L_1K_1 \subseteq I_1$, while $L_1$ and $K_1$ have finite colength in $R$ and in $J$, respectively. Note that

$$K_1ML_1 \subseteq I_1M \subseteq I.$$

Let $X = K_1/K_1M, Y = L_1/L_1M,$ and $V = I_1/I$. We have a bilinear map $X \times Y \to V$ given by

$$(x + ML_1, y + MK_1) \mapsto xy + I.$$ 

For $W \subseteq V$, let $D_W = \{x \in X, xy \subseteq W\}$ like in Lemma 2.1. Let $H$ be a hyperplane of $V$ and $\tilde{H}$ its preimage in $R$. Observe that $\tilde{H}$ is an ideal. Let $D$ be the preimage of $D_H$ in $R$, i.e.

$$D = K_1 \cap (H : L_1).$$

Since $K_1 \subseteq (I_1 : L_1)$ and $R$ does not possess unfolding triples, the length of the quotient

$$(I_1 : L_1)/(\tilde{H} : L_1)$$

is finite. Accordingly, $\ell(K_1/D) < \infty$, which implies that $\text{codim}(D_H) < \infty$. Lemma 2.1 yields the existence of finite-codimension subspaces $L/L_1M$ of $X$ and $K/K_1M$ of $Y$ satisfying $L_K \subseteq I$. The preimages $L$ and $K$ are ideals of finite colength in $R$ and in $J$, respectively. This establishes statement a).

Moving on to b), it is sufficient to establish the following: Given $k$ in $\{1,\ldots,n-1\}$ and an ideal $D$ such that $I \subseteq D \subseteq I_k$ and $\ell(I_k/D) = t < \infty$, there is an ideal $E$ such that $I \subseteq E \subseteq I_{k-1}$, $E \cap I_k = D$, and $\ell(I_{k-1}/E)$ is bounded in terms of $t$, $\ell$, and $n$. In order to achieve this, we consider the homomorphism

$$\psi : M/L \to \text{Hom}(I_{k-1}, I_k/D)$$

given by $\psi(x + L)(y + I) = xy + D$. Let $\tilde{E} = \ker \psi$. Observe that $\ell(I_{k-1}/\tilde{E}) \leq \ell t$. Let $T$ be a system of coset representatives in $I_{k-1}$ of the elements of an $F$-basis of $\tilde{E}/I_k$ and set $E = \langle T, D \rangle$. The ideal $E$ satisfies $E \cap I_k = D$, while $\ell(I_{k-1}/E) \leq t(\ell + 1)$. \[\Box\]

**Remark 3.4.** Retaining the notation of Lemma 3.3, assume that $J = M$. Let $n \in \mathbb{N}$. There is an ideal $Q$ of finite colength $s$ in $R$ satisfying $Q^n \subseteq M^n$ - indeed, if $KL \subseteq M^n$ as in 3.3 a), we simply take $Q = K \cap L$. Since $s$ varies only with $n$, Statement b) in 3.3 yields a function $g : \mathbb{N} \to \mathbb{N}$ such that every
element \( v \) of \( R \setminus M^n \) has nonzero image in a quotient of \( R \) of length bounded by \( g(n) \). Note that such a \( g \) must be monotonically increasing.

**Lemma 3.5.** The following are equivalent:

a) Every subdirectly irreducible quotient of \( R \) has finite length.

b) For every ideal \( I \) of \( R \), \( I = \bigcap_{n \in \mathbb{N}} (M^n + I) \) and \( R \) does not possess an unfolding triple of ideals.

**Proof.** Assume that a) holds. Let \( a \in R \setminus I \). As remarked in the introduction, there exists an ideal \( L \) with \( I \subseteq L \not\supseteq a \) and subdirectly irreducible quotient \( R/L \). So \( R/L \) has finite length and \( M^k \subseteq L \) for some integer \( k \), in particular, \( a \not\in M^k + I \). This establishes the first property in b).

Suppose that \( R \) possesses an unfolding triple \( (H, I, J) \) of ideals. Let \( L \) be an ideal of finite colength in \( R \) satisfying \( H \subseteq L \not\supseteq I \). Setting \( K = (I: J) \), we have

\[
J \cap L \subseteq I \cap L = H \supseteq IM \supseteq JMK.
\]

In particular, \( JM + (J \cap L) \subseteq (H: K) \). The \( F \)-vector space \( V = J/(JM + (J \cap L)) \) is of finite dimension, say \( n \). Since also \( KM \subseteq (H: J) \), we have a homomorphism \( K/KM \to \text{Hom}(V, I/H) \) whose kernel is \( (H: J) \). However, \( K/(H: J) \) is now an \( F \)-vector space of dimension at most \( n \), a contradiction. Therefore b) follows from a).

Conversely, suppose that b) holds. Let \( I \) be an ideal of \( R \) with subdirectly irreducible quotient \( R/I = \bar{R} \). Let \( 0 \neq a \in \mu(\bar{R}) \). Condition b) guarantees the existence of a power \( \bar{M}^n \) with \( a \not\in \bar{M}^n \), so that \( \bar{M}^n = \{0\} \).

Let \( J \) be a nonzero ideal of \( \bar{R} \) of finite length \( k \) and let \( K = (J: \bar{M}) \). We use induction on \( k \) to show that \( K \) has finite length. If \( k = 0 \), then each nonzero element of \( K \) generates a minimal ideal isomorphic to \( R/M \), so that \( K = \mu(\bar{R}) \cong F \). Suppose that \( k > 0 \). Let \( J_1 \subseteq J \) be an ideal with \( J/J_1 \cong F \) and let \( K_1 = (J_1: \bar{M}) \). Since \( \bar{R} \) does not have unfolding triples, \( \ell(K_1/K) < \infty \) and, since \( \ell(J_1) = k - 1 \), induction yields that \( K_1 \) has finite length. So does \( K \).

Recall that \( \bar{M}^n = \{0\} \). Since \( \bar{M}^{n-k-1} \subseteq (\bar{M}^{n-k}: \bar{M}) \) for \( k \in \{0, \ldots, n-1\} \), \( \bar{R} \) has finite length. \( \square \)

**Theorem 3.6.** A local ring \((R,M)\) is an \( N \)-ring if and only if its subdirectly irreducible quotients have finite length.

**Proof.** Lemma 3.1 establishes the necessity of the condition. A subdirectly irreducible quotient of a quotient of \( R \) is one of \( R \). The sufficiency of the condition will thus be established once we have shown that, if the subdirectly irreducible quotients of \( R \) have finite length, then every ascending chain of annihilator ideals stabilises.

Assuming the existence of a non-stabilising ascending chain of annihilator ideals in \( R \), there exist an ascending chain \((I_k)_{k \in \mathbb{N}}\) and a descending chain \((A_k)_{k \in \mathbb{N}}\) of ideals such that, for all \( k \), \( A_k = \text{Ann}_R(I_k) \). \( I_k = \text{Ann}_R(A_k) \) and neither chain stabilises.

Let \( I = \bigcup_{k \in \mathbb{N}} I_k \). By Lemmas 3.5 and 2.3, there is an integer \( n \) with \( I \cap M^n \) contained in a finitely generated subideal of \( I \) which must be contained in some \( I_k \). Thus we have \((I_{k+t} \cap M^n)A_{k} = \{0\}\) whenever \( t \in \mathbb{N} \).

Let \( n > i \geq 0 \). Assume that there is \( k(i) \in \mathbb{N} \) such that, for \( t \in \mathbb{N} \), \((I_{k(i)+t} \cap M^n)A_{k(i)} = \{0\}\). Let \( A = A_{k(i)} \), \( J = \bigcup_{m \geq k(i)} (I_m \cap M^{n-i-1}) \), and \( V = JA \). Note that \( MV = A(JM) = \{0\} \). From here on, we argue along very similar lines to the proof of Lemma 3.3 a): Let \( X = A/MA \) and \( Y = J/MJ \). A
The codimension of $B$ in $A$ is finite, too, so that there is $s \geq r$ such that $A_s = A_{s+t} + (A_s \cap B)$ holds for all $t$. Given that $(I_{r+t} \cap M^{n-i-1}) A_s = \{0\}$ for all $t$, we may set $s = k(i+1)$ and arrive at the desired conclusion $(I_{k(i+1)+t} \cap M^{n-i-1}) A_{k(i+1)} = \{0\}$ ($t \in \mathbb{N}$).

Via induction on $i$, the chain $(I_k)_{k \in \mathbb{N}}$ does stabilise after all, the intended contradiction. \qed

We close the section with a variation on Lemma 2.3 c) available in local $N$-rings.

**Lemma 3.7.** [compare [6, Lemma 3.4]] Let $(R, M)$ be a local $N$-ring. There is a finitely generated ideal $J$ of $R$ such that $M^k \subseteq J^{k-1}M$ holds whenever $2 \leq k \in \mathbb{N}$.

**Proof.** By Lemma 3.3, applied to the section $M/M^3$, there are ideals $L_1$ and $L_2$ of finite co-length with $L_1 L_2 \subseteq M^3$. Let $L = L_1 \cap L_2$. There is a finitely generated ideal $I$ of $R$ with $M = L + J$. We have $M^2 = J^2 + JL + M^3$. If $n \geq 2$ and $M^n \subseteq J^n + L J^{n-1} + M^{n+1}$, then $M^{n+1} \subseteq (J + L) M^n + M^{n+2} = J^{n+1} + L J^n + M^{n+2}$. In particular, $M^k \subseteq \bigcap_{n \in \mathbb{N}} [J^{k-1}M + M^n]$ ($n, k \in \mathbb{N}$) so that statement $a)$ in Lemma 2.3 yields that $M^k = J^{k-1}M$. \qed

### 3.2. Rings whose subdirectly irreducible quotients have finite length.

There is no analogous result to Theorem 3.6 for general rings. This section is devoted to describing the exact relation between finitely length subdirectly irreducible quotients and the $N$-ring property. In analogy with “locally Noetherian” (see [4]), a ring will be called *locally an $N$-ring* or even *locally $N$* if its localisations at maximal ideals are $N$-rings.

We provide an example of a non-$N$-ring that is locally $N$: Let $G$ be an infinite elementary abelian 2-group with basis $\langle x_i \mid i \in \mathbb{N} \rangle$ and let $R = \mathbb{C}G$. For $x \in G$, $(x+1)(x-1) = 0$. Given that $1 = \frac{1}{2}(x+1 - (x-1))$, every (x)-invariant subspace $U$ of $R$ is the direct sum $(x+1)U \oplus (x-1)U$, with $Ann_U(x-1) = (x+1)U$ and $Ann_U(x+1) = (x-1)U$. For $n \in \mathbb{N}$, let $a_n = \prod_{j=1}^{n} (x_j+1)$ and $I_n = \langle x_1-1, \ldots, x_n-1 \rangle$. We have $Ann_R(I_1) = a_1 R$. Suppose that $Ann_R(I_n) = a_n R$; then $Ann_{Ann_R(I_n)}(I_{n+1}) = Ann_{a_n R}(I_{n+1}) = Ann_{a_{n+1} R}(I_{n+1}) = I_n(x_{n+1}+1) = a_{n+1} R$. The ideals $a_n R, n \in \mathbb{N}$, form a descending chain of annihilators that does not stabilise.

For every $x \in G$ and maximal ideal $M$ of $R$, exactly one of the elements $x+1$ and $x-1$ is in $M$, so exactly one of $\frac{x-1}{x}$ and $\frac{x+1}{x}$ is 0 in $R_M$, showing that $R_M \cong \mathbb{C}$. 

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Every group algebra over $\mathbb{C}$ of an Abelian group of finite torsion free rank with infinite torsion subgroup exhibits similar behaviour and could serve as example (see [3],[4]).

**Lemma 3.8.** If $R$ is locally an $N$-ring, then the subdirectly irreducible quotients of $R$ have finite length.

**Proof.** Let $S$ be a subdirectly irreducible quotient of $R$ and let $A = \mu(S)$. If $a \in A \setminus \{0\}$, then $A = aS \cong S/\text{Ann}_S(a)$, so that $D = \text{Ann}(a) = \text{Ann}(A)$ is a maximal ideal of $S$. Unless $S$ is a field and $D = \{0\}$, in which case there is nothing to prove, we have $a \in D$. If $b$ is a zero divisor in $S$, then $A \subseteq \text{Ann}_S(b)$, showing that $D$ is the set of zero divisors of $S$ (see [9]). Let $0 \neq x \in D$. Consisting of zero divisors, $\text{Ann}_S(x)$ is contained in $D$. Hence the homomorphism $S \to S_D$ defined via $a \mapsto \frac{a}{1}$ is an embedding and $S$ may be identified with a subring of $S_D$. A nonzero ideal $J$ of $S_D$ is extended from a nonzero ideal $I$ of $S$, which, since $A \subseteq I$, yields $AS_D \subseteq J$. Thus $S_D$ is subdirectly irreducible with $\mu(S_D) = AS_D$. By Lemma 3.1, $S_D$ is indecomposable of finite length. Thus for $x \in S \setminus D$, $x^{-1}$ is a polynomial in $x$, so $x$ is invertible in $S$. The universal property of localisation yields $S_D = S$. \hfill $\square$

**Lemma 3.9.** Let $R$ be a ring whose subdirectly irreducible quotients have finite length. For any maximal ideal $M$ of $R$, $R_M$ is an $N$-ring.

**Proof.** Let $R$ be a ring with the indicated property; the homomorphic images of $R$ naturally inherit it. Let $M$ be a maximal ideal of $R$. If $I$ is an ideal contained in $M$, then $(R/I)_{M/I} \cong R_I/IR_M$. So it is sufficient to establish that every nonzero element $a$ of $R_M$ has nonzero image in some finite-length quotient of $R_M$. Write $a = \frac{b}{s}$ with $b \in R$ and $s \notin M$. Since $a \neq 0$, $b \notin bM$, and there is an ideal $I \subseteq R$ such that $R/I$ is subdirectly irreducible and $bM \subseteq I \neq b$. Since $R/I$ is indecomposable of finite length and $bM \subseteq I$, the nilpotent radical of $R/I$ must be $M/I$. Thus $M^k \subseteq I$ for some $k$, in particular, $I$ is an $M$-primary ideal. As in the proof of the previous lemma, $R/I \cong R_M/IR_M$; since $I$ is $M$-primary, $a \notin IR_M$. \hfill $\square$

In the common usage of "residual" a ring being residually of finite length means that each nonzero element has nonzero image in some finite-length quotient. The theorem below is a direct consequence of Theorem 3.6 and Lemmas 3.8 and 3.9.

**Theorem 3.10.** The following are equivalent:

a) Every localisation of the ring $R$ at a maximal ideal is an $N$-ring.

b) Every homomorphic image of $R$ is residually of finite length.

Theorem 3.6 extends to semilocal rings.

**Lemma 3.11.** Let $R$ be a semilocal ring; then $R$ is an $N$-ring if and only if it is locally an $N$-ring. In particular, $R$ is an $N$-ring if and only if its subdirectly irreducible quotients have finite length.

**Proof.** In [5, Proposition 2.5], it is shown that an $N$-ring is locally $N$. Suppose that $R$ has finitely many maximal ideals $M_1, \ldots, M_r$. Let $I$ be an ideal of $R$ and let $J_k = (I: A_k)_{k \in \mathbb{N}}$ be an ascending chain of colon ideals. As usual, we may assume $A_k \supset A_{k+1}$ for $k \in \mathbb{N}$. By Theorem 1.1, there is $n$ in $\mathbb{N}$ such that, for $t \in \mathbb{N}$ and $i = 1, \ldots, r$, $(IR_M: A_nR_M) = (IR_M: A_{n+t}R_M)$. Let $t \in \mathbb{N}$, $a \in A_n$ and $b \in (I: A_{n+t})$. For every maximal ideal $M$ of $R$, there exists an element $s \in R \setminus M$ such that $sab \notin I$, but this is possible only if $R = (I: ab)$ and $a \in (I: A_n)$. By Theorem 1.1, $R$ is an $N$-ring. The "in particular"-statement follows from Theorem 3.6 and Lemmas 3.8 and 3.9. \hfill $\square$
The next two lemmas return to local \(N\)-rings and in particular their finite-length quotients.

**Lemma 3.12.** Let \((R, M)\) be a local \(N\)-ring. There is a descending chain \((B_k)_{k \in \mathbb{N}}\) of finite-colength-ideals of \(R\) satisfying:

a) For all \(i\), \(M^i \subseteq B_i\) and \(B_i^2 \subseteq M^{i+1}\).

b) With \(D = \bigcap_{n \in \mathbb{N}} B_n\), we have \(DB_2 = \{0\}\). Furthermore, \(xR\) is a Noetherian \(R\)-module for all \(x\) in \(D\).

c) There is \(m \in \mathbb{N}\) such that \(D \cap M^m = \{0\}\).

d) There is an integer \(N\) such that every nonzero element of \(D\) has nonzero image in a quotient of \(R\) of length at most \(N\).

e) If \(\text{Ann}_D(M)\) has at most countable dimension as an \(F\)-module, then there exists a descending chain \((C_n)\) of finite-codimension ideals of \(R\) with \(\bigcap_{n \in \mathbb{N}} C_n = \{0\}\).

**Proof.** We apply Theorem 3.6 and Lemma 3.5. Let \(A_1 = B_1 = M\). Suppose that, for some \(k \geq 2\), finite-colength ideals \(A_1 \supseteq \ldots \supseteq A_{k-1}\) have been found satisfying \(M^i \subseteq A_i\) and \(A_i^2 \subseteq M^{i+1}\) \((i = 1, \ldots, k-1)\). Lemma 3.3 yields that \(KL \subseteq M^{k+1}\) for suitable finite-colength ideals \(K\) and \(L\) of \(R\). Without loss, \(M^k \subseteq K \cap L\). Let \(A_k = K \cap L \cap A_{k-1}\). We have established the existence of an infinite descending chain \((A_n)\) of finite-colength ideals of \(R\) satisfying

\[
A_n^2 \subseteq M^{n+1} \text{ and } M^n \subseteq A_n.
\]

Let \(C = \bigcap_{n \in \mathbb{N}} A_n\). Note that \(C^2 \subseteq \bigcap_{n \in \mathbb{N}} M^n = \{0\}\). For \(x \in C\) and \(n \in \mathbb{N}\), \(xA_{n-1} \subseteq M^n\). It follows that \(\ell(J/(J \cap M^n))\) is finite for all \(n\) whenever \(J\) is a finitely generated subideal of \(C\). By Lemma 2.3, there is \(m\) such that \(J = M^m \cap C\) is contained in some finitely generated subideal of \(C\). Let \(I \subseteq J\) be an ideal of \(R\). There is \(n \geq m\) such that \(I \cap M^n\) is contained in a finitely generated subideal of \(J\); together with \(\ell(I/I \cap M^n) < \infty\), this renders \(I\) finitely generated, so that \(J\) is a Noetherian \(R\)-module, as claimed.

The sequence \((B_n)\) emerges from \((A_n)\) after some slight modifications. As just seen, \(J = M^m \cap C\) is a Noetherian \(R\)-module for sufficiently large values of \(m\). For all \(k\), \(J/(J \cap M^k)\) has finite length. Let \(n \geq m\). Since the quotient \(J/(J \cap M^{n+1})\) has finite length, Lemma 3.1 supplies a finite-colength ideal \(I_n\) of \(R\) with the property that \(I_n \cap L \subseteq M^{n+1}\). Let \(\tilde{B}_k = A_k\) for \(k = 1, \ldots, m - 1\) and \(\tilde{B}_k = A_k \cap I_k\) whenever \(k \geq m\). Setting \(\tilde{D} = \bigcap_{i \in \mathbb{N}} \tilde{B}_i\), we have \(\tilde{D}^2 = \{0\}\). Moreover, \(\tilde{D} \cap M^m \subseteq C \cap M^m \cap \bigcap_{i \in \mathbb{N}} I_n \subseteq \bigcap_{n \in \mathbb{N}} M^n = \{0\}\).

Lemma 2.4 b) yields finite-colength subideals \(K\) of \(\tilde{D}\) and \(L\) of \(R\) that satisfy \(KL \subseteq \tilde{D} \cap M^m = \{0\}\); part b) of the same Lemma provides a finite-colength subideal \(V\) of \(R\) satisfying \(V \cap \tilde{D} \subseteq K\). Let \(B_i = \tilde{B}_i \cap V \cap L\) whenever \(i \geq 2\). With \(D = \bigcap_{i \in \mathbb{N}} B_i\), \(DB_2 = \{0\}\) and \(B \cap M^m = \{0\}\) as claimed. This proves a), b), and c).

Turning to d), we recall that \(D \cap M^m = \{0\}\) and then apply part Lemma 3.3 b). Let \(E = \text{Ann}_D(M)\). The ideal \(D\) being annihilated by \(M^m\), every nonzero subideal of \(D\) has nonzero intersection with \(E\).

If \(E\) has at most countable dimension (as \(F\)-vector space), there is a descending chain \((E_n)_{n \in \mathbb{N}}\) of finite-codimension subspaces of \(E\) with \(\bigcap_{n \in \mathbb{N}} E_n = \{0\}\). Statement b) in 2.4 yields a descending chain \((L_n)\) of finite-colength ideals satisfying \(L_n \cap E = E_n\) for all \(n\). As our final act, resulting in the sequence \((C_n)\) required in d), we replace \(B_n\) by \(C_n = B_n \cap L_n\) for each \(n\) as required. \(\square\)
The section concludes with a lemma intended to show that the ideal $D$ cannot always be "spirited away", i.e. $Ann_D(M)$ might have uncountable $F$-dimension. Let $R$ be a ring and $V$ an $R$-module; recall that the idealisation $R(+)V$ of $V$ (see e.g. [1]) is the ring defined on the Cartesian product $R \times V$ with componentwise addition and multiplication given by $(x,u)(y,v) = (xy,uy+vx)$.

**Lemma 3.13.** Let $(R, M)$ be a local $N$-ring and $V$ an $R$-module with the property that $Ann_R(V)$ has finite colength in $R$. Then $R(+)V$ is an $N$-ring.

**Proof.** Let $S = R(+)V$. We regard $R$ as a subring of $S$ and $V$ as an ideal. Let $U = M + V$. If $x \in R$ is a unit and $u \in V$, then $(x,u)V = V$ and $xR = R$. It follows that $S$ is a local ring with unique maximal ideal $U$. An ideal $J$ of $S$ that contains $V$ has the form $J = J_0 + V$, where $J_0 = \{x \in R \mid \text{there is } w \in V \text{ such that } (x,w) \in J\}$ is an ideal in $R$. If $(y,u)$ and $(z,w)$ are elements of $S$, then $(yz, yw + zu) \in J$ if and only if $yz \in J_0$. By Theorem 1.1, $S/V$ is an $N$-ring.

Let $I$ be an ideal of $S$ with subdirectly irreducible quotient. We denote the natural homomorphism $S \to S/I$ by a bar. If $V \subseteq I$, then, by Lemma 3.1, $S$ is of finite length. Now assume $V \nsubseteq I$. Let $A = \mu(S)$.

Since $V \nsubseteq I$, there is $v \in V \setminus I$ with $A = \langle \bar{v} \rangle$. If $\bar{U}$ is not a nil ideal, then $\bar{M}$ contains a regular element $x$.

Basic properties of subdirectly irreducible rings - as discussed during the proof of Lemma 3.1 - imply that $\bar{U} = Ann_A(A)$ and $A \subseteq x^n \bar{U}$ for all $n$, in particular $A \subseteq \bigcap_{k \in \mathbb{N}} \bar{U}^k$. The module $V$ is annihilated by some power $M^n$. For $a \in M$ and $u \in V$, $(a,u)^{n+1} = (a^{n+1}, (n+1)a^n u) = (a^{n+1}, 0)$, so $U^{n+1} = M^{n+1}$.

Therefore there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $R$ with the properties that, for all $i, n_i \geq n + 1$,

$$a_{n_i} \in M^{n_i} \text{ and } v - a_{n_i} \in I.$$  

By Lemma 2.3, applicable on account of Lemma 3.5, the ideal $\langle a_{n_i} \mid i \in \mathbb{N} \rangle$ is finitely generated, say by $a_{n_1}, \ldots, a_{n_k}$. We choose $k$ minimal with this property.

For $m > k$, $a_m$ is an $R$-linear combination $\sum_{j=1}^{k} \alpha_j a_{n_j}$. This yields that

$$I \supseteq v - \sum_{j=1}^{k} \alpha_j a_{n_j} - \sum_{j=1}^{k} \alpha_j (v - a_{n_j}) = v(1 - \sum_{j=1}^{k} \alpha_j).$$

Since $v \not\in I$, $\sum_{j=1}^{k} \alpha_j a_{n_j} \not\in M$. Accordingly, $\alpha_i \not\in M$ for at least one $i$ in $\{1, \ldots, k\}$, which yields that $a_i \in \langle a_{n_j} \mid j \in \{1, \ldots, k\} \setminus \{i\} \rangle + M^m$. A pigeonhole argument supplies the existence of $i$ in $\{1, \ldots, k\}$ with the property that $a_{n_i}$ is an element of $\langle a_{n_j} \mid j \in \{1, \ldots, k\} \setminus \{i\} \rangle + M^m$ for infinitely many values of $m$.

Due to Lemma 2.3, this implies that $a_{n_i} \in \langle a_{n_j} \mid j \in \{1, \ldots, k\} \setminus \{i\} \rangle$, contradicting the minimal choice of $k$.

Having established that $\bar{U}$ is a nil ideal, Lemma 2.4 provides an integer $\ell$ with $M^\ell \subseteq I + V$. Since $V$ is annihilated by a power of $M + V$, it emerges that $\bar{U}$ is a nilpotent ideal. Let $T_0 = \{0\}$ and $T_i = \bar{V} \cap (T_{i-1} \cdot \bar{M})$. Note that $T_1$ is generated by $\bar{v}$. The quotients $T_i/T_{i-1}$ may be considered as $F$-vector spaces. If $T_j/T_{j-1}$ has finite $F$-dimension, then, since $Ann_R(V)$ has finite colength in $R$, $T_{j+1}/T_j$ has finite $F$-dimension, too. Consequently, $\bar{V}$ has finite length as an $R$-module. Let $(J_k)_{k \in \mathbb{N}}$ be an ascending chain of colon ideals in $\bar{S}$. Since $\bar{V}$ has finite dimension, the chain $(J_k)$ stabilises if and only if $(J_k + \bar{V})$ stabilises. Since $S/(I + V)$ is a quotient of the $N$-ring $S/V$, it is an $N$-ring, so the
4. Varieties with spectral synthesis

4.1. Basic properties. Throughout this final section, $G$ denotes an Abelian group, $V$ denotes a variety on $G$ and the ring $\mathbb{C}G/V\otimes$ is denoted by $R$. We expand on the definition of spectral synthesis on a variety. The definitions below can be found in [14]:

Spectral analysis holds on $V$ if every subvariety of $V$ contains an exponential, i.e. a homomorphism from $G$ into the multiplicative group of $\mathbb{C}$. Equivalently, spectral analysis means that every subvariety of $V$ contains a one-dimensional subvariety. Appealing to Lemma 1.3, it is easily seen that spectral analysis holds on $V$ if and only every maximal ideal quotient of $R$ is isomorphic to $\mathbb{C}$ (see also [11]).

The variety $V$ is called synthesizable if is the closure of the linear span of its finite-dimensional subvarieties. Using Lemma 1.3 once again, (see also [15]), $V$ is synthesizable if and only if $R$ is residually finite-dimensional. Spectral synthesis holds on $V$ if every subvariety of $V$ is synthesizable.

A function $f$ in $\mathbb{C}G$ is an exponential polynomial if the variety generated by $f$ has finite dimension. The name “exponential polynomial” is due to the fact that the functions in question are polynomials in exponentials on $G$ and homomorphisms from $G$ into the additive group of $\mathbb{C}$. Equivalently, exponential polynomials are matrix coefficient functions arising from finite dimensional complex matrix representations of $G$. Synthesizability of a variety means that the exponential polynomials in the variety span a dense subspace. For an in-depth discussion of exponential polynomials see chapter 12 of L. Székelyhidi’s book [14].

Spectral synthesis is said to hold on the group $G$ if every variety on $G$ is synthesizable, M. Laczkovich and L. Székelyhidi have shown that spectral synthesis holds on $G$ if and only if $G$ has finite torsion free rank (see [11], [12], also [15] for a more algebraic proof based on [4] and a generalisation). Less is known about spectral synthesis on a variety. We are going to formulate a ring-theoretic characterisation of spectral synthesis on $V$. The results are then applied to gauge the impact of spectral synthesis on the overall module structure.

Lemma 4.1. Spectral synthesis holds on $V$ if and only if every subdirectly irreducible quotient of $R$ has finite dimension over $\mathbb{C}$.

Proof. Suppose that spectral synthesis holds on $V$. Let $I$ be an ideal of $R$ with subdirectly irreducible quotient and let $I \neq a + I \in \mu(R/I)$. Since the variety $I^\perp$ is synthesizable, there is an ideal $J$ of $R$ that contains $I$, has finite codimension in $R$, and that $a$ does not belong to. This is impossible unless $J = I$. Conversely, assume that the subdirectly irreducible quotients of $R$ have finite dimension. Let $J \subseteq R$ be an ideal and let $a \in R \setminus J$. As mentioned in the introduction, Zorn’s Lemma supplies an ideal $K$ such that $J \subseteq K \neq a$ and the quotient $R/K$ is subdirectly irreducible. By assumption, $R/K$ is finite-dimensional. □

Theorem 4.2. Spectral synthesis holds on $V$ if and only if $R/M \cong \mathbb{C}$ for all maximal ideals of $R$ and $R$ is locally an $N$-ring.

Proof. Suppose that spectral synthesis holds on $V$. A field is a subdirectly irreducible ring and the only field extension of finite degree over $\mathbb{C}$ is $\mathbb{C}$ itself. The maximal ideal quotients of $R$ are therefore
isomorphic to \( \mathbb{C} \). Let \( M \) be a maximal ideal of \( R \). Lemmas 3.9 and 4.1 say that the subdirectly irreducible quotients of \( R_M \) have finite length, which, according to Theorem 3.6, means that \( R_M \) is an \( N \)-ring.

Suppose that, for all maximal ideals \( M \) of \( R \), \( R/M \cong \mathbb{C} \) and \( R_M \) is an \( N \)-ring. Corollary 3.10 says that the subdirectly irreducible quotients of \( R \) have finite length; given that the simple \( R \)-modules are one-dimensional over \( \mathbb{C} \), finite length means finite \( \mathbb{C} \)-dimension.

4.2. Remarks on the module structure of a variety with spectral synthesis. In this final subsection, \( G \) continues to be an Abelian group, while spectral synthesis is assumed to hold on the variety \( V \). As before, \( \mathbb{C}G/V^\perp = R \). For a maximal ideal \( M \) of \( R \), let \( \varphi_M \) be the natural homomorphism \( R \to R_M \), let \( I_M = \ker \varphi \) and \( W_M = I_M^\perp \). Since the intersection of the ideals \( I_M \) over all maximal ideals of \( R \) is \{0\}, \( V \) is the closure of the linear span of the varieties \( W_M \). Moreover, Theorems 3.6 and 4.2 combine to say that \( I_M = \bigcap \{ M^n \} \), whence it follows that \( W_M \) is the closure of \( \bar{W}_M = \bigcup \{ M^n \}^\perp \). Since the powers of \( M \) are \( M \)-primary ideals, we have \( \bar{W}_{M_1} \cap \bar{W}_{M_2} = \{0\} \) if \( M_1 \) and \( M_2 \) are distinct maximal ideals of \( R \). This reflects the Krull-Schmidt theorem for finite-dimensional \( R \)-modules, of course.

For every \( x \in R \setminus I_M \), there is a power \( M^k \) that \( x \) does not belong to. Let \( \bar{R} = R/M^k \). The map \( \varphi \) induces an isomorphism \( \bar{R} \to R_M/M^kR_M \); indeed, \( \bar{R} \) is a local ring since, for \( y \notin \bar{M} \), there is \( 0 \neq \alpha \in \mathbb{C} \) with \( (y - \alpha)^k = 0 \), from which \( y^{-1} \) emerges as a polynomial in \( y \). By Theorems 4.2 and 3.6, there is a finite-codimension ideal \( J \) of \( R \) with \( M^k \subseteq J \) and \( x \notin J \).

If \( R \) is actually Noetherian, then \( \bar{W}_M \) is the direct limit of finite-dimensional modules and the set of elements of \( W_M \) generating a finite-dimensional subvariety, i.e. the set of exponential polynomials in \( W_M \). The purpose of the following lemma is to find out how much of this desirable state of affairs can be salvaged in the non-Noetherian case.

**Theorem 4.3.** Let \( M \) be a maximal ideal of \( R \). The following hold.

\begin{itemize}
  \item [a)] For any given \( k \in \mathbb{N} \), there exists a constant \( N(k) \) such that the variety \( (M^k)^\perp \) is the closure of the linear span of its subvarieties of dimension at most \( N(k) \).
  \item [b)] There is an ascending sequence \( (U_n) \) of finite-dimensional subvarieties of \( W_M \) with the following property: Denoting the closure of \( \bigcup_{n\in\mathbb{N}} U_n \) by \( U \), there is a power \( M^m \) satisfying \( M^m \cap U^\perp = I_M \).
  \end{itemize}

Furthermore, there is an integer \( \ell \) such that \( W_M \) is the closure of the linear span of \( U \) and the subvarieties of \( W_M \) of dimension at most \( \ell \).

If \( \dim W_M \leq k_0 \), then \( W_M \) is the closure of the union of an ascending chain of finite-dimensional subvarieties.

**Proof.** Statement a) follows directly from Lemma 3.3 b).

Statement b) is a corollary of Lemma 3.12 and we resume the notation established there. For an ideal \( J \) of \( R_M \), the ideal of \( R \) contracted from \( J \) will be indicated by a dash. For every \( n \), \( \dim(R_M/B_n) < \infty \), which implies that the elements of \( R \setminus M \) are invertible modulo \( B_n \) and \( R_M/B_n \cong R/B_n \). Certainly, \( D' = \bigcap_{n\in\mathbb{N}} B_n \), for if \( a \notin \bigcap_{n\in\mathbb{N}} B_n \), then \( a \notin D \). Let \( U_n = (B_n)^\perp \) \( \in \mathbb{N} \) and let \( U \) be the closure of the submodule \( \bigcup_{n\in\mathbb{N}} U_n \). We have seen that \( U^\perp = D' \) or, equivalently, \( U = (D')^\perp \) (see Lemma 1.2). Powers of \( M \) being \( M \)-primary ideals, \( M^n = (M^nR_M)^\perp \) holds for every \( n \). Lemma 3.12 c) hence yields the
existence of a power $M^m$ satisfying $M^m \cap D' = I_M$; for every element $a$ of $M^m$, there is $n \in \mathbb{N}$ such that $a \notin (U_n)^\perp$. If $x \in D'$, then, since $R/M^mR_M \cong R/M^m$, statement $c)$ of Lemma 3.12 yields a constant $\ell$ such that there is a subvariety $W$ of $M$ of dimension at most $\ell$ with $x \notin W^\perp$. If $\dim W \leq \aleph_0$, we may replace the sequence $(B_n)$ by the sequence $(C_n)$ of 3.12 d) and thus verify the final statement. \hfill \Box

4.3. Concluding remark: Let $R$ be a local $N$-ring. We have seen that $R$ is a subdirect product of $R$-modules of finite length. However, a local ring $(R, M)$ may be subdirect product of rings of finite length, yet not an $N$-ring. We provide an example: Let $F$ be a field. For $n \in \mathbb{N}$, let $T_n$ be the set of $2n \times 2n$-matrices $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ over $F$, where $0$ is the $n \times n$ zero matrix and $A$ is any $n \times n$-matrix over $F$. Let $V_n = F^{2n}$ and assume $T_n$ to be represented on $V_n$ as suggested by the matrices. Form the (unrestricted or restricted) direct product $M = \prod_{n \in \mathbb{N}} [T_n (+) V_n]$. To make $S$ into a ring (with unity), let $R = F \oplus M$ with pointwise addition and $(\lambda + x)(\mu + y) = \lambda \mu + \lambda y + \mu x + xy$. Then $M$ (or rather $\{0\} + S$) is the unique maximal ideal of $R$. Clearly, $M^n = \{0\}$ and $R$ is the subdirect product of the rings $F \oplus T_n, n \in \mathbb{N}$. If $0 \neq v \in V_n$ and $I \subset R$ is an ideal with $v \notin I$, then $\dim (R/I) \geq n + 2$. If $M$ is the maximal ideal of $R$, however, the radical length of $R$ is 2 and, according to Remark 3.4, $R$ cannot be an $N$-ring. It might be interesting to find out if a condition like Remark 3.4 is sufficient for a local ring that is a subdirect product of finite-length rings to be an $N$-ring?

References


Email address: bwlklk@gmail.com