GENERALIZED LATTICES OVER ONE-DIMENSIONAL NOETHERIAN DOMAINS

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ABSTRACT. We study direct sum decompositions of pure projective torsion free modules over one-dimensional commutative noetherian domains. Having an inspiration in the representation theory of orders in separable algebras we study when every pure projective torsion free module is a direct sum of finitely generated modules. A satisfactory criterion is given for analytically unramified reduced local rings and for Bass domains.

1. Introduction. The study of lattices over orders is a classical part of integral representation theory. For example, if $G$ is a finite group, then representations of $G$ over the ring of integers, i.e., homomorphisms from $G$ to $\text{Aut}(\mathbb{Z}^n)$ are studied in terms of lattices over $\mathbb{Z}G$. Recall that a $\mathbb{Z}G$-lattice is a $\mathbb{Z}G$-module whose underlying abelian group is finitely generated free.

In [4] the notion of a generalized lattice appeared. We omit the definition here, but let us demonstrate it by an example: A generalized lattice over $\mathbb{Z}G$ is a $\mathbb{Z}G$-module whose underlying abelian group is finitely generated free. Note that generalized $\mathbb{Z}G$-lattices correspond to group homomorphisms from $G$ to $\text{Aut}(\mathbb{Z}^{(\kappa)})$, where $\kappa$ is a cardinal.

Obviously, every direct sum of lattices is a generalized lattice. And it is a natural question whether there exists a generalized lattice which is not a direct sum of lattices. This question appeared in [3] for the particular case when $G$ is a group of order two. Butler et al. [4, Theorem 1.1] proved that if $G$ is a cyclic group of prime order, then every generalized $\mathbb{Z}G$-lattice is a direct sum of lattices. This result was extended by Rump [17], who also gave an algorithmic method how to recognize whether every generalized lattice over a suitable order is a direct sum of lattices.
direct sum of lattices. Using this criterion, Rump showed that if \( G \) is a cyclic group of order \( p^2, p \in \mathbb{P} \), then every generalized \( \mathbb{Z}G \)-lattice is a direct sum of lattices.

In commutative algebra, the term lattice is often used for a finitely generated torsion free module over a reduced commutative noetherian ring of Krull dimension 1. In this setting it seems there are more choices for a meaningful definition of a generalized lattice. We follow the suggestion by G. Puninski and define a generalized lattice over a reduced one-dimensional commutative noetherian ring to be a pure projective torsion free module. In other words, if \( R \) is a commutative noetherian ring of Krull dimension one without nonzero nilpotent elements, then \( M \) is said to be a generalized \( R \)-lattice if and only if there is a family \( L_i, i \in I \) of \( R \)-lattices such that \( M \) is a direct summand of \( \oplus_{i \in I} L_i \) (see [15, Proposition 2.1]). There are at least two reasons for such a definition of a generalized lattice. Firstly, the class of generalized \( R \)-lattices is the smallest class of modules which contains all \( R \)-lattices and is closed under (arbitrary) direct sums and direct summands. Secondly, if \( \Lambda \) is a suitable order of finite lattice type, then every generalized lattice is pure projective (see [4, Theorem 2.1] or its generalization [17, p.112] for precise statements of this result). For example, if \( R = \mathbb{Z}G \), where \( G \) is a cyclic group of square-free order, then \( R \) is a \( \mathbb{Z} \)-order in \( \mathbb{Q}G \) and also a reduced commutative noetherian ring of Krull dimension one. In this case the class of generalized \( \mathbb{Z}G \)-lattices as defined in [4] coincides with the class of pure projective torsion free \( \mathbb{Z}G \)-modules.

Being inspired by [17], we try to give a criterion when every generalized lattice over a commutative noetherian domain of Krull dimension one with module-finite normalization is a direct sum of lattices. A satisfactory result is obtained only in the local case, in the global case we succeeded only for locally lattice finite rings.

Our strategy follows [14] and combines two basic tools. A theory of fair-sized projective modules [12] and package-deal theorems of Levy and Odenthal [10]. We give a brief summary of these in the next section. Let us point out that, in theory, the strategy should work also for the general case of reduced commutative noetherian rings of Krull dimension one with module-finite normalization. But in this case the major problem seems to be to distil a useful criterion from the technique.
The author is indebted to Roger Wiegand for helpful discussions and also for suggesting a nice application of our main result in Corollary 2.

2. The tools. Let $\Lambda$ be a (left and right) noetherian ring satisfying the following condition: There is no infinite strictly descending chain of ideals $I_1 \supset I_2 \supset I_3 \supset \cdots$ such that $I_{k+1} = I_k I_{k+1}$ for every $k \in \mathbb{N}$. We call this condition (*).

If $\Lambda$ is a noetherian ring of Krull dimension one, then (*) is equivalent to d.c.c. on idempotent ideals (see [14, Lemma 2.1]). If $\Lambda$ is a module-finite algebra over a commutative noetherian ring of Krull dimension one, then, by [18, Theorem 3], $\Lambda$ contains only finitely many idempotent ideals. So (*) holds for module-finite algebras over one-dimensional commutative noetherian rings.

If $\Lambda$ is a noetherian ring satisfying (*), countably generated projective left $\Lambda$-modules are classified by pairs $(I, P)$, where $I$ is an idempotent ideal of $\Lambda$ and $P$ is a finitely generated projective left $\Lambda/I$-module. If $Q$ is a countably generated projective left $\Lambda$-module, then the corresponding pair is $(I_Q, Q/I_Q Q)$, where $I_Q$ is the smallest ideal of $\Lambda$ such that $Q/I_Q Q$ is finitely generated. The assignment $Q \mapsto (I_Q, Q/I_Q Q)$ gives a bijection between the set of isoclasses of countably generated projective left $\Lambda$-modules and the set $\bigcup_{I \in \mathcal{I}} V(\Lambda/I)$, where $\mathcal{I}$ is the set of all idempotent ideals of $\Lambda$ and $V(\Lambda/I)$ is the set of isoclasses of all finitely generated projective left $\Lambda/I$-modules (see [12, Theorem 2.12], we just reformulated this result for left modules). So countably generated projective modules over $\Lambda$ can be studied via finitely generated modules.

Recall that for a left (or right) $\Lambda$-module $M$ the trace ideal of $M$ is $\text{Tr}(M) := \sum_{f \in \text{Hom}_\Lambda(M, \Lambda)} f(M)$. If $M$ is projective, then its trace ideal is idempotent and it is the smallest ideal $I$ such that $M = IM$ (or $M = MI$).

The following criterion can be used to prove that every projective module is a direct sum of finitely generated modules.

Fact 1. (See [13, Theorem 4.7]). Let $\Lambda$ be a module-finite algebra over a commutative noetherian ring of Krull dimension one. Then every projective left $\Lambda$-module is a direct sum of finitely generated modules if and only if the following two conditions are satisfied.
(i) Every idempotent ideal of $\Lambda$ is the trace ideal of a finitely generated projective left $\Lambda$-module.

(ii) If $I \subseteq \Lambda$ is an idempotent ideal and $Q$ is a finitely generated projective left $\Lambda/I$-module, then there exists a finitely generated projective left $\Lambda$-module $Q$ such that $Q \simeq Q/IQ$.

Let us give an example how to apply Fact 1:

**Corollary 1.** Let $\Lambda$ be as in Fact 1 and suppose that every projective left $\Lambda$-module is a direct sum of finitely generated modules. Then the same is true for every projective right $\Lambda$-module.

**Proof.** Suppose that every projective left $\Lambda$-module is a direct sum of finitely generated modules. We can also apply the machinery described above to study countably generated projective right $\Lambda$-modules (see [12, Lemma 2.11]). In particular, we have to check that if $I \subseteq \Lambda$ is an idempotent ideal, then $I$ is the trace ideal of a finitely generated projective right $\Lambda$-module and for a given finitely generated projective right $\Lambda/I$-module $P$ there exists a finitely generated projective right $\Lambda$-module such that $P$ isomorphic $P/PI$.

Let $I \subseteq \Lambda$ be an idempotent ideal. By Fact 1, $I$ is the trace ideal of a finitely generated projective left $\Lambda$-module $P$. Its dual $\text{Hom}_\Lambda(P, \Lambda)$ is a finitely generated projective right $\Lambda$-module with the trace ideal $I$ (see [1, Exercise 22.1]).

Let $P$ be a finitely generated projective right $\Lambda/I$-module. Then there exists a finitely generated projective left $\Lambda/I$-module $Q$ such that $P \simeq \text{Hom}_{\Lambda/I}(Q, \Lambda/I)$. By Fact 1, there exists a finitely generated projective left $\Lambda$-module $Q$ such that $Q/IQ \simeq Q$. Then $\text{Hom}_\Lambda(Q, \Lambda/I) \simeq \text{Hom}_\Lambda(Q, \Lambda)/\text{Hom}_\Lambda(Q, I)$ and $\text{Hom}_\Lambda(Q, I) = \text{Hom}_\Lambda(Q, \Lambda)I$ (this equality is a consequence of the fact that every element of $\text{Hom}_\Lambda(Q, I)$ factors through an epimorphism $\pi: \Lambda^k \to I$ for a suitable $k$). Thus if $P := \text{Hom}_\Lambda(Q, \Lambda)$, then $P/PI \simeq \text{Hom}_\Lambda(Q, \Lambda/I) \simeq \text{Hom}_\Lambda(Q/IQ, \Lambda/I) = \text{Hom}_{\Lambda/I}(Q/IQ, \Lambda/I) \simeq P$. □

In order to verify (i),(ii) of Fact 1, we use a package deal theorem of Levy and Odenthal. We reformulate here [10, Theorem 2.9] just for the particular case we need. If $R$ is a commutative ring and $m$ is its prime ideal, then $R_m$ denotes the localization of $R$ at $m$. 
Fact 2. (See [10, Theorem 2.9]). Let $R$ be a one-dimensional commutative noetherian domain and let $\Lambda$ be a module-finite $R$-algebra. Let $S$ be a finite set of maximal ideals of $R$ and let $L$ be a finitely generated $\Lambda$-module. Suppose that for each $m \in S$ an $R_m \otimes_R \Lambda$-module $L(m)$ is given such that $R_0 \otimes_R \Lambda$-modules $R_0 \otimes_R L$ and $R_0 \otimes_{R_m} L(m)$ are isomorphic. Then there exists a finitely generated $\Lambda$-module $M$ such that

(a) $R_m \otimes_R \Lambda$-module $R_m \otimes_R M$ is isomorphic to $L(m)$ for every $m \in S$ and

(b) $R_m \otimes_R \Lambda$-modules $R_m \otimes_R M$ and $R_m \otimes_R L$ are isomorphic for every $m \in \text{maxSpec}(R) \setminus S$.

Also, we need a version of this result for ideals. Although it is possible to prove a general package deal theorem for ideals, we present only the minimum we need since it is an immediate consequence of Fact 2.

Lemma 1. Let $\Lambda$ be a module-finite $R$-algebra, where $R$ is a commutative one-dimensional noetherian domain. Suppose that $I$ is an idempotent ideal of $\Lambda$ and $S$ is a finite set of maximal ideals of $R$. If for every $m \in S$ an idempotent ideal $I(m) \subseteq R_m \otimes_R \Lambda$ is given such that $R_0 \otimes_{R_m} I(m) = R_0 \otimes_R I$ (the equality is after the natural identification of $R_0 \otimes_{R_m} R_m \otimes_R \Lambda$ and $R_0 \otimes_R \Lambda$), then there exists a unique idempotent ideal $L \subseteq \Lambda$ such that $R_m \otimes_R L = I(m)$ whenever $m \in S$ and $R_m \otimes_R L = R_m \otimes_R I$ for every $m \in \text{maxSpec}(R) \setminus S$.

Proof. To simplify the notation, let us define $I(m) := R_m \otimes_R I$ for $m \in \text{maxSpec}(R) \setminus S$. Fact 2 implies the existence of a $\Lambda$-module $X$ such that $R_m \otimes_R X \simeq I(m)$ for every $m \in \text{maxSpec}(R)$. Let $L := \text{Tr}(X)$. Obviously, $L$ is an ideal of $\Lambda$ and there is $t \in \mathbb{N}$ such that $L$ is an epimorphic image of $X^t$. Then $R_m \otimes_R L$ is an epimorphic image of $(R_m \otimes_R X)^t = I(m)(R_m \otimes_R X)^t$ for every $m \in \text{maxSpec}(R)$ and consequently $R_m \otimes_R L \subseteq I(m)$.

Denote $\varphi : R_m \otimes_R X \to R_m \otimes_R \Lambda$ the composition of an isomorphism $R_m \otimes_R X \cong I(m)$ and the embedding $I(m) \subseteq R_m \otimes_R \Lambda$. By [16, Theorem 2.39], $\varphi = s^{-1} \otimes f$ for some $s \in R \setminus m$ and $f \in \text{Hom}_\Lambda(X, \Lambda)$. Therefore $I(m) = \text{Im} \varphi \subseteq R_m \otimes_R L$ and $I(m) = R_m \otimes_R L$. It remains to check that $L$ is idempotent.
Observe that $R_m \otimes_R L/L^2 = 0$ for every $m \in \text{maxSpec}(R)$, therefore $L/L^2 = 0$. \hfill \Box

3. The results. Throughout this section $R$ is a commutative noetherian domain of Krull dimension one (we repeat all assumptions on $R$ in the statements of proven results in order to avoid misunderstanding). Let $K$ be its quotient field and let $\bar{R}$ be the normalization of $R$, that is, the integral closure of $R$ in $K$. We assume that $\bar{R}$ is a finitely generated $R$-module.

An $R$-lattice is a finitely generated right $R$-module $M$ such that the canonical $R$-homomorphism $M \to K \otimes_R M$ is monic or, equivalently, $M$ is a finitely generated torsion free $R$-module.

A generalized $R$-lattice is a direct summand of a direct sum of (possibly infinitely many) lattices. Equivalently, by [15, Theorem 2.1], a generalized $R$-lattice is a pure projective torsion free $R$-module. A generalized lattice is fully decomposable if it is a direct sum of lattices.

A module is strongly indecomposable if it has local endomorphism ring.

We say that $R$ satisfies (FD) if every generalized $R$-lattice is fully decomposable. For example, if $R$ is a complete local ring, then, by [8, Theorem 1.8] and [6, Corollary 2.55], $R$ satisfies (FD). Our aim is to understand when $R$ satisfies (FD). We consider the case when $R$ is local first.

Let us begin with a general technical lemma.

**Lemma 2.** Let $P$ and $Q$ be modules over a ring $S$ such that $P$ is strongly indecomposable and $\text{Hom}_S(P,Q)\text{Hom}_S(Q,P) \subseteq J(\text{End}_S(Q))$. Then every module of $\text{add}(P \oplus Q)$ is of the form $X \oplus Y$, where $X \simeq P^k$ and $Y$ is isomorphic to a direct summand of $Q^l$ for some $k, l \in \mathbb{N}_0$.

**Proof.** Observe that $P$ is not isomorphic to a direct summand of $Q$, otherwise $\text{Hom}_S(P,Q)\text{Hom}_S(Q,P)$ would contain a nonzero idempotent of $\text{End}_S(Q)$.

Let $Z \in \text{add}(P \oplus Q)$ be such that $Z$ does not contain a direct summand isomorphic to $P$. There exist $l \in \mathbb{N}$ and homomorphisms $\alpha: Z \to P^l \oplus Q^l$, $\beta: P^l \oplus Q^l \to Z$ such that $\beta \alpha = 1_Z$. Write these
morphisms as $\alpha = (\alpha_P, \alpha_Q): Z \rightarrow P^l, \alpha_Q: Z \rightarrow Q^l$ and $\beta = (\beta_P, \beta_Q), \beta_P: P^l \rightarrow Z, \beta_Q: Q^l \rightarrow Z$. So $\beta_Q\alpha_Q = 1_Z - \beta_P\alpha_P$. Since $Z$ does not contain a direct summand isomorphic to $P$, it is easy to check that $\alpha_P\beta_P \in J(\text{End}_S(P^l))$. Thus $1_P - \alpha_P\beta_P$ is (left) invertible. By $[11, \text{Lemma 4.2}]$, $1_Z - \beta_P\alpha_P$ is left invertible, so there exists $\gamma \in \text{End}_S(Z)$ such that $1_Z = \gamma(1_Z - \beta_P\alpha_P) = \gamma\beta_Q\alpha_Q$. In particular, $Z$ is isomorphic to a direct summand of $Q^l$.

Now consider $Z \in \text{add}(P \oplus Q)$ arbitrary, let $Z$ be a direct summand of $(P \oplus Q)^k$ for some $k \in \mathbb{N}$. By $[6, \text{Corollary 4.6}]$, modules with semilocal endomorphism rings cancel from direct sums, therefore $Z \simeq X \oplus Y$, where $X$ is isomorphic to a direct sum of (at most $k$) copies of $P$ and $Y$ does not contain a direct summand isomorphic to $P$. The previous part of the proof implies $Y \in \text{add}(Q)$. □

**Proposition 1.** Let $R$ be a commutative local noetherian one-dimensional domain with module-finite normalization. Then $R$ satisfies (FD) if and only if $\tilde{R}$ is a discrete valuation domain. In this case every indecomposable $R$-lattice is strongly indecomposable.

**Proof.** Observe that $\tilde{R}$ is a Dedekind domain, so $\tilde{R}$ is a discrete valuation domain if and only if it is a local ring. Assume that $\tilde{R}$ is not local. Then we claim that $\text{Add}(R \oplus \tilde{R})$ contains a module which is not a direct sum of finitely generated modules. Let $c := (R : \tilde{R})$ be the conductor, i.e., the largest ideal of $\tilde{R}$ contained in $R$. We identify $\text{End}_R(R \oplus \tilde{R})$ and $\Lambda := \begin{pmatrix} R & \tilde{R} \\ c & \tilde{R} \end{pmatrix} \subseteq M_2(\tilde{R})$, in the usual fashion:

The multiplication by $X \in \Lambda$ on the right induces an endomorphism of $R \oplus \tilde{R}$. Recall that if $M$ is a finitely generated $R$-module, then $\text{Add}(M)$ and $\text{End}_R(M)$-$\text{Proj}$ are equivalent categories (see for example $[6, \text{Theorem 4.7}]$). Thus we need to check that there exists a projective $\Lambda$-module which is not a direct sum of finitely generated modules. By Fact 1, it is sufficient to find an idempotent ideal of $\Lambda$ which is not the trace of a finitely generated projective module.

The ring $\tilde{R}/c$ is artinian and not local, so there exists $e \in \tilde{R}$ such that $e + c$ is a nontrivial idempotent in $\tilde{R}/c$. Then it is easy to verify
that
\[ I := \begin{pmatrix} R & \tilde{R} \\ c & c + e\tilde{R} \end{pmatrix} \]

is an idempotent ideal of \( \Lambda \). Let us check that \( I \) cannot be the trace ideal of a finitely generated projective \( \Lambda \)-module: Let \( e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \)
and \( e_2 := 1 - e_1 \). Since \( \Lambda e_1 \) has local endomorphism ring (isomorphic to \( R \)) and \( e_2\Lambda e_1 \Lambda e_2 \subseteq J(e_2\Lambda e_2) \), an application of Lemma 2 gives that every finitely generated projective module over \( \Lambda \) is of the form \( \Lambda e_k \oplus Q \), where \( k \in \mathbb{N}_0 \) and \( Q \) is a direct summand \( \Lambda e_l \) for some \( l \in \mathbb{N} \). Since \( \text{End}_\Lambda(\Lambda e_2) \cong \tilde{R} \) is a commutative noetherian domain (a Dedekind domain in fact), every nonzero finitely generated projective \( \text{End}_\Lambda(\Lambda e_2) \)-module is a generator. Hence \( \Lambda e_2 \) is a direct summand of \( Q^t \) for some \( t \in \mathbb{N} \) provided \( Q \neq 0 \). It follows that the trace ideal of a finitely generated projective \( \Lambda \)-module can be only \( 0, \langle e_1 \rangle, \langle e_2 \rangle \) or \( \Lambda \). Therefore \( I \) is not the trace ideal of a finitely generated projective \( \Lambda \)-module. So we may apply Fact 1 to see that there exists a projective \( \Lambda \)-module which is not a direct sum of finitely generated modules.

Conversely, assume that \( \tilde{R} \) is a discrete valuation domain. We claim that every indecomposable \( R \)-lattice has local endomorphism ring. This follows from much more general results [9, Theorem 1.1, Corollary 2.11], we include an easy direct proof here. Let \( M \) be an \( R \)-lattice, let us view \( M \) as a submodule of \( K M := K \otimes_R M \) and let \( \tilde{R}M \) be the \( \tilde{R} \)-submodule of \( K M \) generated by \( M \). Observe that \( \tilde{R}M \) is a finitely generated free \( \tilde{R} \)-module. We can consider rings \( \text{End}_R(M) \) and \( \text{End}_{\tilde{R}}(\tilde{R}M) \) as subrings of \( \text{End}_K(K M) \): \( \text{End}_R(M) = \{ \varphi \in \text{End}_K(K M) \mid \varphi(M) \subseteq M \} \) and \( \text{End}_{\tilde{R}}(\tilde{R}M) = \{ \varphi \in \text{End}_K(K M) \mid \varphi(\tilde{R}M) \subseteq \tilde{R}M \} \). Under this identification \( \text{End}_R(M) \) becomes a subring of a semiperfect ring \( \text{End}_{\tilde{R}}(\tilde{R}M) \).

Let us consider \( J := \{ \varphi \in \text{End}_K(K M) \mid \varphi(M) \subseteq cM \} = \{ \varphi \in \text{End}_K(K M) \mid \varphi(\tilde{R}M) \subseteq c\tilde{R}M \} \). Obviously, \( J \) is an ideal of \( \text{End}_R(M) \) and also of \( \text{End}_{\tilde{R}}(\tilde{R}M) \) and, by Nakayama’s lemma, it is in fact contained in the Jacobson radicals of these rings. Observe that \( \text{End}_R(M) \) is a module-finite \( R \)-algebra and if \( 0 \neq r \in c \), then \( r\text{End}_R(M) \subseteq J \). It follows that \( \text{End}_R(M)/J \) is a finitely generated module over an artinian ring \( R/rR \), so the ring \( \text{End}_R(M)/J \) is artinian.
Assume that \( \text{End}_R(M) \) is not a local ring. Then there exists \( e \in \text{End}_R(M) \) such that \( e + J \) is a nontrivial idempotent of \( \text{End}_R(M)/J \). Using the embedding \( \text{End}_R(M)/J \hookrightarrow \text{End}_{\hat{R}}(\hat{R}M)/J \) and the fact that \( \text{End}_{\hat{R}}(\hat{R}M) \) is semiperfect, it is easy to see that there is an idempotent \( e' \in \text{End}_{\hat{R}}(\hat{R}M) \) such that \( e' - e \in J \) (note that the proof of \((e) \Rightarrow (a)\) from [6, Theorem 3.6] works also for ideals contained in the Jacobson radical of the ring). Now \( e'(M) \subseteq (e' - e)(M) + e(M) \subseteq eM + M = M \). Therefore \( e' \in \text{End}_R(M) \) and hence \( M \) is not indecomposable.

So we proved that if \( \hat{R} \) is a discrete valuation domain, then every indecomposable \( R \)-lattice has local endomorphism ring. Hence every generalized lattice is a direct summand of \( \bigoplus_{i \in I} M_i \), where every \( M_i \) is a finitely generated module with local endomorphism ring. By [6, Corollary 2.55], every generalized lattice is a direct sum of finitely generated modules. □

For noetherian local reduced rings with module-finite normalization the proof above can be repeated to get:

**Proposition 2.** Let \( R \) be a reduced commutative local noetherian ring of Krull dimension 1 with module-finite normalization. Then \( R \) satisfies (FD) if and only if \( \hat{R} \) is a product of discrete valuation domains.

Let us continue with the case when \( R \) is a commutative noetherian domain of Krull dimension 1 but not necessarily local.

**Lemma 3.** Let \( R \) be a commutative noetherian domain of Krull dimension 1 with module-finite normalization. If \( R \) satisfies (FD), then \( R_m \) satisfies (FD) for every \( m \in \text{maxSpec}(R) \).

**Proof.** Suppose that there is a maximal ideal \( m \subseteq R \) such that \( R_m \) does not satisfy (FD). Let \( M := R \oplus \hat{R} \) and let \( \Lambda := \text{End}_R(M) \). Recall \( \Lambda_m := R_m \otimes_R \Lambda \simeq \text{End}_{R_m}(R_m \otimes_R M) \) and \( \Lambda_0 := K \otimes_R \Lambda \simeq \text{End}_K(K \otimes_R M) \). As in the proof of Proposition 1 we find an idempotent ideal \( I(m) \) of \( \Lambda_m \) which is not the trace ideal of a finitely generated projective \( \Lambda_m \)-module. Observe that \( \Lambda_0 \) is a simple artinian ring, so \( K \otimes_{R_m} I(m) = \Lambda_0 \). By Lemma 1, there exists an idempotent ideal \( I \subseteq \Lambda \) such that \( R_m \otimes_R \Lambda = I(m) \) and \( R_n \otimes_R I = R_n \otimes_R \Lambda \) for every maximal ideal \( n \) of \( R \) different from \( m \). If \( I \) is the trace ideal of a
finitely generated projective module $P$, then $R_m \otimes_R I = I(m)$ is the trace ideal of a finitely generated projective $\Lambda_m$-module $R_m \otimes_R P$ which is not possible. So we conclude by Fact 1. □

Lemma 4. Let $R$ be a commutative noetherian domain of Krull dimension 1. Suppose there are $m \neq n$ maximal ideals of $R$, a strongly indecomposable $R_m$-lattice $M$, and a strongly indecomposable $R_n$-lattice $N$ such that ranks of $M$ and $N$ are not coprime. Then $R$ does not satisfy (FD).

Proof. Let $r$ be the rank of $M$, $s$ be the rank of $N$ and $d := \gcd(r, s) > 1$. By the package deal theorem Fact 2, there exists an indecomposable $R$-lattice $X$ such that $R_m \otimes_R X \simeq M$ and $R_p \otimes_R X \simeq R_p^r$ for all primes of $R$ different from $m$. Similarly, there exists an indecomposable $R$-lattice $Y$ such that $R_n \otimes_R Y \simeq N$ and $R_p \otimes_R Y \simeq R_p^s$ for all primes of $R$ different from $n$. We claim that there exists a module in $\text{Add}(X \oplus Y)$ which is not a direct sum of finitely generated modules.

Put $\Lambda := \text{End}_R(X \oplus Y)$. We find an idempotent ideal $I \subseteq \Lambda$ and a finitely generated projective $\Lambda/I$-module $P$ such that there is no finitely generated projective $\Lambda$-module $Q$ satisfying $Q/IQ \simeq P$. Then, by Fact 1, there exists a projective $\Lambda$-module which is not a direct sum of finitely generated modules and hence $\text{Add}(X \oplus Y)$ contains a generalized $R$-lattice which is not a direct sum of finitely generated modules.

First we construct the idempotent ideal $I$. Note for every prime $p \subseteq R$, 

$$R_p \otimes_R \Lambda \simeq \text{End}_R(R_p \otimes_R X \oplus R_p \otimes_R Y).$$

So the ring $R_p \otimes_R \Lambda$ contains two idempotents corresponding to the projections $R_p \otimes_R X \oplus R_p \otimes_R Y \to R_p \otimes_R X$ and $R_p \otimes_R X \oplus R_p \otimes_R Y \to R_p \otimes_R Y$. Let us call these idempotents $e_{X,p}$ and $e_{Y,p}$. Since $K \otimes_R \Lambda$ is simple, Lemma 1 implies that there is an idempotent ideal $I \subseteq \Lambda$ such that $R_m \otimes_R I = (e_{X,m})$, $R_n \otimes_R I = (e_{Y,n})$ and $R_p \otimes_R I = R_p \otimes_R \Lambda$ for every prime $p \subseteq R$ different from $m, n$.

Note that $I$ and $\Lambda$ have equal ranks, so $\Lambda/I$ is an artinian ring. By [5, Theorem 2.13 b)], the canonical homomorphism

$$\varphi: \Lambda/I \to \Lambda' := (R_m \otimes_R \Lambda)/(R_m \otimes_R I) \times (R_n \otimes_R \Lambda)/(R_n \otimes_R I)$$
is an isomorphism. Let $e$ be an idempotent of $\Lambda/I$ such that $\varphi(e) = (e', 0)$, where $e' \in (R_m \otimes_R \Lambda)/(R_m \otimes_R I)$ is a primitive idempotent. Put $P = (\Lambda/I)e$. It remains to check that there is no finitely generated projective $\Lambda$-module $Q$ such that $Q/IQ \simeq P$.

Suppose there is such a $Q$. Looking at $\Lambda' \otimes_\Lambda Q \simeq \Lambda'(e', 0)$ we see that $R_n \otimes_R Q$ has the trace ideal contained in $R_n \otimes_R I$ and that $(R_m \otimes_R Q)/[(R_m \otimes_R I)(R_m \otimes_R Q)]$ is an indecomposable $(R_m \otimes_R \Lambda)/(R_m \otimes_R I)$-module.

Note again $R_m \otimes_R \Lambda \simeq \text{End}_{R_m}(R_m \otimes_R X \oplus R_m \otimes_R Y)$. Recall that $R_m \otimes_R X \simeq M$ has local endomorphism ring and $R_m \otimes_R Y \simeq R_m$. So there are only two indecomposable finitely generated projective $R_m \otimes_R \Lambda$-modules, namely $P_1 = R_m \otimes_R \text{Hom}_R(X \oplus Y, R)$ and $P_2 = R_m \otimes_R \text{Hom}_R(X \oplus Y, X)$. The later has its trace ideal equal to $R_m \otimes_R I$.

So $R_m \otimes_R Q \simeq P_1 \oplus P_2^l$ for some $l \in \mathbb{N}_0$.

Similarly, $P_3 = R_n \otimes_R \text{Hom}_R(X \oplus Y, R)$ and $P_4 = R_n \otimes_R \text{Hom}_R(X \oplus Y, Y)$ are the only indecomposable finitely generated projective $R_n \otimes_R \Lambda$-modules. It follows that $R_n \otimes_R Q \simeq P_4^k$ for some $k \in \mathbb{N}_0$.

Now let $L$ be the module of add$(X \oplus Y)$ corresponding to $Q$. Then $R_m \otimes_R L \simeq R_m \oplus M^l$ so the rank of $L$ is congruent to 1 modulo $d$. On the other hand, $R_n \otimes_R L \simeq N^k$, so the rank of $L$ is divisible by $d$. Since $d > 1$, no such $L$ exists. □

Now we are ready to prove the main result.

**Theorem 3.** Let $R$ be a commutative noetherian domain of Krull dimension 1 with module-finite normalization. If $R$ satisfies (FD), then the following conditions hold

(i) For every maximal ideal $m \subseteq R$ the normalization of $R_m$ is a discrete valuation domain.

(ii) For every pair of maximal ideals $m \neq n$ and every pair of indecomposable lattices $M \in R_m$-latt and $N \in R_n$-latt the ranks of $M$ and $N$ are coprime.

Moreover, if (i) and (ii) are satisfied, then for every $R$-lattice $L$ any module in $\text{Add}(L)$ is a direct sum of finitely generated modules.
Proof. Let $R$ be a one-dimensional commutative noetherian domain with module-finite normalization satisfying (FD). Then (i) is just a combination of Proposition 1 and Lemma 3. If (i) holds and $m \in \text{maxSpec}(R)$, then every indecomposable $R_m$-lattice is strongly indecomposable. Therefore (ii) is a consequence Lemma 4.

Let us check the ‘moreover’ part. Let $L$ be a nonzero $R$-lattice and consider $\Lambda := \text{End}_R(L)$. In order to verify that every module of $\text{Add}(L)$ is fully decomposable we prove

1. If $I$ is an idempotent ideal of $\Lambda$, then there exists a finitely generated projective $\Lambda$-module $P$ such that $\text{Tr}(P) = I$.
2. If $I$ is an idempotent ideal of $\Lambda$ and $Q$ is a finitely generated projective $\Lambda/I$-module, then there exists a finitely generated projective $\Lambda$-module $P$ satisfying $Q \simeq P/IP$.

In order to verify 1., consider a nonzero idempotent ideal $I \subseteq \Lambda$. Observe that if (i) holds, then for every $m \in \text{maxSpec}(R)$ every projective $R_m \otimes_R \Lambda$-module is a direct sum of finitely generated modules since $R_m \otimes_R \Lambda \simeq \text{End}_{R_m}(R_m \otimes_R L)$. Therefore every idempotent ideal of $R_m \otimes_R \Lambda$ is the trace ideal of a finitely generated projective $R_m \otimes_R \Lambda$-module. Since $I \neq 0$ and $K \otimes_R \Lambda \simeq \text{End}_K(K \otimes_R L)$ is a simple artinian ring, there exists a finite set $S \subseteq \text{maxSpec}(R)$ such that $R_m \otimes_R I = R_m \otimes_R \Lambda$ for every $m \in \text{maxSpec}(R) \setminus S$. For every $m \in S$ let $P(m)$ be a finitely generated projective $R_m \otimes_R \Lambda$-module such that $\text{Tr}(P(m)) = R_m \otimes_R I$. The modules $P(m)$ can be chosen so that there exists $k \in \mathbb{N}$ such that $K \otimes_{R_m} P(m) \simeq (K \otimes_R \Lambda)^k$ for every $m \in S$. By Fact 2, there exists a finitely generated $\Lambda$-module $P$ such that $R_m \otimes_R P \simeq P(m)$ for every $m \in S$ and $R_m \otimes_R P \simeq R_m \otimes_R \Lambda^k$ for every $m \in \text{maxSpec}(R) \setminus S$. Since every localization of $P$ at an arbitrary $m \in \text{maxSpec}(R)$ is $R_m \otimes_R \Lambda$-projective, $P$ is a projective $\Lambda$-module (see for example [16, Corollary 3.23]). Moreover, $R_m \otimes_R I = \text{Tr}(R_m \otimes_R P) = R_m \otimes_R \text{Tr}(P)$ for every $m \in \text{maxSpec}(R)$, hence $I = \text{Tr}(P)$.

Now let us prove 2. Let $I$ be a proper nonzero idempotent ideal of $\Lambda$ and consider a finitely generated indecomposable projective $\Lambda/I$-module $Q$. Since the $R$-modules $I$ and $\Lambda$ have equal ranks, $\Lambda/I$ is an $R$-module of finite length. Let $S := \{m_1, \ldots, m_k\} \subseteq \text{maxSpec}(R)$ be such that for every $m \in \text{maxSpec}(R) \setminus S$ the ring $R_m$ is a discrete valuation domain and $R_m \otimes_R (\Lambda/I) = 0$. In particular, every $R_m$-lattice
is free if $m \in \text{maxSpec}(R) \setminus S$. There are canonical isomorphisms
\[ \Lambda/I \simeq R_{m_1} \otimes_R (\Lambda/I) \times \cdots \times R_{m_k} \otimes_R (\Lambda/I) \]
\[ R_m \otimes_R (\Lambda/I) \simeq (R_m \otimes_R \Lambda)/((R_m \otimes_R I)) \]

Without loss of generality, we may assume that $R_{m_1} \otimes_R Q$ is an indecomposable projective $R_{m_1} \otimes_R (\Lambda/I)$-module and $R_{m_1} \otimes_R Q = 0$ for every $i \in \{2, \ldots, k\}$.

For every $i \in \{1, \ldots, k\}$ there exists an indecomposable $R_{m_i}$-lattice $L_i \in \text{add}(R_{m_i} \otimes_R L)$ such that the corresponding projective $R_{m_i} \otimes_R \Lambda$-module $P_i := \text{Hom}_{R_{m_i}}((R_{m_i} \otimes_R L, L_i)$ has the trace ideal contained in $R_{m_i} \otimes_R I$. The existence of such a lattice is a consequence of (i), since $R_{m_i} \otimes I$ is a trace ideal of a finitely generated projective $R_{m_i} \otimes_R \Lambda$-module $P_i$ and we choose $L_i$ to be the module of $\text{add}(R_{m_i} \otimes_R L)$ corresponding to an indecomposable direct summand of $P_i$.

By Proposition 1, (i) implies that every $R_{m_i}$-lattice is fully decomposable. In particular, every projective module over $R_m \otimes_R \Lambda \simeq \text{End}_{R_m}(R_m \otimes_R L)$ is a direct sum of finitely generated projective modules. So there exists a finitely generated projective $R_{m_i} \otimes_R \Lambda$-module $P_0$ such that $P_0/(R_{m_1} \otimes_R I)P_0 \simeq R_{m_1} \otimes_R Q$. We may assume that $P_0$ is indecomposable and let $L_0$ be the $R_{m_0}$-lattice in $\text{add}(R_{m_0} \otimes_R L)$ corresponding to $P_0$.

Let $r_0$ be the rank of $L_0$, $r_i$ the rank of $L_i$ for $i = 1, \ldots, k$. Because of (ii), $r_1, r_2, \ldots, r_k$ are pair-wise coprime. Therefore there are $n_1, n_2 \in \mathbb{N}_0$ such that $r_0 + n_1 r_1 = n_2 r_2 \cdots r_k$. Let $L(m_1) := L_0 \oplus L_1^{n_1}$ and $L(m_i) := L_1^{r_2 \cdots r_k/r_i}$ for $i \in \{2, \ldots, k\}$. Observe that all $L(m_i)$'s have rank $l = r_0 + n_1 r_1$. Using the package deal theorem Fact 2 we can change localizations of $R^l$ at primes from $S$ to obtain the lattice $L'$ satisfying $R_m \otimes R' \simeq L(m)$ for every $m \in S$ and $R_m \otimes R' \simeq R_{m_i}^l$ for every $m \in \text{maxSpec}(R) \setminus S$.

Observe that for every $m \in \text{maxSpec}(R)$ there exists $t \in \mathbb{N}$ such that $R_m \otimes R L'$ is a direct summand of $R_m \otimes_R L'$ (for $m \in S$ it is by the construction and for the remaining maximal ideals it is true whenever $t \geq l$). Therefore $L' \in \text{add}(L)$.

Finally, let $P := \text{Hom}_R(L, L')$ be the projective $\Lambda$-module corresponding to $L'$. For every $m \in \text{maxSpec}(R)$ we have $R_m \otimes_R P \simeq \text{Hom}_{R_m}(R_m \otimes L, L(m))$. Therefore $R_{m_1} \otimes_R P \simeq P_0 \oplus P_1^{n_1}$ and
$R_{m_i} \otimes_R P$ is a direct sum of copies of $P_i$ if $i \in \{2, \ldots, k\}$. Consequently, $R_{m_i} \otimes_R P/IP = 0$ for $i \in \{2, \ldots, k\}$ and $R_{m_i} \otimes_R P/IP \cong P_0/(R_{m_i} \otimes_R I)P_0 \cong R_{m_2} \otimes_R Q$ (recall $\text{Tr}(P_i) \subseteq R_{m_i} \otimes_R I$ for every $i \in \{1, \ldots, k\}$).

We checked that $R_m \otimes_R P/IP \cong R_m \otimes_R Q$ for every $m \in \text{maxSpec}(R)$. Since $P/IP$ and $Q$ are artinian $R$-modules, we apply [5, Theorem 2.13 b] to conclude $P/IP \cong Q$. □

Let us consider the relation between property (FD) and representation type of a one-dimensional noetherian domain with module-finite normalization. Recall that a commutative domain $R$ is lattice finite if there are only finitely many indecomposable $R$-lattices up to isomorphism. Lattice finite local one-dimensional noetherian domains with module-finite normalization are characterized by the Drozd-Roıter conditions (see [8, p.41]). We say that a commutative domain is locally lattice finite if $R_m$ is lattice finite for every $m \in \text{maxSpec}(R)$. Observe that locally lattice finite does not imply lattice finite even in the class of Dedekind domains. Recall that a commutative domain $R$ is a Bass domain if every ideal of $R$ can be generated by at most 2 elements. By [2, Theorem 7.8], if $R$ is a one-dimensional local noetherian domain with module-finite normalization, then $R$ is Bass if and only if every indecomposable $R$-lattice is of rank 1.

The following corollary was suggested by Roger Wiegand.

**Corollary 2.** Let $R$ be a one-dimensional noetherian domain with module-finite normalization. If $R$ satisfies (FD), then there exists at most one $m \in \text{maxSpec}(R)$ such that $R_m$ is not a Bass domain.

**Proof.** Let $R$ be a one-dimensional noetherian domain with module-finite normalization satisfying (FD). First suppose that $R$ is not locally lattice finite, i.e., there exists $m \in \text{maxSpec}(R)$ such that $R_m$ is not lattice finite. Then, by [8, Theorem 4.10], every positive integer is a rank of an indecomposable $R_m$-lattice. By Theorem 3, if $n \in \text{maxSpec}(R)$, $n \neq m$, then every indecomposable $R_n$-lattice is of rank 1, therefore $R_n$ is a Bass domain.

Now let us assume that $R$ is locally lattice finite. In this case it is sufficient to prove that if $R$ is local, satisfying (FD), lattice finite, and not Bass, then there exists an indecomposable lattice of rank 2.
This result should be present in the literature but we did not found a convenient reference. However, [7, Section 5] gives a clue how to prove it.

Assume that $R$ is local with maximal ideal $m$ and let $\tilde{R}$ be the normalization of $R$. By Proposition 1, $\tilde{R}$ is a discrete valuation domain, let $M = \pi \tilde{R}$ be its maximal ideal. From the Drozd-Roĭter conditions it follows that $R/mR$ is an $R/m$-vector space of dimension at most 3. It is well known that if the dimension of this space is 1 or 2, then $R$ is a Bass domain. Therefore we assume that $\tilde{R}/m\tilde{R}$ has dimension 3. We distinguish 2 cases:

a) $M = m\tilde{R}$, so $R/m \subseteq \tilde{R}/M$ is an extension of degree 3. Let $u, v \in \tilde{R}$ be such that $1 + M, u + M, v + M$ is a basis of the $R/m$-module $R/M$. Consider the lattice

$$L = \{(r + su, t + sv) \mid r, s, t \in R\} \subseteq \tilde{R}^2.$$  

Obviously, $L$ is of rank 2 and $L/mL$ is a 3-dimensional space over $R/m$. If $L$ is decomposable, then $L$ contains a cyclic direct summand which has to be isomorphic to $R$. So if $L$ is decomposable, then there exists an epimorphism $f: L \to R$. Such an epimorphism is given by $(a, b) \in K \oplus K$ satisfying $a(r + su) + b(t + sv) \subseteq R$ for every $r, s, t \in R$. In particular, $a, b \in \tilde{R}$ and $au + bv \in \tilde{R}$ but it is not possible if $a \notin m$ or $b \notin m$. Therefore $f(L) \subseteq R \cap M = m$, therefore $f$ is not onto.

b) $m\tilde{R} = M^3$ and $[\tilde{R}/M : R/m] = 1$. In this case consider the lattice

$$L = \{(r + s\pi, t + s\pi^2) \mid r, s, t \in R\} \subseteq \tilde{R}^2.$$  

Again, $L$ is an $R$-lattice of rank 2 and if $L$ is decomposable, then there are $a, b \in R$ such that $R = \{a(r + s\pi) + b(t + s\pi^2) \mid r, s, t \in R\}$. In particular, $a\pi + b\pi^2 \in R$. Then $a \in m$, otherwise $a\pi + b\pi^2 \in M \setminus M^2$, and consequently $a\pi \in M^4$. Then $b \in m$, otherwise $a\pi + b\pi^2 \in M^2 \setminus M^3$. Then $a(r + s\pi) + b(t + s\pi^2) \in M \cap R = m$, a contradiction. □

Note that Theorem 3 gives a criterion when a lattice finite one-dimensional noetherian domain with module-finite normalization satisfies (FD). Indeed, if $L$ is a direct sum of a representative set of indecomposable lattices, then every generalized $R$-lattice is in $\text{Add}(L)$. In fact, it is possible to extend the criterion to locally lattice finite domains.
Proposition 4. Let $R$ be a locally lattice finite one-dimensional noetherian domain with module-finite normalization. Then there exists an $R$-lattice $L$ such that every generalized lattice is contained in $\text{Add}(L)$.

Proof. First let us recall the following claim: If $M, N$ are $R$-lattices such that $R_m \otimes_R M$ is isomorphic to a direct summand of $R_m \otimes_R N$ for every $m \in \text{maxSpec}(R)$, then there exists $k \in \mathbb{N}$ such that $M$ is a direct summand of $N^k$.

Let $S \subseteq \text{maxSpec}(R)$ be a finite set such that for every $m \in \text{maxSpec}(R) \setminus S$ the ring $R_m$ is integrally closed (and therefore a discrete valuation domain). For every $m \in S$ let $\mathcal{L}_m$ be a representative set of indecomposable $R_m$-lattices. For every $m \in S$ and $X \in \mathcal{L}_m$ there exists an $R$-lattice $L_{X,m}$ such that $R_m \otimes_R L_{X,m} \simeq X$ and $R_n \otimes_R L_{X,m}$ is free for every $n \in \text{maxSpec}(R) \setminus \{m\}$ (just apply Fact 2). Put $L := R \oplus \bigoplus_{m \in S, X \in \mathcal{L}_m} L_{X,m}$.

It remains to prove that for every $R$-lattice $M$ there exists $k \in \mathbb{N}$ such that $M$ is a direct summand of $L^k$. For sure there exists $k_0$ such that $K \otimes_R M$ is a direct summand of $K \otimes_R L^{k_0}$. By [10, Lemma 2.4], $R_m \otimes_R M$ is a direct summand of $R_m \otimes_R L^{k_0}$ for almost all $m \in \text{maxSpec}(R)$. Then it is easy to verify that there exists $k_0 \leq k_1 \in \mathbb{N}$ such that $R_m \otimes_R M$ is a direct summand of $R_m \otimes_R L^{k_1}$ for every $m \in \text{maxSpec}(R)$. Finally we apply the claim mentioned at the beginning of the proof to see that there is $k \in \mathbb{N}$ such that $M$ is a direct summand of $L^k$. \hfill $\Box$

Note that every local Bass domain is lattice finite. Proposition 4 and Theorem 3 give immediately the following characterization of Bass domains satisfying (FD).

Corollary 3. Let $R$ be a Bass domain with module finite normalization. Then $R$ satisfies (FD) if and only if for every $m \in \text{maxSpec}(R)$ the normalization of $R_m$ is a discrete valuation domain.

Example 1. Let $R$ be either $\mathbb{C}[x, y]/(y^2 - x^3)$ or $\mathbb{C}[x, y]/(y^2 - x^2(x+1))$. It is well known and easy to check that $R$ is a Bass domain. By [5, Theorem 4.14], $R$ has module finite normalization. In fact, by [5, Exercise 4.24], the normalization of $R$ is $\tilde{R} := R[\frac{y}{x}]$. If $m \subseteq \tilde{R}$ is a maximal ideal different from $xR + yR$, then $R_m$ is integrally closed.
Assume that $R = \mathbb{C}[x, y]/(y^2 - x^3)$ and $m = xR + yR$. In this case $\tilde{R}/m\tilde{R} \simeq \mathbb{C}[z]/(z^2)$, so there is only one maximal ideal of $\tilde{R}$ containing $m$. Hence the normalization of $R_m$ is local and $R$ satisfies (FD) by Corollary 3.

On the other hand, if $R = \mathbb{C}[x, y]/(y^2 - x^2(x+1))$ and $m = xR + yR$, then $\tilde{R}/m\tilde{R} \simeq \mathbb{C}[z]/(z^2 - 1)$. So there are two different maximal ideals of $\tilde{R}$ containing $m$. Hence the normalization of $R_m$ is not local and $R$ does not satisfy (FD).

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