SURJECTIONS OF UNIT GROUPS AND SEMI-INVERSES

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ABSTRACT. Given a surjective ring homomorphism, we study when the induced group homomorphism on unit groups is surjective. To this end, we introduce notions of generalized inverses and units, as well as a class of rings such that the set of closed points in the spectrum is a closed set. It is shown that any surjection out of such a ring induces a surjection on unit groups.

1. Introduction. Let CRing be the category of commutative rings with $1 \neq 0$, and Ab the category of abelian groups. One of the most natural functors from CRing to Ab is the group of units functor, $(\_)^\times$, associating to any (commutative) ring its (abelian) group of units. Functoriality follows from the fact that a ring homomorphism $\varphi : R \to S$ sends 1 to 1, hence units to units, and thus induces (by set-theoretic restriction) a group homomorphism $\varphi^\times : R^\times \to S^\times$. By definition as a set-theoretic restriction, one sees that $\varphi$ injective implies $\varphi^\times$ injective (i.e., $(\_)^\times$ is “left exact”). The question we consider here is: when does $\varphi$ surjective imply $\varphi^\times$ surjective, i.e., how does $(\_)^\times$ fail to be “right exact”?

Example 1. For any prime number $p$, the natural surjection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ induces a group homomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$, which is a surjection iff $p = 2, 3$.

Example 2. For a field $k$, any ring surjection $\varphi : k \to R$ is necessarily injective, hence an isomorphism, so (by functoriality) $\varphi^\times$ is also an isomorphism.

Example 3. For a field $k$, the surjection $\varphi_1 : k[x] \to k[x]/(x) \cong k$ induces a surjection on unit groups, but $\varphi_2 : k[x] \to k[x]/(x^2)$ does not.

not, as \( \varphi_2(1 + x) \in (k[x]/(x^2))^\times \), but is not the image of any unit of \( k[x] \) (= nonzero constant in \( k \)).

With these examples at hand, we make the following (non-vacuous) definition:

**Definition.** A ring surjection \( \varphi : R \to S \) has (\( \ast \)) if \( \varphi^\times : R^\times \to S^\times \) is surjective. We say that the ring \( R \) has (\( \ast \)) if every ring surjection \( \varphi : R \to S \) (for any ring \( S \)) has (\( \ast \)).

If \( \varphi : R \to S \) is a ring surjection, then \( S \cong R/I \) for some \( R \)-ideal \( I \) (namely \( I = \ker \varphi \)), so one may instead refer to an ideal \( I \) having (\( \ast \)) (i.e. if the canonical surjection \( R \to R/I \) has (\( \ast \))). Thus \( R \) has (\( \ast \)) iff \( I \) has (\( \ast \)) for every \( R \)-ideal \( I \), so in this way property (\( \ast \)) for a ring becomes an ideal-theoretic statement, intrinsic to the ring. The examples above say that any field \( k \) has (\( \ast \)), while \( \mathbb{Z} \) and \( k[x] \) do not.

We begin with some characterizations of (\( \ast \)). Recall that if \( W \) is a multiplicative set, the saturation of \( W \) is defined as \( W^\sim := \{ r \in R \mid \exists s \in R, sr \in W \} \), and \( W \) is called saturated if \( W = W^\sim \).

**Proposition 1.1.** Let \( R \) be a ring, \( I \) an \( R \)-ideal. The following are equivalent:

i) \( I \) has (\( \ast \))

ii) \( R^\times + I \) is saturated

iii) \( R^\times + I = (1 + I)^\sim \)

iv) For any \( a \in R \) such that \( 1 - ab \in I \) for some \( b \in R \), there exists \( u \in R^\times \) with \( 1 - au \in I \).

*Proof.* (ii) \( \iff \) (iii): follows from the containment \( 1 + I \subseteq R^\times + I \subseteq (1 + I)^\sim \) which holds for any ideal \( I \), and the fact that saturation is a closure operation (in particular, is monotonic and idempotent).

(i) \( \implies \) (iii): Suppose that the canonical surjection \( p : R \to R/I \) induces a surjection \( p^\times : R^\times \to (R/I)^\times \), i.e. if \( r \in R \) is such that \( p(r) \) is a unit, then \( p(r) = p(u) \) for some \( u \in R^\times \). Then \( r - u \in \ker p = I \), i.e. \( r \in R^\times + I \). Thus the preimage of the units of \( R/I \) is contained in \( R^\times + I \), but this preimage is exactly
(1 + I)~, since p(r) is a unit ⇔ 1 = p(1) = p(r)p(s) for some s ∈ R ⇔ 1 − rs ∈ I ⇔ rs ∈ 1 + I.

(iii) ⇒ (i): if R× + I = (1 + I)~, then any preimage of a unit of R/I differs from a unit of R by an element of I, so every unit of R/I is the image of a unit of R.

(iii) ⇔ (iv): Notice that a ∈ (1 + I)~ ⇔ 1 − ab ∈ I for some b ∈ R, and a ∈ R× + I ⇔ v − a ∈ I for some v ∈ R× ⇔ 1 − v⁻¹a ∈ I. □

2. Sufficient conditions for (∗). As a first application of Proposition 1.1, one has the following sufficient condition for an ideal to have (∗) (hereafter, the Jacobson radical of R is denoted by rad(R) := ∩ m∈mSpec(R) m, the intersection of all maximal ideals of R).

Corollary 2.1. Let R be a ring, I an R-ideal. If I ⊆ rad(R), then I has (∗).

Proof. If I ⊆ rad(R), then R× + I = R× = {1}~ is saturated, so Proposition 1.1(ii) applies. □

In fact, rather than requiring I to be contained in every maximal ideal, one can allow finitely many exceptions:

Theorem 2.2. Let R be a ring, I an R-ideal. If I is contained in all but finitely many maximal ideals of R (i.e. |mSpec(R) \ V(I)| < ∞), then I has (∗).

Proof. Write mSpec(R) \ V(I) := {m₁, ..., mₙ}, so that {I, m₁, ..., mₙ} are pairwise comaximal (the case n = 0 is Corollary 2.1). Let p : R → R/I be the canonical surjection, pick v ∈ (R/I)×, and write v = p(r) for some r ∈ R. By Chinese Remainder, there exists a ∈ R with a ≡ 0 (mod I), a ≡ 1 − r (mod mᵢ) for i = 1, ..., n. Since r is not contained in any maximal ideal containing I, r + a ∈ R×, and p(r + a) = p(r) = v. □

Corollary 2.3. Let R be a semilocal ring, i.e. |mSpec(R)| < ∞. Then R has (∗).
Proof. If \( R \) is semilocal, then for any \( R \)-ideal \( I \), \( \text{mSpec}(R) \setminus V(I) \) is finite. \( \square \)

Corollary 2.1 lends support to the idea that the Jacobson radical will not play a role in whether or not a ring has \((*)\). This is indeed true, as the following reduction to the \( J \)-semisimple case (i.e. \( \text{rad}(R) = \emptyset \)) will show.

**Proposition 2.4.** Let \( R \) be a ring, \( I \) an \( R \)-ideal, \( p : R \to R/I \) the canonical surjection, and \( \overline{p} : R/\text{rad}(R) \to R/(\text{rad}(R) + I) \) the map obtained by applying \( \otimes_R R/\text{rad}(R) \). Then \( p \) has \((*)\) iff \( \overline{p} \) has \((*)\). In particular, \( R \) has \((*)\) iff \( R/\text{rad}(R) \) has \((*)\).

**Proof.** Consider the commutative diagram of natural maps

\[
\begin{array}{ccc}
R & \xrightarrow{p} & R/I \\
\downarrow{\alpha} & & \downarrow{\beta} \\
R/\text{rad}(R) & \xrightarrow{\overline{p}} & R/(\text{rad}(R) + I)
\end{array}
\]

If \( p^* \) is surjective, then since \( \beta^* \) is surjective (by Corollary 2.1 as \( (\text{rad}(R) + I)/I \subseteq \text{rad}(R/I) \)), so is \( \overline{p}^* \). Conversely, suppose \( \overline{p}^* \) is surjective, and let \( v \in (R/I)^* \). Then \( \beta(v) \in (R/(\text{rad}(R) + I))^* \), so there exists \( \overline{u} \in (R/\text{rad}(R))^* \) with \( \overline{p}(\overline{u}) = \beta(v) \). Again by Corollary 2.1 \( u^* \) is surjective, hence \( u = \alpha(u) \) for some \( u \in R^* \). Then \( \beta(p(u)) = \overline{p}(\alpha(u)) = \beta(v) \), so \( v - p(u) \in \ker \beta \). But \( \ker \beta = p(\text{rad}(R)) \), so \( v - p(u) = p(r) \) for some \( r \in \text{rad}(R) \). Then \( v = p(u + r) \), and \( u + r \in R^* \). \( R \subseteq R^* \).

We can use Proposition 2.4 to give examples of rings with \((*)\) that are not semilocal. We include a proof of the following lemma for completeness (cf. [2], Exercise §4.12):

**Lemma 2.5.** For an arbitrary direct product of rings, \( \text{rad}(\prod_i R_i) = \prod_i \text{rad}(R_i) \).
Proof. \(\supseteq\): let \((a_i) \in \prod_i \operatorname{rad}(R_i)\). Then for each \(i\) and any \(b_i \in R_i\),
\[1 - a_i b_i \in R_i^\times,\]
so every \(b = (b_i) \in \prod_i R_i\) satisfies \(1 - ab = (1 - a_i b_i) \in \prod_i R_i^\times = (\prod_i R_i)^\times\).

\(\subseteq\): for any surjective ring map \(\varphi : R \twoheadrightarrow S\), \(\varphi(\operatorname{rad}(R)) \subseteq \operatorname{rad}(S)\), so applying this to each natural projection \(\pi_j : \prod_i R_i \twoheadrightarrow R_j\) gives \(\pi_j(\operatorname{rad}(\prod_i R_i)) \subseteq \operatorname{rad}(R_j)\). \(\square\)

**Example 4.**

i) If \(R = \prod_i R_i\) is an arbitrary product of semilocal rings, then \(R\) has \((\ast)\) (note that \(\dim R\) could be infinite, even if each \(R_i\) is zero-dimensional, cf. [1], Theorem 3.4). To see this, note that by Proposition 2.4 and Lemma 2.5, it suffices to show that any product of fields has \((\ast)\). Thus, let \(R = \prod_i k_i\), where \(k_i\) are fields. Using Proposition 1.1(iii), let \(I\) be an \(R\)-ideal, and \(a = (a_i) \in (1 + I)^\sim\), such that \(1 - ab \in I\) for some \(b \in R\). Let \(J\) be the set of indices \(j\) such that \(a_j = 0\), and let \(e_J\) be the indicator vector of \(J\), i.e. \(e_J := (e_i) \in R\), where \(e_i := \begin{cases} 1, & i \in J \\ 0, & i \notin J \end{cases}\). Then \(e_J(1 - ab) \in I\) satisfies \((e_J(1 - ab))_i = 0\) if \(i \notin J\) (if \(i \in J\), then \(a_i = 0\) and both \((1 - ab)_i = 1 - a_i b_i\) and \((e_J)_i\) equal 1, whereas if \(i \notin J\), then \((e_J)_i\) is already 0). Thus \((a + e_J(1 - ab))_i\) is nonzero for every \(i\), hence \(a + e_J(1 - ab) \in R^\times \implies a \in R^\times + I\).

ii) Via a different approach, we can also show that \((\ast)\) passes to finite products. Let \(R = \prod_{i=1}^n R_i\), where \(R_i\) have \((\ast)\). Using Proposition 1.1(iv), let \(I\) be an \(R\)-ideal. Then \(I = \prod_{i=1}^n I_i\) for \(R_i\)-ideals \(I_i\). Let \(a = (a_i) \in R\) be such that \(1 - ab \in I\) for some \(b = (b_i) \in R\). Then \(1 - a_i b_i \in I_i\) for each \(i\), so there exists \(u_i \in R_i^\times\) with \(1 - a_i u_i \in I_i\). Thus \(u = (u_i) \in \prod_{i=1}^n R_i^\times = R^\times\), and \(1 - au \in I\).
iii) In view of Example 4(i), as the diagonal map \( \mathbb{Z} \hookrightarrow \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \) is injective, we see that (*) does not pass to subrings. On the other hand, it is easy to see that (*) passes to quotient rings.

We briefly turn to the graded case. Let \( R = \bigoplus_{i \geq 0} R_i \) be a \( \mathbb{Z}_{\geq 0} \)-graded ring, \( I = \bigoplus_{i \geq 0} I_i \) a graded \( R \)-ideal, and \( p : R \to R/I \) the canonical surjection, a graded ring map of degree 0. Let \( p_0 : R_0 \to (R/I)_0 = R_0/I_0 \) be the induced ring map of degree 0 components. In general, the units of \( R \) need not be graded. However, with some primality assumptions we may reduce to the ungraded case, as follows:

**Proposition 2.6.** Suppose \( I \) is prime. If \( p_0 \) has (*), then \( p \) has (*). The converse holds if \( R \) is a domain.

**Proof.** If \( I \) is prime, then \( R/I \) is a positively graded domain, which has units only in degree 0, i.e. \( (R/I)\times \subseteq (R/I)_0 \). Then \( (R/I)^\times = ((R/I)_0)^\times = p_0^\times (R_0^\times) \subseteq p^\times (R^\times) \), and the first statement follows. Conversely, if \( R \) is a domain, then \( R^\times \subseteq R_0 \), so \( R^\times = (R_0)^\times \) and \( p(R^\times) = p_0(R_0^\times) \). \( \square \)

**Corollary 2.7.** If \( I \subseteq R_+ = \bigoplus_{i \geq 1} R_i \) is prime, then \( I \) has (*).

**Proof.** In this case, \( I_0 = 0 \), so \( p_0 : R_0 \to R_0 \) is the identity, hence \( p_0 \) has (*). \( \square \)

To motivate the next section, we briefly summarize the results thus far: we have seen that property (*) for a ring \( R \) depends only on the \( J \)-semisimple reduction \( R/\text{rad}(R) \). Since the \( J \)-semisimple reduction of a semilocal ring is a finite product of fields, this gives an alternate proof of Corollary 2.3. However, being semilocal is not a necessary condition for a ring to have (*), as an infinite product of fields is never semilocal. Despite this, the examples given so far of rings with (*) are quite similar - e.g. they all share the property that the \( J \)-semisimple reduction is 0-dimensional.
From a different angle, one can start with the observation that for any ring \( R \), if \( r \in R \) is a nonunit, then \( R \rightarrow R/(r^2) \) is such that \( 1 + r \) goes to a unit in \( R/(r^2) \), with inverse \( 1 - r \). In particular, if a ring \( R \) is to have \((*)\), then necessarily any element \( r \) must satisfy \( 1 + r \in R^\times \subseteq R \). Recalling that \( \text{rad}(R) = \{ r \in R \mid 1 + (r) \subseteq R^\times \} \), this will certainly be satisfied if for every \( r \in R \), there exists \( s \in R \) with \( r - sr^2 = r(1 - sr) \in \text{rad}(R) \). It is this last condition which we now examine in detail.

3. Semi-inverses. Returning to a general setting (laying aside for now the surjectivity question), let \( R \) be a ring, and \( r \in R \). The failure of \( r \) to be a unit is encoded in the set of maximal ideals which contain \( r \) – namely, \( r \) is a unit iff \( r \) is not contained in any maximal ideal. Furthermore, when this occurs there is a unique element \( r^{-1} \), with \( 1 - r^{-1} \cdot r = 0 \in m \) for every maximal ideal \( m \). Generalizing this basic fact gives an analogous notion for any \( r \in R \):

**Definition.** Let \( R \) be a ring, \( r \in R \). A subset \( S \subseteq R \) is called a semi-inverse set for \( r \) if for every maximal ideal \( m \in \text{mSpec}(R) \), either \( r \in m \), or there exists \( s \in S \) with \( 1 - sr \in m \).

Notice that the two cases in the definition above are exhaustive and mutually exclusive: i.e. for any \( r \in R \) and any \( m \in \text{mSpec}(R) \), it is always the case that either \( r \in m \) or there exists \( s \in R \) with \( 1 - sr \in m \), and both cases cannot occur simultaneously. Notice that existence of semi-inverse sets follows from the Axiom of Choice: for every maximal ideal \( m \) containing \( r \), the image \( \pi \in R/m \) is a unit, so there exists \( \pi \in R/m \) with \( \pi \cdot \pi = 1 \), i.e. \( 1 - sr \in m \). This also shows that for any \( r \in R \), the minimum size of a semi-inverse set for \( r \) is at most \( |\text{mSpec}(R) \setminus V(r)| \), which leads to the following definition:

**Definition.** For a ring \( R \), define a function \( \rho : R \rightarrow \mathbb{N} \cup \{\infty\} \) by

\[
\rho(r) := \begin{cases} 
\min\{|S| : S \text{ semi-inverse set for } r\}, & \text{if } r \text{ has a finite semi-inverse set} \\
\infty, & \text{if } r \text{ has no finite semi-inverse set}
\end{cases}
\]

The possible values that the function \( \rho \) can attain are rather limited:
Proposition 3.1. Let $R$ be a ring, $r \in R$. Then $\rho(r) < \infty$ iff $\rho(r) \in \{0, 1\}$.

Proof. Suppose $\rho(r) \neq \infty$, and let $S = \{s_1, \ldots, s_n\}$ be a finite semi-inverse set for $r$. Now $\prod_{i=1}^{n}(1-s_ir) = 1 - sr$ for some $s \in R$ (since the product is finite). Thus $r(1 - sr) = r\prod_{i=1}^{n}(1-s_ir) \in \text{rad}(R)$, so $\{s\}$ is a semi-inverse set for $r$, and $\rho(r) \leq 1$. □

Proposition 3.2. Let $R$ be a ring, $r \in R$. Then $\rho(r) = 0$ iff $r \in \text{rad}(R)$.

Proof. If $r \in \text{rad}(R)$, then $\emptyset$ is a semi-inverse set for $r$. Conversely, if $r \notin m$ for some $m \in \text{mSpec}(R)$, then if $S$ is any semi-inverse set for $r$, there must exist $s \in S$ with $1 - sr \in m$, so $|S| \geq 1$, hence $\rho(r) \geq 1$. □

Proposition 3.3. Let $R$ be a ring. Then $R^\times \subseteq \rho^{-1}(\{1\})$, and equality holds iff $\text{Spec}(R/\text{rad}(R))$ is connected.

Proof. If $u \in R^\times$, then $\{u^{-1}\}$ is a semi-inverse set for $u$, so $\rho(u) = 1$ (as $u \notin \text{rad}(R) \implies \rho(u) = 0$). For the second statement, suppose $R/\text{rad}(R)$ has no idempotents, and pick $r \in R$, $\rho(r) = 1$. Let $\{s\}$ be a semi-inverse set for $r$, so $r(1 - sr) \in \text{rad}(R)$. Then $\bar{r} = \bar{s} \cdot \bar{r}^2$ in $R/\text{rad}(R)$, so $\bar{s} \cdot \bar{r}$ is idempotent in $R/\text{rad}(R)$. By assumption $\bar{s} \cdot \bar{r} = \bar{0}$ or $\bar{1}$. If $\bar{s} \cdot \bar{r} = \bar{0}$, then $\bar{r} = (\bar{s} \cdot \bar{r})\bar{r} = \bar{0}$, i.e. $r \in \text{rad}(R)$, but this cannot happen if $\rho(r) = 1$. Thus $\bar{s} \cdot \bar{r} = \bar{1}$, so $r$ is a unit modulo $\text{rad}(R)$, hence $r$ is in fact a unit in $R$.

Conversely, suppose $\rho^{-1}(\{1\}) = R^\times$, and let $r \in R$ with $\bar{0} \neq \bar{r}$ idempotent in $R/\text{rad}(R)$. Then $r - r^2 \in \text{rad}(R)$, so $\{1\}$ is a semi-inverse set for $r$, i.e. $\rho(r) = 1$, so $r \in R^\times$. This implies $R/\text{rad}(R)$ has only trivial idempotents, hence has connected spectrum. □

Remark 3.4. i) If $\text{Spec}(R/\text{rad}(R))$ is connected, then $\text{Spec}(R)$ is also connected: if $e \in R$ is idempotent, then $\bar{e} \in R/\text{rad}(R)$ is also idempotent, so (replacing $e$ by $1 - e$ if necessary) $\bar{0} = \bar{e} \implies e \in \text{rad}(R) \implies 1 - e \in R^\times$, hence $e(1 - e) = 0 \implies e = 0$. 

ii) If $R$ is the coordinate ring of an (irreducible) affine variety (i.e. a finitely generated domain over a field), then $\text{Spec}(R/\text{rad}(R))$ is connected.

Proposition 3.1 and Proposition 3.2 indicate that the only interesting behavior occurs for elements $r \in R$ with $\rho(r) = 1$, which motivates the following definition:

**Definition.** Let $R$ be a ring, $r \in R$. If $\rho(r) = 1$, we say that $r$ is a **semi-unit**. In this case, if $\{s\}$ is a semi-inverse set for $r$, we say that $s$ is a **semi-inverse of** $r$. If every element of $R$ is either a semi-unit or in the Jacobson radical (i.e. $\rho(R) \subseteq \{0, 1\}$), we say that $R$ is a **semi-field**.

**Remark 3.5.** According to the definition, only semi-units can have semi-inverses, so although $\{1\}$ (or indeed any singleton set) is a semi-inverse set for 0, 1 is not treated as a semi-inverse of 0. Also, the relation of being a semi-inverse need not be symmetric: e.g. in $\mathbb{Z}/10\mathbb{Z}$, 3 is a semi-inverse of 2 (as $2 \equiv 3 \cdot 2^2 \mod 10$), but 2 is not a semi-inverse of 3 ($3 \not\equiv 2 \cdot 3^2 \mod 10$). However, notice that 2 and 8 are semi-inverses of each other.

The following proposition addresses uniqueness of semi-inverses:

**Proposition 3.6.** Let $R$ be a ring, $r \in R$ a semi-unit. If $s_1, s_2 \in R$ are semi-inverses of $r$, then $s_1 - s_2 \in \text{rad}(R) :_R r$. Conversely, if $s$ is a semi-inverse of $r$ and $a \in \text{rad}(R) :_R r$, then $s + a$ is a semi-inverse of $r$.

**Proof.** If $s_1, s_2$ are semi-inverses of $r$, then $r(1 - s_1r), r(1 - s_2r) \in \text{rad}(R)$, so $r(1 - s_1r) - r(1 - s_2r) = (s_2 - s_1)r^2 \in \text{rad}(R)$, i.e. $s_2 - s_1 \in \text{rad}(R) : r^2$. For the second statement, if $s$ is a semi-inverse of $r$ and $a \in \text{rad}(R) : r^2$, then $r(1 - sr), ar^2 \in \text{rad}(R)$, so $r(1 - (s + a)r) = r(1 - sr) - ar^2 \in \text{rad}(R)$ also.

Finally, notice that $\text{rad}(R) : r^2 = \text{rad}(R) : r$, since if $ar^2 \in \text{rad}(R)$, then $(ar)^2 = a(ar^2) \in \text{rad}(R) \implies ar \in \text{rad}(R)$, as $\text{rad}(R)$ is a radical ideal. \qed
Thus semi-inverses of \( r \) are unique precisely up to cosets of \( \text{rad}(R) : r \). In particular, semi-inverses of non-trivial semi-units are never unique:

**Corollary 3.7.** Let \( R \) be a ring, \( r \in R \) a semi-unit. Then \( r \) has a unique semi-inverse iff \( r \) is a unit and \( \text{rad}(R) = 0 \).

**Proof.** \( \Leftarrow \): if \( r \) is a unit, then \( \text{rad}(R) : r = \text{rad}(R) = 0 \), so \( r^{-1} \) is the only semi-inverse of \( r \). \( \Rightarrow \): if \( r \) has a unique semi-inverse \( s \), then \( \text{rad}(R) = 0 \), and \( r = sr^2 \). But \( 0 = \text{rad}(R) : r = 0 : r \), so \( r \) is a nonzerodivisor, hence \( 1 = sr \), i.e. \( r \in R^\times \). \( \square \)

On the other hand, any semi-unit has a semi-inverse that is a unit. This follows from the following general decomposition theorem:

**Theorem 3.8.** Let \( R \) be a ring, \( r \in R \). Then \( r \) is a semi-unit iff \( r = ue + t \) for some \( t \in \text{rad}(R) \), \( u \in R^\times \), and \( e \in R \) a semi-unit with \( 1 \) as a semi-inverse of \( e \) (\( \iff \overline{e} \text{ idempotent in } R/\text{rad}(R) \)). In particular, \( u^{-1} \) is a semi-inverse of \( r \).

**Proof.** Passing to \( R/\text{rad}(R) \), it suffices to show that \( \overline{r} \) is a product of a unit and an idempotent. Let \( s \) be a semi-inverse of \( r \), so \( \overline{r} = \overline{sr}^2 \). Set \( \overline{e} := \overline{rs} \). Then \( \overline{e}^2 = \overline{e} \), so if \( e \) is any lift of \( \overline{e} \), then \( e \) is a semi-unit in \( R \) with \( 1 \) as a semi-inverse. Notice also that \( \overline{r} = \overline{re} \).

Next, set \( \overline{u} := \overline{re} + (1 - \overline{e}) \). Then \( \overline{ue} = \overline{r}e^2 + (1 - \overline{e})\overline{e} = \overline{r} \). Furthermore,
\[
\overline{u} \cdot (\overline{ue} + (1 - \overline{e})) = (\overline{r}e + (1 - \overline{r})) \cdot (\overline{sr}^2 + (1 - \overline{r}))
\]
\[
= \overline{r}sr^2 + (1 - \overline{r})^2
\]
\[
= \overline{e}^3 + (1 - \overline{e})
\]
\[
= 1
\]
so \( \overline{u} \) is a unit. Lifting to \( R \) gives a unit \( u \in R \), such that \( t := r - ue \in \text{rad}(R) \).

Finally, notice that \( r(1 - u^{-1}r) = (ue + t)(1 - u^{-1}(ue + t)) = ue(1 - e) + t(1 - 2e - u^{-1}t) \in \text{rad}(R) \), so \( u^{-1} \) is a semi-inverse of \( r \). \( \square \)
4. Semi-fields. Having described the structure of semi-units, we now focus on the rings that have as many semi-units as possible, starting with the following criterion:

**Proposition 4.1.** Let $R$ be a ring. Then the following are equivalent:

i) $R$ is a semi-field

ii) $R/\text{rad}(R)$ is von Neumann regular

iii) $\dim R/\text{rad}(R) = 0$.

**Proof.** $R$ is a semi-field $\iff$ for every $r \in R$, there exists $s \in R$ with $r(1 - sr) \in \text{rad}(R)$ $\iff$ for every $\tau \in R/\text{rad}(R)$, there exists $\tau \in R/\text{rad}(R)$ with $\tau = \tau \cdot \tau^2$ $\iff$ $R/\text{rad}(R)$ is von Neumann regular. Since $R/\text{rad}(R)$ is always reduced, this happens iff $\dim R/\text{rad}(R) = 0$. \[\square\]

A geometric reformulation of the semi-field property is given by:

**Proposition 4.2.** Let $R$ be a ring. Then $R$ is a semi-field iff $\text{mSpec } R$ is closed in $\text{Spec } R$.

**Proof.** First, note that the closure of $\text{mSpec } R$ is equal to $V(\text{rad } R)$: for any $p \in \text{Spec } R$, $p$ is in $\overline{\text{mSpec } R}$ $\iff$ for all $f \in R$ with $p \in D(f)$, there exists $m \in \text{mSpec } R$ with $m \in D(f)$ $\iff$ $R - p \subseteq \bigcup_{m \in \text{mSpec } R} (R - m) \iff p \supseteq \bigcap_{m \in \text{mSpec } R} m = \text{rad } R$.

Thus, $\text{mSpec } R = \overline{\text{mSpec } R}$ iff $\text{mSpec } R = V(\text{rad } R)$ iff $\dim R/\text{rad}(R) = 0$, so the conclusion follows from Proposition 4.1. \[\square\]

**Corollary 4.3.** The following are equivalent for a ring $R$:

i) $R$ is semilocal

ii) $R/\text{rad}(R)$ is Artinian

iii) $R$ is a semi-field and $|\text{Min}(\text{rad}(R))| < \infty$

(Here $\text{Min}(\cdot)$ denotes the set of minimal primes).
Proof. iii) $\implies$ ii): If $R$ is a semi-field with $\text{Min}(R/\text{rad}(R)) = \text{Spec}(R/\text{rad}(R))$ finite, then $R/\text{rad}(R)$ is a von Neumann regular ring with finite spectrum, hence is Noetherian.

ii) $\implies$ i): An Artinian ring is semilocal, and $R/\text{rad}(R)$ semilocal $\implies$ $R$ semilocal.

i) $\implies$ iii): If $R$ is semilocal, then by Chinese Remainder $R/\text{rad}(R)$ is a finite direct product of fields. $\square$

We give two ways to produce new semi-fields:

**Proposition 4.4.** The class of semi-fields is closed under quotients and products.

**Proof.** Let $R$ be a semi-field, and $I$ an $R$-ideal. The surjection $p : R \to R/I$ sends $p(\text{rad}(R)) \subseteq \text{rad}(R/I)$, so $(R/I)/\text{rad}(R/I)$ is a quotient of $R/(\text{rad}(R) + I)$, which is itself a quotient of $R/\text{rad}(R)$. Thus $\dim R/\text{rad}(R) = 0$ implies $\dim (R/I)/\text{rad}(R/I) = 0$.

If now $R_i$ are semi-fields, then by Lemma 2.5

\[
(\prod_i R_i)/\text{rad}(\prod_i R_i) = (\prod_i R_i)/(\prod_i \text{rad}(R_i)) = \prod_i R_i/\text{rad}(R_i)
\]

is a product of von Neumann regular rings, hence is von Neumann regular. $\square$

**Remark 4.5.** Geometrically, the first part of Proposition 4.4 says that the semi-field property passes to closed subschemes. However, the semi-field property does not pass to open subschemes – e.g. if $R$ is any Noetherian ring, $x \in \text{rad}(R)$ but $x$ is not contained in any minimal prime of $R$, then $\dim R_x = \dim R - 1$, and if $\dim R < \infty$, then $\text{rad}(R_x) = \text{nil}(R_x)$. Thus any Noetherian local domain $(R, m)$ of dimension $\geq 2$ and $0 \neq x \in m$ gives an example where $R$ is a semi-field (being local), but $R_x$ is not.

Even in light of Proposition 4.4 it is still reasonable to ask for nontrivial examples of semi-fields. One trivial reason for being a semi-field is that the set of closed points is finite, and Corollary 4.3 guarantees that this is the only possibility in the Noetherian case –
thus, one must search among non-Noetherian rings for a nontrivial example.

Now one can easily form non-Noetherian rings by taking infinite products. However, products are an arguably trivial way to construct examples – for finite products, the geometric intuition is that the property of the closed points forming a closed set should pass to disjoint unions. This intuition fails for general von Neumann regular rings though, since not every von Neumann regular ring is a product of fields: e.g. if $k$ is a finite field, then the subring of $\prod_{i \in \mathbb{N}} k$ consisting of eventually constant sequences is non-Noetherian and countable, whereas any product of fields is either Noetherian or uncountable. Despite this, von Neumann regular rings are trivially semi-fields for the same reason any zero-dimensional ring is: the set of closed points is certainly closed if every point is closed!

Nevertheless, there are indeed less trivial examples of semi-fields, which arise formally in a manner similar to Hilbert’s basis theorem and (a general form of) the Nullstellensatz, which say that the Noetherian and Jacobson properties pass to rings of finite type. To emphasize the analogy, for a ring $R$, we say that a ring is of semi-finite type over $R$ if it is of the form $R[[x_1, \ldots, x_n]]/I$.

**Proposition 4.6.** Let $R$ be a semi-field. Then any ring of semi-finite type over $R$ is a semi-field.

**Proof.** By Proposition 4.4, it suffices to show $R$ semi-field \implies $R[[x]]$ semi-field, and by induction it is enough to do the base case $n = 1$. This follows immediately from the fact that $x \in \text{rad}(R[[x]])$, which in turn implies that every maximal ideal of $R[[x]]$ is of the form $\mathfrak{m}R[[x]] + (x)$ for a (uniquely determined) maximal ideal $\mathfrak{m}$ of $R$, so $R/\text{rad}(R) \cong R[[x]]/\text{rad}(R[[x]])$. \qed

**5. Property (*) revisited.** We finally return to the original surjectivity question. Proposition 4.1 shows that every example given earlier of a ring with (*) has been a semi-field. The following theorem gives the general phenomenon:

**Theorem 5.1.** Let $R$ be a semi-field. Then $R$ has (*).
Proof. By Proposition 2.4, we may pass to $R/\text{rad}(R)$, so by Proposition 4.1 it suffices to show any von Neumann regular ring $R$ has $(\ast)$. For this we use Proposition 1.1(iv). Let $I \neq R$ be an ideal, and $a \in R$ such that $1 - ab \in I$ for some $b \in R$. As $R$ is von Neumann regular, $I$ is a radical ideal, so $I = \bigcap_i p_i$ for some primes $p_i \in \text{Spec} R$. Then $1 - ab \in p_i$ implies $a \notin p_i$, for all $i$. Now $a$ is a semi-unit, so by Theorem 3.8, $a$ has a semi-inverse which is a unit, i.e. there exists $u \in R^\times$ with $a = a^2 u$. Then $a(1 - au) = 0 \in p_i$ for all $i$, so $1 - au \in p_i$ for all $i$, hence $1 - au \in I$. \hfill \Box

Remark 5.2. Corollary 4.3, Proposition 4.4, and Theorem 5.1 give an alternate proof of Example 4, that an arbitrary product of semilocal rings has $(\ast)$. We do not know if the class of rings with $(\ast)$ is closed under arbitrary products.

Theorem 5.1 thus generalizes and gives a uniform proof of all the previous sufficient conditions for a ring to have $(\ast)$: Corollary 2.1, Corollary 2.3, and Example 4.

We conclude with an application and a generalization. Although the motivation in determining when an ideal or ring has $(\ast)$ has been mostly intrinsic, one possible application of these results is in constructing rings with trivial unit group.

Proposition 5.3. Let $X \subseteq \mathbb{P}_F^n$ be a projective variety over $F_2$. Then the homogeneous coordinate ring of $X$ has trivial unit group.

Proof. Let $S = F_2[x_0, \ldots, x_n]$ and $R = S/I$, where $I$ is a homogeneous radical ideal. Then $I = p_1 \cap \ldots \cap p_m$, where $p_i$ are homogeneous primes in $S$, so $R \twoheadrightarrow S/p_1 \times \ldots \times S/p_m$. Thus $R^\times \subseteq \prod_{i=1}^m (S/p_i)^\times$, so it suffices to show $(S/p_i)^\times = \{1\}$ for each $i$. Now $S$ is a polynomial ring over $F_2$, so $S^\times = (F_2)^\times = \{1\}$, and each $p_i \subseteq S_+$, so by Corollary 2.7 there is a surjection $\{1\} = S^\times \twoheadrightarrow (S/p_i)^\times$. \hfill \Box

In fact, the above reasoning holds in any number of variables. Thus, if $R = \mathbb{Z}[x_1, \ldots]/I$ is any ring presented as a $\mathbb{Z}$-algebra, then homogenizing the defining ideal $I$ with a new variable $x_0$ gives a
standard graded ring $\tilde{R} := \mathbb{Z}[x_0, x_1, \ldots]/\tilde{I}$, and then $(\tilde{R} \otimes_{\mathbb{Z}} \mathbb{F}_2)_{\text{red}} = \mathbb{F}_2[x_0, x_1, \ldots]/\sqrt{\tilde{I}}$ has trivial unit group.

Conversely, every ring with trivial unit group has characteristic 2 (as 1 = −1) and has trivial Jacobson radical (in particular, is reduced). Thus if $R^\times = \{1\}$, then $R$ is the (affine) coordinate ring of a reduced scheme over $\mathbb{F}_2$, and Proposition 5.3 realizes every (standard) graded ring with trivial unit group.

Finally, one possible generalization is to consider other functors from $\text{CRing}$ to $\text{Grp}$. A natural choice which directly generalizes the group of units functor is $GL_n(\_): \text{CRing} \to \text{Grp}$, which for $n = 1$ coincides with $\_^\times$. Note that if $M_n(\_)$ denotes the matrix ring functor $\text{CRing} \to \text{Ring}$ (sending a commutative ring $R$ to the ring of $n \times n$ matrices over $R$), then $GL_n(\_) = (\_^\times \circ M_n(\_))$.

**Proposition 5.4.** Let $R$ be a ring, $I \subseteq \text{rad}(R)$ an $R$-ideal, and $p : R \to R/I$ the canonical surjection. Then for any $n \in \mathbb{N}$, $\bar{p} : GL_n(R) \to GL_n(R/I)$ is surjective.

**Proof.** Pick $B = (b_{ij}) \in GL_n(R/I)$, and let $A = (a_{ij}) \in M_n(R)$ be any (entrywise) lift of $B$ to $R$, i.e. $p(a_{ij}) = b_{ij}$ for all $i, j$. Since $\det A$ is a polynomial in the entries of $A$, $p(\det A) = \det B$ is a unit in $R/I$. But $I \subseteq \text{rad}(R)$, so $\det A$ is in fact a unit in $R$, i.e. $A \in GL_n(R)$. □

Notice that the proof of Proposition 5.4 shows a stronger fact than preserving surjectivity; namely, any lift of a matrix in $GL_n(R/I)$ is already in $GL_n(R)$. In fact, the analogue of Proposition 2.4 holds for $GL_n(\_)$, which shows that corollary 2.3 holds for $GL_n(\_)$ as well (cf. [2], Exercises §4.21 and §20.7).

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