WELL-COVERED AND COHEN-MACAUŁAY THETA-RING GRAPHS

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ABSTRACT. In this paper we characterize the well-covered property for: theta-ring and ring graphs. Furthermore, we prove that Cohen-Macaulayness, pure shellability and pure vertex decomposability are equivalent for theta-ring graphs. Also, we give a combinatorial characterization of these graphs.

1. Introduction. Let $G$ be a simple graph (without loops and multiplies edges) whose vertex set is $V(G) = \{x_1, ..., x_n\}$ and edge set $E(G)$. Let $R = k[x_1, ..., x_n]$ be a polynomial ring over a field $k$, the edge ideal of $G$, denoted by $I(G)$, is the ideal of $R$ generated by all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. $G$ is a Cohen-Macaulay graph if $R/I(G)$ is a Cohen-Macaulay ring (see [4], [16]). A subset $F$ of $V(G)$ is a stable set or independent set if $e \not\in F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. $G$ is called well-covered if every maximal stable set has the same cardinality. The Stanley-Reisner complex of $I(G)$, denoted by $\Delta_G$, is the simplicial complex whose faces are the stable sets of $G$. The set of facets (maximal faces) of $\Delta$ is denoted by $F(\Delta)$. If $|F_1| = |F_2|$ for every $F_1, F_2 \in F(\Delta)$, then $\Delta$ is called pure. Thus, $\Delta_G$ is pure if and only if $G$ is well-covered. A simplicial complex $\Delta$ is shellable if the facets of $\Delta$ can be ordered $F_1, ..., F_s$ such that for all $1 \leq i < j \leq s$, there exist some $v \in F_j \setminus F_i$ and some $l \in \{1, ..., j - 1\}$ with $F_j \setminus F_l = \{v\}$. A graph $G$ is called shellable if $\Delta_G$ is shellable. $G$ is vertex decomposable if $G$ is a totally disconnected graph or there is a vertex $v$ such that $G \setminus v$ and $G \setminus N_{G}[v]$ are both vertex decomposable, and each stable set in $G \setminus N_{G}[v]$ is not a maximal stable set in $G \setminus v$. We have the following implications (see [4], [14], [16], [17]):

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Pure vertex decomposable ⇒ Pure shellable ⇒ Cohen–Macaulay ⇒ Well–covered

Theta-ring graphs were introduced in [7], these graphs can be constructed recursively by clique-sums of cycles and/or complete graphs. $G$ is theta-ring if and only if the toric ideal associated to each orientation of $G$ is a binomial complete intersection (see [7]). Theta-ring graphs are closed under induced subgraphs. In [6] it is proven that the obstructions of universally signable graphs are the Truemper configurations, i.e., $G$ is universal signable if and only if $G$ does not contain a Trumper configuration as an induced subgraph. Also, in [7] it is proven that the obstructions of theta-ring graphs are the Truemper configurations. Hence, universally signable and theta-ring are equivalent. Ring graphs are the graphs such that the set of the characteristic vectors of their chordless cycles is a base of its cycle space over $\mathbb{Z}_2$. Ring graphs were introduced in [9], furthermore, they have been studied in other contexts and with other applications in [1], [2] and [11]. Therefore, theta-ring graphs and ring graphs are of interest from combinatorial, algebraic and geometry points of view. For more details, see the introduction of [8]. In general, we have the following relations between some families of graphs.

$$
\text{Ring graphs} \subset \Theta \text{-Rings} \supset \text{Chordal graphs} \cup \text{Cactus graphs} \subset \text{Block-Cactuses} \cup \text{Block graphs}
$$

The well-covered property has been characterized for chordal graphs and block graphs in [12]; and for cactus and block-cactus graphs in [13]. Also, vertex decomposable, shellable and sequentially Cohen-Macaulay cactus graphs have been characterized in [10]. Chordal graphs are always shellable and sequentially Cohen-Macaulay (see [15]). In this paper we characterize well-coveredness for theta-ring graphs and ring graphs. Also, we prove that pure vertex decomposability and Cohen-Macaulayness are equivalent for theta-ring graphs and we give a combinatorial characterization of these graphs.

2. $\mathcal{T}$-family, sun-complete and well-covered graphs. If $X \subseteq V(G)$, then the induced subgraph by $X$ in $G$, denoted by $G[X]$, is the
Let \( S \in F \)

Remark 2.3. Let \( K \) be a complete subgraph of \( G \). \( K \) is called simplicial if the induced subgraph \( G[N_G[v]] \) is a complete graph. In this case \( G[N_G[v]] \) is called a simplex. \( S_G \) denotes the set of the simplexes of \( G \) and \( S'_G \) is the set of the sun-complete subgraphs of \( G \).

Remark 2.5. If \( G \) is a graph, then \( S_G \subseteq S'_G \).

Proof. We take \( K = G[N_G[x]] \in S'_G \). If \( V(K) \cap S = \emptyset \) for some \( S \in F(\Delta_G) \), then \( V(K) \subseteq N_G(S) \). Thus, \( x \in N_G(y) \) for some \( y \in S \). But \( G[N_G[x]] = K \), so \( y \in V(K) \). This is a contradiction, since \( V(K) \cap S = \emptyset \), therefore \( K \in S'_G \).

Proposition 2.6. An edge \( e = \{x, y\} \in S'_G \) if and only if \( \{z, z'\} \in E(G) \) for each \( z \in N_G(x) \) and each \( z' \in N_G(y) \).

Proof. \( \Rightarrow \) Suppose \( \{z, z'\} \notin E(G) \), for some \( z \in N_G(x) \) and \( z' \in N_G(x) \). Hence, \( z \neq y \) and \( z' \neq x \). Furthermore, there is \( S \in F(\Delta_G) \) with \( \{z, z'\} \subseteq S \), since \( \{z, z'\} \notin E(G) \) (it is possible that \( z = z' \)). Thus, \( e \cap S = \emptyset \). This is a contradiction, since \( e \in S'_G \). Therefore, \( \{z, z'\} \in E(G) \).

\( \Leftarrow \) We take \( S \in F(\Delta_G) \). If \( e \cap S = \emptyset \), then there are \( z \in N_G(x) \) and \( z' \in N_G(y) \) such that \( z, z' \in S \). But by hypothesis, \( \{z, z'\} \in E(G) \), a contradiction. So \( e \cap S \neq \emptyset \) and \( e \in S'_G \).

Proposition 2.7. Let \( G \) be a well-covered graph. If \( K \in S_G \) and \( K' \in S'_G \), then \( K = K' \) or \( V(K) \cap V(K') = \emptyset \).
Proof. Suppose \( w \in V(K) \cap V(K') \). There is \( S \in \mathcal{F}(\Delta_G) \) such that \( w \in S \). We take a simplicial vertex \( y \) of \( K \), then \( N_G[y] = V(K) \subseteq N_G[w] \). Thus, \( N_G[y] \cap S = \{ w \} \) since \( N_G[w] \cap S = \{ w \} \) and \( S \in \mathcal{F}(\Delta_G) \). Hence, \( S' = (S \setminus w) \cup \{ y \} \in \Delta_G \). Furthermore, \( G \) is well-covered and \( |S'| = |S| \), then \( S' \in \mathcal{F}(\Delta_G) \). Also, \( S \cap V(K') = \{ w \} \), implies \( S' \cap V(K') \subseteq \{ y \} \). Consequently, \( y \in V(K') \) since \( K' \in S' \). So, \( V(K') \subseteq N_G[y] = V(K) \). If \( a \in V(K) \setminus V(K') \), then there is \( S_1 \in \mathcal{F}(\Delta_G) \) with \( a \in S_1 \). This implies \( S_1 \cap V(K') = \emptyset \) since \( S_1 \cap V(K') \subseteq S_1 \cap V(K) = \{ a \} \) and \( a \notin V(K') \), a contradiction. Therefore \( K = K' \).

**Remark 2.8.** A free vertex is simplicial, but the converse is not true.

**Corollary 2.9.** If \( G \) is a well-covered graph, then all its simplexes are pairwise disjoint. In particular, if \( x \in V(G) \), then there are no two free vertices in \( N_G(x) \).

**Proof.** From Remark 2.5 and Proposition 2.7.

**Definition 2.10.** A c-minor of \( G \) is a subgraph \( H = G \setminus N_G[S] \), where \( S \in \Delta_G \).

**Remark 2.11.** ([12], [16]) The well-covered property is closed under taking c-minor.

**Lemma 2.12.** If \( K \in S'_G \) and \( A \) is a stable set of \( V(G) \setminus V(K) \), then \( K' = K \setminus N_G[A] \in S'_G \), where \( G' = G \setminus N_G[A] \).

**Proof.** If \( S' \cap V(K') = \emptyset \) for some \( S' \in \mathcal{F}(\Delta_G) \), then \( S' \cup A \in \Delta_G \) and \( V(K') \subseteq N_G(S') \). Since \( B = V(K) \setminus V(K') \subseteq N_G(A) \), we have \( V(K) \subseteq V(K') \cup N_G(A) \subseteq N_G(S' \cup A) \). This is a contradiction by Remark 2.3, since \( S' \cup A \in \Delta_G \) and \( K \in S'_G \). Therefore, \( K' \in S'_G \).

**Definition 2.13.** A 5-cycle \( C \) of \( G \) is called basic if \( C \) is induced and it does not contain two adjacent vertices (in \( C \)) of degree three or more. A 5-cycle \( C' = (a,b,c,d,e,a) \) is semi-basic if \( \deg_G(a) = \deg_G(c) = 2 \), \( \deg_G(d) = \deg_G(e) = 3 \) and there exists an induced 4-cycle \( Q \) such that \( V(Q) \cap V(C') = \{ d, e \} \). The set of the basic 5-cycles of \( G \) is denoted by \( \mathcal{C}_G \) and \( \mathcal{C}_G' \) is the set of the semi-basic 5-cycles.

**Remark 2.14.** If \( C' \in C_G' \), then \( C' \notin C_G \) and \( C \) is an induced cycle. So, \( C_G \cap C_G' = \emptyset \). Also, \( S'_G \cap (C_G \cup C_G') = \emptyset \).

**Definition 2.15.** We denote \( V(S'_G) = \{ x \in V(G) \mid x \in V(H) \) for some \( H \in S'_G \} \), \( V(C'_G) = \{ x \in V(G) \mid x \in V(H) \) for some \( H \in C'_G \} \) and \( V(C_G) = \{ x \in V(G) \mid x \in V(H) \) for some \( H \in C_G \} \).
Suppose \( C \subseteq K \) and \( C \subseteq K \). Let \( C \subseteq K \) and \( C \subseteq K \). This is a contradiction, since \( \text{deg} G \). We can assume \( \text{deg} G \). Then \( K = G \) \( N_G[z_2] \subseteq S_G \),. By Lemma 2.12, \( K = G \) \( N_G[z_2] \subseteq S_G \). Thus, by Proposition 2.7, \( K = K' \), since \( y \in V(K) \cap V(K') \). But, \( z_4 \in V(K) \cap V(K') \subseteq V(K) \), then \( \{y', z_4\} \subseteq E(G) \). This is a contradiction, since \( \text{deg}_G(z_4) = 2 \). Therefore, \( V(K) \cap V(C') = \emptyset \).

Now, we suppose \( C = \{x_1, x_2, x_3, x_4, x_5, x_1\} \) with \( x_1 \in V(C') \). By Remark 2.14, there is \( z \in V(C') \setminus V(C) \). Without loss of generality, we can assume \( \{z, x_1\} \subseteq E(G) \). Hence, \( \text{deg}_G(x_2) = \text{deg}_G(x_5) = 2 \) and \( \text{deg}_G(x_3) = 2 \) or \( \text{deg}_G(x_4) = 2 \) since \( C \subseteq C_G \). We can suppose \( \text{deg}_G(x_3) = 2 \). If \( \{x_2, x_3\} \subseteq V(C') \setminus \emptyset \), then \( C' = \{z, x_1, x_2, x_3, x_4, z\} \). This is a contradiction, since \( \text{deg}_G(x_2) = \text{deg}_G(x_3) = 2 \), \( \text{deg}_G(x_1) \geq 3 \), \( \text{deg}_G(x_4) \geq 3 \) and \( C' \subseteq C' \). So, \( \{x_2, x_3\} \subseteq V(C') \subseteq \emptyset \). If \( x_5 \subseteq V(C') \), then \( C' = \{z, x_1, x_3, x_4, z', z\} \) where \( z' \subseteq V(C') \setminus V(C) \). Thus \( \text{deg}_G(z) = 2 \) or \( \text{deg}_G(z') = 2 \), since \( \text{deg}_G(x_1) \geq 3 \) and \( \text{deg}_G(x_4) \geq 3 \).

We can assume \( \text{deg}_G(z) = 2 \). Consequently, \( \text{deg}_G(x_4) = \text{deg}_G(z') = 3 \) and \( \{z', x_1, x_3, x_2, z'\} \) is a 4-cycle. This is a contradiction, since \( \text{deg}_G(x_2) = 2 \). Hence, \( V(C) \cap V(C') \subseteq \{x_1, x_4\} \). This implies, \( C' = \{x_1, z_1, z_2, z_3, x_1\} \), where \( z_3 \notin V(C) \), since \( C \) is induced. Thus, \( \text{deg}_G(x_1) \geq 4 \), so \( \text{deg}_G(z) = \text{deg}_G(z_3) = 2 \) since \( C' \subseteq C' \). Also, \( z_1 \notin V(C) \) or \( z_2 \notin V(C) \), since \( V(C) \cap V(C') \subseteq \{x_1, x_4\} \). We can suppose \( z_1 \notin V(C) \). So, \( N_{G_2}(x_1) \) has two free vertices, \( x_3 \) and \( z_3 \),

**Figure 1.** \( K \in S_G, C \subseteq C_G \) and \( C \subseteq C' \).
in $G_2 = G \setminus N_G[x_3, z_1]$. This is a contradiction by Corollary 2.9 and Remark 2.11. Therefore, $V(C) \cap V(C') = \emptyset$. 

**Definition 2.17.** A graph $G$ is in the $\mathcal{T}$-family if $\{V(H) \mid H \in S'_G \cup C_G \cup C'_G\}$ is a partition of $V(G)$, and we denote it by $G \in \mathcal{T}$.

**Figure 2.** $C_1, C_2 \in C_G$ and $\{V(C_1), V(C_2)\}$ is a partition of $V(G)$, then $G \in \mathcal{T}$

**Figure 3.** $K_1, K_2, K_3, K \in S'_G$ and $\{V(K_1), V(K_2), V(K_3), V(K)\}$ is a partition of $V(G)$, then $G \in \mathcal{T}$

**Remark 2.18.** Let $G$ be a graph, then $G \in \mathcal{T}$ if and only if each connected component of $G$ is in $\mathcal{T}$.

**Lemma 2.19.** If $S \in \mathcal{F}(\Delta_G)$ and $C \in C_G \cup C'_G$, then $|S \cap V(C)| = 2$.

**Proof.** We set $C = (a, b, c, d, e, a)$, then $|S \cap V(C)| \leq 2$. We can suppose $\deg_G(a) = \deg_G(e) = 2$. If $a, c \in S$, then $|S \cap V(C)| = 2$. Now, assume $a \notin S$. So, $e \in S$ or $b \in S$, since $S \in \mathcal{F}(\Delta_G)$ and $\deg_G(a) = 2$. If $b \notin S$, then $e \in S$, $d \notin S$ and $c \in S$, since $N_G(c) = \{b, d\}$. Hence, $|S \cap V(C)| = 2$. Now, assume $b \in S$. If $S \cap V(C) = \{b\}$, then $(N_G(d) \cap S) \setminus V(C) \neq \emptyset$ and $(N_G(e) \cap S) \setminus V(C) \neq \emptyset$. Thus, $\deg_G(d) = \deg_G(e) = 3$ and $C \in C'_G$. Consequently, $f, g \in S$ where $Q = (d, e, f, g, d)$ is a 4-cycle. This is a contradiction, since $\{f, g\} \in E(G)$. Therefore, $|S \cap V(C)| = 2$. 

**Proposition 2.20.** If $G \in \mathcal{T}$, then $G$ is well-covered.
Definition 3.1. A chorded-theta subgraph $T$ of $G$ is an induced subgraph by three paths $L_1$, $L_2$, $L_3$ each one between two non adjacent vertices $x$ and $y$ such that $V(L_i) \cap V(L_j) = \{x, y\}$ for $1 \leq i < j \leq 3$. The edges that do not belong to any of the sets $E(L_1)$, $E(L_2)$ and $E(L_3)$ are called the chords of $T$.

Definition 3.2. Let $G$ be a chorded-theta graph with $L_1(T) = (x, x_1, ..., x_{r_1}, y)$, $L_2(T) = (x, y_1, ..., y_{r_2}, y)$ and $L_3(T) = (x, z_1, ..., z_{r_3}, y)$. A transversal triangle $H$ of $T$ is a triangle in $G$ such that $V(H) = \{x, y, z_k\}$ for some $i, j, k$. A graph $G$ is called a theta-ring graph if every chorded-theta of $G$ has a transversal triangle.

Remark 3.3. ([7], Corollary 4 and Theorem 6) Theta-rings are closed under induced subgraphs.

Lemma 3.4. Let $G$ be a theta-ring graph. If $C = (z_1, ..., z_k, z_1)$ is an induced cycle and $P$ is a path such that $P \cap V(C) = \{z_i, z_j\}$ with $i \neq j$, then $\{z_i, z_j\} \in E(G)$.

Proof. We can assume $i \notin \{1, k\}$ and $i < j$. Suppose that $\{z_i, z_j\} \notin E(G)$, then $j \notin \{i - 1, i + 1\}$. So, $k \geq 4$ and there is a chorded-theta $T$ with $L_1(T) = P$, $L_2(T) = (z_i, z_{i+1}, ..., z_j)$ and $L_3(T) = (z_1, z_1, ..., z_{i-1}, z_{i+1}, z_k, z_{k-1}, ..., z_1)$. Since $C$ is an induced cycle, $T$ does not contain transversal triangles. This is a contradiction, since $G$ is theta-ring.

Corollary 3.5. Let $G$ be a theta-ring graph. If $C = (x_1, ..., x_s, x_1)$ is an induced cycle in $G' = G \setminus N_G[x]$, where $x \in V(G)$ and $s \geq 4$, then $\deg_G(x_i) = \deg_{G'}(x_i)$ or $\deg_G(x_i) = \deg_{G'}(x_j)$ for $1 \leq i \leq j - 2 \leq s - 3$.

Proof. By contradiction, assume there are $z_1, z_2 \in N_G(x)$ such that $\{z_1, x_1\}$, $\{z_2, x_s\} \in E(G)$. We take $P = (x_i, z_1, x_j)$ if $z_1 = z_2$ or $P = (x_i, z_1, x, z_2, x_j)$ if $z_1 \neq z_2$, a contradiction by Lemma 3.4.

Definition 3.6. Let $A$ and $B$ be connected subgraphs of $G$. If $G = A \cup B$ such that $E(G) = E(A) \cup E(B)$ and $A \cap B$ is a complete graph (it can be a single vertex), then $G$ is called the clique-sum of
A and B. In this case, G is denoted by $A \oplus B$. Furthermore, if $|V(A) \cap V(B)| = k$, then G is called the k-clique-sum of A and B. Let $A_1, \ldots, A_r$ be subgraphs of G, then $(((A_1 \oplus A_2) \oplus A_3) \cdots) \oplus A_r$ is denoted by $A_1 \oplus A_2 \oplus \cdots \oplus A_r$.

**Proposition 3.8.** ([7], Theorem 4) A graph G is a theta-ring graph if and only if each connected component of G can be constructed by clique-sums of complete graphs and/or cycles.

**Remark 3.7.** If $G = H_1 \oplus H_2$ is a connected graph where $H_2$ is a cycle, then $1 \leq |V(H_1) \cap V(H_2)| \leq 2$.

**Proposition 3.8.** ([7], Theorem 4) A graph G is a theta-ring graph if and only if each connected component of G can be constructed by clique-sums of complete graphs and/or cycles.

![Figure 4. G is a theta-ring graph](image)

**Remark 3.9.** The graph in Figure 2 is not a theta-ring graph.

**Lemma 3.10.** Let $G = H_1 \oplus H_2$ be a connected well-covered theta-ring graph. If $H_2 = (x_1, x_2, \ldots, x_k, x_1)$ is a cycle with $V(H_1) \nsubseteq V(H_2)$ and $\{x_1\} \subseteq V(H_1) \cap V(H_2) \subseteq \{x_1, x_k\}$, then $k \leq 5$. Furthermore:

(a) If $H_2$ is a 5-cycle with $\deg_G(x_1) \geq 3$ and $\deg_G(x_5) \geq 3$, then $H_2 \in C_5^r$.

(b) If $H_2$ is a 4-cycle, then $\deg_G(x_1) \geq 3$, $\deg_G(x_4) \geq 3$ and $\oplus$ is a 2-clique-sum.

**Proof.** Since $V(H_1) \cap V(H_2) \subseteq \{x_1, x_k\}$, $\deg_G(x_i) = 2$ for $i \in \{2, \ldots, k - 1\}$. We can assume there is $y_1 \in V(G) \setminus V(H_2)$ such that $\{x_1, y_1\} \in E(G)$, since $V(H_1) \nsubseteq V(H_2)$ and G is connected. If $k = 6$, then $N_{G_1}(x_4)$ has two free vertices, $x_3$ and $x_5$, in $G_1 = G \setminus N_G[x_1]$. If $k = 7$, then $N_{G_2}(x_3)$ has two free vertices, $x_2$ and $x_4$, in $G_2 = G \setminus N_G[y_1, x_6]$. If $k \geq 8$, then $N_{G_3}(x_4)$ has two free vertices, $x_3$ and $x_5$, in $G_3 = G \setminus N_G[x_1, x_7]$. In the three cases we have a contradiction by Corollary 2.9 and Remark 2.11. Therefore, $k \leq 5$.

(a) Now, assume $k = 5$ and $\deg_G(x_5) \geq 3$. Thus, there is $y_2 \in V(G) \setminus V(H_2)$ such that $\{y_2, x_3\} \in E(G)$. If $\{y_1, y_2\} \notin E(G)$, then
Lemma 3.12. Let $G$ be a well-covered theta-ring graph. If $G' = G \setminus y \in \mathcal{T}$ where $y \in N_G(x)$ and $x$ is a simplicial vertex, then $G \in \mathcal{T}$.

Proof. We take $H \in S'_G \cup C_G \cup C'_G \setminus \{K_1\}$, where $K_1 = G[N_G(x)] \subseteq S_G$. By Proposition 2.7 and Proposition 2.16, $V(K_1) \cap V(H) = \emptyset$, since $K_1 \subseteq S_G$. So, $y \notin V(H)$ and $H \subseteq G'$. By Proposition 3.11, $H \in S'_G \cup C_G \cup C'_G$. Furthermore, $K_2 = K_1 \setminus y = G'[N_G[y]] \in S'_G$. By Remark 2.3, $K_2 \subseteq S'_G$, since $K_2 \subseteq N_G(y)$. Hence, $S'_G \cup C_G \cup C'_G \setminus \{K_1\} \subseteq S'_G \cup C_G \cup C'_G \setminus \{K_2\}$. Thus, the elements of $S'_G \cup C_G \cup C'_G$ are disjoint since $G' \in \mathcal{T}$. 

Proposition 3.11. Let $G'$ be an induced subgraph of $G$.

(1) If $H \in S'_G \cup C_G \cup C'_G$, and $N_G[H] \subseteq V(G')$, then $H \subseteq S'_G \cup C_G \cup C'_G$.

(2) If $H \in S'_G \cup C_G \cup C'_G$ and $H \subseteq G'$, then $H \in S'_G \cup C_G \cup C'_G$.

Proof. (1) If $x \in V(H)$, then $\deg_G(x) = \deg_{G'}(x)$ since $N_G[H] \subseteq V(G')$. Consequently, if $H \in C_G \cup C'_G$, then $H \subseteq S'_G \cup C_G \cup C'_G$. Now, suppose $H \in S'_G \setminus S'_G$, then by Remark 2.3 there is $S_1 \in \mathcal{F}(\Delta_G)$ such that $H \subseteq N_G(S_1)$. Hence, if $A = S_1 \cap N_G(H)$, then $H \subseteq N_G(A)$ and $A \in \Delta_G$. Furthermore, $A \subseteq G'$, since $N_G[H] \subseteq V(G')$, a contradiction by Remark 2.3, since $H \in S'_G$, and $A \in \Delta_G$. Therefore $H \in S'_G$.

(2) If $H \in C_G$, then $H \subseteq C_G$, since $\deg_G(x) \geq \deg_{G'}(x)$ for each $x \in V(H)$. Now, if $H = (x_1, x_2, x_3, x_4, x_5, x_1) \in C_G$, then we can suppose $Q = (x_1, x_5, y_1, y_2, x_1)$ is an induced cycle of $G$. If $Q \subsetneq G'$, then $H \subseteq C_G$. Also, if $Q \subseteq G'$, then $H \in C'_G$, since $\deg_G(x_1) = 2$ or $\deg_{G'}(x_1) = 2$. Now, if $H \subseteq S'_G \setminus S'_G$, then by Remark 2.3, there is $S_3 \in \Delta_G$ such that $V(H) \subseteq N_G(S_3) \subseteq N_G(S_3)$. This is a contradiction by Remark 2.3, since $H \in S'_G$. Therefore, $H \in S'_G$. 

$N_G(x_3)$ has two free vertices, $x_2$ and $x_4$, in $G_4 = G \setminus N_G[y_1, y_2]$, a contradiction by Corollary 2.9. Consequently, $\{y_1, y_2\} \notin E(G)$ and $\{y_1, x_5\}, \{y_2, x_1\} \notin E(G)$. So, $y_1 \neq y_2$ and $C = (y_1, y_2, x_5, x_1, y_1)$ is an induced cycle. Now, if $\deg_G(x_1) \geq 4$, then there is $y' \in N_G(x_1) \setminus (V(H_2) \cup \{y_1\})$. By the last argument, $\{y', y_2\} \notin E(G)$. Hence, $(x_1, y', y_2)$ is a path, a contradiction by Lemma 3.4, since $C$ is induced. Therefore, $\deg(x_1) = \deg(x_3) = 3$ and $H_2 \in C'_G$. 

(b) Now, assume $k = 4$. If $\deg_G(x_4) = 2$, then $N_G(x_3)$ has two free vertices, $x_2$ and $x_4$, in $G_5 = G \setminus N_G[y_1]$, a contradiction by Corollary 2.9, so $\deg(x_4) \geq 3$. Therefore, $\oplus$ is a 2-clique-sum. 

□
Now, we take $K' \in S_G' \setminus \{K_3\}$, then $V(K') \cap V(K_2) = \emptyset$, since $G' \in \mathcal{T}$. If there is $S \in \mathcal{F}(\Delta_G)$ such that $V(K') \cap S = \emptyset$, then $y \in S$ since $K' \in S_G'$. Since $N_G[x] \subseteq N_G[y]$ and $G$ is well-covered, $S' = (S \setminus y) \cup \{x\} \in \mathcal{F}(\Delta_G)$. Then, $S' \in \mathcal{F}(\Delta_{G'})$ and $S' \cap V(K') = \emptyset$, a contradiction. Then $K' \in S'_G$. Therefore, $S'_G = (S_G' \setminus \{K_3\}) \cup \{K_1\}$.

Now, assume $C = \{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq C_{G'}$, then $x \notin V(C)$, since $G' \in \mathcal{T}$. If $C \notin C_G$, then we can assume $\deg_G(a_1) \geq 3$, $\deg_G(a_2) \geq 3$ and $\deg_{G'}(a_1) = 2$. So, $N_G(a_1) = \{y, a_2, a_3\}$. By Lemma 3.4, $a_3, a_4 \notin N_G(\{x, y\})$, since $C$ is induced. If $\{y, a_2\} \in E(G)$, then $G_1[N_G[x]]$ and $G_1[N_G[a_1]]$ are simplexes in $G_1 = G \setminus N_G[a_4]$. But $y \in N_G[x] \cap N_G[a_1]$, a contradiction by Corollary 2.9 and Remark 2.11. Consequently, $\{y, a_2\} \notin E(G)$. Similarly, $\{y, a_5\} \notin E(G)$. We take $w \in N_{G'}(a_2) \setminus V(C)$. Suppose $w, y \notin E(G)$, then $w \notin N_G[x]$. By Lemma 3.4, $\{w, a_4\} \notin E(G)$. Thus, $G_2[N_G[x]]$ and $G_2[N_G[a_3]]$ are simplexes in $G_2 = G \setminus N_G[w, a_4]$ with $y \in N_G[x] \cap N_G[a_1]$, a contradiction by Corollary 2.9. Then $\{y, w\} \in E(G)$ and $N_{G'}(a_2) \setminus V(C) \subseteq N_G(y)$. Similarly, $N_{G'}(a_5) \setminus V(C) \subseteq N_G(y)$. Since $y \notin N_G(a_2)$, we have $\deg_{G'}(a_2) \geq 3$, so there is $\tilde{w} \in N_G(a_2) \setminus N_G(a_3) \cap V(C)$ such that $Q = (a_1, y, \tilde{w}, a_2, a_1)$ is an induced 4-cycle, since $\deg_G(a_1) = 3$. Since $C \subseteq C_{G'}$, $\deg_{G'}(a_3) = 2$ implies $\deg_G(a_3) = 2$, since $y \notin N_G(a_3)$. If $w' \in N_G(a_2) \setminus \{a_1, a_3, \tilde{w}\}$, then $(y, w', a_2)$ is a path. This is a contradiction by Lemma 3.4, since $Q$ is induced. This implies, $\deg_G(a_2) = 3$. Assume $z \in N_G(a_5) \setminus V(C)$, then $z \in N_G(y)$. So, $(a_5, z, a_2)$ or $(a_5, z, y, \tilde{w}, a_2)$ is a path if $z = \tilde{w}$ or $z \neq \tilde{w}$, respectively. This is a contradiction by Lemma 3.4, since $C$ is induced. Therefore, $\deg_G(a_5) = 2$ and $C \subseteq C_G$. Now, we assume $C^1 = \{a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq C_{G'}$, where $Q_1 = (a_1, a_5, a_6, a_7, a_4)$ is an induced 4-cycle. Thus, $\deg_{G'}(a_1) = \deg_{G'}(a_3) = 2$ and $\deg_{G'}(a_4) = \deg_{G'}(a_5) = 3$. If $\{y, a_1\} \in E(G)$, then by Lemma 3.4 $a_3, a_6 \notin N_G(y)$, since $C^1$ and $Q_1$ are induced. Consequently, $a_3, a_6 \notin N_G[y]$, implying $G_3[N_G[x]]$ and $G_3[N_G[a_1]]$ are simplexes in $G_3 = G \setminus N_G[a_3, a_6]$. But $y \in N_G[x] \cap N_G[a_1]$, a contradiction by Corollary 2.9. This implies, $\{y, a_1\} \notin E(G)$. Similarly, $\{y, a_3\} \notin E(G)$, if $\{y, a_3\} \in E(G)$, then by Lemma 3.4, $a_2 \notin N_G(y)$. Furthermore, $y, a_2 \notin N_G(a_7)$, since $Q_1$ is induced. So, $G_4[N_G[x]]$ and $G_4[N_G[a_5]]$ are simplexes in $G_4 = G \setminus N_G[a_2, a_7]$. But $y \in N_G[x] \cap N_G[a_5]$, a contradiction. So $a_5 \notin N_G[y]$. Similarly, $a_4 \notin N_G[y]$. Thus, $C^1 \subseteq C_G$. Hence, $C_G \cup C'_G = C_G \cup C_{G'}$.

This implies $S_G \cup C_G \cup C'_G = (S_G \cup C_G \cup C_{G'}) \setminus \{K_2\} \cup \{K_1\}$.
Therefore, $G \in \mathcal{T}$, since $G' \in \mathcal{T}$.

**Lemma 3.13.** Let $G = G' \oplus H_s$ be a connected well-covered theta-ring graph, where $H_s$ is a 4-cycle and $V(G') \not\subseteq V(H_s)$. If $G' \in \mathcal{T}$, then $G \in \mathcal{T}$.

**Proof.** By Lemma 3.10, we can assume $H_s = (x_1, x_2, x_3, x_4, x_1)$ with $\deg_G(x_1) \geq 3$, $\deg_G(x_4) \geq 2$ and $V(H_s) \setminus V(G') = \{x_2, x_3\}$, since $H_s$ is a 4-cycle. By Proposition 2.20, $G'$ is well-covered. Furthermore by Proposition 2.6, $K_1 = G[[x_2, x_3]] \in S'_G$. We take $H = (S'_G \cup C_G \cup C_G') \setminus \{K_1\}$. Suppose $V(H) \cap V(K_1) \neq \emptyset$, then by Proposition 2.16, $H \in S'_G$. We can assume $x_3 \in V(H)$ and $x_1 \in V(H)$, since $H \neq K_1$. Now, we take $S = G[[x_1, x_2]]$, since $\{x_1, x_2\} \notin E(G)$ and $\deg_G(x_2) = 2$. There is $a \in N_G(x_1) \setminus \{x_2, x_4\}$, since $\deg_G(x_1) \geq 3$. This is a contradiction by Proposition 2.6, since $\{a, x_3\} \notin E(G)$. Hence, $V(H) \cap V(K_1) = \emptyset$ and $H \subseteq G'$. Therefore, by Proposition 3.11, $(S'_G \cup C_G \cup C_G') \setminus \{K_1\} \subseteq S'_G \cup C_G' \cup C_G'$.

Now, suppose $K \in S'_G$ with $S_2 \cap V(K) = \emptyset$ for some $S_2 \in F(\Delta_G)$. Then $S_2 \cap \{x_2, x_3\} \neq \emptyset$ since $K \in S'_G$. We can assume $x_2 \in S_2$. We take $S_3 \in F(\Delta_G)$ such that $S'_2 = S_2 \setminus x_2 \subseteq S_3$, then $S_3 \subseteq S_2 \cup \{x_1\}$, since $N_G(x_2) = \{x_1, x_3\}$. Then $S_3 \cap V(K) = \{x_1\}$, since $K \in S'_G$. This implies, $S_3 = S_2' \cup \{x_1\}$ and $N_G(x_1) \cap S_3' = \emptyset$. So, $x_4 \notin S_3'$ and $x_3 \notin N_G(S_3')$. Thus, $N_G(x_2)$ has two free vertices, $x_1$ and $x_3$, where $G' = G \setminus N_G(S_3')$, a contradiction, by Corollary 2.9 and Remark 2.11. Hence, $K \in S'_G$. Therefore, $S'_G = S'_G \cup \{K_1\}$.

Now, we take $C = (w_1, w_2, w_3, w_4, w_5, w_1) \in C_G' \cup C_G'$. If $\{x_1, x_4\} \cap V(C) = \emptyset$, then by Proposition 3.11, $C \subseteq C_G \cup C_G'$. So, we can assume $x_1 = w_1$. Now, suppose $x_4 \notin \{w_2, w_5\}$, then $x_4 \notin V(C)$, since $C$ is induced. Thus, $\deg_{G'}(w_i) = \deg_G(w_i)$ for $i = 2, \ldots, 5$ and $4 \leq \deg_G(x_1) = \deg_G(x_4) + 1$, since $N_G[x_2, x_3] = V(H_s)$. If $\deg_G(w_2) = \deg_G(w_3) = 2$, then $C \subseteq C_G \cup C_G'$. Hence, we can assume $\deg_G(w_5) > 2$. Consequently, $C \subseteq C_G'$, $\deg_G(x_1) = \deg_G(w_5) = 3$ and $(x_1, x_4, b, w_5, x_1)$ is a 4-cycle, where $b \in V(G') \setminus V(C)$. By Lemma 3.4, $\{w_3, b\} \notin E(G)$, since $C$ is induced. This implies, $N_G(x_2)$ has two free vertices, $x_1$ and $x_3$ in $G' = G \setminus N_G[w_3, b]$, a contradiction by Corollary 2.9 and Remark 2.11. Now, we assume $x_4 = w_2$. Suppose $\deg_{G'}(w_1) > 3$, then there is $y \in N_G(w_1) \setminus V(C)$. Lemma 3.4 implies $\{y, w_1\} \notin E(G)$. Also, $\deg_G(w_2) = 2$ or $\deg_G(w_2) = 3$. If $\deg_G(w_2) = 3$, then $C \subseteq C_G'$ and $(x_1, x_4, y_1, y, x_1)$ is a 4-cycle with $y_1 \in V(G') \setminus V(C)$. So, in both cases, $N_G(x_3)$ has two free vertices, $x_2$.
and \( w_2 \), in \( G_2 = G \setminus N_G[y, w_2] \), a contradiction. Then \( \deg_G(w_1) = 3 \). Similarly, \( \deg_G(w_2) = 3 \). Now, if there is \( y' \in N_{G'}(w_3) \setminus V(C) \), then by Lemma 3.4, \( \{y', w_3\} \notin E(G) \). Consequently, \( N_{G_4}(x_2) \) has two free vertices, \( w_1 \) and \( x_3 \), in \( G_4 = G \setminus N_G[y', w_3] \), a contradiction. Thus \( \deg_G(w_3) = 2 \). Similarly, \( \deg_G(w_3) = 2 \). This implies \( C \in C_G \). Hence, \( C_G \cup C_G' = C_G \cup C_G' \).

Therefore, \( G \in \mathcal{T} \), since \( G' \in \mathcal{T} \).

**Definition 3.14.** A vertex \( v \) is shedding if for each stable set \( S \subseteq V(G) \setminus N_G[v] \), there is \( x \in N_G(v) \) such that \( S \cup \{x\} \in \Delta_G \setminus v \).

**Remark 3.15.** If \( N_G(x) \subseteq N_G(S) \) where \( x \notin S \subseteq \Delta_G \), then \( x \) is not a shedding vertex.

**Proof.** If \( y \in N_G(x) \), then \( y \in N_G(S) \). So \( S \cup \{y\} \notin \Delta_G \setminus x \) and \( y \notin S \), since \( S \subseteq \Delta_G \). Thus, \( S \subseteq V(G) \setminus N_G[x] \) since \( x \notin S \). Therefore, \( x \) is not shedding, since \( S \cup \{y\} \notin \Delta_G \setminus x \) for each \( y \in N_G(x) \).

**Remark 3.16.** ([5], Corollary 13) If \( G \) is well-covered and \( v \) is shedding, then \( G \setminus v \) is well-covered.

**Remark 3.17.** If \( v, w \in V(G) \) such that \( N_G[w] \subseteq N_G[v] \), then \( v \) is a shedding vertex of \( G \) ([17], Lemma 6). In particular, if \( w \) is a simplicial vertex, then any \( v \in N_G(w) \) is a shedding vertex ([17], Corollary 7).

**Lemma 3.18.** Let \( G \) be a theta-ring graph and \( C = (y, x_1, x_2, x_3, x_4, y) \in C_G \cup C_G' \). So, \( y \) is shedding if and only if \( \deg_G(x_1) = \deg_G(x_4) = 2 \).

**Proof.** \( \Rightarrow \) We suppose \( \deg_G(x_1) \geq 3 \), then there is \( z \in N_G(x_1) \setminus V(C) \). By Lemma 3.4, \( \{z, x_3\} \notin E(G) \). If \( \deg_G(y) = 2 \), then \( N_G(y) = \{x_1, x_4\} \subseteq N_G(z, x_3) \). A contradiction by Remark 3.15, then \( \deg_G(y) = 3 \). Thus, \( C \in C_G \) and \( (z, x_1, y, w, z) \) is an induced 4-cycle where \( w \in N_G(y) \setminus V(C) \). So, \( N_G(y) \subseteq N_G(z, x_3) \), a contradiction by Remark 3.15. Hence, \( \deg_G(x_1) = 2 \). Similarly, \( \deg_G(x_4) = 2 \).

\( \Leftarrow \) We take \( S \in \Delta_G \setminus N_G[y] \), then \( x_2 \notin S \) or \( x_3 \notin S \) since \( \{x_2, x_3\} \in E(G) \). We can suppose \( x_2 \notin S \), then \( S \cup \{x_1\} \in \Delta_G \setminus y \), since \( \deg_G(x_1) = 2 \). Hence, \( y \) is a shedding vertex.

**Theorem 3.19.** Let \( G \) be a connected theta-ring graph. \( G \) is well-covered if and only if \( G \in \{C_4, C_5\} \cup \mathcal{T} \).
Proof. \( \Rightarrow \) By induction on \(|V(G)|\). By Proposition 3.8, \( G = H_1 \oplus H_2 \oplus \cdots \oplus H_n \) where \( H_i \) is a cycle or a complete graph. We can assume \( H_s \not\subseteq H_1 \oplus H_2 \oplus \cdots \oplus H_{s-1} \).

If \( H_s \) is complete, then \( G \) has a simplicial vertex \( x \in V(H_s) \). If \( y \in N_G(x) \), then by Remark 3.17, \( y \) is a shedding vertex. Thus, by Remark 3.16, \( G' = G \setminus y \) is well-covered. Furthermore, by Remark 3.3, \( G' \) is theta-ring. We take a connected component \( G_i' \) of \( G' \).

By induction hypothesis, \( G_i' \in \{C_4, C_7\} \cup T \). If \( G_i' \in \{C_4, C_7\} \), then we can assume \( G_i' = (a_1, a_2, \ldots, a_l) \) and \( \{a_4, y\} \in E(G) \) with \( l \in \{4, 7\} \).

We take \( G'' = G \setminus N_G[a_3] \) if \( l = 4 \) or \( G'' = G \setminus N_G[a_3, a_4] \) if \( l = 7 \). By Lemma 3.4, \( y \in V(G'') \). Furthermore, \( x \not\in V(G_i') \), since \( x \) is simplicial. So, \( G[N_G[x]] \in S_{G''} \) and \( y \in N_G[x] \cap V(H) \), where \( H = G[y, a_1] \). This is a contradiction by Corollary 2.9, since \( a_1 \not\in N_G[x] \). Consequently, \( G_i' \in T \), implies \( G' \in T \). Therefore, by Lemma 3.12, \( G \in T \).

Now, assume \( H_s \) is a cycle. If \( s = 1 \), then by Remark 2.1 \( G \in \{C_3, C_4, C_5, C_7\} \) and \( C_3, C_5 \in T \). Now, assume \( s \geq 2 \), then by Lemma 3.10, \( H_s = (x_1, x_2, \ldots, x_k, x_1) \) with \( k \in \{4, 5\} \) (since, if \( k = 3 \), then \( H_s \) is complete). We can suppose \( \deg_G(x_i) = 2 \) for \( i \in \{2, \ldots, k-1\} \) and there is \( y_1 \in V(G) \setminus V(H_s) \) such that \( \{x_1, y_1\} \in E(G) \). First, assume \( k = 5 \). If \( \deg_G(x_3) = 2 \), then \( H_s \in C_G \). Furthermore, by Lemma 3.10, if \( \deg_G(x_5) \geq 3 \), then \( H_s \in C_G \). By Lemma 3.18, \( x_3 \) is a shedding vertex, since \( \deg_G(x_2) = \deg_G(x_4) = 2 \). By Remark 3.16, \( G_1 = G \setminus x_3 \) is a connected well-covered theta-ring graph. If \( K_1 = G_1[\{x_1, x_2\}] \) and \( K_2 = G_1[\{x_4, x_5\}] \), then \( K_1, K_2 \in S_G \), since \( \deg_G(x_2) = \deg_G(x_1) = 1 \). Hence, by induction hypothesis, \( G_1 \in T \).

So, if \( H' \in S_G' \cup C_G \cup C'_{G_1} \setminus \{K_1, K_2\} \), then \( V(H') \cap V(K_i) = \emptyset \) for \( i \in \{1, 2\} \). Furthermore, by Proposition 3.11, \( H' \in S_G' \cup C_G \cup C'_{G_1} \setminus \{H_s\} \), since \( x_3 \not\in N_G[H] \). Now, if \( H'' \in S_G' \cup C_G \cup C'_{G_1} \setminus \{H_s\} \), then \( H'' \subseteq G_1 \), since \( \deg_G(x_3) = 2 \). By Proposition 3.11, \( H'' \in S_G' \cup C_G \cup C'_{G_1} \setminus \{K_1, K_2\} \), since \( K_1, K_2 \notin S_G \). This implies \( S_G' = S_G' \setminus \{K_1, K_2\} \) and \( C_G \cup C'_{G_1} = C_G \cup C'_{G_1} \cup \{H_s\} \). Therefore, \( G \in T \).

Now, assume \( k = 4 \), then \( \oplus_{i=1}^{s-1} \) is a 2-clique-sum from Lemma 3.10. If \( G_2 = G \setminus \{x_2, x_3\} \), then \( G_2 = H_1 \oplus H_2 \oplus \cdots \oplus H_{s-1} \). Thus, \( G_2 \) is a connected theta-ring graph. Now, we take \( S \in F(\Delta G_2) \). Consequently, \(|\{x_1, x_4\} \cap S| \leq 1 \), implies \( S \cup \{x_2\} \in F(\Delta G) \) or \( S \cup \{x_3\} \in F(\Delta G) \). Since \( |S \cup \{x_2\}| = \beta(G) \) or \( |S \cup \{x_3\}| = \beta(G) \), since \( G \) is well-covered. Then, \( |S| = \beta(G) - 1 \) implying \( G_2 \) is well-covered. Hence, by induction, \( G_2 \in \{C_4, C_7\} \cup T \). If \( G_2 \in \{C_4, C_7\} \), then \( G = G_2 \oplus H_s = H_s \oplus G_2 \).
Then, by Lemma 3.10, $G_2 \simeq C_4$ and $G_2 = (x_1, y_1, y_2, x_4, x_1)$. But $N_{G_3}(x_3)$ has two free vertices, $x_2$ and $x_4$, in $G_3 = G \setminus N_G[y_1]$, a contradiction, by Corollary 2.9. Thus, $G_2 \in T$. Therefore, by Lemma 3.13, $G \in T$.

$\Leftarrow$ By Remark 2.1 and Proposition 2.20, $G$ is well-covered. □

**Remark 3.20.** A graph $G$ is well-covered if and only if each connected component of $G$ is well-covered. Hence, by Theorem 3.19, a theta-ring graph $G$ is well-covered if and only if each connected component of $G$ is contained in $\{C_4, C_7\} \cup T$.

**Remark 3.21.** The graphs in the Figures 1 and 3 are well-covered theta-ring graphs and the graph in the Figure 4 is a theta-ring graph but it is not well-covered.

**Remark 3.22.** By Corollary 2.15 in [9], ring graph are closed under induction subgraphs. Also, by Theorem 3.8 in [8], the complete graph with four vertices is not a ring graph. Hence, a complete graph with $n$ vertices is a ring graph if and only if $n \leq 3$.

**Remark 3.23.** ([8], Remark 3.3) $G$ is a ring graph if and only if $G$ can be constructed by 0,1,2-clique-sums of cycles, edges and vertices. Consequently, ring graphs are theta-rings

**Corollary 3.24.** A ring graph $G$ is well-covered if and only if each connected component of $G$ is contained in $\{C_4, C_7\} \cup T'$, where $T' = \{G \in T \mid |V(H)| \leq 3 \text{ for each } H \in S'_G\}$.

Proof. $\Rightarrow$ If $H$ is a connected component of $G$, then $H \in \{C_4, C_7\} \cup T$, by Remark 3.20. Therefore, $G \in \{C_4, C_7\} \cup T'$ from Remark 3.22.

$\Leftarrow$ By Remark 3.20. □


**Definition 4.1.** A graph $G$ is vertex decomposable if $G$ is a totally disconnected graph or there is a shedding vertex $v$, such that $G \setminus v$ and $G \setminus N_G[v]$ are both vertex decomposable.

**Remark 4.2.** ([3], [15], [16]) The following properties: shellable, Cohen-Macaulay and vertex decomposable are closed under taking c-minors.
Definition 4.3. Let $K$ be a sun-complete subgraph of $G$. An induced 4-cycle $C = (y, a, a', y', y)$ is a $K$-4-cycle if $V(C) \cap V(K) = \{y, y', y\}$.

Proposition 4.4. Let $G$ be a theta-ring graph with $K \in S_G$. If $C = (y_1, z_1, z_2, y_2, y_1)$ is a $K$-4-cycle with $y_1, y_2, y_2 \in V(K)$ and $K' = K \setminus y_1$, then $K' = K \setminus N_G[z_1] \in S_G$, where $G' = G \setminus N_G[z_1]$. Also,

(a): $\deg_G(y_2) = \deg_G(y_2) + 2$ and $\deg_G(y) = \deg_G(y) + 1$ for $y \in V(K) \setminus \{y_1, y_2\}$.

(b): The simplicial vertex of $K'$ is $y_2$ if $k \notin S_G$ and $K' \in S_G$.

Proof. If $\tilde{y} \in N_G[z_1] \cap V(K) \setminus \{y_1\}$, then $\tilde{y} \neq y_2$ and $(z_1, \tilde{y}, y_2)$ is a path, since $C$ is induced, a contradiction by Lemma 3.4. Hence, $K \setminus N_G[z_1] = K'$. This implies, by Lemma 2.12, $K' \in S_G$.

(a) Now, we take $y \in V(K) \setminus y_1$. If $z \in N_G(z_1) \setminus \{y_1, z_2\}$ and $\{y, z\} \in E(G)$, then $(z_1, z, y, y_2)$ is a path, a contradiction by Lemma 3.4. Thus, $(z_1) \subseteq N_G(y) \cap N_G(z_1) \subseteq \{z_2, y_1\}$. Since $(z_2, y_2) \in E(G)$, $\deg_G(y_2) = \deg_G(y_2) + 2$. Now, assume $y \neq y_2$. If $(z_2, y) \in E(G)$, then $(z_2, y, y_1)$ is a path, a contradiction. Then $\deg_G(y) = \deg_G(y) + 1$.

(b) Now, assume $K' = N_G[a] \in S_G$ and $K \notin S_G$, then $\deg_G(a) = |V(K')| - 1$. If $y_2 \neq a$, then $\deg_G(a) = \deg_G(a) + 1 = |V(K)| - 1$. This is a contradiction, since $K \notin S_G$. Therefore, $a = y_2$. □

Definition 4.5. A sun-complete subgraph $K$ of $G$ is a $C_4$-sun-complete if $x$ is in an induced 4-cycle and $\deg_G(x) = |V(K)|$ for each $x \in V(K)$.

Remark 4.6. In Definition 4.5, if $x \in V(C)$ and $C = (x, y_1, y_2, y_3, x)$ is an induced 4-cycle, then $(y_1, y_1) \in E(G)$, since $C$ is induced. Thus, $y_1 \notin V(K)$ or $y_3 \notin V(K)$. We can assume $y_1 \notin V(K)$. Hence, $y_3 \in V(K)$, since $\deg_G(x) = |V(K)|$. Furthermore, $(x, y_2) \notin E(G)$, since $C$ is induced. Therefore, $V(C) \cap V(K) = \{x, y_3\}$ implying $C$ is a $K$-4-cycle.

Lemma 4.7. Let $G$ be a theta-ring graph with a complete subgraph $K$ and $B \in \Delta_G$ such that $N_G(z) \cap V(K) \neq \emptyset$ for each $z \in B$. Furthermore, $K' = K \cap G_1 \neq \emptyset$ is a $C_4$-sun-complete of $G_1$, where $G_1$ is a connected induced subgraph of $G' = G \setminus G_B$. If $x \in V(G_1) \setminus V(K')$ with $\deg_G(x) = \deg_G(x)$, then $\deg_G(x) = \deg_G(x)$.

Proof. Since $x \in V(G_1) \setminus V(K')$, there is a minimal path $P$ from $x$ to $K'$ in $G_1$. We can assume $V(P) \cap V(K') = \{y\}$. Since $K'$ is a $C_4$-sun-complete of $G_1$, by Remark 4.6, there is a $K$-4-cycle $C = (y, z_1, z_2, y', y)$
with $V(K') \cap V(C) = \{y,y'\}$ and $\deg_{G_1}(y) = \deg_{G_1}(y') = |V(K')|$. So, $z_1 \in V(P)$, since $V(P) \cap V(K') = \{y\}$. Consequently, $P' = P \setminus y$ is a path between $z_1$ and $x$. If $z_2 \in P$, then there is a path $P''$ from $x$ to $z_2$ such that $P'' \subseteq P \setminus \{z_1,y\}$. Thus, $P'' \cup (z_2,y')$ is a path from $x$ to $K'$. This is a contradiction, since $P$ is minimal. Hence, $z_2 \notin P$. By contradiction suppose $\deg_{G}(x) \neq \deg_{G_1}(x)$, then there is $w \in B$ and $w' \in N_G(w) \cap N_G(x)$, since $\deg_{G}(x) = \deg_{G_1}(x)$. By hypothesis there is $y \in N_G(w) \cap V(K)$. We take $P_1 = P' \cup (x,w',w,y,y')$ if $w' \neq y$ or $P_1 = P' \cup (x,y,y')$ if $w' = y$. Then, $V(P_1) \cap V(C) = \{z_1,y'\}$, since $w',w,y \in N_G[B]$, a contradiction by Lemma 3.4. Therefore, $\deg_{G}(x) = \deg_{G_1}(x)$.

\[\square\]

**Lemma 4.8.** Let $G$ be a theta-ring graph. If $K \in S_G \setminus S_G$, then $K$ has at least one $K$-4-cycle and $G$ satisfies one of the following conditions:

1. there is a $K$-4-cycle $C = (y,x,x',y')$ with $y,y' \in V(K)$ such that $\deg_G(y) = |V(K)|$, $\deg_G(y') > |V(K)|$ and if $z \in N_G(y') \setminus (V(K) \cup \{x\})$, then there is a path $P$ from $z$ to $K \setminus y'$ with $V(P) \cap V(C) = \emptyset$, or

2. there is a stable set $L$ of $G$ such that $K' = K \setminus N_G[L] \neq \emptyset$ is a $C_4$-sun-complete in $G' = G \setminus N_G[L]$. Furthermore, if $G_1'$ is the connected component of $G'$ such that $K' \subseteq G_1'$, then $\deg_{G_1'}(y) = \deg_{G}(y)$ for each $y \in V(G_1') \setminus V(K')$.

**Proof.** By induction on $|V(K)|$. We have $N_G(\bar{w}) \cap (V(G) \setminus V(K)) \neq \emptyset$ for each $\bar{w} \in V(K)$, since $K \notin S_G$. So, there is a minimal set $D \subseteq V(G) \setminus V(K)$ such that $V(K) \subseteq N_G(D)$. By Remark 2.3, $D \notin \Delta_G$ since $K \in S_G$. Then, there are $x_1,x_1' \in D$ with $\{x_1,x_1'\} \in E(G)$. Since $D$ is minimal, there are $y_1,y_2 \in V(K)$ such that $y_1 \in N_G(x_1) \setminus N_G(x_1')$ and $y_2 \in N_G(x_1') \setminus N_G(x_1)$. Thus, $(y_1,x_1,x_1',y_2,y_1)$ is a $K$-4-cycle. If $E_4$ denotes the set of the $K$-4-cycles, then $E_4 \neq \emptyset$. We take

$$A = \left\{ C \in E_4 \mid \begin{array}{c} \text{If } C = (y,x,x',y',y) \text{ with } y,y' \in V(K), \\
\text{then } \deg_G(y) = \deg_G(y') = |V(K)| \end{array} \right\}.$$ 

First, assume $A = E_4$. We take $A_1 = \{ y \in V(K) \mid \text{there is } C \in E_4 \text{ such that } y \in V(C) \}$ and $A_2 = V(K) \setminus A_1$. Since $K \notin S_G$, there is a minimal set $B \subseteq V(G) \setminus V(K)$ such that $A_2 \subseteq N_G(B)$. If $B \notin \Delta_G$, then there is $(\bar{y}_1, \bar{x}_1, \bar{x}_2, \bar{y}_2, \bar{y}_1) \in E_4$ with $\bar{x}_1, \bar{x}_2 \in B$ and $\bar{y}_1, \bar{y}_2 \in A_2$, since $B$ is minimal, a contradiction, since $\bar{y}_1 \notin A_1$. This implies $B \in \Delta_G$. Now, we take $C^1 \in A$. Suppose $z \in B \cap V(C^1)$, then $C^1 = (y,z,z',y',y)$
where \( y, y' \in A_1 \), since \( B \cap V(K) = \emptyset \). Also, there is \( y'' \in A_2 \) such that \( \{ y'', z \} \in E(G) \). This is a contradiction by Lemma 3.4, since \( \{ z, y'', y' \} \) is a path. So, \( B \cap V(C^1) = \emptyset \). Now, suppose \( w \in N_G(B) \cap V(C^1) \), then there are \( b \in B \) and \( a_3 \in A_2 \) such that \( w, a_3 \in N_G(b) \). If \( w \in A_1 \), then \( b \in V(C^1) \) since \( \deg_G(w) = \lvert V(K) \rvert \), a contradiction, since \( B \cap V(C^1) = \emptyset \). Then, \( w \notin A_1 \) and \( C^2 = (a_1, w, w', a_2, a_1) \) where \( a_1, a_2 \in A_1 \). This is a contradiction by Lemma 3.4, since \( \{ w, b, a_3, a_2 \} \) is a path. Hence, \( N_G(B) \cap V(C) = \emptyset \) for each \( C \in A \). Thus, \( K = K \setminus N_G(B) \neq \emptyset \) is a \( C_4 \)-sun-complete in \( G = G \setminus N_G(B) \), since \( V(K) = A_1 \). We take \( \tilde{G}_1 \) the connected component of \( \tilde{G} \) such that \( \tilde{K} \subseteq \tilde{G}_1 \). If \( \tilde{z} \in V(G_1) \setminus V(K) \), then \( \deg_{\tilde{G}_1}(\tilde{z}) = \deg_G(\tilde{z}) \) by Lemma 4.7, since \( \deg_{\tilde{G}_1}(\tilde{z}) = \deg_G(\tilde{z}) \). Therefore, \( G \) satisfies (2).

Now, we assume \( A \neq C_4 \), then there is a \( K \)-4-cycle \( C^1 = (y_1, x_1, x'_1, y_2, y_1) \) with \( y_1, y_2 \in V(K) \) and \( \deg_G(y_2) > \lvert V(K) \rvert \). We take \( G_1 = G \setminus N_G[x_1] \) and \( K_1 = K \setminus y_1 \). By Remark 3.3 and Proposition 4.4, \( G_1 \) is theta-ring and \( K_1 \subseteq S_{G_1} \). If \( K_1 \subseteq S_{G_1} \), then by Proposition 4.4, \( y_2 \) is the simplicial vertex of \( K_1 \) and \( \deg_G(y_2) = \deg_{G_1}(y_2) + 2 = |V(K)| - 1 + 2 = |V(K)| \), a contradiction. Then \( K_1 \notin S_{G_1} \). By induction hypothesis, \( K_1 \) satisfies (1) or (2) in \( G_1 \). If \( K_1 \) satisfies (2) in \( G_1 \), then there is a stable set \( \mathcal{L}' \) of \( G_1 \) such that \( K' = K_1 \setminus N_{G_1}[\mathcal{L}'] \neq \emptyset \) is a \( C_4 \)-sun-complete in \( G' = G_1 \setminus N_{G_1}[\mathcal{L}'] \). Also, \( \deg_{G'}(w) = \deg_{G_1}(w) \) for each \( w \in V(G') \setminus V(K') \) where \( G' \) is the connected component of \( G' \) such that \( K' \subseteq G' \). So, \( G = G \setminus N_G[\mathcal{L}] \) and \( K' = K \setminus N_G[\mathcal{L}] \) where \( \mathcal{L} = \mathcal{L}' \cup \{ x_1 \} \). By Lemma 4.7 \( \deg_{G_1}(w) = \deg_{G'}(w) \), since \( \{ x_1 \} \) is a stable set and \( N_G(x_1) \cap V(K) \neq \emptyset \). Hence, \( K \) satisfies (2). Now, suppose \( K_1 \) satisfies (1) in \( G_1 \), then there is a \( K_1 \)-4-cycle \( C^2 = (q, p, p', q', q) \) with \( q, q' \in V(K) \) such that \( \deg_{G_1}(q) = |V(K)| \) and \( \deg_{G_1}(q') > |V(K)| \). By Proposition 4.4, \( \deg_{G_1}(q') > |V(K)| \). If \( q \neq y_2 \), then by Proposition 4.4 \( \deg_{G_1}(q) = \deg_{G_1}(q) + 1 = |V(K)| + 1 = |V(K)| \). If \( C^2 \) does not satisfy (1) in \( G \), then there is \( z \in N_G(q') \setminus (V(K) \cup \{ p' \}) \) such that \( z \notin N_G(x_1) \), since \( C^2 \) satisfies (1) in \( G_1 \). Thus, \( (x_1, z, q', y_2) \) if \( q' \neq y_2 \) or \( (x_1, z, y_2) \) if \( q' = y_2 \) is a path. This is a contradiction, by Lemma 3.4, since \( C^2 \) is induced, then \( K \) satisfies (1). Consequently, we can assume \( q = y_2 \) and \( \deg_G(y_2) = \deg_{G_1}(y_2) + 2 = |V(K)| + 1 \). We take \( x_2 = p \), \( x'_2 = p' \) and \( y_3 = q' \), then \( V(C^1) \cap V(C^2) = \{ y_2 \} \), since \( C_2 \subseteq G_1 \).

Since \( V(K) \) is finite, there is a maximal \( r \) such that \( C^1, \ldots, C^r = (y_1, x_1, x'_1, y_1, y_2, \ldots, y_r, x_r, x'_r) \) are \( K \)-4-cycles such that \( y_1, \ldots, y_r+1, x_1, x'_1, \ldots, x_r, x'_r \) are different vertices, \( \deg_G(y_2) = \cdots = \deg_G(y_r) = |V(K)| + 1 \).
and \( \deg_G(y_{r+1}) > |V(K)| \), where \( y_1, \ldots, y_{r+1} \in V(K) \). By a similar argument over \( C^1 \) (now over \( C^r \)). We can assume there is a \( K \)-cycle \( C^{r+1} = (y_{r+1}, x_{r+1}, x'_{r+1}, u', y_{r+1}) \) with \( V(C^r) \cap V(C^{r+1}) = \{y_{r+1}\} \), \( u' \in V(K) \), \( \deg_G(u') \geq |V(K)| \) and \( \deg_G(y_{r+1}) = |V(K)| + 1 \), since in other case \( K \) satisfies (1) or (2). By Lemma 3.4, \( \{x_{r+1}, x'_{r+1}\} \cap \{x_1, x'_1, \ldots, x_{r-1}, x'_{r-1}\} = \emptyset \), since \( C^1, \ldots, C^{r-1} \) are induced. Hence, \( u' \notin \{y_2, \ldots, y_r\} \), since \( \deg_G(y_2) = \cdots = \deg_G(y_r) = |V(K)| + 1 \). Thus, \( u' = y_1 \), since \( r \) is maximal. By Lemma 3.4, \( B' = \{x_1, x'_1, \ldots, x_{r+1}\} \in \Delta_G \), since \( C^1, \ldots, C^r \) are induced. By Remark 2.3, \( V(K) \notin N_G(B') \), so \( K_H = K \setminus N_G(B') \neq \emptyset \). We take \( H = G \setminus N_G(B') \).

By Proposition 4.4, if \( x \in V(K_H) \), then \( \deg_G(x) = \deg_{G_{l}}(x) + 1 \) where \( G_l = G \setminus N_G[x] \) implies \( N_G[x] \cap N_G[x_i] = \{y_i\} \) for each \( 1 \leq i \leq r + 1 \). Consequently, \( N_G(x) = N_H(x) \cup \{y_1, \ldots, y_{r+1}\} \) and \( \deg_H(x) = \deg_G(x) - (r + 1) > |V(K)| - (r + 1) = |V(K_H)| \), since \( K \notin S_G \). This implies, \( K_H \notin S_{H} \). Furthermore, by Remark 3.3 and Proposition 4.4, \( K_H \in S_{H} \) and \( H \) is a theta-ring graph. By induction hypothesis, \( K_H \) satisfies (1) or (2) in \( H \). Assume \( C' = (c, d', e', c) \) is a \( K_H \)-4-cycle with \( c, c' \in V(K_H) \) and \( \deg_{H}(c) = |V(K_H)| \) such that \( C' \) satisfies (1) in \( H \). Therefore, \( C' \) is a \( K \)-4-cycle satisfying (1) in \( G \), since \( \deg_G(c) = \deg_H(c) + (r + 1) = |V(K)| + (r + 1) = |V(K)| \) and \( N_G(c') \setminus V(K) = (N_H(c') \setminus \{y_1, \ldots, y_{r+1}\}) \setminus V(K) = N_H(c') \setminus V(H) \).

Now, if there is \( \mathcal{L}^{*} \in \Delta_H \) such that \( K_H \subseteq K_H \setminus N_H[\mathcal{L}^{*}] \neq \emptyset \) satisfies (2) in \( H' = H \setminus N_H[\mathcal{L}^{*}] \). So, if \( w \in V(H_1) \setminus V(K') \), then \( \deg_{H}(w) = \deg_{H_1}(w) \), where \( H_1 \) is the connected component of \( H' \) such that \( K_H \subseteq H_1 \). Also, \( K_H = K \setminus N_G[B''] \) and \( H' = G \setminus N_G[B''] \) where \( B' = B' \cup \mathcal{L}^{*} \), then by Lemma 4.7 \( \deg_{H}(w) = \deg_{H_1}(w) \), since \( N_G(x_i) \cap V(K) \neq \emptyset \) for each \( x_i \in B' \). Therefore, \( K \) satisfies (2). 

**Remark 4.9.** Let \( G = (w_1, w_2, \ldots, w_r, w_1) \) be a cycle. If \( S_G' \neq \emptyset \), then \( G = C_4 \). In this case, each sun-complete graph is a \( C_4 \)-sun complete.

**Proof.** Suppose \( S \in S_G' \), so \( |V(S)| = 2 \), since \( K_3 \notin G \). We can assume \( S = G[(w_1, w_2)] \), then by Proposition 2.6 \( \{w_3, w_r\} \in E(G) \). Thus, \( r = 4 \) since \( G \) is a cycle. Now, if \( G = C_4 \) and \( e \in E(G) \), then by Proposition 2.6 \( e \in S_G' \). Also, if \( x \in e \), then \( \deg_S(x) = 2 = |e| \). Hence, \( e \) is a \( C_4 \)-sun complete.

**Lemma 4.10.** Let \( G \) be a well-covered theta-ring graph with a connected c-minor \( G' \) of \( G \). If \( S_G' \) is a partition of \( V(G) \), then \( G' = C_4 \) or \( S_{G'}' \) is a partition of \( V(G') \) and \( C_{G'} \cup C_{G'}' = \emptyset \).
Proof. We have $G' = G \setminus N_G[B]$, where $B \in \Delta_G$. If $K \in S'_G$ and $V(G) \cap B = \emptyset$, then by Lemma 2.12, $K \setminus N_G[B] \in S'_G$. Thus, $V(G') = V(S'_G)$, since $S'_G$ is a partition of $V(G)$. By Remark 2.11 and Remark 3.3, $G'$ is well-covered theta-ring. Consequently, by Theorem 3.19 and Remark 4.9, $G' \in \{C_4\} \cup T$. If $G' \in T$, then $S'_G$ is a partition of $V(G')$ and $C_{G'} \cup C_{G'} = \emptyset$. \hfill $\Box$

Definition 4.11. Let $G$ be a connected well-covered theta-ring graph such that each sun-complete subgraph is a $C_4$-sun-complete. We say $G$ is in $N'$, if $G = C_4$ or $S'_G$ is a partition of $V(G)$.

Remark 4.12. If $G \in N'$, then $\deg_G(x) = |V(K)|$ for each $x \in V(K)$ and $K \in S'_G$. Hence, $S_G = \emptyset$. Also, by Theorem 3.19 and Remark 4.9, $G \in \{C_4, y, x, x' \}$ when $S_G = \emptyset$. We can assume $S'_G \setminus S_G$ is a partition of $V(G)$. We take $K \in S'_G \setminus S_G$, then $K$ satisfies (1) or (2) of Lemma 4.8.

First, suppose there is a $K$-4-cycle $C = (y, x, x' y', y)$ with $\deg_G(y) = |V(K)|$, $\deg_G(y') > |V(K)|$ and $y, y' \in V(K)$ such that $C$ satisfies (1) of Lemma 4.8. We take $G_1 = G \setminus N_G[x)$, $K_1 = K \setminus N_G[x]$ and $G'_1$ the connected component of $G_1$ such that $K_1 \subseteq G'_1$. If $G'_1 = C_4$, then $G$ has a c-minor of $N'$. Thus, by Lemma 4.10, we can suppose $S'_G \setminus S_G$ is a partition of $V(G'_1)$ and $S'_G \cup S'_G = \emptyset$. By Remark 4.9, $G'_1 \neq C_4$. Also, by Proposition 4.4, $K_1 = K \setminus y \in S'_G$. Furthermore, by (b) in Proposition 4.4 $K_1 \notin S'_G$, since $\deg_{G_1}(y') > |V(K)|$. We take $w \in V(G'_1) \setminus V(K_1)$. Suppose there is $w' \in N_G(w) \cap N_G(x)$. If $w' \in V(K)$, then $w' = y$ since $K_1 \subseteq G'_1$. This implies $w \in N_G(y)$, a contradiction since $\deg_{G_1}(y) = |V(K)|$. So, $w' \notin V(K)$. Since $G'_1$ is connected, there is a minimal path $P$ from $w$ to $K_1$ in $G'_1$. Then $N_G[x] \cap V(P) = \emptyset$. We can assume $V(P) \cap V(K_1) = \{y''\}$. If $w' \neq x'$, then $V(P_1) \cap V(C) = \{x, y'\}$ where $P_1 = (x, w', w, P, y', y')$ when $y' \neq y'$ or $P_1 = (x, w', P, y', y')$ when $y'' = y'$. This is a contradiction by Lemma 3.4, since $C$ is induced. Hence, $w' = x'$. If $y'' \neq y'$, then $V(P_2) \cap V(C) = \{x', y\}$, where $P_2 = (x', w, P, y'', y)$, a contradiction.
by Lemma 3.4. Then $y'' = y'$. Consequently, there is $b \in N_{G_1}(y')$ such that $P_3 = P_1 \setminus y'$ is a path from $w$ to $b$. Since $C$ satisfies (1) from Lemma 4.8, there is a path $P''$ from $b$ to $K \setminus y'$ such that $V(P'') \cap V(C) = \emptyset$. Thus, there is a path $P_4$ from $x'$ to $y$ such that $V(P_4) \subseteq \{x', y\} \cup V(P_3) \cup V(P'')$ and $V(P_4) \cap V(C) = \{x', y\}$, a contradiction by Lemma 3.4. So, $deg_G(w) = deg_{G_1}(w)$. Hence $S_{G_1} = \emptyset$, since $S_G = \emptyset$. This implies, $S_{G_1} \cup C_{G_1} \cup C_{G_1}' = \emptyset$ and by induction hypothesis $G_1'$ has a c-minor in $N''$, since $G_1' \neq C_7$. Therefore, $G$ has a c-minor in $N''$.

Now, we assume $K$ satisfies (2) of Lemma 4.8, thus there is a stable set $L_K$ of $G$ such that $K_2 = K \setminus N_G[L_K] \neq \emptyset$ is a $C_4$-sun-complete in $G_2 = G \setminus N_G[L_K]$. Let $G_2'$ be the connected component of $G_2$ such that $K_2 \subseteq G_2'$. If $G_2' = C_4$, then $G$ has a c-minor in $N''$. Consequently, by Lemma 4.10, we can suppose $S_{G_1}'$ is a partition of $V(G_2')$ and $C_{G_2}' \cup C_{G_2}' = \emptyset$. Then by Remark 4.9, $G_2' \neq C_7$. Since $K$ satisfies (2), $N_G[w] = N_{G_2}[w]$ for each $w \in V(G_2') \setminus V(K_2)$. So $S_{G_2}' = \emptyset$, since $S_G = \emptyset$. If $L_{K} \neq \emptyset$, then by induction hypothesis $G_2'$ has a c-minor in $N''$. This implies, $G$ has a c-minor in $N''$. Now, we can assume $K$ satisfies (2) of Lemma 4.8 and $L_K = \emptyset$ for each $K \in S_G \setminus S_G$, then $K$ is a $C_4$-sun-complete. Therefore, $G$ is in $N''$. □

**Definition 4.15.** If $\Delta$ is a simplicial complex with $d = \dim(\Delta)$ and $f_i = |\{ F \in \Delta \mid |F| = i + 1 \}|$, then the $f$-vector of $\Delta$ is the $(d+1)$-tuple $f(\Delta) = (f_0, \ldots, f_d)$. Note $f_{-1} = 1$. If $\Delta = \Delta_G$, then $f(\Delta)$ is denoted by $f(G)$.

**Proposition 4.16.** ([14], pg 58) Let $\Delta$ be a simplicial complex with $f$-vector $f(\Delta) = (f_0, \ldots, f_d)$. The $h$-vector of $\Delta$ is $h(\Delta) = (h_1, \ldots, h_{d+1})$, where $h_k = 0$ for $k > d + 1$ and

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{k-i} \quad \text{for} \quad 0 \leq k \leq d + 1.$$

**Theorem 4.17.** ([16], Theorem 6.7.7) If $\Delta$ is Cohen-Macaulay, then $h_k(\Delta) \geq 0$ for $0 \leq k \leq \dim(\Delta) + 1$.

**Lemma 4.18.** If $G \in N''$, $K \in S_G'$ and $x \in V(K)$, then there is a unique $K$-4-cycle $Q$ such that $x \in V(Q)$. Hence, $|V(K)|$ is even.

*Proof.* Suppose $Q = (x, z, z', x')$ and $Q' = (x, z_1, z_2, x, x')$ are $K$-4-cycles with $x', x'' \in V(K)$. Thus $z = z_1$, since $deg_G(x) = |V(K)|$. If
\[ x' = x'', \text{ then } z' = z'', \text{ since } \deg_G(x') = |V(K)|, \text{ a contradiction, since } Q \neq Q'. \]

Consequently, \( x' \neq x'' \). We take \( P = (z', x'', x) \) if \( z'' = z' \) or \( P = (z, z'', x'', x') \) if \( z'' \neq z' \). This is a contradiction by Lemma 3.4, since \( Q \) is induced. Hence, \( x \) is exactly in one \( K-4 \)-cycle, since \( K \) is a \( C_4 \)-sun-complete. Therefore, \(|V(K)|\) is even. \( \square \)

**Lemma 4.19.** If \( K \) is a complete subgraph of \( G \) with a \( K-4 \)-cycle \( Q = (x, y, y', x', x) \), where \( x, x' \in V(K) \) and \( \deg_G(x) = \deg_G(x') = |V(K)| \), then \( K \in \mathcal{S}'_G \).

**Proof.** If \( S \in \mathcal{F}(\Delta_G) \) with \( S \cap V(K) = \emptyset \), then \( N_G(x) \cap S \neq \emptyset \) and \( N_G(x') \cap S \neq \emptyset \). So, \( y, y' \in S \), since \( \deg_G(x) = \deg_G(x') = |V(K)| \), a contradiction, since \( \{y, y'\} \in E(G) \). Therefore, \( K \in \mathcal{S}'_G \). \( \square \)

**Proposition 4.20.** If \( G \in \mathcal{N}' \), then \( h_{\beta(G)}(\Delta_G) = -1 \).

**Proof.** By induction on \(|V(G)|\). By Proposition 3.8, we can assume \( G = H_1 \oplus \cdots \oplus H_s \) such that \( H_s \not\subseteq G' = H_1 \oplus \cdots \oplus H_{s-1} \) where \( H_i \) is a complete graph or a cycle. By Remark 4.12 \( S_G = \emptyset \), then \( H_s = \emptyset \). If \( s = 1 \), then by Remark 4.9, \( G = C_4 \). In this case, \( f_1(\Delta_G) = 2, f_0(\Delta_G) = 4 \) and \( f_{-1}(\Delta_G) = 1 \), therefore \( h_2(\Delta_G) = (-1)^2(1) + (-1)^1(4) + (-1)^0(2) = -1 \). Now, assume \( s \geq 2 \). By Remark 4.12, \( S_G \subseteq C_G \). Also, \( \mathcal{S}'_G = \emptyset \). Then, by Lemma 3.10, we can suppose \( H_s = (a, b, c, d, e) \) with \( \deg_G(a) = \deg_G(b) = 2, \deg_G(e) \geq 3 \) and \( \deg_G(d) \geq 3 \). Furthermore, \( c \in V(K) \) for some \( K \in \mathcal{S}'_G \), since \( \mathcal{S}'_G \) is a partition of \( V(G) \). From Proposition 2.6, \( H = G'[a, b] \in \mathcal{S}'_G \), so \( b \notin V(K) \). Also, \(|V(K)| = \deg_G(c) \geq 3 \), then \( d \in V(K) \) and \( K \neq G'[c, d] \). Consequently, \( G' = G' \setminus \{a, b\} \). We take \( q = \dim(\Delta_G) \), then \( \beta(G) = q + 1 \) and \( |S'_G| = q + 1 \), since \( G \in \mathcal{N}' \). Also, \( S_G \subseteq \{H\} \) is a partition of \( V(G') \), then \( \beta(G') = q \). If \( S \in \mathcal{F}(\Delta_G) \), then \( S' = S \setminus V(H) \subseteq \Delta_G \) and \( |S'| = q \), so \( \beta(G') = q \). We take \( G'' = G' \setminus \{c, d\} \). By Lemma 4.18, \(|V(K')|\) is even, then \(|V(K')| \geq 2 \) where \( K' = K \setminus \{c, d\} \) since \(|V(K')| \geq 3 \). We take \( V(K) = \{c, d, x_1, x_1', x_2, x_2', \ldots, x_{r-1}, x_{r-1}'\} \) such that \( \deg_G(x_i) = \deg_G(x_i') = |V(K)| \) and \( x_i, x_i' \in V(Q_i) \) where \( Q_i \) is a \( K-4 \)-cycle. Since \( \deg_G(a) = \deg_G(b) = 2 \) and \( \deg_G(c) = \deg_G(d) = |V(K)| \), we have that \( \deg_G(x_i) = \deg_G(x_i') = |V(K')| \) and \( Q_i \subseteq G'' \). Hence, \( K' \subseteq \mathcal{S}'_G \), by Lemma 4.19. If \( \hat{H} \in \mathcal{S}'_G \setminus \{K, H\} \), then \( \hat{H} \subseteq G'' \). Thus, by Proposition 3.11, \( \hat{H} \in \mathcal{S}'_G \). This implies, \( \mathcal{S}'_G \supseteq (\mathcal{S}'_G \cup \{K'\}) \setminus \{K, H\} \). Also if \( H'' \subseteq \mathcal{S}'_G \setminus \mathcal{S}'_G \), then \( N_G[H'' \cap V(H_s) \neq \emptyset \). Consequently, \( H'' \subseteq K' \), since \( N_G(H_s \setminus V(H_s)) \subseteq V(K') \). If \( y' \in V(K') \setminus V(H'') \), then \( V(H'') \subseteq N_G(x'), \) a contradiction by Remark 2.3. Then \( H'' = K' \). So
We calculate $h_q(G')$. We take $S \in \Delta_G$ such that $|S| = j + 1 \leq q$, then $c \in S$, $d \in S$ or $c, d \notin S$. Thus, $f_j(G') = f_{j-1}(G_1) + f_{j-1}(G_2) + f_j(G'')$ where $G_1 = G' \setminus N_G[c]$, and $G_2 = G' \setminus N_G[d]$. Furthermore, $G_1 = G' \setminus K = G_2$, since $N_G[c] = K = N_G[d]$. Also, $\beta(G_1) = \beta(G_2) = q - 1$, since $S_G \setminus \{H,K\}$ is a partition of $V(G_1) = V(G_2)$. Thus, $h_q(G') = \sum_{i=0}^q (-1)^q f_i(G')$, since dim$(\Delta_G) = \beta(G') - 1 = q - 1$. So, $h_q(G') = (-1)^q + \sum_{i=1}^q (-1)^q G f_{i-1}(G'') - i(2 f_{i-2}(G_1) + f_{i-1}(G'')) = (-1)^q + 2 \sum_{i=0}^q (-1)^q f_{i-1}(G_1) + \sum_{i=0}^q (-1)^q f_{i-1}(G'') + (-1)^q f_{i-1}(G'')$ where $q' = q - 1$. This implies, $h_q(G') = 2 h_{q-1}(G_1) + h_q(G'').$

Now, we calculate $h_{q+1}(G)$. We have $G \setminus N_G[a,c] = G_1 = G \setminus N_G[b,d]$ and $G \setminus (N_G[a] \cup \{c\}) = G'' = G \setminus (N_G[b] \cup \{d\})$. We take $S_1 \in \Delta_G$ such that $|S_1| = j + 1$. If $a, c \in S_1$, then $S_1 \setminus \{a,c\} \in \Delta_{G_1}$. If $a, c \notin S_1$, then $S_1 \setminus \{a,c\} \in \Delta_{G''}$. If $b, d \in S_1$, then $S_1 \setminus \{b,d\} \in \Delta_{G_1}$. If $b, d \notin S_1$, then $S_1 \setminus \{b,d\} \in \Delta_{G''}$. Hence, $f_j(G) = 2 f_{j-2}(G_1) + 2 f_{j-2}(G'') + f_j(G')$ by $1 \leq j \leq q$. Since dim$(\Delta_G) = q$, $h_{q+1}(G) = \sum_{i=0}^{q+1} (-1)^{q+1-i} f_{i-1}(G)$, then $h_{q+1}(G) = (-1)^{q+1} + (-1)^q f_0(G) + \sum_{i=2}^q (-1)^{q+1-i}(2 f_{i-3}(G_1) + 2 f_{i-2}(G'') + f_i(G')) + 2(f_{q-2}(G) + f_q(G''))$. Consequently, $h_{q+1}(G) = (-1)^{q+1} + (-1)^q f_0(G) + 2 \sum_{i=0}^{q} (-1)^{q+1-i} f_{i-1}(G_1) + 2 \sum_{i=0}^{q} (-1)^{q+1-i} f_{i-1}(G'') - 2(-1)^q f_{i-1}(G'') - \sum_{i=0}^{q} (-1)^{q+1-i} f_{i-1}(G') + (-1)^q f_{i-1}(G') + (-1)^q f_0(G')$. So, $h_{q+1}(G) = 2 h_{q-1}(G_1) + 2 h_q(G'') - h_q(G)$, since $f_0(G) = f_0(G) + 2$. Hence, $h_{q+1}(G) = h_q(G'')$, since $h_{q+1}(G) = 2 h_{q-1}(G_1) + h_q(G'')$. By induction hypothesis, $h_q(G'') = -1$ since $G'' \in \mathcal{N}^\prime$. Therefore, $h_{q+1}(G) = -1$.

Remark 4.21. If $G \in \mathcal{N}^\prime$, then by Theorem 4.17 and Proposition 4.20, $G$ is not Cohen-Macaulay, since $\beta(G) = \dim(\Delta_G) + 1$.

Lemma 4.22. Let $G$ be a theta-ring graph such that $G \in \mathcal{T}$. If $y$ is a shedding vertex with $y \in V(G \cup C_G)$ or $y \in N_G(x)$ with $G[N_G(x)] \in \mathcal{S}_G$, then $G \setminus y \notin \mathcal{T}$.

Proof. By Proposition 2.20, $G$ is well-covered, since $G \in \mathcal{T}$. Furthermore, $G' = G \setminus y$ is an induced subgraph of $G$. Then, by Remark 3.3 and Remark 3.16, $G'$ is well-covered theta-ring, since $y$ is a shedding vertex. We take a connected component $H$ of $G'$, then
Let $G$ be a theta-ring graphs. The following conditions are equivalent:

(i) $G$ is well-covered vertex decomposable.

(ii) $\Delta_G$ is pure shellable.

(iii) $R/I(G)$ is Cohen-Macaulay.

(iv) $G \in \mathcal{T}$ and $G$ does not have a c-minor in $N'$.

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii) It is known ([14], [16], [17]).

(iii)$\Rightarrow$(iv) $G$ is well-covered, then by Remark 3.20, each connected component of $G$ is in $\{C_4, C_7\} \cup \mathcal{T}$. Since $C_4$ and $C_7$ are not Cohen-Macaulay, we have $G \in \mathcal{T}$. Furthermore, by Remark 4.2 and Remark 4.21, $G$ does not have a c-minor in $N'$.

(iv)$\Rightarrow$(i) By induction on $|V(G)|$. By Proposition 2.20, $G$ is well-covered and each connected component of $G$ is in $\mathcal{T}$. By Theorem 4.14, there is $H \in S_G \cup C_G \cup C''_G$, since $C_7 \notin \mathcal{T}$. By Remark 3.17 and Lemma 3.18, $G$ has a shedding vertex $y \in V(H)$. By Lemma
4.22. $G_1 = G \setminus y \in T$. Now, assume $G_2 = G \setminus N_{G_1}(A) \in N'$ for some $A \in \Delta_G$. If $N_G(y) \cap A \neq \emptyset$, then $G \setminus N_G[A] = G_2 \in N'$, a contradiction. Then $N_G(y) \cap A = \emptyset$. If $H = G[N_G[x]] \in S_G$, then $N_G[x] \cap A \subseteq N_G[y] \cap A = \emptyset$, since $y \notin A$. So, $x \notin N_G[A]$ and $G_2[N_G[x]] \in S_{G_2}$, a contradiction, by Remark 4.12. Consequently, $H = (y, x_1, x_2, x_3, x_4, y) \in C_G \cup C'_G$ and $\{x_1, x_4\} \cap A \subseteq N_G(y) \cap A = \emptyset$. By Lemma 3.18, $\deg_G(x_1) = \deg_G(x_4) = 2$. Furthermore, $x_2 \notin A$ or $x_3 \notin A$, since $\{x_2, x_3\} \in E(G)$. We can assume $x_2 \notin A$, then $x_1 \in V(G_2)$ and $G_2[N_{G_2}[x_1]] \in S_{G_2}$, a contradiction, by Remark 4.12. Hence, $G_1$ does not have a $c$-minor in $N'$. Therefore, by induction hypothesis, $G_1$ is well-covered vertex decomposable.

Now, if $G_3 = G \setminus B$ where $B = N_G[y]$, then $G_3$ does not have a $c$-minor in $N'$. By Remark 2.11 and Remark 3.3, $G_3$ is well-covered theta-ring. If $H_1$ is a connected component of $G_3$, then by Remark 3.20, $H_1 \in \{C_4, C_7\} \cup T$. Furthermore, $H_1 \neq C_4$, since $H_1$ is a $c$-minor of $G_3$ and $C_4 \in N'$. Now, if $H_1 = C_7$, then $H_1$ is not a connected component of $G$, since $G \in T$. This implies, $N_G(B) \cap V(H_1) \neq \emptyset$. If $b_1, b_2 \in N_G(B) \cap V(H_1)$ with $b_1 \neq b_2$, then there are $c_1 \in N_G(y) \cap N_G(b_1)$ and $c_2 \in N_G(y) \cap N_G(b_2)$. Thus, $L = (b_1, c_1, b_2)$ if $c_1 = c_2$ or $L = (b_1, c_1, y, c_2, b_2)$ if $c_1 \neq c_2$ is a path. By Lemma 3.4, $\{b_1, b_2\} \in E(H_1)$. Consequently, $1 \leq |V(H_1) \cap N_G[B]| \leq 2$. If $G'$ is the connected component of $G$ such that $H_1 \subseteq G'$, then there is $G''$ such that $G'' = G'' \oplus H_1$, a contradiction by Lemma 3.10. Then $H_1 \in T$. Hence, $G_3 \in T$. By induction hypothesis, $G_3$ is well-covered vertex decomposable. Therefore, $G$ is well-covered vertex decomposable, since $y$ is a shedding vertex, $G_1 = G \setminus y$ and $G_3 = G \setminus N_G[y]$ are vertex decomposable and $G$ is well-covered.

\[\square\]

**Remark 4.24.** The graph in Figure 1 is a Cohen-Macaulay theta-ring graph.

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