1. Introduction

In this note, by the term “group” we will mean an abelian $p$-group, where $p$ is a fixed prime. Except where specifically noted, the notation and terminology will be standard and follow [6] or [7].

For the group $G$, $|x|$ will denote the height of $x \in G$; so $|x|$ will be an ordinal or the symbol $\infty$. Therefore, $p^0G = \{x \in G : |x| \geq \alpha\}$, and $G$ is separable if $p^0G = 0$. If $X$ is a subgroup of $G$, then the restriction of the height function of $G$ to $X$ is sometimes called a valuation on the subgroup. And if $\alpha$ is an ordinal, then $X(\alpha) := \{x \in X : |x| \geq \alpha\}$.

The height sequence of $x \in G$ is $||x|| := (|x|, |px|, |p^2x|, \ldots)$. Clearly, if $\mu : G \to H$ is a homomorphism, then for every $x \in G$ we have $||x|| \leq ||\mu(x)||$ (using the natural coordinate-wise ordering of these sequences). Using language that goes back to Kaplansky’s famous “little red book” ([16]), $G$ is said to be fully transitive if, whenever $x, y \in G$ satisfy $||x|| \leq ||y||$, there is an endomorphism $\phi : G \to G$ such that $\phi(x) = y$. Kaplansky showed that the separable groups, as well as the countable groups, are fully transitive.

The subgroup $T \subseteq G$ is said to be fully invariant if for every endomorphism $\phi : G \to G$ we have $\phi(T) \subseteq T$. For example, if $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots)$ is a strictly increasing sequence of ordinals (where we agree that $\infty < \infty$), then $G(\overline{\alpha}) := \{x \in G : ||x|| \geq \overline{\alpha}\}$ is clearly fully invariant. If $G$ is fully transitive (e.g., separable or countable), then every fully invariant subgroup is of this form.

Perhaps the most remarkable class of groups consists of the so-called totally projective groups. Not only do they include the direct sums of countable reduced groups, but they can be characterized in multiple ways (e.g., as the simply presented groups, as the balanced-projective groups and as the groups with a nice composition series - see [6], Chapter XII). In a justifiably famous result, Paul Hill showed that the totally projective groups can be completely classified by cardinal invariants, the so-called Ulm countably totally projective abelian $p$-groups have minimal full inertia

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ABSTRACT. A new class of abelian $p$-groups is introduced, the countably totally projective groups, that contains the well-known class of totally projective groups. A countably totally projective group is shown to have the property that every fully inert subgroup is commensurable with a fully invariant subgroup. This generalizes results of Goldsmith, Salce and Zanardo (2014), who proved that a direct sum of cyclic $p$-groups has this property. It also answers affirmatively two questions recently posed in the literature.
We will say the reduced group $G$ is fully transitive, so that any fully invariant subgroup of $G$ will again be of the form $G(\overline{\gamma})$ (see [12]).

A notion related to the above, that of fully inert subgroups, came from an apparently unrelated source, the study of algebraic entropy (see [4], [9], et al), which is arguably the most significant addition to the theory of abelian groups of the last decade.

To describe this notion we review some basic ideas and terminology. The subgroups $B, C$ of $G$ are said to be commensurable, written $B \sim C$, if both $[B + C]/B \cong C/(B \cap C)$ and $[B + C]/C \cong B/(B \cap C)$ are finite. It is easy to verify that $\sim$ is an equivalence relation. The subgroup $X \subseteq G$ is said to be fully inert if for every endomorphism $\phi : G \to G$ we have $\phi(X) + X \sim X$, i.e., $[\phi(X) + X]/X$ is finite.

It is elementary that if $T$ is fully invariant in $G$ and $X \sim T$, then $X$ is fully inert. Conversely, if $G$ has the property that for every fully inert subgroup $X \subseteq G$ there is a fully invariant subgroup $T \subseteq G$ such that $X \sim T$, then we say $G$ has minimal full inertia. Again, this concept was initially introduced as a result of a study of the concept of algebraic entropy, even though, perhaps paradoxically, that more general theory is not really needed to understand the idea, or to appreciate its significance.

To date, there are relatively few groups that have been shown to have minimal full inertia. In [11] it was shown that a direct sums of cyclic groups has minimal full inertia. Later, this result was extended when it was verified that the torsion-complete groups also have this property ([10]). It is perhaps surprising that a simple direct sum of a torsion-complete group and a direct sum of cyclics may fail to have minimal full inertia (see [8]).

The purpose of this note is to describe a class of groups that properly contains the totally projective groups, and to show that the members of this class have minimal full inertia.

**Definition 1.1.** We will say the reduced group $G$ is countably totally projective if whenever $C \subseteq G$ is a countable subgroup, then there is a totally projective group $H$ and a homomorphism $\mu : G \to H$ such that $\mu$ restricts to a (valuated) embedding on $C$ (i.e., $|x|_G = |\mu(x)|_H$ for all $x \in C$).

Our principal objective is to prove the following:

**Theorem 1.2.** Every countably totally projective group has minimal full inertia.

The class of countably totally projective groups is quite large, as illustrated by the following.

**Proposition 1.3.** Let $\mathcal{C}$ be the class of countably totally projective groups.

(a) Every totally projective group is in $\mathcal{C}$.

(b) $\mathcal{C}$ is closed under isotype subgroups.

(c) $\mathcal{C}$ is closed under direct sums.

(d) If $\lambda$ is a limit ordinal of uncountable cofinality, $p^\lambda G = 0$ and for every $\gamma < \lambda$, $G/p^\gamma G \in \mathcal{C}$ (e.g., each $G/p^\gamma G$ is totally projective), then $G \in \mathcal{C}$.

(e) If $G$ is a reduced group and every countable subgroup $C \subseteq G$ is contained in a summand $G_C \in \mathcal{C}$, then $G$ itself is in $\mathcal{C}$.

(f) If $G \in \mathcal{C}$, then $G$ is fully transitive. Consequently, every fully invariant subgroup of $G$ is of the form $G(\overline{\gamma})$.

**Proof.** Throughout, we let $C$ be a countable subgroup of $G$. If $G$ is totally projective, then (a) follows by letting $H = G$ and $\mu$ be the identity map.
For (b), if $G'$ is an isotype subgroup of $G \in \mathcal{C}$ and $C \subseteq G'$, then there is a totally projective group $H$ and a homomorphism $\mu : G \to H$ that restricts to an embedding on $C$. Restricting $\mu$ to $G'$ shows that $G'$ too, is countably totally projective.

For (c), suppose $G := \bigoplus_{i \in I} G_i$, where each $G_i \in \mathcal{C}$. For each $i \in I$, let $\pi_i : G \to G_i$ be the usual projection. For each $i \in I$, it follows that there is a totally projective group $H_i$ and a homomorphism $\mu_i : G_i \to H_i$ that restricts to an embedding on $\pi_i(C)$. Let $\mu : G \to H := \bigoplus_{i \in I} H_i$ be the sum of these $\mu_i$. So $H$ is totally projective and it is readily checked that $\mu$ restricts to an embedding on $C$.

As to (d), choose $\gamma$ large enough so that $p^\gamma G \cap C = 0$. If $\pi : G \to G/p^\gamma G$ is the canonical surjection, then $\pi$ restricts to an embedding on $C$. By hypothesis there is a totally projective group $H$ and a homomorphism $\mu' : G/p^\gamma G \to H$ that restricts to an embedding on $\pi(C)$. Letting $\mu = \mu' \circ \pi$ gives the result.

Turning to (e), if $G_C \in \mathcal{C}$ is a summand containing $C$, then we know there is a homomorphism $\mu : G_C \to H$ that restricts to an embedding on $C$. Letting $\mu$ equal 0 on a complementary summand of $G_C$ gives the result.

Finally, as to (f), suppose $x, y \in G$ with $\|x\| \leq \|y\|$. If we let $C = \langle x \rangle$, then there is a totally projective group $H$ and a homomorphism $\mu : G \to H$ that restricts to an embedding on $C$. However, since $\|\mu(x)\| = \|x\| \leq \|y\|$, it follows easily from the theory of totally projective groups that there is a homomorphism $\rho : H \to G$ such that $\rho(\mu(x)) = y$. Letting $\phi = \rho \circ \mu : G \to G$, we have $\phi(x) = y$, as desired. □

So, not only is every totally projective group also countably totally projective, but by (b), so is every IT-group ($G$ is an IT-group if it is isomorphic to an isotype subgroup of a totally projective group - see, for example, [14]). This means that members of such well-known classes of IT-groups as the S-groups and the A-groups will be countably totally projective. Also, by (d), if $G$ is a $p^{\alpha}$-bounded $C_{o_1}$-group (i.e., $G/p^\alpha G$ is totally projective for all $\alpha < o_1$), then $G$ is again countably totally projective. There are many examples of groups satisfying this last condition that fail to be IT-groups (such as the groups constructed in [3] and [13]).

Notice as well that if $G$ is reduced and has the property that every countable subgroup is contained in a countable summand, then since every reduced countable group is totally projective, $G$ satisfies (e), and so is countably totally projective. In particular, if $G$ is also separable, then this summand will be a direct sum of cyclic groups. In [17] these groups were termed sufficiently projective, and so the following immediate Corollary of Theorem 1.2 affirmatively answers the problem posed at the end of Section 4 of [8].

**Corollary 1.4.** Every sufficiently projective group has minimal full inertia.

In fact, the class of separable groups with the above property has long been of interest in abelian groups, especially to investigators using set-theoretic techniques. In particular, such groups may be far from being direct sums of cyclic groups themselves (see, for example, [5]).

As another application of our results, in [1], the group $G$ was said to be fully inert socle-regular if for every fully inert subgroup $X \subseteq G$ there is an ordinal $\alpha$ such that $X[p] \sim (p^\alpha G)[p]$. It was asked (Problem 2) if every totally projective group has this property. It follows from our results that this, in fact, holds for the larger class of countably totally projective groups (Corollary 2.9).
The observation in (f) actually implies that a countably totally projective group is, in the language of [2], *universally fully transitive*. As was noted in that work, this means that $G$ will be isomorphic to an isotype subgroup of the torsion subgroup of the direct product of a collection of totally projective groups. On the other hand, an unbounded torsion-complete group will be universally fully transitive and of minimal full inertia, but it is readily checked that such a group will never be countably totally projective. In fact, any homomorphism from a torsion-complete group to a totally projective group is necessarily small (i.e., its kernel contains an unbounded fully invariant subgroup).

Outlining our exposition, Section 2 starts with a quick review of standard facts regarding commensurable and fully inert subgroups. Then, given a fully inert subgroup $X$ of a countably totally projective group $G$, we construct a fully invariant subgroup $A$ such that $X[p^n] \sim A[p^n]$ for all $n < \omega$. In Section 3 we show that for a sufficiently large $n$ there is a strictly increasing sequence of ordinals $\overline{p}$ such that $p^nX = G(\overline{p})$. Combining $A$ and $\overline{p}$ allows us to construct the required fully invariant subgroup $T$ such that $X \sim T$.

### 2. Preliminaries and $p^n$-Socles

We begin by reviewing some well known consequences of these definitions.

**Lemma 2.1.** Suppose $L$ is a group, $B, C$ and $Y$ are subgroups of $L$ and $B \sim C$. Then the following hold:

(a) $B \cap Y \sim C \cap Y$.

(b) If $\alpha$ is some ordinal, then $B(\alpha) \sim C(\alpha)$.

(c) If $k < \omega$, then $B[p^k] \sim C[p^k]$ and $p^kB \sim p^kC$.

(d) For some $k < \omega$ we have $p^kB = p^kC$.

**Proof.** It is straightforward to verify (a). For (b) and half of (c), just use (a) with $Y = p^\alpha L$ and $Y = L[p^k]$, respectively. If $F_1$ and $F_2$ are finite subgroups of $L$ with $B + F_1 = B + C = C + F_2$, then the other half of (c) together with (d) follows from $p^kB + p^kB_1 = p^kB + p^kC = p^kC + p^kF_2$. $\square$

Recall that if $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots)$ is a strictly increasing sequence of ordinals and $k < \omega$, then $p^k\overline{\alpha} := (\alpha_k, \alpha_{k+1}, \ldots)$; it is readily checked that $p^kL(\overline{\alpha}) = L(p^k\overline{\alpha})$. With this idea, part (d) of the last result means that if $B \sim C$, then $B[p^k]$ and $C[p^k]$ may differ, but eventually, their “infinite tails” $p^kB$ and $p^kC$ must agree.

**Lemma 2.2.** Suppose $L$ is a group and $X \subseteq L$ is a fully inert subgroup.

(a) If $Y \subseteq L$ is fully invariant, then $X \cap Y$ is fully inert.

(b) If $\alpha$ is some ordinal, then $X(\alpha)$ is fully inert.

(c) If $k < \omega$, then $X[p^k]$ and $p^kX$ are fully inert.

**Proof.** Let $\phi$ be an arbitrary endomorphism of $L$, and $\hat{\phi}(X) := (\phi(X) + X)/X$.

For (a), it is easily checked that there is an injection $\hat{\phi}(X \cap Y) \rightarrow \hat{\phi}(X)$; (b) and the first half of (c) follow directly from (a).

For the rest of (c), multiplication by $p^k$ gives a surjection $\hat{\phi}(X) \rightarrow \hat{\phi}(p^kX)$ $\square$

**Lemma 2.3.** Suppose $L$ is a group and $B, C$ are subgroups of $L$.

(a) If $B \subseteq C$ and $k < \omega$, then $B \sim C$ if and only if $B[p^k] \sim C[p^k]$ and $p^kB \sim p^kC$. 

(a) $B \subseteq C$ and $k < \omega$, then $B \sim C$ if and only if $B[p^k] \sim C[p^k]$ and $p^kB \sim p^kC$. 

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Suppose first that $B \subseteq C$, then $B = C$ if and only if $(p^kC)[p] = (p^kC)[p]$ for all $k < \omega$.

**Proof.** For any $k < \omega$, multiplication by $p^k$ determines a short exact sequence

$$0 \to C[p^k]/B[p^k] \to C/B \to p^kC/p^kB \to 0.$$  

This immediately implies (a). Similarly, (b) follows by inducting on $k < n$ using the corresponding sequences

$$0 \to C[p^k]/B[p^k] \to C[p^k+1]/B[p^k+1] \to (p^kC)[p]/(p^kB)[p] \to 0.$$  

Finally, (c) follows from inducting on the same sequences showing that $B[p^n] = C[p^n]$ for all $n < \omega$. \hfill \Box

We now review some important material on Ulm factors.

For the group $G$ and ordinal $\alpha$, we fix the notation $S_{\alpha} = (p^\alpha G)[p]$ and let $f_G$ denote the Ulm function of $G$. So $f_G(\alpha)$ is the dimension of $S_{\alpha}/S_{\alpha+1}$.

We will utilize a perspective on Ulm invariants for subgroups of $G$ that goes back at least to [15]. If $X$ is a subgroup of $G$, then the $\alpha$th Ulm factor of the valued group $X$ is

$$U_\alpha(X) := \{x \in X(\alpha) : px > \alpha + 1\}/X(\alpha + 1).$$

So $U_\alpha(X)$ reflects the presence of elements of $X$ with a “jump” or “gap” in their height at $\alpha$. If $X = G$, then inclusion determines a natural isomorphism $S_{\alpha}/S_{\alpha+1} \cong U_\alpha(G)$, and it is the former quotient of socles that is the usual definition. This construction is clearly functoral, so any homomorphism $\nu : G \to L$ will induce homomorphisms $\nu_{\alpha} : U_\alpha(G) \to U_\alpha(L)$. The inclusion $X \subseteq G$ determines an injection $U_\alpha(X) \to U_\alpha(G)$, which will generally be interpreted as an inclusion. It is readily checked that the cokernel of this homomorphism is naturally isomorphic to the usual idea of the *relative* Ulm invariant of $X$ in $G$ (see [15]). Let $\mathcal{U}_G = \{\alpha : f_G(\alpha) \neq 0\}$ and $\mathcal{U}_X = \{\alpha : U_\alpha(X) \neq 0\}$.

Our proofs will be based upon the idea that if $G$ is totally projective and $L$ is some group (even non-totally projective) and we are given any collection of homomorphisms $\nu_{\alpha} : U_\alpha(G) \to U_\alpha(L)$, then we can find a homomorphism $\nu : G \to L$ which induces each of the $\nu_{\alpha}s$. In fact, if $\{N_\gamma\}_{\gamma < \kappa}$ is a nice composition series for $G$, then the usual Kaplansky-Mackey technique (Lemma 11.4.5 of [7]) can be used to successively build up valued (i.e., non-height decreasing) homomorphisms $\nu^T : N_\gamma \to L$ that induce $\nu_{\alpha}$ on each $U_\alpha(N_\gamma) \subseteq U_\alpha(G)$ (see, for example, [7], Lemma 11.4.5 and Corollary 11.4.6).

The following shows how we can describe the countably totally projective groups using Ulm factors.

**Proposition 2.4.** The reduced group $G$ is countably totally projective if and only if for every countable subgroup $C \subseteq G$, if $L$ is a group and for every ordinal $\alpha$ we have a homomorphism $\nu_\alpha : U_\alpha(C) \to U_\alpha(L)$, then there is a homomorphism $\nu : G \to L$ which induces each of the $\nu_{\alpha}$ on $U_\alpha(C) \subseteq U_\alpha(G)$.

**Proof.** Suppose first that $G$ is countably totally projective, $C \subseteq G$ is countable, and we are given $L$ and the $\nu_{\alpha}$ as above. Suppose $H$ is a totally projective group such that there is a homomorphism $\mu : G \to H$ that restricts to an embedding on $C$. It follows that $\mu$ induces injective homomorphisms $\mu_{\alpha} : U_\alpha(C) \to U_\alpha(H)$. This clearly implies that for each ordinal $\alpha$ we can define a homomorphism...
\[ \chi_\alpha : U_\alpha(H) \to U_\alpha(L) \text{ such that } \chi_\alpha \circ \mu_\alpha = \nu_\alpha. \] Since \( H \) is totally projective, the \( \chi_\alpha \) will all be induced by a single homomorphism \( \chi : H \to L \). Letting \( \nu = \chi \circ \mu \) will satisfy our requirements.

Conversely, suppose \( G \) satisfies this condition for all groups \( L \) and homomorphisms \( \nu_\alpha \). If \( C \) is a countable subgroup of \( G \), there is clearly a totally projective group \( H \) such that for each ordinal \( \alpha \) there is an injective homomorphism \( \mu_\alpha : U_\alpha(C) \to U_\alpha(H) \). By hypothesis, these \( \mu_\alpha \) will be induced by a single homomorphism \( \mu : G \to H \).

We need to show that \( \mu \) restricted to \( C \) is an embedding. If \( x \in C \), then it certainly follows that \( |\mu(x)| \geq |x| \). Suppose we can find an \( x \in C \) such that this inequality is strict. Replacing \( x \) by \( p^k x \) for some \( k \) if necessary, we may assume that \( |\mu(x)| > |x| =: \alpha \) and \( |\mu(p x)| = |px| \). It follows that \( \alpha + 1 = |x| + 1 \leq |\mu(x)| < |\mu(p x)| = |\mu(px)| \). Therefore, \( x \) represents a non-zero element of \( U_\alpha(C) \), but \( \mu(x) \) represents the 0 element of \( U_\alpha(H) \). This contradiction shows that \( \mu \) must restrict to an embedding on \( C \), as required.

For the remainder of this note, \( G \) will denote a fixed countably totally projective group of length \( \lambda \).

Lemma 2.5. Suppose \( L \) is a group, \( \xi \) is an ordinal, \( \{u_k\}_{k < \omega} \subseteq p^\omega G \) are linearly independent modulo \( p^{\xi + 1}G \). If for each \( k < \omega \) we have an element \( y_k \in (p^\xi L)[p] \), then there is a homomorphism \( \phi : G \to L \) such that \( \phi(u_k) = y_k \) for each \( k < \omega \).

Proof. Let \( C \) be a countable subgroup of \( G \) containing each \( u_k \). By hypothesis there is a totally projective group \( H \) and a homomorphism \( \mu : G \to H \) that restricts to an embedding on \( C \). Let \( \nu \) be the composition of \( \mu \) with the canonical epimorphism \( H \to \hat{H} := H/p^{\xi + 1}H \); so the images \( \nu(u_k) \) will be linearly independent elements of \( p^\xi \hat{H} \).

There is clearly a homomorphism \( \rho : p^\xi \hat{H} \to p^\xi L \) such that \( \rho(\nu(u_k)) = y_k \) for each \( k < \omega \). Since \( \hat{H} \) is totally projective, this extends to a homomorphism \( \rho : \hat{H} \to L \). Clearly, \( \phi := \rho \circ \nu \) has the desired properties. \( \square \)

Note that the last proof works equally well if there are only finitely many \( u_k \)'s.

Lemma 2.6. Suppose \( L \) is a group, \( \xi_0 < \xi_1 < \xi_2 < \cdots \) are ordinals, and for each \( k < \omega \), \( u_k \) represents a non-zero element of \( U_{\xi_k}(G) \). If for each \( k < \omega \) we have an element \( y_k \in (p^{\xi_k} L)[p] \), then there is a homomorphism \( \phi : G \to L \) and an infinite subsequence \( k_j (j < \omega) \) such that for each \( j < \omega \) we have \( \phi(u_{k_j}) = y_j \).

Proof. Again, let \( C \subseteq G \) be a countable subgroup containing all the \( u_k \)'s.

For each \( k \), let \( B_k \) be a totally projective group such that \( p^{\xi_k} B_k = \langle z_k \rangle \cong \mathbb{Z}_p \). By Proposition 2.4, there is a homomorphism \( \mu : G \to H := \bigoplus_{k < \omega} B_k \) such that for each \( k < \omega \), \( \mu_{\xi_k} \) takes the element of \( U_{\xi_k}(G) \) represented by \( u_k \) to the element of \( U_{\xi_k}(H) \) represented by \( z_k \).

We can conclude that \( \mu(u_k) \equiv z_k \pmod{p^{\xi_k + 1}H} \). In other words,

\[ \mu(u_k) = z_k + w_k \]

where \( w_k \in \bigoplus_{i < \omega} (p^{\xi_k + 1} B_i) \).\( \bigoplus_{i < \omega} (p^{\xi_k + 1} B_i) \).

Note that there is a subsequence \( \xi_{k_j} \) such that for each \( j < \omega \),

\[ w_{k_j} \in \bigoplus_{k_i < j < k_{j+1}} (p^{\xi_{k_j} + 1} B_i) \].
Every countably totally projective group is fully inert socle-regular.

If we let $\pi$ be the natural projection $H = \bigoplus_{k<\omega} B_k \to \bigoplus_{j<\omega} B_{k_j}$, then letting $\phi = \rho \circ \pi \circ \mu$ gives the result.

From here on, we will assume $X \subseteq G$ is fully inert.

Obviously, $S_\lambda/X(\lambda)[p] = 0$ is finite. Define $\alpha_X \leq \lambda$ to be the least ordinal such that $X(\alpha_X)[p] \sim S_\alpha_X$, i.e., $S_\alpha_X/X(\alpha_X)[p]$ is finite. So for all $\xi < \alpha_X$, $X(\xi)[p] \not\sim S_\xi$, i.e., $S_\xi/X(\xi)[p]$ is infinite. And for all $\xi \geq \alpha_X$, since $S_\xi/X(\xi)[p]$ naturally embeds in $S_{\alpha_X}/X(\alpha_X)[p]$, we can conclude that $S_\xi/X(\xi)[p]$ is finite. Of course, for $n < \omega$, $p^nX$ will also be fully inert and we will have occasion to refer to $\alpha_{p^nX}$ as well.

The following is a fundamental observation. It allows us to translate information from the socle of $X$ to the entire subgroup.

Lemma 2.7. $X \sim X(\alpha_X)$, i.e., $X/X(\alpha_X)$ is finite.

Proof. For simplicity, we will write $\alpha$ for $\alpha_X$. Suppose that $X/X(\alpha)$ is infinite. Our objective is to construct an infinite subset $\{u_0, u_1, \ldots\} \subseteq X$ whose members represent distinct elements of $(G/p^\alpha G)[p]$ and an endomorphism $\phi : G \to G$ such that $\{\phi(u_0), \phi(u_1), \ldots\}$ represent distinct elements of $G/X$. This contradicts that $X$ is fully inert and completes the proof.

If $X' = \{x \in X : px \in X(\alpha)\}$, then $X'/X(\alpha) := \hat{X}$ is the socle of $X/X(\alpha)$. Therefore, $\hat{X}$ will be an infinite valued vector space. Let $\mathcal{V}$ be the value spectrum of $\hat{X}$ (i.e., $\{|x| : x \in (X' \setminus X(\alpha))\}$). We divide the argument into two cases depending upon whether or not $\mathcal{V}$ is finite.

$\mathcal{V}$ is finite: It is easy to see that for some $\xi \in \mathcal{V}$, $X'(\xi)/X'(\xi + 1) \subseteq p^\xi G/p^{\xi+1}G$ is infinite. Suppose $\{u_k\}_{k<\omega} \subseteq X'(\xi)$ are linearly independent modulo $p^{\xi+1}G$. Since $\xi < \alpha = \alpha_X$, we can find $\{y_k\}_{k<\omega} \subseteq S_\xi$ that are pairwise non-congruent modulo $X$. By Lemma 2.5, there is an endomorphism $\phi : G \to G$ such that $\phi(u_k) = y_k$, which gives our forbidden endomorphism.

$\mathcal{V}$ is infinite: In this case, we can clearly choose $\{u_0, u_1, \ldots\} \subseteq X'$ such that if $\xi_k := |u_k|$, then $\xi_0 < \xi_1 < \cdots < \alpha$. Since $pu_k \in X(\alpha)$, we can conclude that $|u_k| + 1 = \xi_k + 1 \leq \xi_{k+1} < \alpha \leq |pu_k|$. It follows that $u_k$ represents a non-zero element of $U_{\xi_k}(X) \subseteq U_{\xi_k}(G)$.

We now inductively choose elements $y_k \in S_{\xi_k}$ which are all distinct modulo $X$. First, let $y_0 \in S_{\xi_0}$ be arbitrary (e.g., let $y_0 = 0$). Having chosen $y_0, \ldots, y_{k-1}$, since $S_{\xi_k}/X(\xi_k)[p]$ is infinite and naturally embeds in $G/X$, we must be able to find our next $y_k \in S_{\xi_k}$ not congruent modulo $X$ to any of the preceding elements in our list.

By Lemma 2.6, we can find an endomorphism $\phi : G \to G$ and a subsequence $k_j$ such that $\phi(u_{j_k}) = y_{j_k}$, again giving us the forbidden endomorphism.

We begin our proof of Theorem 1.2 with an immediate consequence of the above.

Corollary 2.8. If $X \subseteq G[p]$ is a subsocle of $G$, then $X \sim X(\alpha_X) \sim S_{\alpha_X}$.

Recall that in [1], the group $G$ was said to be fully inert socle-regular if for every fully inert subgroup $X \subseteq G$ there is an ordinal $\alpha$ such that $X[p] \sim (p^\alpha G)[p]$. The following gives an affirmative answer to Problem 2 of that work.

Corollary 2.9. Every countably totally projective group is fully inert socle-regular.
Proof. Since $X[p] \subseteq G[p]$ is fully inert, by Corollary 2.8 we have $X[p] \sim S_{\alpha X[p]} = S\alpha X$. \qed

We also note the following consequence of the above construction.

Corollary 2.10. If $\alpha X = \lambda$, then $X$ is finite. And if $p^\xi G$ infinite for all $\xi < \lambda$, then the converse also holds.

Proof. If $\alpha X = \lambda$, then by Lemma 2.7, $X \cong X/X(\lambda)$ is finite. And if $p^\xi G$ is infinite for all $\xi < \lambda$, then $S_\xi$ is infinite for all $\xi < \lambda$. So, if $X$ is finite, then $S_\xi \not\sim X(\xi)[p]$ for all these $\xi$, so that $\alpha X$ must be finite. \qed

Returning to our proof of Theorem 1.2, we want to extend Corollary 2.8 via induction. The next result states that for any fully inert subgroup of a countably totally projective group there is a fully invariant group, and a formula for defining it, such that the two subgroups have commensurable $p^n$-socles for all $n$.

Theorem 2.11. Define $\alpha_n$ inductively as follows: $\alpha_0 = \alpha_X$ and for $n \geq 1$, let

$$\alpha_n = \max\{\alpha_{n-1} + 1, \alpha_{p^n X}\}.$$

If $\alpha = (\alpha_0, \alpha_1, \ldots)$ and $A = G(\alpha)$, then for all $n \geq 0$ the following hold:

1. $p^n X \sim (p^n X)(\alpha_n)$ (i.e., $(p^n X)/(p^n X)(\alpha_n)$ is finite).
2. $\alpha_n \sim (p^n X)(\alpha_n)[p]$ and $(p^n X)(\alpha_n)[p]$ are finite.
3. If $\alpha_{n-1} + 1 > \alpha_n$, then $S_\xi \not\sim (p^n X)(\xi)[p]$ (i.e., $S_\xi/(p^n X)(\xi)[p]$ is infinite) for all $\xi < \alpha_n$.
4. $X \sim X_n := X \cap G(\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_{n-1} + 1, \alpha_{n-1} + 2, \ldots)$.
5. $X[p^n] \sim A[p^n] = G(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \infty, \infty, \ldots)$.

Proof. We first induct on $n$ to prove (1-3) and handle (4-5) separately. Suppose $n = 0$ or $\alpha_{n-1} + 1 < \alpha_n$, so that $\alpha_n = \alpha_{p^n X}$. Part (1) is Lemma 2.7, which also implies that $(p^n X)(\alpha_n)[p] \sim (p^n X)[p]$, which is half of (2)

The other half of (2), namely $S_{\alpha_n} \sim (p^n X)(\alpha_n)[p]$, follows from the definition of $\alpha_{p^n X}$, as does (3).

So we need only prove (1-3) when $n \geq 1$, assuming they hold for $n - 1$ and $\alpha_{p^n X} \leq \alpha_{n-1} + 1 = \alpha_n$.

Clearly, (3) holds vacuously.

Considering (1), note that

$$p((p^{n-1} X)(\alpha_{n-1})) \subseteq (p^n X)(\alpha_{n-1} + 1) = (p^n X)(\alpha_n).$$

So multiplication by $p$ determines a surjection

$$(p^{n-1} X)/(p^{n-1} X)(\alpha_{n-1}) \rightarrow (p^n X)/(p^n X)(\alpha_n).$$

By induction, the first is assumed to be finite; therefore, so is the second.

Considering (2), $(p^n X)(\alpha_n)[p] \sim (p^n X)[p]$ again follows from (1), which we just verified. And since $\alpha_{p^n X} \leq \alpha_n$, the fact that $S_{\alpha_{p^n X}} \sim (p^n X)(\alpha_{p^n X})[p]$ implies that $S_{\alpha_n} \sim (p^n X)(\alpha_n)[p]$, as required.

Turning to (4), if $n = 1$, this just says $X \sim X(\alpha)$, which is again just Lemma 2.7. We now show that for $n \geq 1$ we have $X_n \sim X_{n+1}$, which will complete the argument. Note that if $\alpha_n = \alpha_{n-1} + 1$, then $X_n = X_{n+1}$ and this follows trivially.
Suppose that $\alpha_{n-1} + 1 < \alpha_n = \alpha_{p^n}$. Consider the homomorphism $\nu$ given by the composition

$$X_n \times p^n \rightarrow p^n X_n \subseteq p^n X \rightarrow (p^n X)/(p^n X)(\alpha_{p^n} X).$$

It readily follows that $X_{n+1}$ is the kernel of $\nu$. And since by Lemma 2.7, its codomain is finite, we have $X_n \sim X_{n+1}$, as desired.

Finally, turning to (5), by (4) it will suffice to show for each $n < \omega$ that $X_n[p^n] \sim A[p^n]$. But since it is clear that $X_n[p^n] \subseteq A[p^n]$, this follows from (4), (2), Lemma 2.3(b) and the fact that for all $k < n$ we have $(p^k X_n)[p] \sim (p^k X)[p] \sim S_\alpha_k = (p^k A)[p]$. □

For the duration, $\alpha$ and $A$ will always represent the sequence of ordinals and the corresponding fully invariant subgroup defined in Theorem 2.11.

The fully invariant subgroup $A = G(\alpha)$ can be characterized as the largest such subgroup with property (5) from Theorem 2.11.

**Proposition 2.12.** If $\alpha' = (\alpha'_0, \alpha'_1, \ldots)$ and $A' := G(\alpha')$ satisfies

$$A'[p^n] \sim X[p^n] \sim A[p^n]$$

for all $n < \omega$, then $A' \subseteq A$.

**Proof.** We show by induction that $\alpha'_n \geq \alpha_n$ for all $n < \omega$. For $n = 0$, we have $S_{\alpha'_0} = A'[p] \sim X[p]$; so by the minimality of $\alpha$, we have $\alpha'_0 \geq \alpha_0 = \alpha_0$.

Assume now that $n - 1 \geq 0$ and $\alpha'_{n-1} \geq \alpha_{n-1}$. If $\alpha_{n-1} + 1 = \alpha_n$, then

$$\alpha'_n \geq \alpha'_{n-1} + 1 \geq \alpha_{n-1} + 1 = \alpha_n.$$

And if $\alpha_{n-1} + 1 < \alpha_n = \alpha_{p^n}$, then since

$$S_{\alpha'_n} = (p^n A')[p] = p^n (A'[p^{n+1}]) \sim p^n (X[p^{n+1}]) = (p^n X)[p] \sim S_{\alpha_{p^n}},$$

by the minimality of $\alpha_{p^n}$, we have $\alpha'_n \geq \alpha_n$, completing the argument. □

Though the subgroup $A$ works nicely for the $p^n$-socles of $X$, it is insufficiently precise to determine an infinite tail of $X$. As an illustration of this, we consider one particularly simple case.

**Proposition 2.13.** Suppose $G$ is an unbounded separable group and $f_G(n)$ is finite for all $n < \omega$ (i.e., $G$ is “semi-standard”).

1. If $X$ is unbounded, then $A = G$.
2. If $X$ is bounded and $n < \omega$ is the smallest integer such that $p^n X$ is finite, then $A = G[p^n]$.

**Proof.** The fact that $G$ is semi-standard implies that $S_n \sim S_0 = G[p]$ for all $n < \omega$. And Corollary 2.10 implies that $\alpha_{p^n}$ is $\omega$ exactly when $p^n X$ is finite, so that $\alpha_{p^n} X$ is $0$ exactly when $p^n X$ is infinite. This readily implies the result. □

Proposition 2.13 implies that for semi-standard separable groups, many non-commensurable fully inert subgroups will determine the same subgroup $A$. 
3. Tails

Since $X[p^n] \sim A[p^n]$, we need to focus on the tails of our height sequences.

For the rest of the paper, we will assume $X$ is unbounded.

Again, as illustrated by Proposition 2.13, we are forced to amend our construction to get a fully invariant subgroup $T \subseteq G$ such that for some $n$, not only does $X[p^n] \sim T[p^n]$, but we also have $p^nX = p^nT$.

The next result is a key step in this program. Like Lemma 2.7, it relates information regarding the socle of $X$ to the structure of the entire subgroup. It is straightforward to verify that the equivalence it describes would be true for any fully invariant subgroup.

**Lemma 3.1.** There is an $N$ such that for all $n \geq N$ and all $\alpha \in \mathcal{U}_G$, the following are equivalent:

1. $\alpha \in \mathcal{U}_{p^nX}$;
2. $S\alpha = (p^nX)(\alpha)[p]$.

**Proof.** Clearly, (2) always implies (1) and does not require that we select $N$ in some special way. The critical part of the proof, therefore, is to select $N$ in a manner so that (1) implies (2) for all $n \geq N$.

We assume this fails and derive a contradiction. This means that we can find a strictly increasing sequence $n_k$ of integers and ordinals $\xi_k \in \mathcal{U}_{p^nX}$ such that $(p^{n_k}X)(\xi_k)[p] \not\equiv S\xi_k$.

For each $k$, choose $v_k \in X$ so that $u_k := p^{n_k}v_k$ represents a non-zero element of $U\xi_k(p^{n_k}X)$. Our objective is, after again possibly replacing the original sequence with some subsequence, to find an endomorphism $\phi : G \to G$ such that $y_k := \phi(u_k) \in S\xi_k \setminus (p^{n_k}X)[p]$ for every $k$. Before constructing such a $\phi$, let us see how finding it completes the proof.

We claim that the $\phi(v_k) + X$ are all distinct elements of $G/X$, contrary to our assumption that $X$ is fully inert. Suppose otherwise, and choose $j < k$ with $\phi(v_k) - \phi(v_j) \in X$. It would follow that

$$y_k = y_j - p^{n_k-n_j}y_j = \phi(p^{n_k}v_k) - \phi(p^{n_k}v_j) = p^{n_k}(\phi(v_k) - \phi(v_j)) \in (p^{n_k}X)[p],$$

contrary to hypothesis.

To construct our endomorphism, we first observe that we can replace $n_k$ by some subsequence $n_{k_j}$ so that the $\xi_{k_j}$ are also increasing (i.e., $\xi_{k_j} \leq \xi_{k_{j+1}}$ for all $j$): Choose $k_0$ so that $\xi_{n_{k_0}} = \min\{\xi_{n_k} : k < \omega\}$. Having chosen $k_0, k_1, \ldots, k_j$, choose $k_{j+1} > k_j$ so that $\xi_{n_{k_{j+1}}} = \min\{\xi_{n_k} : k < k < \omega\}$.

A bit of thought shows this works. So we assume the original sequence has been replaced by this subsequence. We now break the argument into two parts.

Case 1 – $\{\xi_k : k < \omega\}$ is finite: Restricting to a subsequence, there is no loss of generality in assuming that for $j \neq k$ that $\xi_j = \xi_k := \xi$. And again restricting to another subsequence, we may
assume that the \( u_k \) all represent the same non-zero element of \( p^k \hat{X} / p^{k+1} G \), or else they form a linearly independent subset of this quotient.

Let \( y \in S_{\hat{X}} \setminus (p^n X)[p] \). In either of the two cases mentioned in the last paragraph, Lemma 2.5 implies that there is an endomorphism \( \phi : G \to G \) such that

\[
\phi(u_k) = y \in S_{\hat{X}} \setminus (p^n X)[p] \subseteq S_{\hat{X}} \setminus (p^n X)[p],
\]

for every \( k < \omega \), as required.

Case 2 – \( \{ \xi_k : k < \omega \} \) is infinite: Restricting to a subsequence, there is no loss of generality in assuming that \( \xi_k \) is strictly increasing.

For each \( k \), let \( y_k \in S_{\hat{X}} \setminus (p^n X)[p] \). By Lemma 2.6 there is a subsequence \( k_j \) and an endomorphism \( \phi : G \to G \) such that \( \phi(u_{k_j}) = y_{k_j} \) for each \( j \). Replacing \( \{ u_k \}_{k<\omega} \) by the subsequence \( \{ u_{k_j} \}_{j<\omega} \) gives the result.

For the rest of this note we will assume \( N \) is chosen so that the conclusion of Lemma 3.1 is valid.

We now introduce a second natural sequence of ordinals defined for \( n \geq N \) by

\[
\beta_n = \min\{ |x| : x \in (p^n X)[p] \} \in \mathcal{U}_{p^X} \subseteq \mathcal{U}_G.
\]

So \( f_G(\beta_n) \neq 0 \) and \( \beta_n \leq \beta_{n+1} \). The next statement follows easily from Lemma 3.1.

**Lemma 3.2.** For all \( n \geq N \) we have \( (p^n X)[p] = S_{\beta_n} \).

We next observe that for every \( n \geq N \) there is an \( n' > n \) such that \( \beta_{n'} > \beta_n \); Otherwise, \( p^n X \) would have the property that for every \( k < \omega \), \( (p^k (p^n X))[p] = (p^n X)[p] \). This, however, would imply that \( p^n X \) is divisible, which cannot be true.

As it stands, the sequence of \( \beta_n \)s will often not be strictly increasing, and as such, will not be totally adequate in defining a fully invariant subgroup. To address that limitation, we will develop an amended sequence, which we will label by \( \gamma_n \), that will do the trick.

We define \( \mathcal{E} = \{ n \geq N : \beta_n = \beta_{n+1} \} \) and \( \mathcal{F} = \{ n \geq N : \beta_n < \beta_{n+1} \} \). We now define a third set that will prove useful:

\[
\mathcal{M} = \{ m > N : \beta_{m-1} = \beta_m < \beta_{m+1} \}
\]

\[
= \{ m > N : (m-1 \in \mathcal{E}) \land (m \in \mathcal{F}) \} \subseteq \mathcal{F}.
\]

To say \( m \in \mathcal{M} \) means that the \( \beta_n \)s are constant for a number of values up to \( m \), where they once again start to strictly increase.

If \( m \in \mathcal{M} \), let \( \hat{m} = \min\{ i : (N \leq i \leq m) \land (\beta_i = \beta_m) \} \). Clearly, \( n \in \mathcal{E} \) if and only if \( \hat{m} \leq n < m \) for some unique \( m \in \mathcal{M} \); this \( m \) will be the smallest element of \( \{ i \in \mathcal{F} : i > n \} \) and we will have \( \beta_n = \beta_m \).

If \( m \in \mathcal{M} \), then we can find \( v_m \in X \) such that \( u_m := p^m v_m \in (p^m X)[p] \) satisfies \( |u_m| = \beta_m \); so \( u_m \) represents a non-zero element of \( U_{\beta_m}(G) \). If \( \hat{m} \leq n < m \), \( \alpha = |p^n v_m| \) and \( |p^m v_m| + 1 < |p^{n+1} v_m| \leq |p^m v_m| = \beta_m \), it would follow that \( 0 \neq U_{\alpha}(p^n X) \subseteq U_{\alpha}(G) \). However, since \( (p^n X)[p] = S_{\beta_n} = S_{\beta_m} \) has no elements of value \( \alpha < \beta_m \), this would contradict Lemma 3.1. Therefore, the sequence
There is an $N_2.6$ has no gaps; in other words, for each $\hat{m} \leq n \leq m$, that is $n = \hat{m} + k$ where $0 \leq k \leq m - \hat{m}$, we have

$$|p^{\hat{m}}v_m| = |p^{\hat{m}+k}v_m| = |p^{\hat{m}}v_m| + k = \beta_m - (m - \hat{m}) + (n - \hat{m}) = \beta_m - (m - n).$$

For the remainder of this note, we assume these $u_m$, $v_m$ for $m \in \mathcal{M}$ are fixed.

We now alter the $\beta_n$ for $n \in \mathcal{E}$ to get our $\gamma_n$. If $n \in \mathcal{F}$, let $\gamma_n = \beta_n$. And if $n \in \mathcal{E}$, and $m \in \mathcal{M}$ with $\hat{m} \leq n < m$, then let $\gamma_n = |p^n v_m| = |p^n v_m| + k = \beta_m - (m - n)$.

Our next objective is to show that the sequence $\gamma$ is eventually well behaved, and in particular, strictly increasing.

First, for all $n \in \mathcal{E}$ and $\hat{m} \leq n < m$, we have already observed that $\gamma_{n+1} = |p^{n+1} v_m| = |p^{n} v_m| + 1 = \gamma_n + 1$. On the other hand, if both $n, n + 1 \in \mathcal{F}$, then $\gamma_n = \beta_n < \beta_{n+1} = \gamma_{n+1}$. So the only way that we could have $\gamma_n \geq \gamma_{n+1}$ is if $n \in \mathcal{F}$ and $n + 1 \in \mathcal{E}$. In other words, if $\beta_n < \beta_{n+1} = \beta_{n+2}$, i.e., if there is an $m \in \mathcal{M}$ such that $n + 1 = m$.

The following variation on Lemma 3.1 accomplishes what we need.

**Lemma 3.3.** There is an $N' \geq N$ such that for all $n \geq N'$ we have

1. $n \in \mathcal{E}$ implies $f_G(\gamma_n) = 0$;
2. $\gamma_n < \gamma_{n+1}$.

**Proof.** In fact, it is (1) that is the key to the result. Before proving (1), let’s show how it implies (2).

Assume (2) fails at $n$; that is $\gamma_n \geq \gamma_{n+1}$. By our previous discussion, we must have $n \in \mathcal{F}$ and $n + 1 \in \mathcal{E}$. Choosing $m \in \mathcal{M}$ such that $\hat{m} = n + 1$, we have

$$\gamma_{n+1} \leq \gamma_n = \beta_n < \beta_{n+1} = \beta_m = \gamma_m.$$

Since $n \in \mathcal{F}$, we know $f_G(\gamma_n) = f_G(\beta_n) \neq 0$. On the other hand, (1) implies that $f_G(\gamma) = 0$ when $\gamma_{n+1} \leq \gamma < \gamma_m$. This contradiction completes the argument.

Turning to (1), we again assume it fails and derive a contradiction. This means that we can construct strictly increasing sequences $n_k, m_k$, for $k \geq 0$, such that

$$\hat{m}_k \leq n_k < m_k \text{ and } f_G(\gamma_{n_k}) \neq 0.$$ We assume that $n_k$ is chosen so that $f_G(\gamma) = 0$ whenever $\gamma_{n_k} < \gamma < \gamma_{n_k}$.

If $k > 0$, then $m_{k-1}, m_k \in \mathcal{F}$ implies $f_G(\gamma_{m_{k-1}}) \neq 0$ and $\gamma_{m_{k-1}} = \beta_{m_{k-1}} < \beta_{m_k} = \gamma_{m_k}$. So by our choice of $n_k$ we must have $\gamma_{m_{k-1}} \leq \gamma_{n_k}$.

So we have an ascending sequence

$$\gamma_0 < \gamma_{n_0} \leq \gamma_{n_1} < \gamma_{n_2} < \gamma_{m_2} \leq \cdots.$$

If we restrict our attention to the terms where $k$ is even (or odd, for that matter), we will have a strictly ascending sequence

$$\gamma_0 < \gamma_{n_0} < \gamma_{n_1} < \gamma_{n_2} < \gamma_{m_2} < \cdots.$$ For each $k < \omega$ let $e_k = m_k - n_k > 0$ and $u_{e_k} = p^{e_k} v_{m_k}$; so $u_{m_k} = p^{m_k v_{m_k}} = p^{e_k u_{e_k}}$ and $|u_{e_k}| = \gamma_{n_k}$.

We next want to slightly amend the technique of Lemma 2.6. Let $H = \bigoplus_{k < \omega} B_k$ be totally projective, where for each $k < \omega$ we have $p^{\hat{m}_k} B_k = \langle z_k \rangle$ is cyclic of order $p^{e_k + 1}$; so $p^{\hat{m}_k} B_k = \langle p^{e_k} z_k \rangle \cong \mathbb{Z}/p$. Again,
we can find a homomorphism \( \mu : G \to H \) such that for all \( k < \omega \), the element of \( \text{U}_{\rho k}(G) \) represented by \( u_{mk} \) is taken to the element of \( \text{U}_{\rho k}(H) \) represented by \( p^k z_k \).

For each \( k < \omega \) consider the image of \( \mu(u_k') \in H \). Since \( p^{\rho k} \bigoplus_{0 \leq j < k} B_j = 0 \), we can conclude that

\[
\mu(u_k') = rz_k + w_k \in rz_k + \bigoplus_{k < j} B_j
\]

where \( r \in \mathbb{Z} \) is relatively prime to \( p \). Replacing \( z_k \) by \( rz_k \) if need be, we may assume \( r = 1 \). And again, after possibly restricting to a subsequence and projecting onto the corresponding summand of \( H \), we may assume that each \( w_k = 0 \).

Since \( f_G(\gamma_k) \neq 0 \), we can find \( y_k \in S_{\rho k} \setminus S_{\rho k} + 1 \). So the assignment \( z_k \mapsto y_k \) extends to a homomorphism \( \rho_k : B_k \to G \) which sum up to a homomorphism \( \rho : H \to G \). Again, letting \( \phi = \rho \circ \mu \) we can conclude that \( \phi(p^{\rho k} v_{mk}) = \phi(u_k') = y_k \) for each \( k \).

Mimicking Lemma 3.1, we claim that the \( \phi(v_{mk}) + X \) are all distinct elements of \( G/X \), contrary to our assumption that \( X \) is fully inert. Suppose otherwise, and choose \( j < k \) with \( \phi(v_{mk}) - \phi(v_{mj}) \in X \). It would follow that

\[
y_k = y_k - 0 = y_k - p^{\rho k - n} y_j = \phi(p^{\rho k} v_{mk}) - \phi(p^{\rho k} v_{mj}) = p^{\rho k} (\phi(v_{mk}) - \phi(v_{mj})),
\]

is in

\[(p^{\rho k} X)[p] = S_{p n k} = S_{p m k} \subseteq S_{p m k + 1},\]

contrary to its construction. This contradiction completes the proof.

From here on we will assume \( N \) is chosen large enough so that the conclusions of Lemmas 3.1 AND 3.3 both hold.

This leads to a key step:

**Proposition 3.4.** For every \( n \geq N \) we have \( (p^n X)[p] = S_{\gamma n} \).

**Proof.** We divide this argument into two cases:

1. \( n \in \mathcal{F} \): We know \( \gamma_n = \beta_n \), so that \( (p^n X)[p] = S_{\beta n} = S_{\gamma n} \).
2. \( n \in \mathcal{E} \): We can find an \( m \in \mathcal{M} \) such that \( m \leq n < m \). Since \( f_G(\gamma) = 0 \) whenever \( \gamma_n \leq \gamma < \gamma_m = \beta_m \), we can conclude that \( (p^n X)[p] = S_{\beta n} = S_{\beta m} = S_{\gamma n} \), as required.

We now see that we have found the correct sequence of ordinals to generate an appropriate tail of a fully invariant subgroup.

**Proposition 3.5.** If \( \gamma = (\gamma_0, \gamma_{N+1}, \gamma_{N+2}, \ldots) \), then \( p^N X = G(\gamma) \).

**Proof.** Suppose first that \( x \in X \); we need to show that \( \gamma_n \leq |p^n x| \) for all \( n \geq N \). Assume this fails and let \( n \geq N \) be the greatest integer such that \( \gamma_n > |p^n x| \equiv \alpha \) (such an \( n \) exists since \( p^N X \) has finite order). It follows that \( |p^{n+1} x| \geq \gamma_{n+1} > \gamma_n > |p^n x| = \alpha \). Therefore, \( p^n x \) represents a non-zero element
of $U_\alpha(p^nX)$; i.e., Lemma 3.1 (1) holds for $\alpha$. However, since $\alpha < \gamma_n$, we have $(p^nX)[p] = S_{\gamma_n} \subseteq S_{\alpha + 1}$, and it follows that Lemma 3.1 (2) fails for $\alpha$. This contradiction completes the proof of this inclusion.

Conversely, since we now know $p^NX \subseteq G(\gamma)$ and for every $k < \alpha$, we have

$$(p^k(p^NX))[p] = (p^{N+k}X)[p] = S_{\gamma_{N+k}} = (p^kG(\gamma))[p],$$

by Lemma 2.3(c) we can conclude that $p^NX = G(\gamma)$, as required. □

The following observation is exactly what is needed to complete the proof.

**Proposition 3.6.** There is an $n \geq N$ such that $\gamma_n \geq \alpha_n > \alpha_{n-1}$.

Before attacking this last statement, we show how it completes the argument.

**Proof.** (of Theorem 1.2) Using the $n \geq N$ from Proposition 3.6, if

$$\bar{\gamma} = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots),$$

then $\bar{\gamma}$ is strictly increasing, and we will show that

$$X \sim T := G(\bar{\gamma}).$$

By Proposition 3.5, $p^NX = G(\bar{\gamma})$; so $p^nX = G(\gamma_n, \gamma_{n+1}, \gamma_{n+2}, \ldots) = p^nT$.

Using the notation of Theorem 2.11(4), we have $p^nX_n \sim p^nX = p^nT$. We claim that $X_n \subseteq T$.

Certainly, if $x \in X_n$, then for all $j < n$ we have $|p^jx| \geq \alpha_j = \tau_j$. And if $j \geq n$, then $p^jx \in p^nX = p^nT$, so we can again conclude that $|p^jx| \geq \tau_j$.

By Theorem 2.11(5) we have

$$X_n[p^n] \sim X[p^n] \sim A[p^n] = G(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n, \ldots) = T[p^n].$$

And since $X_n \subseteq T$, by Lemma 2.3(a) we can conclude $X \sim X_n \sim T$, as stated. □

So, the following will complete our argument:

**Proof.** (of 3.6) As usual, we argue indirectly and assume $\gamma_n < \alpha_n$ for all $n \geq N$ and derive a contradiction.

We break into two cases:

Case 1 - There is an $n > N$ such that $\alpha_n - 1 + 1 < \alpha_n = \alpha_{n+1}$. From Proposition 3.4 we have $S_{\gamma_n} = (p^nX)[p] = (p^{n+1}X)[\gamma_n][p]$. On the other hand, from Theorem 2.11(3) with $\xi = \gamma_n < \alpha_n = \alpha_{n+1}$ we have

$$(p^nX)[\gamma_n][p] \not\sim S_{\gamma_n},$$

which is clearly a contradiction.

Case 2 - For every $n \geq N$, $\alpha_n + 1 = \alpha_{n+1}$, i.e., $p^N\alpha$ has no gaps.

From Theorem 2.11(2) and Proposition 3.4 we can conclude $S_{\gamma_n} \sim (p^NX)[p] = S_{\gamma_N}$. Therefore, $S_{\gamma_n}/S_{\gamma_N}$ is finite, which implies that $\alpha_N = \gamma_N + s$ where $s$ is a positive integer.

It follows that $\bar{\gamma}$ has at most $s - 1$ gaps, so we can choose $n \geq N$ such that neither $p^N\alpha$ nor $p^n\tau = p^{n-N}\bar{\gamma}$ has any gaps. Suppose $t$ is positive and $\alpha_n = \gamma_n + t$.

It follows that

$$p^nX = G(\gamma_n, \gamma_n + 1, \gamma_n + 2, \ldots) = p^n\alphaG.$$

By Theorem 2.11(1), we know that

$$p^nX = (p^nX)(\alpha_n) = p^{nN}G/(p^{nN}G)(\gamma_n + t) = p^{nN}G/p^t(p^{nN}G)$$

and derive a contradiction.
For the countably totally projective group $G$, the following are equivalent:

Assume (a), i.e.,

A. Chekhlov, P. Danchev, and B. Goldsmith,

A. Chekhlov, P. Danchev, and P. Keef,

2.3

L. Fuchs,

D. Cutler,

B. Goldsmith and L. Salce,


Proposition 3.7. For the countably totally projective group $G$, the following are equivalent:

(a) For each ordinal $\alpha$, $f_G(\alpha)$ is either 0 or infinite, i.e., the Ulm function of $G$ never takes on finite, non-zero values.

(b) For each fully inert subgroup $X \subseteq G$ there is a unique fully invariant subgroup $T \sim X$.

(c) For each fully inert subgroup $X \subseteq G$, if $A$ is the subgroup from Theorem 2.11, $T$ is fully invariant and $T \sim X$, then $T = A$.

(d) For each ordinal $\alpha$, if $T$ is fully invariant and $T \sim p^\alpha G$, then $T = p^\alpha G$.

Proof. Assume (a), i.e., $f_G$ never takes on non-zero, finite values. Suppose $T := G(\tau) \sim X \sim T' := G(\tau')$; we want to show $T = T'$.

Suppose first that $T \subseteq T'$. If this inclusion is strict, then by Lemma 2.3(c) there must be an $n < \omega$ such that $(p^n T)[p] \neq (p^n T')[p]$. On one hand we have

$$S_{\tau n} = (p^n T)[p] \sim (p^n T')[p] = S_{\tau' n}.$$ 

But our hypotheses on $f_G$ implies that $S_{\tau n} / S_{\tau n}$ is 0 or infinite, giving our required contradiction.

If $T$ is not contained in $T'$, then let $T'' = T \cap T'$, so $T'' \sim X \cap X = X$. Clearly, $T''$ is also fully invariant and $T'' \subseteq T$. So, by the first part of the argument, $T = T'' = T'$.

We now assume (b) holds and verify (c), so assume $X$, $A$ and $T$ are as stated. For each $n < \omega$, we can conclude $T[p^n] \sim X[p^n] \sim A[p^n]$. So, by (b) applied to $X[p^n]$, we can conclude $T[p^n] = A[p^n]$.

Since this holds for all $n < \omega$, we have $T = A$.

Conversely, if (c) holds, then (b) is immediate: If $T \sim X \sim T'$, then $T = A = T'$. The fact that (b) implies (d) is also immediate.

Finally, if (a) fails; we will use it to produce a counterexample to (d), completing the proof.

Let $\beta$ be some ordinal such that $f_G(\beta) \neq 0$ is finite. If $\alpha = \beta + 1$, then it readily follows that $T := G(\beta, \alpha + 1, \alpha + 2, \alpha + 3, \ldots) \sim p^\alpha G$ but $T \neq p^\alpha G$, as required.

References

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