A LOCALLY F-FINITE NOETHERIAN DOMAIN THAT IS NOT F-FINITENOT F-FINE

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Abstract. Using an old example of Nagata, we construct a Noetherian ring of prime characteristic \( p \), whose Frobenius morphism is locally finite, but not finite.

1. Introduction

Let \( p > 0 \) be a prime number. We assume that all rings have characteristic \( p \), that is they contain a field of characteristic \( p \). For such a ring \( A \), one can define the Frobenius morphism:

\[
F_A : A \rightarrow A, \quad F_A(a) = a^p, \quad a \in A.
\]

The Frobenius morphism is playing a major role in studying the properties of \( A \). In [6, Th. 2.1] Kunz proved that a Noetherian local ring of characteristic \( p \) is regular if and only if the Frobenius morphism of \( A \) is flat. This was the starting point in the study of singularities in characteristic \( p \). An important property of Frobenius is its finiteness. A ring \( A \) is called \( F \)-finite if the Frobenius morphism of \( A \) is a finite morphism, that is \( A \) is a finite \( A \)-module via \( F \). Rings that are \( F \)-finite have important properties. For example such rings are excellent, as it is proved by Kunz [7, Th. 2.5]. Conversely, a reduced Noetherian ring with \( F \)-finite total quotient ring is excellent if and only if it is \( F \)-finite [3, Cor. 2.6]. Recall that an excellent ring is a Noetherian ring \( A \) such that (see [8, Def. p. 260]):

i) the formal fibers of \( A \) are geometrically regular;

ii) any \( A \)-algebra of finite type has open regular locus;

iii) \( A \) is universally catenary.

Excellent rings were invented by Grothendieck, in order to avoid pathologies in the behaviour of Noetherian local rings.

In [2] Datta and Murayama asked the following:

Question 1. Let \( A \) be a Noetherian domain of prime characteristic \( p > 0 \). Suppose that for any prime ideal \( \mathfrak{p} \) of \( A \), the localization \( A_\mathfrak{p} \) is \( F \)-finite. Does it follow that \( A \) is \( F \)-finite?

We show that a certain specialization of Nagata’s example [9, §5], gives a negative answer to the above question for any prime \( p \).

All the rings will be commutative and unitary. Moreover, we fix a prime number \( p > 0 \).

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2. The example

Assume first that \( p \) is odd. Let \( K \) be an algebraically closed field of characteristic \( p \) and \( X \) an indeterminate. Consider the ring

\[
A := K[X, \frac{1}{\sqrt{(X+a)^3 + \sqrt{b^3}}}; \ a, b \in K, b \neq 0].
\]

For each square radical above, we choose one of its two values. Hence the denominator \( \sqrt{(X+a)^3 - \sqrt{b^3}} \) is not in our list.

**Remark 2.** i) Note that

\[
\sqrt{(X+a)^3} = \frac{1}{\sqrt{(X+a)^3 + \sqrt{b^3}}}((X+a)^3 - b^3) + \sqrt{b^3} \in A.
\]

ii) It follows from i) that \( A \) is a fraction ring of \( K[X, \sqrt{(X+a)^3}; \ a \in K] \), which in turn is an integral extension of \( K[X] \).

iii) By ii) \( A \) is at most one dimensional.

**Lemma 3.** The factor ring \( A/(X, \sqrt{X^3}) \) is isomorphic to \( K \), while the factor ring \( A/(X) \) is isomorphic to \( K[Y]/(Y^2) \), where \( Y \) is an indeterminate.

**Proof.** From Kneser’s Theorem [5, Th. 5.1] we obtain that

\[
\sqrt{X+c} \notin K(X, \sqrt{X+a}, \ a \in K, a \neq c)
\]

for every \( c \in K \) (this also follows by adapting the well-known argument used to prove that \( \sqrt{p_n} \notin Q(\sqrt{p_1}, \ldots, \sqrt{p_{n-1}}) \), when \( p_1, \ldots, p_n \) are distinct primes). Consequently, we get a ring isomorphism

\[
\frac{K[X, T_a, a \in K]}{(T_a - (X+a)^3, a \in K)} \simeq K[X, \sqrt{(X+a)^3}, a \in K]
\]

sending each indeterminate \( T_a \) into \( \sqrt{(X+a)^3} \), which extends to an isomorphism

\[
\frac{K[X, T_a, (T_a + \sqrt{b^3})^{-1}, a, b \in K, b \neq 0]}{(T_a^2 - (X+a)^3, a \in K)} \simeq A.
\]

It follows that

\[
A/(X) \simeq \frac{K[T_a, (T_a + \sqrt{b^3})^{-1}, a, b \in K, b \neq 0]}{(T_a^2 - a^3, a \in K)} \simeq \frac{K[T_a, (T_a + \sqrt{b^3})^{-1}, a, b \in K, b \neq 0]}{(T_a^2, T_a - \sqrt{a^3}, a \in K, a \neq 0)}
\]

\[
\simeq \frac{K[T_0, (\sqrt{a^3} + \sqrt{b^3})^{-1}, (T_0 + \sqrt{b^3})^{-1}, a, b \in K - \{0\}]}{(T_0^2)} \simeq \frac{K[T_0]}{(T_0^2)}
\]

so \( A/(X, \sqrt{X^3}) \) is isomorphic to \( K \). Note that \( \sqrt{a^3} + \sqrt{b^3} \) is nonzero for \( a, b \in K - \{0\} \), by our initial one-value-choice for \( \sqrt{b^3} \). Also note that \( T_0 + \sqrt{b^3} \) is a unit modulo \( T_0^2 \).

**Lemma 4.** The nonzero prime ideals of \( A \) are \((X + a, \sqrt{(X + a)^3})A \) with \( a \in K \). In particular, \( A \) is a Noetherian domain of dimension one.
Proof. By Lemma 3, \((X, \sqrt[3]{X^3})\) is the only prime ideal of \(A\) containing \(X\). By part ii) of Remark 2, every nonzero prime ideal of \(A\) contains some \(X + a\), so our assertion follows from Lemma 3, performing a translation in \(K\). The final assertion follows from Cohen’s Theorem [8, Th. 3.4].

Proposition 5. \(A_m\) is F-finite for each maximal ideal \(m\) of \(A\).

Proof. Performing a translation in \(K\), it suffices to work with \(m = (X, \sqrt[3]{X^3})\). As noted in Remark 2, ii), \(A\) is a fraction ring of \(B := K[X, \sqrt[3]{(X + a)^3}]\), where \(k\) is the integer \(3(p - 1)/2\). Hence

\[B \subseteq (A_m)^p[X, \sqrt[3]{X^3}] \subseteq A_m\]

Let \(n = mA_m \cap B\). Since \(A\) is a fraction ring of \(B\) and \(1/b^p \in (A_m)^p\) for each \(b \in B - n\), we get

\[A_m = B_n \subseteq (A_m)^p[X, \sqrt[3]{X^3}],\]

therefore \(A_m = (A_m)^p[X, \sqrt[3]{X^3}]\).

Lemma 6. The regular locus of \(A\) is \(\{(0)\}\).

Proof. Suppose that \(A_{m_0}\) is regular for some maximal ideal \(m_0\) of \(A\). By our translation argument, it follows that \(A_m\) is regular for each maximal ideal \(m\) of \(A\). Then \(A\) is normal, so \(\sqrt{X} \in A\). Hence \(X\) divides \(\sqrt[3]{X^3}\) in \(A\), which contradicts Lemma 3.

Proposition 7. \(A\) is not F-finite.

Proof. If \(A\) is F-finite, by [6, Cor. 2.3] the regular locus of \(A\) is open, but this contradicts Lemma 6.

From Propositions 5 and 7 we get:

Theorem 8. \(A\) is a one-dimensional Noetherian domain that is not F-finite, such that \(A_p\) is F-finite for any prime ideal \(p\) of \(A\).

Remark 9. The ring \(A\) constructed above is not excellent, as its regular locus is not open. On the other hand, for any prime ideal \(p\) the localization \(A_p\) is F-finite, hence excellent. A ring that is locally excellent but not excellent was constructed in any characteristic by Hochster [4, Example 1]. Indeed, Hochster’s example has non-open regular locus, while its localizations are essentially of finite type over a field. In characteristic zero, Abhyankar and Heizner [1, Th. 6.3 and Th. 7.3] constructed two-dimensional Noetherian normal domains which are locally excellent but not globally excellent. One should remark that in the definition of excellent rings conditions i) and iii) are of local nature, while condition ii) is not. Thus a Noetherian ring \(A\) that is locally F-finite satisfies conditions i) and iii) globally. Moreover, if \(A\) is reduced then \(A\) has F-finite total quotient ring, hence by the result cited above [3, Cor. 2.6] \(A\) is F-finite if and only if \(A\) is excellent, that is if and only if \(A\) satisfies condition ii).
Remark 10. In characteristic two, a similar example can be constructed as follows. Let $K$ be an algebraically closed field of characteristic two and $X$ an indeterminate. Consider the ring
\[ B := K[X, \sqrt[3]{(X + a)^4}, \frac{1}{\sqrt[3]{(X + a)^8 + \sqrt[4]{(X + a)^4} \sqrt[8]{b^4 + \sqrt[8]{b^8}}}}; \ a, b \in K, b \neq 0] \]
For each cube radical above, we choose one of its three values. By Kneser’s Theorem [5, Th. 5.1], it follows that
\[ [K(X, \sqrt[3]{X + a_1}, \ldots, \sqrt[3]{X + a_n}) : K(X)] = 3^n \]
whenever $a_1, \ldots, a_n$ are distinct elements of $K$, so we get a ring isomorphism
\[ \frac{K[X, T_a, (T_a^3 + T_a \sqrt[8]{b^4 + \sqrt[8]{b^8}})^{-1}; \ a, b \in K, b \neq 0]}{(T_a^3 - (X + a)^4, a \in K)} \sim B \]
sending each indeterminate $T_a$ into $\sqrt[3]{(X + a)^4}$. It easily follows that $B/(X) \simeq K[T_0]/(T_0^3)$ and $B/(X, \sqrt[3]{X^4}) \simeq K$. Now all arguments used above can be adapted to show that $B$ is locally $F$-finite but not $F$-finite.

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References